# One Dimensional Magnetized TG Gas Properties in an External Magnetic Field

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With Girardeau's Fermi-Bose mapping, we have constructed the eigenstates of a TG gas in an external magnetic field. When the number of bosons $N$ is commensurate with the number of potential cycles $M$, the probability of this TG gas in the ground state is bigger than the TG gas raised by Girardeau in 1960. Through the comparison of properties between this TG gas and Fermi gas, we find that the following issues are always of the same: their average value of particle's coordinate and potential energy, system's total momentum, single-particle density and the pair distribution function. But the reduced single-particle matrices and their momentum distributions between them are different.

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## I. INTRODUCTION

The problem of a one-dimensional (1D) hard-core gas was first studied classically by Tonks [1]. Girardeau [2, 3] continued the research in quantum, and put forward the Fermi-Bose mapping to solve energy spectrum and the corresponding wave function of a one dimensional Tonks-Girardeau (TG) gas. After that, Lieb and Liniger [4] considered $N$ Bose particles interacting via a repulsive $\delta$-function potential with a coupling constant $\gamma$. In fact, TG gas can be considered as the Lieb-Liniger gas when $\gamma \to \infty$.

During the past decade, the study on TG gas has undergone a rapid development in theoretical and experimental [5–17]. In particular, A. Lenard [5] indicated that the off-diagonal parts of the one-body density matrix and the momentum distribution show distinct differences in both the TG gas and the Fermi gas. G. J. Lapeyre et al. [8] studied the momentum distribution of a harmonically trapped gas. The most important progress is the realization of the strongly interacting bosons [16] in a one-dimensional optical-lattice trap and the TG gas [17] by trapping $^{87}$Rb atoms by trapping them with a combination of two light traps.

We have known some properties of one dimensional TG gas, and we are interested in its properties and features in external magnetic field. Consequently, we first reveal some properties of both gases in the same external magnetic field and compared them. With the Fermi-Bose mapping [2, 3], we obtained all the eigenenergies and eigenstates of this TG gas. When the number of bosons $N$ is commensurate with the number of potential cycles $M$, the TG system in an external magnetic field will be in the ground state with a bigger probability than the TG gas raised by Girardeau in 1960. Many properties of the TG gas and the Fermi gas are always of the same even if it's time-dependent, such as their average value of particle's coordinate and potential energy, system's total momentum, single-particle density and the pair distribution function. But both their reduced single-particle matrices and momentum distributions are different.

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## II. TG GAS IN AN EXTERNAL MAGNETIC FIELD

In a TG gas, the boson is assumed to have an "impenetrable" hard core characterized by a radius of $a$. From Girardeau's work, with the hard core radius $a \to 0$, the interparticle interaction is given by

$$U(x_i, x_j) = \begin{cases} 0, & x_i \neq x_j, \\ \infty, & x_i = x_j. \end{cases} \quad (1)$$

Such an interparticle interaction could be represented by the following subsidiary condition on the wave function $\psi$ :

$$\psi(x_1, \cdots, x_N, t) = 0 \ \text{if} \ x_i = x_j, 1 \leqslant i < j \leqslant N. \quad (2)$$

In order to let the TG gas wave function, which must be symmetric with respect to the permutations of any $x_i$ and $x_j$, satisfy this condition (2), we need the ideas of Fermi-Bose mapping proposed by Girardeau [2, 3]. That is, using the fact that the Fermi wave function, denoted by $\psi^F$, satisfies condition (2) naturally and is antisymmetric, a bosonic wave function $\psi^B$ of a TG gas can be constructed by

$$\psi^B(x_1, \cdots, x_N, t) = A(x_1, \cdots, x_N)\psi^F(x_1, \cdots, x_N, t) \quad (3)$$

in which

$$A(x_1, \cdots, x_N) \equiv \prod_{i>j}^{N} \mathrm{sgn}(x_i - x_j), \quad (4)$$

and

$$\mathrm{sgn}(x) \equiv \frac{x}{|x|} = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

