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TENSORS OF FINITE ROTATIONS AND SMALL STRAINS ON THE MIDDLE SURFACE OF A SHELL

Imre Kozák
Department of Mechanics, University of Miskolc
3515 Miskolc-Egyetemváros, Hungary
mechkoz@uni-miskolc.hu

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Dedicated to István Páczelt on the occasion of his 65th birthday

Abstract. On the middle surface of the shell, the displacements as well as the rotations of the base vectors are finite, the strains are, however, considered to be infinitesimally small. The rotation of the base vectors is described by three rotation tensors that define three special, geometrically well identified rotations. This paper derives the three-dimensional Green-Lagrange strain tensor and the symmetric right Jaumann strain tensor on the middle surface.

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1. **Introduction**

Shell theories are two-dimensional theories that have an approximate nature with respect to the three-dimensional theories of solid bodies. This approximate nature arises from the special geometry of shells and from the description of the deformation and boundary conditions of the shell-like body, which are usually expressed through different hypotheses and neglections.

Topics related to the theories of shells seem to be always topical, especially from the point of view of the numerical analysis of shell problems. This is well demonstrated by the following two citations: "*Shell structures may be called the prima donnas of structures.*" by Chapelle and Bathe (1998, [1]), and "*The modelling of shell structures represents a challenging task since the early developments of the finite element method.*" by Valente et al. (2003, [2]).

The topic of this paper is related to the kinematics of non-linear shell theories in which the middle surface of the shell has a distinguished role. This is expressed through the description of both the deformation of the middle surface and the deformations along the normal to the middle surface.

The main goal of the present paper is to derive the Green-Lagrange and the Jaumann strain tensors on the middle surface of the shell. The derivation is based on the non-linear theory of three-dimensional deformation of continuum mechanics and on the introduction of kinematical hypotheses.

In the investigations it is assumed that the middle surface of the shell is sufficiently smooth, quantities defined on it are continuous and continuously differentiable with respect to the surface parameters as many times as required.

The independent variables on the middle surface of the shell in this paper are the displacements, the rotation tensors (including the drilling rotation, i.e. the rotation about the normal to the middle surface), and the transverse normal strain, which represent seven independent parameters in all.

The primary goal is to derive the three-dimensional deformation gradient tensor and the three-dimensional Green-Lagrange strain tensor *on the middle surface of the shell* with the assumption that the displacements as well as the rotations of the base vectors are finite, the strains at the points of the middle surface and across the thickness of the shell are, however, infinitesimally small.

The finite rotations of the base vectors on the middle surface are described by three rotation tensors. These tensors define three *special rotations* which are geometrically well identified. Two rotation tensors describe the finite rotation of the base vectors (including the drilling rotation) in such a case when the middle surface normal to, and the tangential base vectors of, the reference middle surface are mapped into the middle surface normal to, and the tangential base vectors of the deformed middle surface. The third rotation tensor describes an infinitesimal rotation of the already deformed base vectors, ensuring that the transverse shear strains be, according to our primary goals, infinitesimally small.

Considering the above assumptions, the secondary goal is to derive the three-dimensional, symmetric right Jaumann strain tensor *on the middle surface of the shell*, utilizing the polar decomposition of the three-dimensional deformation gradient tensor.

The third goal is to present, in an exemplary manner, the derivation of the three-dimensional deformation gradient tensor and the Green-Lagrange strain tensor *at an arbitrary point of the shell*, making use of the previous results and from the point of view of shell theories based on the Reissner-Mindlin hypothesis.

In the majority of shell theories applying finite rotations to describe the deformation of the shell, the rotation tensors are defined, just like in this paper, on the middle surface. In those theories the middle surface strains can be finite or infinitesimally small. The transverse shear strains are usually taken into account, which is not, however, the case for transverse normal strains. In what follows, a brief review of shell theories without completeness is given from the point of view of finite rotations.

The concept of finite rotation tensor to develop a nonlinear shell theory has been introduced by Simmonds and Danielson [3],[4]. Wriggers and Gruttmann [5] discuss a finite element model for shells subjected to finite rotations. A detailed analysis on the role of drilling rotations in non-linear shell theories has been given in the significant papers by Ibrahimbegovich [6], Ibrahimbegovich and Frey [7],[8]. Pietraszkiewicz [9],[10] has developed a general, geometrically non-linear shell theory using the drilling rotations.

Wisniewski [11] investigated the rotation tensor with respect to the derivation of the Green-Lagrange strain tensor, obtained directly from the deformation gradient tensor, and the Jaumann strain tensor, obtained through the polar decomposition of the deformation gradient tensor.

Brank et al. [12],[13], Ibrahimbegovich et al. [14], Brank and Ibrahimbegovich [15], Ibrahimbegovich et al. [16], Lee and Lee [17] derived geometrically exact shell theories without using the drilling rotations.

Atluri et al. [18] pointed out that the use of the drilling rotation as an independent variable in shell structures with faceted joints has special significance.

Campello et al. [19] applied a geometrically exact six-parameter non-linear shell theory to develop a shell finite element.

Bertóti [20] developed a three-dimensional non-linear shell theory using the equilibrated stress field and the rotation field.

Aside from the above aspects, the possibilities and requirements of the application of the finite element method are not investigated in this paper. In this respect we refer to the papers by Parisch [21], Basar and Ding [22], Bischoff and Ramm [23], Sansour and Kollmann [24], as well as the papers by Bucalem and Bathe [25] and Bathe at al. [26].

Section 2 introduces the notation and gives a summary of the basic relations applied in the paper, among them the representation of the rotation tensor using the Rodrigues formula. Section 3 introduces important assumptions with respect to the rotation of the base vectors and the measures of strains on the middle surface, then defines three rotation tensors and gives their geometric interpretations. Making use of the above results, Section 4 derives the three-dimensional deformation gradient tensor, the Green-Lagrange strain tensor and the symmetric right Jaumann strain tensor obtained through the polar decomposition of the deformation gradient tensor. Section 5 presents, as an example, the derivation of the three-dimensional deformation gradient tensor and the Green-Lagrange strain tensor at arbitrary points of the shell, employing the Reissner-Mindlin hypothesis.

2. **Notations. Fundamental relations**

2.1. Both invariant and indicial notations of tensor analysis are used. Scalar variables are denoted by italic normal letters. When *invariant notation* is applied, vector variables are denoted by upright boldface letters, tensor variables by italic boldface letters. The dyadic (tensorial) product has no special sign (e.g. $\mathbf{a}^1\,\mathbf{a}_1$), the scalar product is denoted by a dot (e.g. $\boldsymbol{R}\cdot\mathbf{a}_1$), the vector product is denoted by a cross sign (e.g. $\mathbf{a}_1\times\mathbf{a}_2$). When *indicial notation* is applied, the range of Latin indices is

$1, 2, 3$, the range of Greek indices is $1, 2$. The usual summation convention is applied over the repeated indices. A comma followed by an index in the right subscript denotes partial differentiation, whereas a semicolon indicates covariant differentiation. The so called *marking indices* are underlined and they do not have the range $1, 2, 3$ (e.g. $e_{\underline{s}}^{m}$). The unit tensor is denoted by $\boldsymbol{I}$, δ_l^k is the Kronecker delta, ϵ_{klm} is the covariant permutation tensor and e_{klm} is the covariant permutation symbol. The transpose of a tensor is denoted by a "T" in the right superscript (e.g. $\boldsymbol{R}^{\mathrm{T}}$).

2.2. Let the shell, as a three-dimensional solid body, be denoted by (B) in the *reference configuration*, and by $(\overline{B})$ in the *deformed configuration*. The reference configuration is assumed to be stress- and deformation-free. The middle surface of the shell and its surface element in (B) are denoted by (S_{o}) and $(\mathrm{d}S_{\mathrm{o}})$, whereas in $(\overline{B})$ they are denoted by $(\overline{S}_{\mathrm{o}})$ and $(\mathrm{d}\overline{S}_{\mathrm{o}})$.

A convected (x^1, x^2, x^3) *coordinate system* is employed, where x^1, x^2 are the surface coordinates on the middle surface (S_{o}). The coordinate line x^3 is perpendicular to (S_{o}) in (B) and $x^3 = 0$ on both (S_{o}) and $(\overline{S}_{\mathrm{o}})$. In $(\overline{B})$ x^3 is not necessarily straight and perpendicular to $(\overline{S}_{\mathrm{o}})$.

Total Lagrangian description is applied throughout this paper.

Quantities and geometrical forms in the deformed configuration are distinguished by a bar, in the reference configuration they have no special sign.

On (S_{o}) and $(\overline{S}_{\mathrm{o}})$ (at $x^3 = 0$), the value of a quantity defined at an arbitrary point of coordinate x^3 is denoted by a sign "o" in the right subscript, i.e.,

$$\mathbf{u} = \mathbf{u}(x^1, x^2, x^3), \quad \text{but} \quad \mathbf{u}_{\mathrm{o}} = \mathbf{u}\,(x^1, x^2, x^3 = 0).$$

Quantities defined only at $x^3 = 0$ are written without the sign "o", e.g.,

$$\mathbf{a}_\alpha = \mathbf{a}_\alpha(x^1, x^2), \quad \text{or} \quad \overline{\mathbf{n}} = \overline{\mathbf{n}}\,(x^1, x^2).$$

2.3. The middle surface of the shell together with the basic geometrical quantities are shown in Figure 1 (see also Subsection **5.1** and Figure 4).

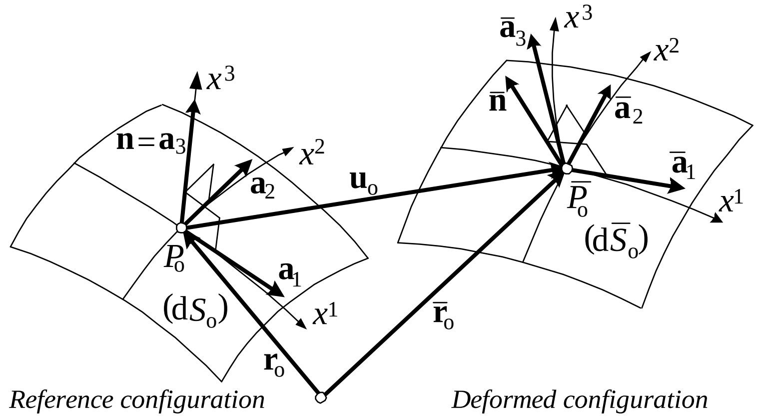

Figure 1.

