Turk J Math
26 (2002) , 53 – 68

On 1-point Gromov-Witten invariants of the Hilbert schemes of points on surfaces

Wei-Ping Li, Zhenbo Qin

Abstract

We compute certain 1-point genus-0 Gromov-Witten invariants of the Hilbert scheme of points on a simply-connected smooth projective surface.

1. Introduction

The Hilbert scheme $X^{[n]}$ of points in a smooth projective surface X is the set of length-n 0-dimensional closed subschemes of X. On one hand, $X^{[n]}$ is the moduli space of rank-1 torsion free sheaves V on X such that the first and second Chern classes of V are equal to 0 and n respectively. It is the simplest one among the moduli spaces of rank-r stable vector bundles (or sheaves in general) on a projective surface, which are isomorphic to the moduli spaces of anti-self-dual Yang-Mills connections on some principle bundles over X. Mathematicians as well as physicists showed great interest in these moduli spaces. One area of interest is the Gromov-Witten invariants of the Hilbert scheme $X^{[n]}$. On the other hand, the Hilbert scheme $X^{[n]}$ is smooth [Fo1]. Hence it is the desingularization of the n-th symmetric product $X^{(n)}$ of X. In fact, the Hilbert-Chow map

$$\rho\colon X^{[n]} \to X^{(n)}. \tag{1}$$

sending an element in $X^{[n]}$ to its support in $X^{(n)}$ is a crepant resolution of the orbifold $X^{(n)}$. Recently, Ruan [Ru2] formulated some conjecture on the relation between the cohomology rings of crepant resolutions of orbifolds and the orbifold cohomology rings of the orbifolds themselves. It turns out that the Gromov-Witten invariants of the crepant resolutions appear in a very interesting way in Ruan's conjecture. In this paper, we shall compute the 1-point Gromov-Witten invariants of $X^{[n]}$ with respect to some special degree-2 homology cycles on $X^{[n]}$. Our result partially verifies Ruan's conjecture for the crepant resolution $\rho\colon X^{[n]} \to X^{(n)}$.

Throughout the paper, we assume that X is a simply-connected smooth projective surface. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. Let $x_1, \ldots, x_{n-1} \in X$ be distinct but fixed points. Let $M_2(x_1) =$

1991 *Mathematics Subject Classification.* Primary 14C05, 14N35.
Key words and phrases. Hilbert schemes, projective surfaces, Gromov-Witten invariants.
Partially supported by the grant HKUST6170/99P.
Partially supported by an NSF grant.

$\{\xi \in X^{[2]} \mid \mathrm{Supp}(\xi) = \{x_1\}\}$ be the punctual Hilbert scheme parametrizing length-2 0-dimensional subschemes supported at x_1. It is known that $M_2(x_1) \cong \mathbb{P}^1$. Let β_n be the smooth rational curve in $X^{[n]}$ defined by

$$\big\{\, \xi + x_2 + \ldots + x_{n-1} \in X^{[n]} \mid \xi \in M_2(x_1) \big\}. \tag{2}$$

Clearly, the curve β_n is mapped to a point by the Hilbert-Chow map ρ.

Let d be a positive integer, and let $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ be the moduli space of 1-point stable maps $\mu\colon (D; p) \to X^{[n]}$ from a genus-0 nodal curve D with one marked point p to $X^{[n]}$ such that $\mu_*(D)$ is homologous to $d\beta_n$. A point in $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ is denoted by $[\mu\colon (D; p) \to X^{[n]}]$. The expected complex dimension of the moduli space $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ is given by

$$\mathfrak{d} = -K_{X^{[n]}} \cdot d\beta_n + \dim X^{[n]} - 3 + 1 = 2n - 2. \tag{3}$$

Here we used the fact that $K_{X^{[n]}} \cdot \beta_n = 0$ since the canonical class $K_{X^{[n]}}$ of $X^{[n]}$ is the pullback of a divisor on $X^{(n)}$ via the Hilbert-Chow map.

Take a cohomology class $\alpha \in H^{4n-4}(X^{[n]}, \mathbb{C})$. Consider the evaluation map

$$ev_1\colon \overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n) \to X^{[n]}, \quad ev_1([\mu\colon (D; p) \to X^{[n]}]) = \mu(p) \tag{4}$$

Let $[\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)]^{vir}$ be the virtual fundamental class. The main result of the paper is the computation of the 1-point Gromov-Witten invariant

$$\langle \alpha \rangle_{0,d\beta_n} \stackrel{\text{def}}{=} \int_{[\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)]^{vir}} ev_1^*(\alpha). \tag{5}$$

We refer to Theorem 3.5 for the detailed statement of the main result.