As the ground state of a bose system is non-negative [19], mapping (3) for the stationary ground state of a system reduces to a simplified form [2]

$$\psi^B(x_1, \cdots, x_N) = |\psi^F(x_1, \cdots, x_N)| \tag{5}$$

To illustrate our general ideas, we now study the case that magnetized bosons in an external magnetic field $B(x) = -B\cos(2\omega x)$. For simplicity, we take $B(x)$ independent of $t$. therefore,

$$\hat{H} = \sum_{i=1}^{N} \left[ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + V(x_i) \right]. \tag{6}$$

in which

$$V(x) = \begin{cases} \mu B\cos(2\omega x), & 0 \leqslant x_i \leqslant \frac{L}{\omega}, \\ 0, & \text{else.} \end{cases} \tag{7}$$

We suppose $L = M\pi$, where $M$ is an integer and $\mu$ is the magnetic moment of atom. For the sake of simplicity, we assume that both $N$ and $M$ are odd (As a result of the introduction of $A(x_1, \cdots, x_N)$, $\psi^B$ is periodic if $N$ is odd and antiperiodic if $N$ is even [2, 12]). From Bloch theory, the corresponding energy spectrum of system in periodic potentials has the energy band structure. We use the notation $n$ as the band index and $k$ as the Bloch wave vector in the first Brillouin zone. For convenience, we use $\alpha = \{n, k\}$ to denote both the band index and the Bloch wave vector. Not considering the interparticle interaction, every particle in $x \in [0, L/\omega]$ is governed by

$$\left[ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu B\cos(2\omega x) \right] \varphi_\alpha(x) = E_\alpha \varphi_\alpha(x). \tag{8}$$

Substitute $z = \omega x$, $q = \dfrac{m\mu B}{\hbar^2\omega^2}$ and $\lambda = \dfrac{2mE}{\hbar^2\omega^2}$ into equation (8), then this equation becomes

$$\frac{d^2\varphi(z)}{dz^2} + [\lambda - 2q\cos(2z)]\varphi(z) = 0, \tag{9}$$

$$z \in [0, L].$$

It is called Mathieu's Differential Equation, which can be solved with Randall B. Shirts's [18] method. Using Bloch's theorem, we first set

$$\varphi(z) = \exp(i\nu z)u(z), \tag{10}$$

$$\varphi(z) = \varphi(z + \pi), \tag{11}$$

$$u(z) = u(z + L), \tag{12}$$

$$\nu = \frac{2l}{M}, \quad \left( l \in 0, \pm 1, \cdots, \pm\frac{M-1}{2} \right). \tag{13}$$

Since $u(z)$ is periodic with period $\pi$, it can be expanded in Fourier series:

$$\varphi(z) = \exp(i\nu z) \sum_n c_n \exp(i2nz). \tag{14}$$

Then substitute (14) into (9), we obtain an infinite symmetric tridiagonal matrix equation, that is, eigen equation.

$$\begin{pmatrix} \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & (\nu-4)^2 & q & 0 & 0 & 0 & \cdots \\ \cdots & q & (\nu-2)^2 & q & 0 & 0 & \cdots \\ \cdots & 0 & q & \nu^2 & q & 0 & \cdots \\ \cdots & 0 & 0 & q & (\nu+2)^2 & q & \cdots \\ \cdots & 0 & 0 & 0 & q & (\nu+4)^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} \times \begin{pmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix}. \tag{15}$$

By truncating the matrix in each direction (centered at the smallest diagonal element $\nu^2$) at sufficiently large dimensions, approximations to the desired eigenvalues can be obtained to any desired precision. We could obtain the eigenvalue and eigenfunction easily using Mathematica or Matlab. In practice, the eigenvalue and eigenfunction converge quickly as the dimension increases when $q$ is not too large. For example, if we choose $\nu = 6/7$ and $q = 1$, the relative difference of the lowest eigenvalue between truncating the matrix at 21-D and 2001-D is only $10^{-38}$. For large values of $q$ and for high orders, we can use asymptotic expansions instead [21, 22]. since the matrix dimension that needed to get accurate eigenvalues and eigenfunctions becomes large.