At point P_o of the *middle surface* (S_o) in the *reference configuration*, the unit normal vector is denoted by $\mathbf{n}$, $|\mathbf{n}| = 1$, the position vector is $\mathbf{r}_\mathrm{o}(x^1, x^2)$, the covariant base vectors are $\mathbf{a}_\alpha = \mathbf{r}_{\mathrm{o},\alpha}$ and $\mathbf{a}_3 = \mathbf{n}$, the metric tensor is $a_{kl} = \mathbf{a}_k \cdot \mathbf{a}_l$ $(a_{\alpha 3} = 0,\ a_{33} = 1)$ and its determinant is denoted by a, the curvature tensor of the middle surface is $b_{\kappa\lambda} = -\ \mathbf{a}_\kappa \cdot \mathbf{a}_{3,\alpha}$, the contravariant base vectors are $\mathbf{a}^m$, and the inverse of the metric tensor is $a^{mn} = \mathbf{a}^m \cdot \mathbf{a}^n$ $(a^{\mu 3} = 0,\ a^{33} = 1)$. The following relations hold:

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} = \mathbf{n}, \quad |\mathbf{a}_3| = 1, \quad a = \det a_{\alpha\beta} = a_{11}a_{22} - a_{12}a_{21}, \tag{2.1}$$

$$|\mathbf{a}_1 \times \mathbf{a}_2|^2 = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = a, \quad \mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta, \quad \mathbf{a}^3 = \mathbf{a}_3. \tag{2.2}$$

At point $(\overline{P}_\mathrm{o})$ of the *middle surface* $(\overline{S}_\mathrm{o})$ in the *deformed configuration*, the unit normal vector is $\overline{\mathbf{n}}$, $|\overline{\mathbf{n}}| = 1$, the position vector is $\overline{\mathbf{r}}_\mathrm{o}(x^1, x^2)$, the covariant base vectors are $\overline{\mathbf{a}}_\alpha = \overline{\mathbf{r}}_{\mathrm{o},\alpha}$ and $\overline{\mathbf{a}}_3$ (it is assumed that $\overline{\mathbf{a}}_3 \neq \overline{\mathbf{n}}$), the surface part of the metric tensor is $\overline{a}_{\alpha\beta} = \overline{\mathbf{a}}_\alpha \cdot \overline{\mathbf{a}}_\beta$ and its determinant is denoted by $\overline{a}$, the curvature tensor of the middle surface is $\overline{b}_{\kappa\lambda} = -\ \overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{n}}_{,\lambda}$. The following relations hold:

$$\overline{\mathbf{n}} = \frac{\overline{\mathbf{a}}_1 \times \overline{\mathbf{a}}_2}{|\overline{\mathbf{a}}_1 \times \overline{\mathbf{a}}_2|}, \quad \overline{a} = \det \overline{a}_{\alpha\beta} = \overline{a}_{11}\overline{a}_{22} - \overline{a}_{12}\overline{a}_{21}, \tag{2.3}$$

$$|\overline{\mathbf{a}}_1 \times \overline{\mathbf{a}}_2|^2 = (\overline{\mathbf{a}}_1 \times \overline{\mathbf{a}}_2) \cdot (\overline{\mathbf{a}}_1 \times \overline{\mathbf{a}}_2) = [(\overline{\mathbf{a}}_1 \times \overline{\mathbf{a}}_2) \times \overline{\mathbf{a}}_1] \times \overline{\mathbf{a}}_2 = \overline{a}. \tag{2.4}$$

2.4. Let $\mathbf{u}_\mathrm{o}(x^1, x^2) = \underset{\mathrm{o}}{u}^k\, \mathbf{a}_k$ be the displacement field on the middle surface (S_o). Then

$$\overline{\mathbf{r}}_\mathrm{o} = \mathbf{r}_\mathrm{o} + \mathbf{u}_\mathrm{o} \quad \text{and} \quad \overline{\mathbf{a}}_\alpha = \overline{\mathbf{r}}_{\mathrm{o},\alpha} = \mathbf{a}_\alpha + \mathbf{u}_{\mathrm{o},\alpha} = \left(\delta^m_\alpha + u^m_{\mathrm{o};\alpha}\right)\mathbf{a}_m \tag{2.5}$$

on $(\overline{S}_\mathrm{o})$. The deformation gradient tensor is

$$\boldsymbol{F}_\mathrm{o} = \overline{\mathbf{a}}_k\, \mathbf{a}^k = \overline{\mathbf{a}}_\kappa\, \mathbf{a}^\kappa + \overline{\mathbf{a}}_3\, \mathbf{a}^3, \quad \boldsymbol{F}_\mathrm{o} \cdot \mathbf{a}_k = \overline{\mathbf{a}}_k\,, \tag{2.6}$$

i.e.

$$\mathrm{d}\overline{\mathbf{r}}_\mathrm{o} = \boldsymbol{F}_\mathrm{o} \cdot \mathrm{d}\mathbf{r}_\mathrm{o} = \boldsymbol{F}_\mathrm{o} \cdot \mathbf{a}_k\, \mathrm{d}x^k = \overline{\mathbf{a}}_k\, \mathrm{d}x^k\,. \tag{2.7}$$

The Green-Lagrange strain tensor assumes the form

$$\begin{aligned}\boldsymbol{E}_\mathrm{o} &= E_{\mathrm{o}kl}\, \mathbf{a}^k\mathbf{a}^l = \frac{1}{2}\left(\boldsymbol{F}^\mathrm{T}_\mathrm{o} \cdot \boldsymbol{F}_\mathrm{o} - \boldsymbol{I}\right) = \frac{1}{2}\left[(\overline{\mathbf{a}}_k \cdot \overline{\mathbf{a}}_l) - (\mathbf{a}_k \cdot \mathbf{a}_l)\right]\mathbf{a}^k\mathbf{a}^l = \\ &= \frac{1}{2}\left[(\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_\lambda - a_{\kappa\lambda})\,\mathbf{a}^\kappa\mathbf{a}^\lambda + (\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_3)\,\mathbf{a}^\kappa\mathbf{a}^3 + (\overline{\mathbf{a}}_3 \cdot \overline{\mathbf{a}}_\lambda)\,\mathbf{a}^3\mathbf{a}^\lambda + (\overline{\mathbf{a}}_3 \cdot \overline{\mathbf{a}}_3 - 1)\,\mathbf{a}^3\mathbf{a}^3\right].\end{aligned} \tag{2.8}$$

According to (2.3)-(2.5), the unit normal vector to the middle surface $(\overline{S}_\mathrm{o})$ is

$$\overline{\mathbf{n}} = \overline{n}_m\, \mathbf{a}^m = \frac{1}{\sqrt{\overline{a}}}\,\overline{\mathbf{a}}_1 \times \overline{\mathbf{a}}_2 = \frac{1}{\sqrt{\overline{a}}}\,(\mathbf{a}_1 + \mathbf{u}_{\mathrm{o},1}\,) \times (\mathbf{a}_1 + \mathbf{u}_{\mathrm{o},2}\,)\,. \tag{2.9}$$

Components of the normal vector $\overline{\mathbf{n}}$ can be expressed by the components of the gradient of displacement vector, $\mathbf{u}_{\mathrm{o},\alpha}$, as:

$$\overline{n}_1 = \overline{\mathbf{n}} \cdot \mathbf{a}_1 = \frac{1}{\sqrt{\overline{a}}} \left[(\mathbf{a}_1 + \mathbf{u}_{\mathrm{o},1}) \times (\mathbf{a}_2 + \mathbf{u}_{\mathrm{o},2}) \right] \cdot \mathbf{a}_1 = -\frac{\sqrt{a}}{\sqrt{\overline{a}}} \left[\left(1 + u^2_{\mathrm{o};2}\right) u^3_{\mathrm{o};1} - u^2_{\mathrm{o};1} u^3_{\mathrm{o};2} \right], \tag{2.10}$$

$$\overline{n}_2 = \overline{\mathbf{n}} \cdot \mathbf{a}_2 = \frac{1}{\sqrt{\overline{a}}} \left[(\mathbf{a}_1 + \mathbf{u}_{\mathrm{o},1}) \times (\mathbf{a}_2 + \mathbf{u}_{\mathrm{o},2}) \right] \cdot \mathbf{a}_2 = -\frac{\sqrt{a}}{\sqrt{\overline{a}}} \left[\left(1 + u^1_{\mathrm{o};1}\right) u^3_{\mathrm{o};2} - u^1_{\mathrm{o};2} u^3_{\mathrm{o};1} \right], \tag{2.11}$$

$$\overline{n}_3 = \overline{\mathbf{n}} \cdot \mathbf{a}_3 = \frac{1}{\sqrt{\overline{a}}} \left[(\mathbf{a}_1 + \mathbf{u}_{\mathrm{o},1}) \times (\mathbf{a}_2 + \mathbf{u}_{\mathrm{o},2}) \right] \cdot \mathbf{a}_3 =$$

$$= \frac{\sqrt{a}}{\sqrt{\overline{a}}} \left(1 + u^1_{\mathrm{o};1} + u^2_{\mathrm{o};2} + u^1_{\mathrm{o};1} u^2_{\mathrm{o};2} - u^1_{\mathrm{o};2} u^2_{\mathrm{o};1}\right). \tag{2.12}$$

2.5. Any rotation tensor $\boldsymbol{R}$ ($\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$; $\det|\boldsymbol{R}| = 1$), defined at point P_{o} of the middle surface (S_{o}) can be given by the Rodrigues-formula:

$$\boldsymbol{R} = R^k{}_l\, \mathbf{a}_k\, \mathbf{a}^l = \cos\vartheta\, \boldsymbol{I} + (1 - \cos\vartheta)\, \mathbf{e}\,\mathbf{e} + \sin\vartheta\, \mathbf{e} \times \boldsymbol{I}, \tag{2.13}$$

$$R^k{}_l = \cos\vartheta\, \delta^k_l + (1 - \cos\vartheta) e^k e_l + \sin\vartheta\, a^{ks} \varepsilon_{sml} e^m, \tag{2.14}$$

where $\mathbf{e} = \mathbf{e}(x^1, x^2) = e^m\, \mathbf{a}_m$, $|\mathbf{e}| = 1$ is the unit vector of the axis of rotation and $-\pi \leq \vartheta(x^1, x^2) \leq \pi$ is the angle of rotation. The rotation tensor is a proper orthogonal tensor. There exist other representations of the rotation tensor $\boldsymbol{R}$ in the specialist literature.

The rotation tensor $\boldsymbol{R}$ rotates the arbitrary vector $\mathbf{c}$ into the vector

$$\boldsymbol{R} \cdot \mathbf{c} = \cos\vartheta\, \mathbf{c} + (1 - \cos\vartheta)\, (\mathbf{e} \cdot \mathbf{c})\, \mathbf{e} + \sin\vartheta\, \mathbf{e} \times \mathbf{c}, \tag{2.15}$$

while its transpose, $\boldsymbol{R}^{\mathrm{T}}$, rotates $\mathbf{c}$ into the vector

$$\boldsymbol{R}^{\mathrm{T}} \cdot \mathbf{c} = \mathbf{c} \cdot \boldsymbol{R} = \cos\vartheta\, \mathbf{c} + (1 - \cos\vartheta)\, (\mathbf{e} \cdot \mathbf{c})\, \mathbf{e} - \sin\vartheta\, \mathbf{e} \times \mathbf{c}. \tag{2.16}$$

3. Finite rotations and small strains on the middle surface (S_{o})

3.1. Let us introduce, in advance, the rotation tensors $\boldsymbol{R}^*(x^1, x^2)$ and $\overline{\boldsymbol{R}}_s(x^1, x^2)$ as well as the following fundamental assumptions:

1. let $\boldsymbol{R}^*$ rotate the base vectors $\mathbf{a}_k$ into vectors $\overline{\mathbf{k}}^*_k$ by a *finite rotation*:
$$\overline{\mathbf{k}}^*_k = \boldsymbol{R}^* \cdot \mathbf{a}_k, \quad \text{and let} \quad \overline{\mathbf{k}}^*_3 = \boldsymbol{R}^* \cdot \mathbf{a}_3 = \overline{\mathbf{n}}, \tag{3.1}$$
2. let $\overline{\boldsymbol{R}}_s$ rotate vectors $\overline{\mathbf{k}}^*_k$ into vectors $\overline{\mathbf{k}}_k$ by an *infinitesimal rotation*:
$$\overline{\mathbf{k}}_k = \overline{\boldsymbol{R}}_s \cdot \overline{\mathbf{k}}^*_k = \overline{\boldsymbol{R}}_s \cdot \boldsymbol{R}^* \cdot \mathbf{a}_k, \tag{3.2}$$
3. let the rotated base vectors $\overline{\mathbf{k}}_k$ and the *infinitesimal strain vectors* $\overline{\boldsymbol{\alpha}}_k(x^1, x^2)$ give the base vectors $\overline{\mathbf{a}}_k$:
$$\overline{\mathbf{a}}_k = \overline{\mathbf{k}}_k + \overline{\boldsymbol{\alpha}}_k, \tag{3.3}$$