Our motivation for computing the 1-point Gromov-Witten invariant (5) comes from the above-mentioned Ruan's conjecture for a crepant resolution $\rho : Y \to Z$ of an orbifold Z. An essential ingredient in Ruan's conjecture is the quantum corrections which are related to the 3-point Gromov-Witten invariants $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta}$ in which $\beta \neq 0$ and $\rho_*(\beta) = 0$. In our case, the symmetric product $X^{(n)}$ is an orbifold, and the Hilbert-Chow map $\rho\colon X^{[n]} \to X^{(n)}$ is a crepant resolution of $X^{(n)}$. Moreover, if $\beta \neq 0$ and $\rho_*(\beta) = 0$ for some $\beta \in H_2(X^{[n]}; \mathbb{Z})$, then necessarily $\beta = d\beta_n$ for some positive integer d. Even though it remains to be a challenge to compute all the 3-point Gromov-Witten invariants $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,d\beta_n}$ for $X^{[n]}$ at the present time, we are able to perform computations in some special cases. In particular, we are successful in computing all the 1-point Gromov-Witten invariants $\langle \alpha \rangle_{0,d\beta_n}$. Our Theorem 3.5 partially verifies Ruan's conjecture for the crepant resolution $\rho\colon X^{[n]} \to X^{(n)}$. We remark that when $n = 2$, all the 3-point Gromov-Witten invariants of $X^{[2]}$ can be reduced to 1-point Gromov-Witten invariants of $X^{[2]}$. Indeed, our result for $n = 2$ has been used by Ruan [Ru2] to verify his conjecture for the crepant resolution $\rho\colon X^{[2]} \to X^{(2)}$ of the symmetric product $X^{(2)}$.

The key step in computing the 1-point Gromov-Witten invariants $\langle \alpha \rangle_{0,d\beta_n}$ is to determine the obstruction bundle over the moduli space $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$. Even though the curves homologous to $d\beta_n$ in $X^{[n]}$ are complicated, when we compute $\langle \alpha \rangle_{0,d\beta_n}$, we only need to deal with those stable maps $[\mu\colon (D;p) \to X^{[n]}]$ such that $\mu(D)$ is of the form (2). Using the earlier work [LQZ] concerning rational curves of degree-1 in $X^{[n]}$, we are able to determine the obstruction bundle over a Zariski open subset of $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$, which is sufficient for us to compute $\langle\alpha\rangle_{0,d\beta_n}$.

Finally, this paper is organized as follows. In section two, we review Gromov-Witten invariants and virtual fundamental classes. In addition, we discuss some basics of the Hilbert scheme $X^{[n]}$, and determine a basis of $H_4(X^{[n]}, \mathbb{C})$ by using the results of Göttsche, Grojnowski, and Nakajima [Got, Gro, Nak]. In section three, we study the obstruction bundle, and prove Theorem 3.5.

2. Preliminaries

In this section, we shall review the notions of stable maps and Gromov-Witten invariants. In addition, we shall recall some basic facts and notations for the Hilbert scheme of points on a smooth projective surface.

2.1. Stable maps and Gromov-Witten invariants

Let Y be a smooth projective variety. An k-point *stable map* to Y consists of a complete nodal curve D with k distinct ordered smooth points $p_1, \ldots, p_k$ and a morphism $\mu : D \to Y$ such that the data $(\mu, D, p_1, \ldots, p_k)$ has only finitely many automorphisms. In this case, the stable map is denoted by $[\mu : (D; p_1, \ldots, p_k) \to Y]$. For a fixed homology class $\beta \in H_2(Y, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ be the coarse moduli space parameterizing all the stable maps $[\mu : (D; p_1, \ldots, p_k) \to Y]$ such that $\mu_*[D] = \beta$ and the arithmetic genus of D is g. Then, we have the evaluation map:

$$ev_k\colon \overline{\mathfrak{M}}_{g,k}(Y, \beta) \to Y^k \tag{6}$$

defined by $ev_k([\mu : (D; p_1, \ldots, p_k) \to Y]) = (\mu(p_1), \ldots, \mu(p_k))$. It is known [F-P, LT1, LT2, B-F] that the coarse moduli space $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ is projective and has a virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\text{vir}} \in A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,k}(Y, \beta))$ where

$$\mathfrak{d} = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + k \tag{7}$$

is the expected complex dimension of $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$, and $A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,k}(Y, \beta))$ is the Chow group of $\mathfrak{d}$-dimensional cycles in the moduli space $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$.