With Girardeau's Fermi-Bose mapping (3), the eigenstates of the hard-core Bose system is given by the Slater determinant

$$\psi^B(x_1, \cdots, x_N) = \frac{A(x_1, \cdots, x_N)}{\sqrt{N!}} \times \begin{vmatrix} \varphi_{\alpha_1}(x_1) & \varphi_{\alpha_2}(x_1) & \cdots & \varphi_{\alpha_N}(x_1) \\ \varphi_{\alpha_1}(x_2) & \varphi_{\alpha_2}(x_2) & \cdots & \varphi_{\alpha_N}(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{\alpha_1}(x_N) & \varphi_{\alpha_2}(x_N) & \cdots & \varphi_{\alpha_N}(x_N) \end{vmatrix}. \tag{16}$$

The total energy of the system is

$$E = \sum_{i=1}^{N} E_{\alpha_i}. \tag{17}$$

As a result of the Bose-Fermi mapping, the energy spectrum of the Bose and corresponding Fermi system are identical. When the temperature is 0, the system will be on the ground state — $N$ particles on the $N$ lowest eigenstates, respectively.

Calculations show that the energy gap $\triangle E$ between the first and the second Bloch bands decreases with the decrease of $M$, and when $M$ is large, the energy gap $\triangle E$ is independent of $M$. The energy gap between the first and the second Bloch bands as a function of magnetic field $B$ and frequency $\omega$ are plotted in Fig. 1, respectively. At $B = 0$, raised by Girardeau [2], the energy gap of TG gas is the smallest (not 0). The energy gap increases as $B$ and $\omega$ increase. As the Boltzman Distribution Law notes, $\frac{n_k}{n_{k'}} \propto \exp(-\frac{\triangle E}{kT})$, with $\triangle E = E_k - E_{k'}$, $n_k$ and $n_{k'}$ are particle numbers on the corresponding states. So the system have a bigger probability in the ground state with large $B$ and $\omega$ than the case $B = 0$ when the total number $N = M$. That is to say, one could realize ground-stated TG system more simple with large $q$ and $\omega$ than $q = 0$ when the total number $N = M$.

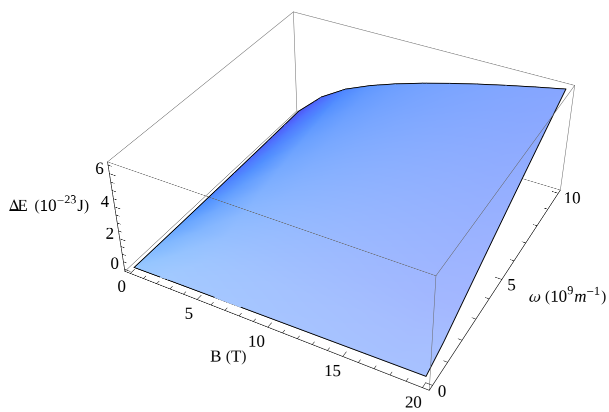


FIG. 1: Plots of the energy gap between the first and the second Bloch bands as a function of magnetic induction $B$ and frequency $\omega$. $M = 9$, $m = 1.44 \times 10^{-25}$kg, $\mu = 9.274 \times 10^{-24}\mathrm{J}\cdot\mathrm{T}^{-1}$.

## III. PROPERTIES OF THE TG GAS COMPARED WITH THE FERMI GAS

In this section, we further study properties of the 1D $N$ magnetized hard-core Bosons in the external magnetic field (7), and then compared with the properties of 1D $N$ magnetized Fermions in the "same" state which hard-core Bosons occupy.