4. let $\overline{\boldsymbol{\alpha}}_3 = \varepsilon_{o3}\,\overline{\mathbf{k}}_3$, i.e., let the base vector $\overline{\mathbf{a}}_3$ be given by

$$\overline{\mathbf{a}}_3 = \overline{\mathbf{k}}_3 + \varepsilon_{o3}\overline{\mathbf{k}}_3\,, \tag{3.4}$$

where ε_{o3} is the *infinitesimal* stretch in the normal direction to the middle surface,

5. let the rotation tensor $\overline{\boldsymbol{R}}_s$, defining an infinitesimal rotation, be given by

$$\overline{\boldsymbol{R}}_s = \boldsymbol{I} + \vartheta_s\,\overline{\mathbf{e}}_s \times \boldsymbol{I}, \tag{3.5}$$

where $|\vartheta_s| \ll 1$ holds for the angle of rotation and the unit vector of the axis of rotation, $\overline{\mathbf{e}}_s$, is perpendicular to $\overline{\mathbf{n}}$:

$$\overline{\mathbf{e}}_s = e^{\mu}_{\underline{s}}\,\overline{\mathbf{k}}^*_{\mu} = \boldsymbol{R}^* \cdot \left(e^{\mu}_{\underline{s}}\,\mathbf{a}_{\mu}\right) = \boldsymbol{R}^* \cdot \mathbf{e}_s; \qquad \mathbf{e}_s = e^{\mu}_{\underline{s}}\,\mathbf{a}_{\mu}\,. \tag{3.6}$$

In the following Subsections **3.2–3.6**, we investigate first the rotation tensor $\boldsymbol{R}^*$ that fulfills condition $(3.1)_2$, then, in view of the results of this investigation, $\boldsymbol{R}^*$ is obtained by the (scalar) product of two rotation tensors ($\boldsymbol{R}^* = \boldsymbol{R}_2 \cdot \boldsymbol{R}_1$), and the geometrical interpretation of these two rotation tensors is given. Next, in Subsection **3.7**, the description of the rotation tensor $\overline{\boldsymbol{R}}_s$ defined in (3.5) is detailed.

3.2. For a given $\overline{\mathbf{n}}$, we are looking for a rotation tensor $\boldsymbol{R}^*$ (the angle of rotation ϑ^* and the axis of rotation $\mathbf{e}^*$) that satisfies assumption (1) with equation $(3.1)_2$:

$$\overline{\mathbf{k}}^*_3 = \boldsymbol{R}^* \cdot \mathbf{a}_3 = \cos\vartheta^*\mathbf{a}_3 + (1-\cos\vartheta^*)e^*_3\mathbf{e}^* + \sin\vartheta^*\mathbf{e}^* \times \mathbf{a}_3 = \overline{\mathbf{n}} = \overline{n}_p\mathbf{a}^p.$$

The solution satisfies the following scalar equations:

$$\mathbf{a}_p \cdot \boldsymbol{R}^* \cdot \mathbf{a}_3 = \cos\vartheta^* a_{3p} + (1-\cos\vartheta^*)\,e^*_3 e^*_p + \sqrt{a}\sin\vartheta^* e_{ps3}e^{*s} = \overline{n}_p,$$

which, after inserting $e^*_p = a_{pq}\,e^{*q}$, can be written as

$$(1-\cos\vartheta^*)\,e^*_3\left(a_{11}e^{*1} + a_{12}e^{*2}\right) + \sqrt{a}\sin\vartheta^* e^{*2} = \overline{n}_1, \tag{3.7}$$

$$(1-\cos\vartheta^*)\,e^*_3\left(a_{21}e^{*1} + a_{22}e^{*2}\right) - \sqrt{a}\sin\vartheta^* e^{*1} = \overline{n}_2, \tag{3.8}$$

$$\cos\vartheta^* + (1-\cos\vartheta^*)\left(e^*_3\right)^2 = \overline{n}_3. \tag{3.9}$$

With components $\overline{n}_p$ given and, in addition, with an e^*_3 chosen, equation system (3.7)-(3.9) has a unique solution for ϑ^*, e^{*1} and e^{*2} (the trivial solution of $\vartheta^* = 0$ is not considered). Indeed, from (3.9) it immediately follows that

$$\cos\vartheta^* = \frac{\overline{n}_3 - \left(e^*_3\right)^2}{1-\left(e^*_3\right)^2}\,, \tag{3.10}$$

whereas e^{*1} and e^{*2} can be obtained from the transformed equations (3.7)-(3.8), using the Cramer-rule:

$$\begin{array}{rcl} (1-\cos\vartheta^*)\,e^*_3 a_{11}e^{*1} + \left[(1-\cos\vartheta^*)\,e^*_3 a_{12} + \sqrt{a}\sin\vartheta^*\right]e^{*2} & = & \overline{n}_1 \\ \left[(1-\cos\vartheta^*)\,e^*_3 a_{21} - \sqrt{a}\sin\vartheta^*\right]e^{*1} + (1-\cos\vartheta^*)\,e^*_3 a_{22}e^{*2} & = & \overline{n}_2 \end{array}.$$

The determinant of this equation system is $d = a\left[(1-\cos\vartheta^*)^2 (e_3^*)^2 + \sin^2\vartheta^*\right]$, and its solution reads:

$$e^{*1} = \frac{(1-\cos\vartheta^*)\, e_3^* \overline{n}^1 - \frac{1}{\sqrt{a}}\sin\vartheta^* \overline{n}_2}{(1-\cos\vartheta^*)^2 (e_3^*)^2 + \sin^2\vartheta^*}, \qquad e^{*2} = \frac{(1-\cos\vartheta^*)\, e_3^* \overline{n}^2 + \frac{1}{\sqrt{a}}\sin\vartheta^* \overline{n}_1}{(1-\cos\vartheta^*)^2 (e_3^*)^2 + \sin^2\vartheta^*}. \tag{3.11}$$

Selection of the possible values for e_3^* is restricted by the fact that $-1 \le \cos\vartheta^* \le 1$, i.e. after taking into account (3.10),

$$-1 \le \frac{\overline{n}_3 - (e_3^*)^2}{1-(e_3^*)^2} \le 1.$$

This means that, beside the evidently satisfied conditions $|\overline{n}_3| \le 1$, condition

$$e_3^* \le \sqrt{\frac{1+\overline{n}_3}{2}} \tag{3.12}$$

should also be satisfied.

Taking into account the above constraint, for a given normal vector $\overline{\mathbf{n}}$ of the middle surface $(\overline{S}_\mathrm{o})$ and for a given component e_3^* of the rotation axis $\mathbf{e}^*$, the rotation angle ϑ^* can be determined using (3.10) and the other two components $e^{*\mu}$ of $\mathbf{e}^*$ can then be obtained from (3.11).

3.3. The results of Subsection **3.2** can now be summarized as follows: For a given normal vector $\overline{\mathbf{n}} = \overline{n}^p\, \mathbf{a}_p = \overline{n}_q\, \mathbf{a}^q$, $|\overline{\mathbf{n}}| = 1$ of the middle surface $(\overline{S}_\mathrm{o})$, there exists an infinite number of solutions to equation $\boldsymbol{R}^* \cdot \mathbf{a}_3 = \bar{\mathbf{k}}_3^* = \overline{\mathbf{n}}$ for the rotation tensor $\boldsymbol{R}^*$. These solutions differ from each other in one component of the unit vector $\mathbf{e}^*$ of the rotation axis, namely the component e_3^*. After selecting the component $e_3^* = e^{*3}$ of vector $\mathbf{e}^*$, with satisfied constraint (3.12), equations (3.10) and (3.11) give a unique solution for ϑ^* and for the other two components of $\mathbf{e}^*$, i.e. for the rotation tensor $\boldsymbol{R}^*$. It can be pointed out that this solution satisfies the requirement $|\mathbf{e}^*| = 1$.

It can also be seen that vectors $\boldsymbol{R}^* \cdot \mathbf{a}_\alpha = \bar{\mathbf{k}}_\alpha^*$, obtained with different rotation tensors $\boldsymbol{R}^*$, lie in the tangent plane, perpendicular to the normal $\bar{\mathbf{k}}_3^* = \overline{\mathbf{n}}$ of the middle surface $(\overline{S}_\mathrm{o})$, in such a way that the different vectors $\bar{\mathbf{k}}_\alpha^*$ belonging to e_3^* (to $\mathbf{e}^*$) can be rotated into each other about vector $\bar{\mathbf{k}}_3^* = \overline{\mathbf{n}}$. This follows from the evidently satisfied equation $\bar{\mathbf{k}}_\alpha^* \cdot \bar{\mathbf{k}}_3^* = \mathbf{a}_\alpha \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot \boldsymbol{R}^* \cdot \mathbf{a}_3 = \mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0$.

Another important consequence of equations (3.10) and (3.11) is that for every normal vector $\overline{\mathbf{n}}$, $|\overline{\mathbf{n}}| = 1$, there exists a finite rotation tensor with the axis of rotation lying in the tangent plane to the middle surface (S_o). Let this rotation tensor be denoted by $\boldsymbol{R}_2$ (the angle of rotation is ϑ_2, the unit vector of the rotation axis is $\mathbf{e}_2$):

$$\boldsymbol{R}_2 = R^k_{\underline{2}\,l}\, \mathbf{a}_k \mathbf{a}^l = \cos\vartheta_2 \boldsymbol{I} + (1-\cos\vartheta)\mathbf{e}_2\mathbf{e}_2 + \sin\vartheta_2\, \mathbf{e}_2 \times \boldsymbol{I}. \tag{3.13}$$

According to relations (3.10) and (3.11):

$$\cos\vartheta_2 = \overline{n}_3, \qquad \sin\vartheta_2 = \sqrt{1-(\overline{n}_3)^2}, \tag{3.14}$$

$$e_{\underline{2}}^1 = -\frac{1}{\sqrt{a}}\frac{\overline{n}_2}{\sin\vartheta_2}, \quad e_{\underline{2}}^2 = \frac{1}{\sqrt{a}}\frac{\overline{n}_1}{\sin\vartheta_2} = -\frac{\overline{n}_1}{\overline{n}_2}e_{\underline{2}}^1. \tag{3.15}$$

3.4. Making use of the results of Subsection **3.3**, the rotation tensor $\boldsymbol{R}^*$ is obtained first by the product of two rotation tensors: $\boldsymbol{R}^* = \overline{\boldsymbol{R}}_n \cdot \boldsymbol{R}_2$, i.e. we apply two rotations, one after the other, described by expression

$$\bar{\mathbf{k}}_k^* = \boldsymbol{R}^* \cdot \mathbf{a}_k = \overline{\boldsymbol{R}}_n \cdot \boldsymbol{R}_2 \cdot \mathbf{a}_k = \overline{\boldsymbol{R}}_n \cdot (\boldsymbol{R}_2 \cdot \mathbf{a}_k)\,, \tag{3.16}$$

where $\boldsymbol{R}_2$ is defined by (3.13)-(3.15) and the rotation tensor $\overline{\boldsymbol{R}}_n$ describes a rotation about the unit normal $\bar{\mathbf{n}}$, being the axis of rotation, with rotation angle ϑ_1:

$$\overline{\boldsymbol{R}}_n = \cos\vartheta_1 \boldsymbol{I} + (1-\cos\vartheta_1)\bar{\mathbf{n}}\,\bar{\mathbf{n}} + \sin\vartheta_1 \bar{\mathbf{n}} \times \boldsymbol{I}. \tag{3.17}$$

The vectors $\bar{\mathbf{k}}_\kappa^* = \boldsymbol{R}^* \cdot \mathbf{a}_\kappa$, rotated into the tangent plane of the middle surface $(\overline{S}_o)$, can thus be obtained by rotating first the base vectors $\mathbf{a}_k$ into vectors $\boldsymbol{R}_2 \cdot \mathbf{a}_\kappa$ lying in the tangent plane, then these vectors are being rotated again in the tangent plane about the normal $\bar{\mathbf{n}}$: $\bar{\mathbf{k}}_\kappa^* = \overline{\boldsymbol{R}}_n \cdot (\boldsymbol{R}_2 \cdot \mathbf{a}_k)$. The vector $\mathbf{a}_3$ is rotated only once by $\bar{\mathbf{k}}_3 = \boldsymbol{R}_2 \cdot \mathbf{a}_3 = \bar{\mathbf{n}}$, because $\overline{\boldsymbol{R}}_n \cdot \bar{\mathbf{n}} = \bar{\mathbf{n}}$. The geometric interpretation of the described rotations is shown in Figure 2.

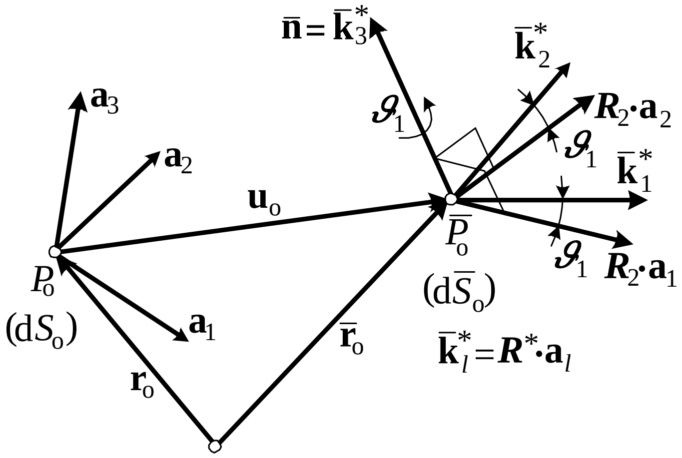

Figure 2.

3.5. The components of the unit normal vector $\bar{\mathbf{n}}$ can be obtained from the gradient $\mathbf{u}_{o,\alpha}$ of the displacement vector of the middle surface (S_o), according to (2.10)-(2.12), and the parameters of the rotation tensor $\boldsymbol{R}_2$ can then be computed from (3.14) and

(3.15) as

$$\cos\vartheta_2 = \overline{n}_3 = \frac{\sqrt{a}}{\sqrt{\overline{a}}}\left(1 + u^1_{\mathrm{o};1} + u^2_{\mathrm{o};2} + u^1_{\mathrm{o};1}u^2_{\mathrm{o};2} - u^1_{\mathrm{o};2}u^2_{\mathrm{o};1}\right), \tag{3.18}$$

$$\sqrt{a}\sin\vartheta_2 e^1_{\underline{2}} = -\overline{n}_2 = \frac{\sqrt{a}}{\sqrt{\overline{a}}}\left[\left(1 + u^1_{\mathrm{o};1}\right)u^3_{\mathrm{o};2} - u^1_{\mathrm{o};2}u^3_{\mathrm{o};1}\right], \tag{3.19}$$

$$\sqrt{a}\sin\vartheta_2 e^2_{\underline{2}} = \overline{n}_1 = -\frac{\sqrt{a}}{\sqrt{\overline{a}}}\left[\left(1 + u^2_{\mathrm{o};2}\right)u^3_{\mathrm{o};1} - u^2_{\mathrm{o};1}u^3_{\mathrm{o};2}\right]. \tag{3.20}$$

3.6. Let us introduce, secondly, the rotation tensor

$$\boldsymbol{R}_1 = \cos\vartheta_1 \boldsymbol{I} + (1 - \cos\vartheta_1)\,\mathbf{a}_3\mathbf{a}_3 + \sin\vartheta_1\,\mathbf{a}_3{\times}\boldsymbol{I} \tag{3.21}$$

on the middle surface (S_o). It can be pointed out (see Appendix A) that

$$\boldsymbol{R}^* = \overline{\boldsymbol{R}}_n \cdot\ \boldsymbol{R}_2 = \boldsymbol{R}_2 \cdot \boldsymbol{R}_1, \tag{3.22}$$

i.e. the following relations hold:

$$\bar{\mathbf{k}}^*_\kappa = \boldsymbol{R}^*\cdot\,\mathbf{a}_\kappa = \overline{\boldsymbol{R}}_n\cdot\boldsymbol{R}_2\cdot\mathbf{a}_\kappa = \boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot\mathbf{a}_\kappa, \qquad \bar{\mathbf{k}}^*_3 = \boldsymbol{R}^*\cdot\mathbf{a}_3 = \boldsymbol{R}_2\cdot\mathbf{a}_3 = \overline{\mathbf{n}}. \tag{3.23}$$

In view of equation (3.23), the finite rotation of the base vectors $\mathbf{a}_\kappa$, given by the rotation tensor $\boldsymbol{R}^*$, will be described *in the following by two rotation tensors*, $\boldsymbol{R}_2$ and $\boldsymbol{R}_1$. Then the base vectors $\bar{\mathbf{k}}^*_\kappa = \boldsymbol{R}^*\cdot\mathbf{a}_\kappa$, rotated into the tangent plane of the middle surface $(\overline{S}_\mathrm{o})$, can be obtained in two steps: first the base vectors $\mathbf{a}_\kappa$ are rotated about $\mathbf{a}_3$ by angle ϑ_1 into vectors $\boldsymbol{R}_1\cdot\mathbf{a}_\kappa$ lying in the tangent plane of (S_o), and, next, these vectors are rotated about axis $\mathbf{e}_2$, lying in the tangent plane of (S_o), by the angle ϑ_2 as $\bar{\mathbf{k}}^*_\kappa = \boldsymbol{R}_2\cdot(\boldsymbol{R}_1\cdot\mathbf{a}_\kappa)$. The base vector $\mathbf{a}_3$ is rotated, however, only once, according to $\bar{\mathbf{k}}^*_3 = \boldsymbol{R}_2\cdot\mathbf{a}_3 = \overline{\mathbf{n}}$, as $\boldsymbol{R}_1\cdot\mathbf{a}_3 = \mathbf{a}_3$. The geometrical interpretation of these rotations is seen in Figure 3.

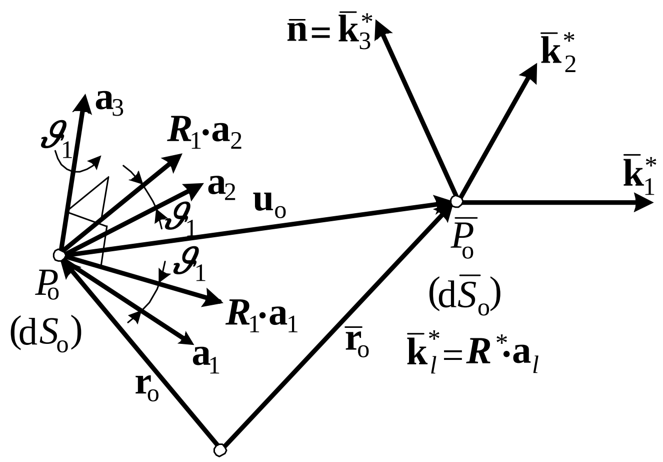

Figure 3.

The parameters of the rotation tensor $\boldsymbol{R}^*$ can be obtained from the parameters of the rotation tensors $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, defined in (3.21) and (3.13), respectively, according to the following expressions:

$$1+\cos\vartheta^* = \frac{1}{2}\left(1+\cos\vartheta_1\right)\left(1+\cos\vartheta_2\right),$$

$$\sin\vartheta^*\mathbf{e}^* = \frac{1}{2}\left[(1+\cos\vartheta_1)\sin\vartheta_2\mathbf{e}_2 + (1+\cos\vartheta_2)\sin\vartheta_1\mathbf{a}_3 + (\sin\vartheta_2\mathbf{e}_2)\times(\sin\vartheta_1\mathbf{a}_3)\right].$$

3.7. The rotation tensor $\overline{\boldsymbol{R}}_s$, describing an infinitesimal rotation, is defined by equations (3.5) and (3.6). According to (3.2), the rotated base vectors $\overline{\mathbf{k}}_k$ and $\overline{\mathbf{k}}_k^*$ can be related to each other through

$$\overline{\mathbf{k}}_k = \overline{\boldsymbol{R}}_s\cdot\overline{\mathbf{k}}_k^* = \overline{\mathbf{k}}_k^* + \vartheta_s\overline{\mathbf{e}}_s\times\overline{\mathbf{k}}_k^* = \boldsymbol{R}^*\cdot\mathbf{a}_k + \vartheta_s e_{\underline{s}}^{\mu}\left(\boldsymbol{R}^*\cdot\mathbf{a}_\mu\right)\times\left(\boldsymbol{R}^*\cdot\mathbf{a}_k\right), \tag{3.24}$$

and, in view of equation (B.1) of Appendix B, we have

$$\overline{\mathbf{k}}_k = \overline{\boldsymbol{R}}_s\cdot\overline{\mathbf{k}}_k^* = \boldsymbol{R}^*\cdot\left(\mathbf{a}_k + \vartheta_s\mathbf{e}_s\times\mathbf{a}_k\right) = \boldsymbol{R}^*\cdot\boldsymbol{R}_s\cdot\mathbf{a}_k = \boldsymbol{R}\cdot\mathbf{a}_k, \tag{3.25}$$

where

$$\boldsymbol{R}_s = \boldsymbol{I} + \vartheta_s\mathbf{e}_s\times\boldsymbol{I}; \qquad \boldsymbol{R} = \boldsymbol{R}^*\cdot\boldsymbol{R}_s = \boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot\boldsymbol{R}_s. \tag{3.26}$$

The resultant rotation tensor $\boldsymbol{R} = \boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot\boldsymbol{R}_s$ has three independent parameters, since $\boldsymbol{R}_2$ $\left[\text{and the two components of its vector } \boldsymbol{\vartheta}_2 = \vartheta_2\,\mathbf{e}_2, (\vartheta_2 = |\boldsymbol{\vartheta}_2|\,)\right]$ is uniquely defined by the displacement field $\mathbf{u}_{\mathrm{o}}$, according to (3.18)-(3.20) (zero parameter in number), $\boldsymbol{R}_1$ is described, according to (3.21), by the rotation angle ϑ_1 (1 parameter), and $\boldsymbol{R}_s$ is described, according to (3.25), by the two components of the vector $\boldsymbol{\vartheta}_s = \vartheta_{\underline{s}}\mathbf{e}_s$ $\left(\vartheta_{\underline{s}} = |\boldsymbol{\vartheta}_s|\right)$ (2 parameters).

The above equations can be supplemented by the following relations for the rotated base vectors $\overline{\mathbf{k}}_k^*$: from $(3.23)_2$ and (2.3) we have:

$$\overline{\mathbf{a}}_\kappa\cdot\overline{\mathbf{k}}_3^* = \overline{\mathbf{a}}_\kappa\cdot\overline{\mathbf{n}} = 0, \tag{3.27}$$

and from (3.6) and (3.27) we have:

$$\left(\overline{\mathbf{e}}_s\times\overline{\mathbf{k}}_\kappa^*\right)\cdot\overline{\mathbf{a}}_\lambda = 0, \quad \text{since} \quad \left(\overline{\mathbf{e}}_s\times\overline{\mathbf{k}}_\kappa^*\right)\parallel\overline{\mathbf{n}}. \tag{3.28}$$

4. The deformation gradient, the Green-Lagrange and the Jaumann strain tensors on the middle surface (S_{o})

4.1. The deformation gradient tensor and its transpose are defined in (2.6) as

$$\boldsymbol{F}_{\mathrm{o}} = \overline{\mathbf{a}}_k\,\mathbf{a}^k = \overline{\mathbf{a}}_\kappa\,\mathbf{a}^\kappa + \overline{\mathbf{a}}_3\,\mathbf{a}^3 \quad \text{and} \quad \boldsymbol{F}_{\mathrm{o}}^{\mathrm{T}} = \mathbf{a}^k\,\overline{\mathbf{a}}_k = \mathbf{a}^\kappa\,\overline{\mathbf{a}}_\kappa + \mathbf{a}^3\,\overline{\mathbf{a}}_3\,,$$

where the base vectors $\overline{\mathbf{a}}_\kappa$ and $\overline{\mathbf{a}}_3$ are given, respectively, by expressions (3.3) and (3.4).