### A. Reduced single-particle density matrix and Single-particle density

The reduced single-particle density matrix with normalization $\int \rho(x,x)\,\mathrm{d}x = N$ is given by

$$\rho^B(x,x',t) \equiv N \int \psi^{B*}(x,x_2,\cdots,x_N,t) \times \psi^B(x',x_2,\cdots,x_N,t)\,\mathrm{d}x_2\cdots\mathrm{d}x_N. \quad (18)$$

For the Fermi gas, the reduced single-particle density matrix can be simplified to

$$\begin{aligned}\rho^F(x,x',t) &= N \int \psi^{F*}(x,x_2,\cdots,x_N,t) \\ &\quad \times \psi^F(x',x_2,\cdots,x_N,t)\,\mathrm{d}x_2\cdots\mathrm{d}x_N \\ &= \sum_{\substack{\alpha=all\ states \\ occupyed}} \varphi_\alpha(x,t)^*\varphi_\alpha(x',t).\end{aligned} \quad (19)$$

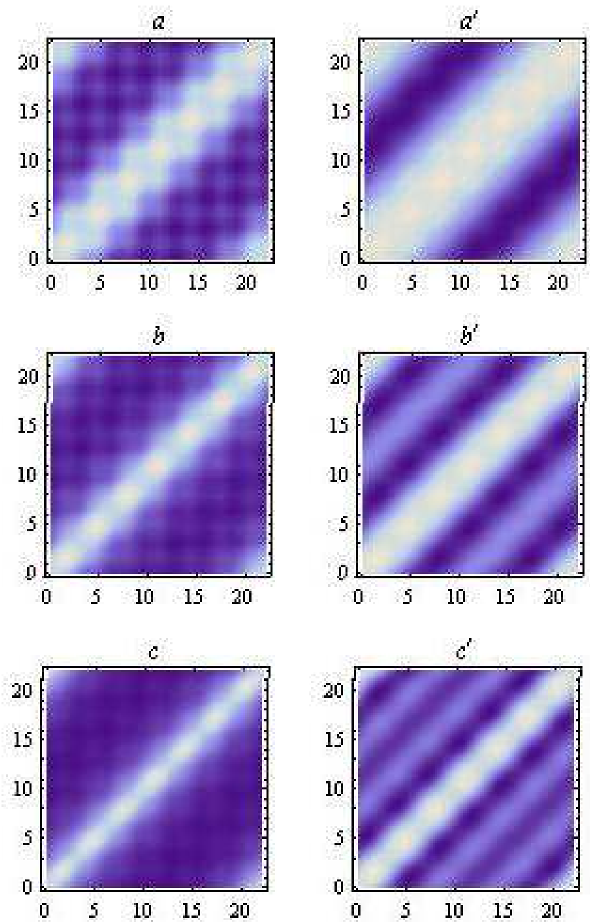


FIG. 2: Density-plots of $\rho(z,z')$ of the hard-core TG gas ($a$,$b$,$c$) and Fermi gas ($a'$,$b'$,$c'$), $M = 7$. Abscissa is $z_1$, Ordinate is $z_2$. ($a$,$a'$) $N = 3$; ($b$,$b'$) $N = 5$; ($c$,$c'$) $N = 7$.

Both the reduced single-particle matrices of hard-core bosons and fermions in the ground state in the external magnetized field (7) are plotted separately in Fig. 2. The multidimensional integral in equation (18) is evaluated numerically by Monte Carlo integration using Mathematica. Because the relationship between $x$ and $z$ is proportional, for the sake of simplicity, we take $z$ as a variable. From the figure we can see the off-diagonal elements of the matrix of both the two kinds of gases decay quickly as $N$ increases; The bright diagonal stripe of TG gas is thinner than Fermi gas under the same condition. This reflects the condensate properties of Bose gas; And the shock of the off-diagonal elements $\rho(z, M\pi - z)$ of TG gas is smaller than Fermi gas. To see this more clearly, we have plotted the off-diagonal elements $\rho(z, M\pi - z)$ in Fig. 3. Sudden rise in the border is the result of using periodic boundary conditions.