The detailed Green-Lagrange strain tensor is given by (2.8). When the different products of the base vectors of the middle surface $(\overline{S}_{\mathrm{o}})$, appearing in equation (2.8) and later on, are computed, it will be taken into account that the strain vectors $\overline{\boldsymbol{\alpha}}_k$, the angle of rotation ϑ_s as well as the stretch ε_{o3} are infinitesimally small, according to the assumptions introduced in Subsection **3.1**. Then we can write:

$$\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_\lambda = \left(\overline{\mathbf{k}}_\kappa + \overline{\boldsymbol{\alpha}}_\kappa\right) \cdot \left(\overline{\mathbf{k}}_\lambda + \overline{\boldsymbol{\alpha}}_\lambda\right) = \overline{\mathbf{k}}_\kappa \cdot \overline{\mathbf{k}}_\lambda + \overline{\mathbf{k}}_\kappa \cdot \overline{\boldsymbol{\alpha}}_\lambda + \overline{\boldsymbol{\alpha}}_\kappa \cdot \overline{\mathbf{k}}_\lambda + \overline{\boldsymbol{\alpha}}_\kappa \cdot \overline{\boldsymbol{\alpha}}_\lambda \approx$$
$$\approx a_{\kappa\lambda} + \overline{\mathbf{k}}_\kappa \cdot \overline{\boldsymbol{\alpha}}_\lambda + \overline{\boldsymbol{\alpha}}_\kappa \cdot \overline{\mathbf{k}}_\lambda, \tag{4.1}$$
$$\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_3 = (1+\varepsilon_{o3})\, \overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{k}}_3 \approx \overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{k}}_3, \tag{4.2}$$
$$\overline{\mathbf{a}}_3 \cdot \overline{\mathbf{a}}_3 = (1+\varepsilon_{o3})^2\, \overline{\mathbf{k}}_3 \cdot \overline{\mathbf{k}}_3 \approx 1 + 2\varepsilon_{o3}. \tag{4.3}$$

In obtaining (4.1) and (4.3), equation (B.2) of Appendix B has been utilized.

Taking into account (3.24) and (3.28), the product $\overline{\mathbf{k}}_\kappa \cdot \overline{\boldsymbol{\alpha}}_\lambda$ appearing in (4.1) can be written in a different way:

$$\overline{\mathbf{k}}_\kappa \cdot \overline{\boldsymbol{\alpha}}_\lambda = \overline{\mathbf{k}}_\kappa \cdot \left(\overline{\mathbf{a}}_\lambda - \overline{\mathbf{k}}_\lambda\right) = \overline{\mathbf{k}}_\kappa \cdot \overline{\mathbf{a}}_\lambda - a_{\kappa\lambda} = \left(\overline{\mathbf{k}}^*_\kappa + \vartheta_s \overline{\mathbf{e}}_s \times \overline{\mathbf{k}}^*_\kappa\right) \cdot \overline{\mathbf{a}}_\lambda - a_{\kappa\lambda} =$$
$$= \overline{\mathbf{k}}^*_k \cdot \overline{\mathbf{a}}_\lambda - a_{\kappa\lambda} = \mathbf{a}_\kappa \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot (\mathbf{a}_\lambda + \mathbf{u}_{o,\lambda}) - a_{\kappa\lambda}. \tag{4.4}$$

The product $\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_3$ of (4.2) can also be modified using (3.24), (3.27) and equation (B.1) of Appendix B:

$$\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_3 \approx \overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{k}}_3 = \overline{\mathbf{a}}_\kappa \cdot \left(\overline{\mathbf{k}}^*_3 + \vartheta_s \overline{\mathbf{e}}_s \times \overline{\mathbf{k}}^*_3\right) = \left(\overline{\mathbf{k}}_\kappa + \overline{\boldsymbol{\alpha}}_\kappa\right) \cdot \left(\vartheta_s \overline{\mathbf{e}}_s \times \overline{\mathbf{k}}^*_3\right) \approx$$
$$\approx \overline{\mathbf{k}}^*_\kappa \cdot \left(\vartheta_s \overline{\mathbf{e}}_s \times \overline{\mathbf{k}}^*_3\right) = \mathbf{a}_\kappa \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot \boldsymbol{R}^* \cdot (\vartheta_s \mathbf{e}_s \times \mathbf{a}_3) = \vartheta_s \mathbf{a}_\kappa \cdot (\mathbf{e}_s \times \mathbf{a}_3) = \vartheta_{\underline{s}} \sqrt{a}\, e_{\kappa\mu 3} e^\mu_{\underline{s}}\,. \tag{4.5}$$

4.2. The components of the Green-Lagrange strain tensor on the middle surface (S_o) are thus the following:

$$E_{o\kappa\lambda} = \mathbf{a}_\kappa \cdot \boldsymbol{E}_o \cdot \mathbf{a}_\lambda = \frac{1}{2}\left(\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_\lambda - a_{\kappa\lambda}\right) \approx$$
$$\approx \frac{1}{2}\left[\mathbf{a}_\kappa \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot (\mathbf{a}_\lambda + \mathbf{u}_{o,\lambda}) + (\mathbf{a}_\kappa + \mathbf{u}_{o,\kappa}) \cdot \boldsymbol{R}^* \cdot \mathbf{a}_\lambda\right] - a_{\kappa\lambda}, \tag{4.6}$$
$$E_{o\kappa 3} = E_{o3\kappa} = \mathbf{a}_\kappa \cdot \boldsymbol{E}_o \cdot \mathbf{a}_3 = \frac{1}{2}\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{a}}_3 \approx \frac{1}{2}\vartheta_s\, \mathbf{a}_3 \cdot (\mathbf{a}_\kappa \times \mathbf{e}_s) = \frac{1}{2}\vartheta_{\underline{s}} \sqrt{a}\, e_{\kappa\mu 3} e^\mu_{\underline{s}}, \tag{4.7}$$
$$E_{o33} = \mathbf{a}_3 \cdot \boldsymbol{E}_o \cdot \mathbf{a}_3 = \frac{1}{2}\left(\overline{\mathbf{a}}_3 \cdot \overline{\mathbf{a}}_3 - 1\right) \approx \varepsilon_{o3}. \tag{4.8}$$

Components E_{okl} are infinitesimal.

It is noted that to compute the components $E_{o\kappa\lambda}$, only the displacement field $\mathbf{u}_o$ and the rotation tensor $\boldsymbol{R}^* = \boldsymbol{R}_2 \cdot \boldsymbol{R}_1$ are needed. On the other hand, components $E_{o\kappa 3}$ depend on the rotation tensor $\boldsymbol{R}_s$ (on the rotation vector $\boldsymbol{\vartheta}_s = \vartheta_s \mathbf{e}_s$), and component E_{o33} depends only on the stretch ε_{o3}.

4.3. The surface part $\overline{a}_{\alpha\beta} = \overline{\mathbf{a}}_\alpha \cdot \overline{\mathbf{a}}_\beta$ of the metric tensor at point $\overline{P}_o$ can be written using (2.8) as follows:

$$\overline{a}_{\alpha\beta} = 2E_{o\alpha\beta} + a_{\alpha\beta}.$$

Taking into account that the strain components $E_{o\alpha\beta}$ are infinitesimally small, the approximate value of the determinant of the metric tensor in $(2.3)_2$ is:

$$\overline{a} = \det \overline{a}_{\alpha\beta} = \overline{a}_{11}\overline{a}_{22} - \overline{a}_{12}\overline{a}_{21} \approx a. \tag{4.9}$$

4.4. The curvature tensor of the middle surface $(\overline{S}_o)$ reads:

$$\overline{b}_{\kappa\lambda} = -\,\overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{n}}_{,\lambda} = -\,\left(\overline{\mathbf{k}}_\kappa + \overline{\boldsymbol{\alpha}}_\kappa\right)\cdot \overline{\mathbf{k}}_{3,\lambda} \approx -\,\overline{\mathbf{k}}_\kappa \cdot \overline{\mathbf{k}}_{3,\lambda}\,. \tag{4.10}$$

4.5. The symmetric right Jaumann strain tensor, $\boldsymbol{H}_o$, can be obtained using the polar decomposition of the deformation gradient tensor:

$$\boldsymbol{F}_o = \boldsymbol{R}\cdot\left(\boldsymbol{H}_o + \boldsymbol{I}\right),$$

$$\boldsymbol{H}_o = H_{okl}\,\mathbf{a}^k\mathbf{a}^l = \frac{1}{2}\left(\boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{F}_o + \boldsymbol{F}_o^{\mathrm{T}}\cdot\boldsymbol{R}\right) - \boldsymbol{I}. \tag{4.11}$$

In view of (3.3) and (4.1), the components of this tensor are given by

$$\begin{aligned}
H_{o\kappa\lambda} &= \mathbf{a}_\kappa\cdot\boldsymbol{H}_o\cdot\mathbf{a}_\lambda = \frac{1}{2}\left(\mathbf{a}_\kappa\cdot\boldsymbol{R}^{\mathrm{T}}\cdot\boldsymbol{F}_o\cdot\mathbf{a}_\lambda + \mathbf{a}_\kappa\cdot\boldsymbol{F}_o^{\mathrm{T}}\cdot\boldsymbol{R}\cdot\mathbf{a}_\lambda\right) - a_{\kappa\lambda} = \\
&= \frac{1}{2}\left(\overline{\mathbf{k}}_\kappa\cdot\overline{\mathbf{a}}_\lambda + \overline{\mathbf{a}}_\kappa\cdot\overline{\mathbf{k}}_\lambda\right) - a_{\kappa\lambda} = \frac{1}{2}\left[\overline{\mathbf{k}}_\kappa\cdot\left(\overline{\mathbf{k}}_\lambda + \overline{\boldsymbol{\alpha}}_\lambda\right) + \left(\overline{\mathbf{k}}_\kappa + \overline{\boldsymbol{\alpha}}_\kappa\right)\cdot\overline{\mathbf{k}}_\lambda\right] - a_{\kappa\lambda} \approx \\
&\approx \frac{1}{2}\left[\overline{\mathbf{k}}_\kappa\cdot\overline{\boldsymbol{\alpha}}_\lambda + \overline{\boldsymbol{\alpha}}_\kappa\cdot\overline{\mathbf{k}}_\lambda\right] \approx \frac{1}{2}\left(\overline{\mathbf{a}}_\kappa\cdot\overline{\mathbf{a}}_\lambda - a_{\kappa\lambda}\right) = E_{o\kappa\lambda},
\end{aligned} \tag{4.12}$$

and, in addition, taking into account (4.5) as well, we obtain:

$$\begin{aligned}
H_{o\kappa 3} &= \mathbf{a}_\kappa\cdot\boldsymbol{H}_o\cdot\mathbf{a}_3 = \frac{1}{2}\left(\overline{\mathbf{k}}_\kappa\cdot\overline{\mathbf{a}}_3 + \overline{\mathbf{a}}_\kappa\cdot\overline{\mathbf{k}}_3\right) = \\
&= \frac{1}{2}\left[\left(1+\varepsilon_{o3}\right)\overline{\mathbf{k}}_\kappa\cdot\overline{\mathbf{k}}_3 + \overline{\mathbf{a}}_\kappa\cdot\overline{\mathbf{k}}_3\right] \approx \frac{1}{2}\overline{\mathbf{a}}_\kappa\cdot\overline{\mathbf{a}}_3 = E_{o\kappa 3},
\end{aligned} \tag{4.13}$$

$$H_{o33} = \mathbf{a}_3\cdot\boldsymbol{H}_o\cdot\mathbf{a}_3 - 1 = \frac{1}{2}\left(\overline{\mathbf{k}}_3\cdot\overline{\mathbf{a}}_3 + \overline{\mathbf{a}}_3\cdot\overline{\mathbf{k}}_3\right) - 1 = \left(1+\varepsilon_{o3}\right)\overline{\mathbf{k}}_3\cdot\overline{\mathbf{k}}_3 - 1 = \varepsilon_{o3} \approx E_{o33}. \tag{4.14}$$

It can be concluded, that using the assumptions (neglections) introduced in the previous sections, the Green-Lagrange and the Jaumann strain tensors are identical on the middle surface (S_o).

5. The deformation gradient and the Green-Lagrange strain tensor at an arbitrary point of the reference configuration (B)

5.1. Let P denote an arbitrary point on the normal (on the coordinate line x^3) to the middle surface (S_o) of *the reference configuration* (B). The base vectors and the metric tensor at point P are:

$$\mathbf{r} = \mathbf{r}\left(x^1, x^2, x^3\right) = \mathbf{r}_o + \mathbf{a}_3 x^3, \tag{5.1}$$

$$\mathbf{g}_\kappa = \mathbf{r}_{,\kappa} = \mathbf{r}_{o,\kappa} + \mathbf{a}_{3,\kappa}\,x^3 = \mathbf{a}_\kappa - \mathbf{a}_\alpha\, b_\kappa^\alpha\, x^3 = \left(\delta_\kappa^\alpha - b_\kappa^\alpha\, x^3\right)\mathbf{a}_\alpha, \qquad \mathbf{g}_3 = \mathbf{r}_{,3} = \mathbf{a}_3 = \mathbf{n}, \tag{5.2}$$

$$g_{\kappa\lambda} = \mathbf{g}_\kappa\cdot\mathbf{g}_\lambda = a_{\kappa\lambda} - 2b_{\kappa\lambda}x^3 + b_{\kappa\alpha}b_\lambda^\alpha\left(x^3\right)^2, \quad g_{\kappa 3} = 0, \quad g_{33} = 1. \tag{5.3}$$

The contravariant base vectors are denoted by $\mathbf{g}^m$, the inverse of the metric tensor is $g^{mn} = \mathbf{g}^m \cdot \mathbf{g}^n$ $\left(g^{\mu 3} = 0,\ g^{33} = 1\right)$. If the shifter is denoted by μ_k^m, $\mu_\kappa^\mu = \delta_\kappa^\mu - b_\kappa^\mu x^3$, $\mu_3^\mu = 0$, $\mu_3^3 = 1$ and its inverse is $\overset{-1}{\mu}{}_k^m$, then the following relations hold:

$$\mathbf{g}_k = \mu_k^m \, \mathbf{a}_m, \qquad \mathbf{g}^m = \overset{-1}{\mu}{}_k^m \, \mathbf{a}^k. \tag{5.4}$$

Let $\overline{P}$ denote an arbitrary point on the coordinate line x^3 of *the deformed configuration* $(\overline{B})$. The position vector of point $\overline{P}$ and the base vectors at $\overline{P}$ are:

$$\overline{\mathbf{r}} = \overline{\mathbf{r}}\left(x^1, x^2, x^3\right) = \overline{\mathbf{r}}_\mathrm{o} + \overline{\mathbf{h}}, \qquad \overline{\mathbf{g}}_\kappa = \overline{\mathbf{r}}_{\mathrm{o},\kappa} + \overline{\mathbf{h}}_{,\kappa} = \overline{\mathbf{a}}_\kappa + \overline{\mathbf{h}}_{,\kappa}\,, \qquad \overline{\mathbf{g}}_3 = \overline{\mathbf{h}}_{,3}\,, \tag{5.5}$$

where $\overline{\mathbf{h}} = \overline{\mathbf{h}}\left(x^1, x^2, x^3\right)$. The geometrical setting with an enlarged scale in the thickness direction is shown in Figure 4. Geometrically non-linear shell theories differ from each other in the assumptions for the form of $\overline{\mathbf{h}}$ (for instance, in shell theories based on the Kirchhoff-Love hypothesis, $\overline{\mathbf{h}} = \overline{\mathbf{n}}\, x^3$).

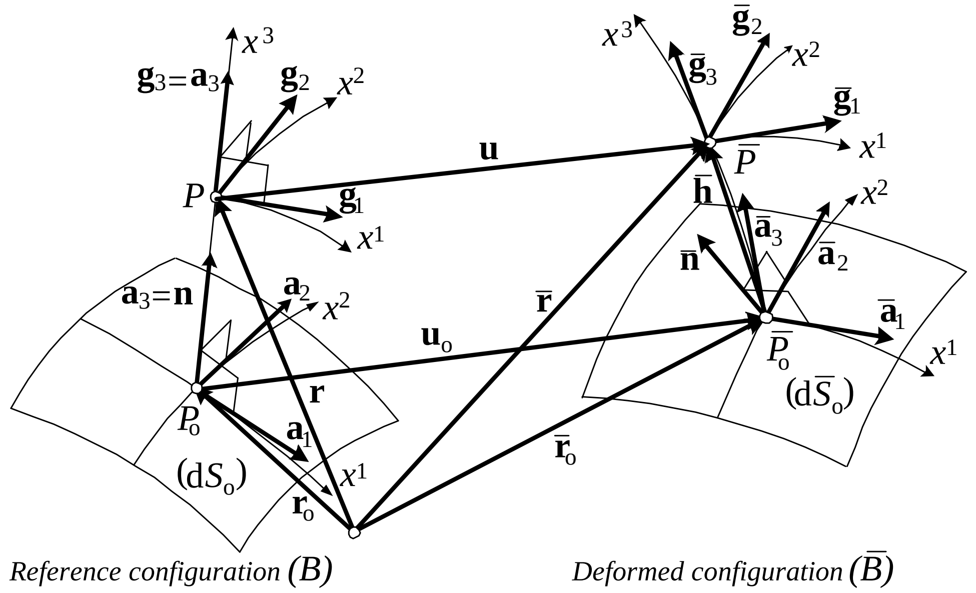

Figure 4.

As a continuation of Sections 3 and 4, the following part of this section derives the deformation gradient and the Green-Lagrange strain tensor at an arbitrary point P of the configuration (B), considering the relatively simple hypothesis of Reissner-Mindlin:

$$\overline{\mathbf{h}} = \overline{\mathbf{k}}_3\, x^3 = \boldsymbol{R}^* \cdot (\mathbf{a}_3 + \vartheta_s \mathbf{e}_s \times \mathbf{a}_3)\, x^3. \tag{5.6}$$

5.2. The deformation gradient and its transpose at the arbitrary point P are:

$$\boldsymbol{F} = \overline{\mathbf{g}}_k\, \mathbf{g}^k = \overline{\mathbf{g}}_\kappa\, \mathbf{g}^\kappa + \overline{\mathbf{g}}_3\, \mathbf{g}^3, \qquad \boldsymbol{F}^\mathrm{T} = \mathbf{g}^k\, \overline{\mathbf{g}}_k = \mathbf{g}^\kappa\, \overline{\mathbf{g}}_\kappa + \mathbf{g}^3\, \overline{\mathbf{g}}_3. \tag{5.7}$$

At point $\overline{P}$, the base vectors can be written on the basis of equations $(5.5)_{2,3}$ and (5.6) as

$$\overline{\mathbf{g}}_\kappa = \overline{\mathbf{a}}_\kappa + \overline{\mathbf{h}}_{,\kappa} = \overline{\mathbf{k}}_\kappa + \overline{\alpha}_\kappa + \overline{\mathbf{k}}_{3,\kappa}\, x^3, \qquad \overline{\mathbf{g}}_3 = \overline{\mathbf{h}}_{,3} = \overline{\mathbf{k}}_3, \tag{5.8}$$

where

$$\overline{\mathbf{k}}_{3,\kappa} = \left[\boldsymbol{R}^* \cdot (\mathbf{a}_3 + \vartheta_s \mathbf{e}_s \times \mathbf{a}_3)\right]_{,\kappa} = \boldsymbol{R}^*_{,\kappa} \cdot (\mathbf{a}_3 + \vartheta_s \mathbf{e}_s \times \mathbf{a}_3) + \boldsymbol{R}^* \cdot (\mathbf{a}_3 + \vartheta_s \mathbf{e}_s \times \mathbf{a}_3)_{,\kappa}\,. \tag{5.9}$$

5.3. The Green-Lagrange strain tensor at the arbitrary point P is given by

$$\boldsymbol{E} = E_{kl}\, \mathbf{g}^k \mathbf{g}^l = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} - \boldsymbol{I}\right) = \frac{1}{2}\left[(\overline{\mathbf{g}}_k \cdot \overline{\mathbf{g}}_l) - (\mathbf{g}_k \cdot \mathbf{g}_l)\right] \mathbf{g}^k \mathbf{g}^l =$$
$$= \frac{1}{2}\left[(\overline{\mathbf{g}}_\kappa \cdot \overline{\mathbf{g}}_\lambda - g_{\kappa\lambda})\, \mathbf{g}^\kappa \mathbf{g}^\lambda + (\overline{\mathbf{g}}_\kappa \cdot \overline{\mathbf{g}}_3)\, \mathbf{g}^\kappa \mathbf{g}^3 + (\overline{\mathbf{g}}_3 \cdot \overline{\mathbf{g}}_\lambda)\, \mathbf{g}^3 \mathbf{g}^\lambda + (\overline{\mathbf{g}}_3 \cdot \overline{\mathbf{g}}_3 - 1)\, \mathbf{g}^3 \mathbf{g}^3\right]. \tag{5.10}$$

Assume that beside the strain vectors $\overline{\boldsymbol{\alpha}}_\kappa$ and the angle of rotation ϑ_s, the derivative $\vartheta_{s,\kappa}$ is also infinitesimally small. Then, using the transformations of Subsection **4.1**, the scalar product of the base vectors of configuration $(\overline{B})$, appearing in (5.10), can be written as

$$\overline{\mathbf{g}}_\kappa \cdot \overline{\mathbf{g}}_\lambda = \left(\overline{\mathbf{k}}_\kappa + \overline{\boldsymbol{\alpha}}_\kappa + \overline{\mathbf{k}}_{3,\kappa}\, x^3\right) \cdot \left(\overline{\mathbf{k}}_\lambda + \overline{\boldsymbol{\alpha}}_\lambda + \overline{\mathbf{k}}_{3,\lambda}\, x^3\right) \approx$$
$$\approx a_{\kappa\lambda} + \overline{\mathbf{k}}_\kappa \cdot \overline{\boldsymbol{\alpha}}_\lambda + \overline{\boldsymbol{\alpha}}_\kappa \cdot \overline{\mathbf{k}}_\lambda + \left(\overline{\mathbf{k}}_\kappa \cdot \overline{\mathbf{k}}_{3,\lambda} + \overline{\mathbf{k}}_{3,\kappa} \cdot \overline{\mathbf{k}}_\lambda\right) x^3 + \overline{\mathbf{k}}_{3,\kappa} \cdot \overline{\mathbf{k}}_{3,\lambda} \left(x^3\right)^2, \tag{5.11}$$

$$\overline{\mathbf{g}}_\kappa \cdot \overline{\mathbf{g}}_3 = \overline{\mathbf{g}}_3 \cdot \overline{\mathbf{g}}_\kappa = \left(\overline{\mathbf{a}}_\kappa + \overline{\mathbf{k}}_{3,\kappa}\, x^3\right) \cdot \overline{\mathbf{k}}_3 = \overline{\mathbf{a}}_\kappa \cdot \overline{\mathbf{k}}_3 \approx \vartheta_s \mathbf{a}_\kappa \cdot (\mathbf{e}_s \times \mathbf{a}_3)\,, \tag{5.12}$$

$$\overline{\mathbf{g}}_3 \cdot \overline{\mathbf{g}}_3 = \overline{\mathbf{k}}_3 \cdot \overline{\mathbf{k}}_3 = 1. \tag{5.13}$$