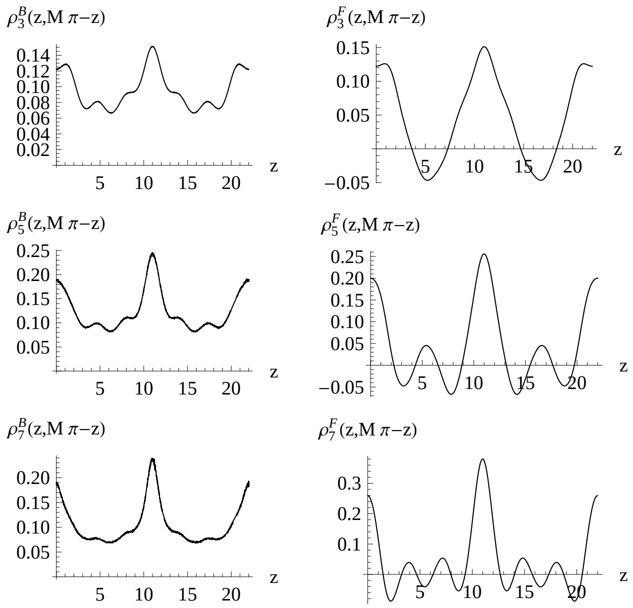


FIG. 3: Plots of $\rho(z, M\pi - z)$, $M = 7$. $\rho_n^B(z, M\pi - z)$ stands for $\rho(z, M\pi - z)$ of $n$ hard-core Boson particles, so as $\rho_n^F(z, M\pi - z)$

.

Using the mapping (3), we can prove that the single-particle density $\rho(x,t)$, normalized to $N$, of both hard-core Bosons and Fermions are equal, no matter whether they are on the ground-state or not.

$$\begin{aligned}\rho^B(x,t) &= N \int |\psi^B(x, x_2, \cdots, x_N, t)|^2 \, \mathrm{d}x_2 \cdots \mathrm{d}x_N \\ &= N \int |\psi^F(x, x_2, \cdots, x_N, t)|^2 \, \mathrm{d}x_2 \cdots \mathrm{d}x_N \\ &= \sum_{\substack{\alpha = all\ states \\ occupyed}} |\varphi_\alpha(x,t)|^2 \\ &= \rho^F(x,t)\end{aligned} \tag{20}$$

$\rho(x,t)$ is just the element of the reduced single-particle matrix $\rho(x, x', t)$ when $x = x'$. Fig. 2 shows that $\rho(x)$ is cyclical, which indicates that particles tend to stay cycle at where its potential energy is low.

### B. Pair distribution function

The pair distribution function, which is the probability of finding a second particle as a function of distance from an initial particle, normalized to $N(N-1)$, is defined as

$$\begin{aligned}&D(x_1, x_2, t) \\ &= N(N-1) \int |\psi^B(x_1, x_2, \cdots, x_N, t)|^2 \, \mathrm{d}x_3 \cdots \mathrm{d}x_N \\ &= N(N-1) \int |\psi^F(x_1, x_2, \cdots, x_N, t)|^2 \, \mathrm{d}x_3 \cdots \mathrm{d}x_N = \\ &\frac{1}{2} \sum_{\substack{\alpha,\alpha' = all\ states \\ occupied}} |\varphi_\alpha(x_1, t)\varphi_{\alpha'}(x_2, t) - \varphi_\alpha(x_2, t)\varphi_{\alpha'}(x_1, t)|^2.\end{aligned} \tag{21}$$

From the above equation we can see that the pair distribution functions of both hard-core Boson gas and Fermi gas are identical. In fact, the average of particle's coordinate and potential of both the two gases are identical, too. Because they involve absolute values of the wave functions only. Also we can say this feature is determined by the fact that the single particle density of both hard-core Boson gas and Fermi gas are exactly of the same. For the ground state of a system with $N$ particles governed by Hamilton (6), the pair distribution function is

$$D(x_1, x_2) = \frac{1}{2} \sum_{\alpha,\alpha'=0}^{N-1} |\varphi_\alpha(x_1)\varphi_{\alpha'}(x_2) - \varphi_\alpha(x_2)\varphi_{\alpha'}(x_1)|^2. \tag{22}$$

Figure 4 shows density-plots of the pair distribution function of the ground state for different number of particles. when $x_1 = x_2$, $D(x_1, x_2) = 0$. It is the result of the impenetrable hard-core interparticle interaction. Just like the single particle density, the pair distribution function reflects the same periodicity and particles tending to tarry at the low potential energy. As the number of particle increases, the black strip becomes thinner. One of the reason is that the averaged distance between particles reduces as the number of particles increases while the scope of the potential field is certain. In two distant regions, away from the diagonal in Fig. 4, $D(z_1, z_2)$ is cyclical. This reflects the hard-core interaction between particles is local.