Making use of (3.24), the scalar product $\overline{\mathbf{k}}_\kappa \cdot \overline{\mathbf{k}}_{3,\lambda}$ in (5.11) can be approximated as

$$\overline{\mathbf{k}}_\kappa \cdot \overline{\mathbf{k}}_{3,\lambda} = \left(\overline{\mathbf{k}}^*_k + \vartheta_s \overline{\mathbf{e}}_s \times \overline{\mathbf{k}}^*_k\right) \cdot \left(\overline{\mathbf{k}}^*_3 + \vartheta_s \overline{\mathbf{e}}_s \times \overline{\mathbf{k}}^*_3\right)_{,\lambda} \approx \overline{\mathbf{k}}^*_\kappa \cdot \overline{\mathbf{k}}^*_{3,\lambda}\,,$$

and then we can write:

$$\overline{\mathbf{k}}_\kappa \cdot \overline{\mathbf{k}}_{3,\lambda} \approx -\,\overline{b}_{\kappa\lambda} \approx \overline{\mathbf{k}}^*_\kappa \cdot \overline{\mathbf{k}}^*_{3,\lambda} \approx \mathbf{a}_\kappa \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot (\boldsymbol{R}^*{}_{,\lambda} \cdot \mathbf{a}_3 + \boldsymbol{R}^* \cdot \mathbf{a}_{3,\lambda}) =$$
$$= \mathbf{a}_\kappa \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot \boldsymbol{R}^*{}_{,\kappa} \cdot \mathbf{a}_3 + \mathbf{a}_\kappa \cdot \mathbf{a}_{3,\lambda}\,, \tag{5.14}$$

$$\overline{\mathbf{k}}_{3,\kappa} \cdot \overline{\mathbf{k}}_{3,\lambda} \approx \overline{b}_{\kappa\mu} \overline{b}^\mu_\lambda \approx \overline{\mathbf{k}}^*_{3,\kappa} \cdot \overline{\mathbf{k}}^*_{3,\lambda} \approx (\boldsymbol{R}^*{}_{,\kappa} \cdot \mathbf{a}_3 + \boldsymbol{R}^* \cdot \mathbf{a}_{3,\kappa}) \cdot (\boldsymbol{R}^*{}_{,\lambda} \cdot \mathbf{a}_3 + \boldsymbol{R}^* \cdot \mathbf{a}_{3,\lambda}) =$$
$$= \mathbf{a}_3 \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\kappa} \cdot \boldsymbol{R}^*{}_{,\lambda} \cdot \mathbf{a}_3 + \mathbf{a}_3 \cdot \left(\boldsymbol{R}^{*\mathrm{T}}{}_{,\kappa} \cdot \boldsymbol{R}^* \cdot \mathbf{a}_{3,\kappa} + \boldsymbol{R}^{*\mathrm{T}}{}_{,\lambda} \cdot \boldsymbol{R}^* \cdot \mathbf{a}_{3,\kappa}\right) + \mathbf{a}_{3,\kappa} \cdot \mathbf{a}_{3,\lambda}\,, \tag{5.15}$$

In obtaining (5.12), (4.6) has also been used.

5.4. In view of the previous results, the following are the scalar components of the Green-Lagrange strain tensor:

$$E_{\kappa\lambda} = \mathbf{g}_\kappa \cdot \boldsymbol{E} \cdot \mathbf{g}_\lambda = \frac{1}{2}\left(\overline{\mathbf{g}}_\kappa \cdot \overline{\mathbf{g}}_\lambda - g_{\kappa\lambda}\right) \approx \frac{1}{2}\left[a_{\kappa\lambda} + \overline{\mathbf{k}}_\kappa \cdot \left(\overline{\mathbf{a}}_\lambda - \overline{\mathbf{k}}_\lambda\right) + \left(\overline{\mathbf{a}}_\kappa - \overline{\mathbf{k}}_\kappa\right) \cdot \overline{\mathbf{k}}_\lambda\right] +$$
$$+ \frac{1}{2}\left[\mathbf{a}_\kappa \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot \boldsymbol{R}^*{}_{,\lambda} \cdot \mathbf{a}_3 + \mathbf{a}_\lambda \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot \boldsymbol{R}^*{}_{,\kappa} \cdot \mathbf{a}_3 - 2b_{\kappa\lambda}\right] x^3 + \frac{1}{2}\left[\mathbf{a}_3 \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\kappa} \cdot \boldsymbol{R}^*{}_{,\lambda} \cdot \mathbf{a}_3 + \right.$$
$$\left. + \mathbf{a}_3 \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\kappa} \cdot \boldsymbol{R}^* \cdot \mathbf{a}_{3,\lambda} + \mathbf{a}_3 \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\lambda} \cdot \boldsymbol{R}^* \cdot \mathbf{a}_{3,\kappa} + b^\mu_\kappa\, b_{\mu\lambda}\right] \left(x^3\right)^2 - \frac{1}{2} g_{\kappa\lambda},$$

i.e.,

$$E_{\kappa\lambda} = E_{o\kappa\lambda} + E_{\underline{1}\kappa\lambda} x^3 + E_{\underline{2}\kappa\lambda} \left(x^3\right)^2, \tag{5.16}$$

$$E_{\underline{1}\kappa\lambda} = -\left(\overline{b}_{\kappa\lambda} - b_{\kappa\lambda}\right) = \frac{1}{2}\left[\mathbf{a}_\kappa \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot \boldsymbol{R}^*{}_{,\lambda} \cdot \mathbf{a}_3 + \mathbf{a}_\lambda \cdot \boldsymbol{R}^{*\mathrm{T}} \cdot \boldsymbol{R}^*{}_{,\kappa} \cdot \mathbf{a}_3\right], \tag{5.17}$$

$$E_{\underline{2}\kappa\lambda} = \frac{1}{2}\left(\overline{b}^{\mu}_{\kappa}\,\overline{b}_{\mu\lambda} - b^{\mu}_{\kappa}\,b_{\mu\lambda}\right) =$$

$$= \frac{1}{2}\left[\mathbf{a}_3 \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\kappa} \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\lambda} \cdot \mathbf{a}_3 + \mathbf{a}_3 \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\kappa} \cdot \boldsymbol{R}^* \cdot \mathbf{a}_{3,\lambda} + \mathbf{a}_3 \cdot \boldsymbol{R}^{*\mathrm{T}}{}_{,\lambda} \cdot \boldsymbol{R}^* \cdot \mathbf{a}_{3,\kappa}\right], \tag{5.18}$$

and, furthermore,

$$E_{\kappa 3} = E_{3\kappa} = \mathbf{g}_\kappa \cdot \boldsymbol{E} \cdot \mathbf{g}_3 = \frac{1}{2}\,\overline{\mathbf{g}}_\kappa \cdot \overline{\mathbf{g}}_3 \approx \frac{1}{2}\vartheta_s \mathbf{a}_3 \cdot (\mathbf{a}_\kappa \times \mathbf{e}_s) = \frac{1}{2}\,\vartheta_{\underline{s}}\sqrt{a}\,e_{\kappa\mu 3} e^{\mu}_{\underline{s}} \approx E_{o\kappa 3}, \tag{5.19}$$

$$E_{33} = \mathbf{g}_3 \cdot \boldsymbol{E} \cdot \mathbf{g}_3 = \frac{1}{2}\left(\overline{\mathbf{g}}_3 \cdot \overline{\mathbf{g}}_3 - g_{33}\right) \approx 0. \tag{5.20}$$

It is noted that the scalar components of the Green-Lagrange strain tensor obtained above at the arbitrary point P are related to the contravariant basis $\mathbf{g}^k \mathbf{g}^l$. Naturally, these components can also be written in the contravariant basis $\mathbf{a}^k \mathbf{a}^l$, using the inverse shifter $\overset{-1}{\mu}{}^{m}_{k}$.

5.5. To summarize the above results it is worth mentioning that when the above shell kinematics based on the Reissner-Mindlin hypothesis is employed, we obtain that $E_{33} \approx 0$ and $E_{\kappa 3} \approx E_{o\kappa 3}$, i.e. neither the change in the thickness, nor the variation of the transverse shear deformations across the thickness of the shell can be taken into account.

6. Concluding remarks

This paper investigates the middle surface of a shell in connection with the kinematical description of nonlinear shell theories. It is assumed that on the reference middle surface the displacements and the rotations of the base vectors are finite, whereas the strains are infinitesimal. A convected coordinate system attached to the middle surface is employed.

On the deformed middle surface, the tangential base vectors and, therefore, the unit normal are uniquely determined by the displacement field. To describe the rotation of the base vectors on the middle surface in the deformation process, this paper applies three rotation tensors.

There exist an infinitely large number of rotation tensors that rotate the unit normal vector and the tangential base vectors of the reference surface into the unit normal vector and tangent base vectors of the deformed middle surface. In this paper, one of the rotation tensors describing the above mentioned rotation ($\boldsymbol{R}_2$) is chosen in such a way that *the axis of the rotation lies in the tangent plane of the reference middle surface.* Such a rotation tensor is uniquely defined by the displacement field of the middle surface. The second rotation tensor ($\overline{\boldsymbol{R}}_n$) defined in the paper rotates the tangent base vectors obtained by rotation ($\boldsymbol{R}_2$) *in the tangent plane of deformed middle surface about its normal.* This rotation is usually called drilling rotation in the specialist literature. Both rotations mentioned are finite and yield a single rotation ($\boldsymbol{R}^* = \overline{\boldsymbol{R}}_n \cdot \boldsymbol{R}_2$). The *axis of the third rotation tensor ($\overline{\boldsymbol{R}}_s$) lies in the tangent plane of the deformed middle surface* and performs an infinitesimally small rotation on the base vectors obtained after the previous two rotations. The resultant rotation tensor thus describes three, geometrically well identified (two finite and an infinitesimal) rotations and can be given by the product of the three rotation tensors ($\boldsymbol{R} = \overline{\boldsymbol{R}}_{\mathbf{s}} \cdot \boldsymbol{R}^* = \overline{\boldsymbol{R}}_s \cdot \overline{\boldsymbol{R}}_n \cdot \boldsymbol{R}_2$). All the three rotation tensors are transformed onto the reference middle surface in the paper ($\boldsymbol{R} = \boldsymbol{R}^* \cdot \boldsymbol{R}_{\mathbf{s}} = \boldsymbol{R}_2 \cdot \boldsymbol{R}_1 \cdot \boldsymbol{R}_s$). The transverse shear strains on the middle surface are obtained from the third, infinitesimally small rotation, whereas the description of the transverse normal strain requires the introduction of another parameter.

After describing the rotations of the base vectors, the complete three-dimensional deformation gradient tensor, the Green-Lagrange strain tensor as well as the symmetric right Jaumann strain tensor on the middle surface of the shell are determined, using the three-dimensional theory of deformation of solids. Due to the assumptions (neglections) introduced by the paper, the Green-Lagrange and the Jaumann strain tensors are identical on the middle surface.

The shell theory presented by this paper to investigate the middle surface of the shell leads hence to a seven-parameter shell model. These parameters consist of the three components of the displacement vector of the middle surface, one parameter of the rotation tensor $\overline{\boldsymbol{R}}_n$, two parameters of the rotation tensor $\overline{\boldsymbol{R}}_s$, and the seventh parameter is the transverse normal strain.

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REFERENCES

1. Chapelle, D. and Bathe, K.: Fundamental considerations for finite element analysis of shell structures. *Int. J. Num. Meths. Engng.*, **66**, (1998), 19–36.
2. Valente, R. F., Jorge, R. N., Cardoso, R., de Sa, J. C., and Gracio, J.: On the use an enhanced transverse shear strain shell element for problem involving large rotation. *Computational Mech.*, **30**, (2003), 286–196.
3. Simmonds, J. and Danielson, D.: Nonlinear shell theory with a finite rotation vector. *Prceedings, Kon. Ned. Ak. Wet., Series B.*, **73**, (1970), 460–478.
4. Simmonds, J. and Danielson, D.: Nonlinear shell theory with finite rotation and stress-function vectors. *J. Appl. Mech.*, **29**, (1972), 1085–1090.
5. Wriggers, P. and Gruttmann, F.: Thin shells with finite rotations formulated in biot stresses: theory and finite element formulation. *Int. J. Num. Meths. Engng.*, **36**, (1993), 2049–2071.
6. Ibrahimbegovich, A.: Stress resultant geometrically nonlinear shell theory with drilling rotations. part i. a consistent formulation. *Comput. Methods Appl. Mech. Engrg.*, **118**, (1994), 265–284.

7. Ibrahimbegovich, A. and Frey, F.: Stress resultant geometrically nonlinear shell theory with drilling rotations. part ii. computational aspects. *Comput. Methods Appl. Mech. Engng.*, **118**, (1994), 285–308.

8. Ibrahimbegovich, A. and Frey, F.: Stress resultant geometrically nonlinear shell theory with drilling rotations. part iii. linearized kinematics. *Int. J. Num. Meths. Engng.*, **37**, (1994), 3659–3683.

9. Pietraszkiewicz, W.: Geometrically nonlinear theories of thin elastic shells. *Advances in Mechanics*, **12**, (1989), 51–130.

10. Pietraszkiewicz, W.: On using rotations as primary variables in the non-linear theory of thin irregular shells. In D. G. D. Durban and J. Simmonds (eds.), *Advances in the mechanics of plates and shells*, pp. 245–258, Kluver Academic Publisher, 2001.

11. Wisniewski, K.: A shell theory with independent rotations for relaxed Biot stress and right stretch tensor. *Computational Mechanics*, **21**, (1998), 101–122.

12. Brank, B., Peric, D., and B.Damjanich, F.: On large deformations of thin elasto-plastic shells: implementation of finite rotation model for quadrilateral shell element. *Computers and Structures*, **40**, (1997), 689–726.

13. Brank, B., Brisghella, L., Peric, N. T., and Damjanich, F. B.: On non-linear dynamics of shells: implementation of energy-momentum conserving algorithm a finite rotation shell model. *Int. J. Num. Meths. Engng.*, **42**, (1998), 409–442.

14. Ibrahimbegovich, A., Brank, B., and Courtois, P.: Stress resultant geometrically exact form of classical shell model and vector-like parametrization of constrained finite rotations. *Int. J. Num. Meths. Engng.*, **52**, (2001), 1235–1252.

15. Brank, B. and Ibrahimbegovich, A.: On the relation between different parametrizations of finite rotations for shell. *Engineering Computations*, **18**, (2001), 950–971.

16. Ibrahimbegovich, A., Mamouri, S., and Brank, B.: Finite rotation shell elements for nonlinear statics and dynamics. In *European Congress on Computational Methods in Applied Sciences and Engineering. ECCOMAS*, 11–14 September 2000, p. 21.

17. Lee, W. and Lee, B.: An effective finite rotation formulation for geometrical non-linear shell structures. *Computational Mechanics*, **27**, (2001), 360–368.

18. Atluri, S., Iura, M., and Suetake, Y.: A consistent theory of, and a variational principle for, thick elastic shells undergoing finite rotations. In D. G. D. Durban and J. Simmonds (eds.), *Advances in the mechanics of plates and shells*, pp. 17–32, Kluver Academic Publisher, 2001.

19. Campello, E., Pimenta, P., and Wriggers, P.: Triangular finite shell element based on a fully nonlinear shell formulation. *Computational Mechanics*, **31**, (2003), 505–518.

20. Bertóti, E.: *Non-linear shell theory in terms of stress-functions and rotations.* Dr.Sc. dissertation, Hungarian Academy of Sciences, 2003. In Hungarian.

21. Parisch, H.: A continuum based shell theory for nonlinear applications. *Int. J. for Num. Meths. in Engng.*, **38**, (1995), 1855–1883.

22. Basar, Y. and Ding, Y.: Shear deformation models for large strain shell analysis. *Int. J. Solids Structures*, **34**, (1997), 1687–1708.

23. Bischoff, M. and Ramm, E.: Shear deformation shell element for large strains and rotations. *Int. J. Numer. Mech. Engng.*, **40**, (1997), 4427–4449.

24. Sansour, C. and Kollmann, F.: Families of 4-node and 9-node finite elements for a finite deformation shell theory. An assesment of hybrid stress, hybrid strain elements. *Computational Mechanics*, **24**, (2000), 435–447.

25. Bucalem, M. and Bathe, K.-J.: Higher-order MITC general shell elements. *Int. J. Num. Meths. Engng.*, **36**, (1993), 3729–3754.

26. Bathe, K.-J., Iosilevich, A., and Chapelle, D.: An evaluation of the MITC shell elements. *Computers and Structures*, **75**, (2000), 1–30.

Appendix A.

This appendix contains the proof of equation $\overline{\boldsymbol{R}}_n \cdot \boldsymbol{R}_2 = \boldsymbol{R}_2 \cdot \boldsymbol{R}_1$, where

$$\overline{\boldsymbol{R}}_n = \cos\vartheta_1 \boldsymbol{I} + (1-\cos\vartheta_1)\,\overline{\mathbf{n}}\,\overline{\mathbf{n}} + \sin\vartheta_1\,\overline{\mathbf{n}}\times\boldsymbol{I},$$
$$\boldsymbol{R}_2 = \cos\vartheta_2\,\boldsymbol{I} + (1-\cos\vartheta_2)\,\mathbf{e}_2\,\mathbf{e}_2 + \sin\vartheta_2\,\mathbf{e}_2\times\boldsymbol{I},$$
$$\boldsymbol{R}_1 = \cos\vartheta_1 \boldsymbol{I} + (1-\cos\vartheta_1)\,\mathbf{a}_3\,\mathbf{a}_3 + \sin\vartheta_1\,\mathbf{a}_3\times\boldsymbol{I},$$
$$\overline{\mathbf{n}} = \boldsymbol{R}_2\cdot\mathbf{a}_3 = \cos\vartheta_2\,\mathbf{a}_3 + \sin\vartheta_2\,\mathbf{e}_2\times\mathbf{a}_3.$$

Since the arbitrary vector $\mathbf{c}$ can be written in terms of three orthogonal unit vectors $\mathbf{e}_2\times\mathbf{a}_3,\ \mathbf{e}_2,\ \mathbf{a}_3$ as $\mathbf{c} = c_1\mathbf{e}_2\times\ \mathbf{a}_3 + c_2\,\mathbf{e}_2 + c_3\,\mathbf{a}_3$, it is to be pointed out that

$$\overline{\boldsymbol{R}}_n\cdot\boldsymbol{R}_2\cdot(\mathbf{e}_2\times\mathbf{a}_3) = \boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot(\mathbf{e}_2\times\mathbf{a}_3)\,, \tag{A.1}$$
$$\overline{\boldsymbol{R}}_n\cdot\boldsymbol{R}_2\cdot\mathbf{e}_2 = \boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot\mathbf{e}_2, \tag{A.2}$$
$$\overline{\boldsymbol{R}}_n\cdot\boldsymbol{R}_2\cdot\mathbf{a}_3 = \boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot\mathbf{a}_3. \tag{A.3}$$

Indeed, the transformations can be detailed on the one hand as

$$\overline{\boldsymbol{R}}_n\cdot\boldsymbol{R}_2\cdot(\mathbf{e}_2\times\mathbf{a}_3) = \overline{\boldsymbol{R}}_n\cdot[\cos\vartheta_2\,(\mathbf{e}_2\times\mathbf{a}_3) - \sin\vartheta_2\mathbf{a}_3\,] =$$
$$= \cos\vartheta_1\,[\cos\vartheta_2\,(\mathbf{e}_2\times\mathbf{a}_3) - \sin\vartheta_2\mathbf{a}_3\,] +$$
$$+(1-\cos\vartheta_1)\,[\sin\vartheta_2\,(\cos\vartheta_2\mathbf{a}_3 + \sin\vartheta_2\mathbf{e}_2\times\mathbf{a}_3) - \sin\vartheta_2\,(\cos\vartheta_2\mathbf{a}_3 + \sin\vartheta_2\mathbf{e}_2\times\mathbf{a}_3)\,] +$$
$$+\sin\vartheta_1\left(\cos^2\vartheta_2\,\mathbf{e}_2 + \sin^2\vartheta_2\,\mathbf{e}_2\right),$$

$$\overline{\boldsymbol{R}}_n\cdot\boldsymbol{R}_2\cdot\mathbf{e}_2 = \overline{\boldsymbol{R}}_n\cdot\mathbf{e}_2 =$$
$$= \cos\vartheta_1\mathbf{e}_2 + \sin\vartheta_1\,[\cos\vartheta_2\,(\mathbf{e}_2\times\mathbf{a}_3) + \sin\vartheta_2\mathbf{a}_3\,]\,,$$
$$\overline{\boldsymbol{R}}_n\cdot\boldsymbol{R}_2\cdot\mathbf{a}_3 = \overline{\boldsymbol{R}}_n\cdot\overline{\mathbf{n}} = \overline{\mathbf{n}},$$

and, on the other hand, as

$$\boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot(\mathbf{e}_2\times\mathbf{a}_3) = \boldsymbol{R}_2\cdot[\cos\vartheta_1\,(\mathbf{e}_2\times\mathbf{a}_3) + \sin\vartheta_1\mathbf{e}_2\,] =$$
$$= \cos\vartheta_2\,[\cos\vartheta_1\,(\mathbf{e}_2\times\mathbf{a}_3) + \sin\vartheta_1\mathbf{e}_2\,] +$$
$$(1-\cos\vartheta_2)\sin\vartheta_1\mathbf{e}_2 - \sin\vartheta_2\cos\vartheta_1\mathbf{a}_3\,,$$

$$\boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot\mathbf{e}_2 = \boldsymbol{R}_2\cdot[\cos\vartheta_1\mathbf{e}_2 + \sin\vartheta_1\,(\mathbf{a}_3\times\mathbf{e}_2)\,] =$$
$$= \cos\vartheta_2\,[\cos\vartheta_1\mathbf{e}_2 + \sin\vartheta_1\,(\mathbf{a}_3\times\mathbf{e}_2)\,] +$$
$$+(1-\cos\vartheta_2)\cos\vartheta_1\mathbf{e}_2 + \sin\vartheta_2\sin\vartheta_1\mathbf{a}_3\,,$$
$$\boldsymbol{R}_2\cdot\boldsymbol{R}_1\cdot\mathbf{a}_3 = \boldsymbol{R}_2\cdot\mathbf{a}_3 = \overline{\mathbf{n}},$$

i.e. conditions (A.1)-(A.3) hold.

Appendix B.

The positions of the base vectors with respect to each other do not change when they are rotated. Hence, the following relations hold (for example):

$$\overline{\mathbf{k}}_m^* \times \overline{\mathbf{k}}_k^* = (\boldsymbol{R}^* \cdot \mathbf{a}_m) \times (\boldsymbol{R}^* \cdot \mathbf{a}_k) = \boldsymbol{R}^* \ \cdot (\mathbf{a}_m \times \mathbf{a}_k)\,, \tag{B.1}$$

$$\begin{aligned} \overline{\mathbf{k}}_m \cdot \overline{\mathbf{k}}_k &= \left(\boldsymbol{R}_{\underline{s}} \cdot \overline{\mathbf{k}}_m^*\right) \cdot \left(\boldsymbol{R}_{\underline{s}} \cdot \overline{\mathbf{k}}_k^*\right) = \\ = \overline{\mathbf{k}}_m^* \cdot \overline{\mathbf{k}}_k^* &= (\boldsymbol{R}^* \cdot \mathbf{a}_m) \cdot (\boldsymbol{R}^* \cdot \mathbf{a}_k) = \mathbf{a}_m \cdot \mathbf{a}_k = a_{mk}. \end{aligned} \tag{B.2}$$