### C. Momentum distribution

The momentum distribution, related to the reduced density matrix, is defined as

$$n(k) = \frac{1}{2\pi} \int \rho(x, x') e^{-ik(x-x')} \, \mathrm{d}x \, \mathrm{d}x', \tag{23}$$

with the normalization

$$\int n(k) \, \mathrm{d}k = N.$$

Actually, the momentum distribution is just the Fourier transformation of the reduced density matrix (18). We

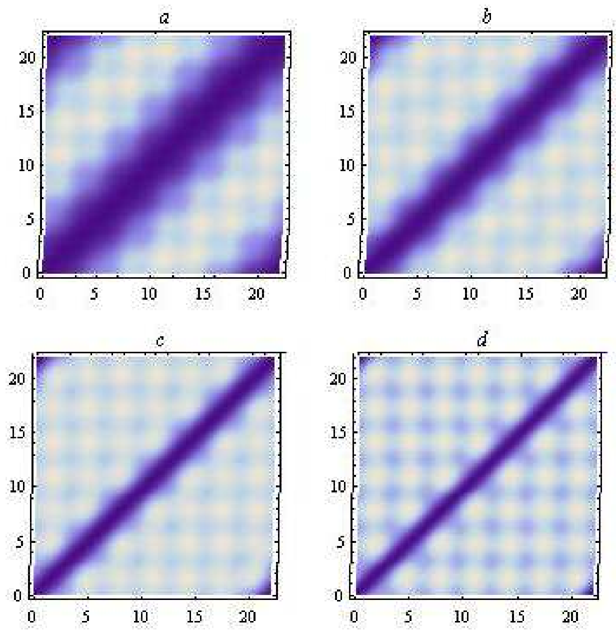


FIG. 4: Density-plots of the pair distribution function $D(z_1, z_2)$. Abscissa is $z_1$, Ordinate is $z_2$. $M = 7$. (a) $N = 2$; (b) $N = 3$; (c) $N = 5$; (d) $N = 7$.

know that the Fourier transform of the original function and the transformed function is one-to-one, as the difference of the reduced density matrix $\rho(x, x')$ of the hard-core Bose gas and Fermion gas, the momentum distribution is surely not the same. Yuan Lin and Biao Wu [10] have plotted the momentum distributions for both the TG gas and the free Fermi gas in a periodic Kronig-Penney potential, which shows the difference — Even a free Fermi gas has a broader momentum distribution than the most strongly interacting boson gas. We would like to add that, although their momentum distribution functions are different, their total momentum are of the same. With the equation

$$(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})\mathrm{sgn}(x_2 - x_1) = 0, \tag{24}$$

we can prove

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A(x_1, \cdots, x_N) = 0. \tag{25}$$

And then, go ahead, we can obtain

$$\bar{P}(t)^B = \bar{P}(t)^F, \text{ where } P = -i\hbar \sum_{i=1}^{N} \frac{\partial}{\partial x_i}. \tag{26}$$

## IV. SUMMARY AND CONCLUSIONS

On solving the eigenfunction of the TG gas in the external magnetic field, we found that the TG system with large $B$ and $\omega$ has a bigger probability in the ground state than the TG gas raised by Girardeau in 1960 when the number of bosons $N$ is commensurate with the number of potential cycles $M$. It reveals that we can realize ground-stated TG system more easier. And then we have studied properties of magnetized TG gas and Fermi gas in an external magnetic field. comparing with each other when they are in the "same" state reveals that it is impossible to distinguish them just from their average value of particle's coordinate and potential energy, system's otal momentum, single-particle density and pair distribution function. But we could distinguish them from their reduced single-particle matrices or their momentum distributions. In order to illustrate our results, we have plotted their reduced single-particle matrices and the momentum distributions when they are in the ground state. Although their momentum distribution functions are different, their total momentum are of the same. These results will be deeply helpful for understanding the TG gas.

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