The Gromov-Witten invariants are defined by using the virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\text{vir}}$. Recall that an element $\alpha \in H^*(Y, \mathbb{C}) \stackrel{\text{def}}{=} \bigoplus_{j=0}^{2\dim_{\mathbb{C}}(Y)} H^j(Y, \mathbb{C})$ is *homogeneous* if $\alpha \in H^j(Y, \mathbb{C})$ for some j; in this case, we take $|\alpha| = j$. Let $\alpha_1, \ldots, \alpha_k \in H^*(Y, \mathbb{C})$ such that every α_i is homogeneous and

$$\sum_{i=1}^{k} |\alpha_i| = 2\mathfrak{d}. \tag{8}$$

Then, we have the k-point Gromov-Witten invariant defined by:

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{g,\beta} = \int_{[\overline{\mathfrak{M}}_{g,k}(Y,\beta)]^{\rm vir}} ev_k^*(\alpha_1 \otimes \ldots \otimes \alpha_k). \tag{9}$$

Next, we summarize certain properties concerning the virtual fundamental class. To begin with, we recall that *the excess dimension* is the difference between the dimension of $\overline{\mathfrak{M}}_{g,k}(Y,\beta)$ and the expected dimension $\mathfrak{d}$ in (7). Let T_Y stand for the tangent sheaf of Y. For $0 \le i < k$, we shall use

$$f_{k,i} : \overline{\mathfrak{M}}_{g,k}(Y,\beta) \to \overline{\mathfrak{M}}_{g,i}(Y,\beta) \tag{10}$$

to stand for the forgetful map obtained by forgetting the last $(k-i)$ marked points and contracting all the unstable components. It is known that $f_{k,i}$ is flat when $\beta \neq 0$ and $0 \le i < k$. The following can be found in [LT1, Beh, Get, C-K, LiJ].

Proposition 2.1. *Let $\beta \in H_2(Y,\mathbb{Z})$ and $\beta \neq 0$. Let e be the excess dimension of $\overline{\mathfrak{M}}_{g,k}(Y,\beta)$, and $\mathfrak{M} \subset \overline{\mathfrak{M}}_{g,k}(Y,\beta)$ be a closed subscheme. Then,*

(i) $[\overline{\mathfrak{M}}_{g,k}(Y,\beta)]^{vir} = (f_{k,0})^*[\overline{\mathfrak{M}}_{g,0}(Y,\beta)]^{vir}$;

(ii) $[\overline{\mathfrak{M}}_{g,k}(Y,\beta)]^{vir} = c_e(R^1(f_{k+1,k})_*(ev_{k+1})^*T_Y)$ *if* $R^1(f_{k+1,k})_*(ev_{k+1})^*T_Y$ *is a rank-e locally free sheaf over the moduli space* $\overline{\mathfrak{M}}_{g,k}(Y,\beta)$;

(iii) $[\overline{\mathfrak{M}}_{g,k}(Y,\beta)]^{vir}|_{\mathfrak{M}} = c_e((R^1(f_{k+1,k})_*(ev_{k+1})^*T_Y)|_{\mathfrak{M}})$ *if there exists an open subset* $\mathfrak{U}$ *of* $\overline{\mathfrak{M}}_{g,k}(Y,\beta)$ *such that* $\mathfrak{M} \subset \mathfrak{U}$ *(i.e,* $\mathfrak{U}$ *is an open neighborhood of* $\mathfrak{M}$*) and the restriction* $(R^1(f_{k+1,k})_*(ev_{k+1})^*T_Y)|_{\mathfrak{U}}$ *is a rank-e locally free sheaf over* $\mathfrak{U}$.

2.2. Basic facts on the Hilbert scheme of points on a surface

Let X be a simply-connected smooth projective surface, and $X^{[n]}$ be the Hilbert scheme of points in X. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ be the corresponding sheaf of ideals. In $X^{[n]} \times X$, we have the universal codimension-2 subscheme:

$$\mathcal{Z}_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \mathrm{Supp}\,(\xi)\} \subset X^{[n]} \times X. \tag{11}$$

In $X^{[n-1]} \times X^{[n]}$, we have the $2n$-dimensional smooth incidence subscheme:

$$X^{[n-1,n]} = \{(\xi, \eta) \in X^{[n-1]} \times X^{[n]} \mid I_\xi \supset I_\eta\}. \tag{12}$$

For a subset $Y \subset X$, we define the subset $M_n(Y)$ in the Hilbert scheme $X^{[n]}$:

$$M_n(Y) = \{\xi \in X^{[n]} \mid \mathrm{Supp}(\xi) \text{ is a point in } Y\} \subset X^{[n]}. \tag{13}$$

In particular, for a fixed point $x \in X$, $M_n(x)$ is just the punctual Hilbert scheme of points on X at x. It is known that the punctual Hilbert schemes $M_n(x)$ are isomorphic for all the surfaces X and all the points $x \in X$.

The definitions and properties of the maps listed below can be found in [E-S].

Notation. There exist various morphisms:

$$\begin{array}{rcl} f_n & : & X^{[n-1,n]} \to X^{[n-1]} \text{ with } f_n(\xi,\eta) = \xi. \\ g_n & : & X^{[n-1,n]} \to X^{[n]} \text{ with } g_n(\xi,\eta) = \eta. \\ \psi_n & : & X^{[n-1,n]} \to \mathcal{Z}_n \text{ with } \psi_n(\xi,\eta) = \big(\eta, \mathrm{Supp}(I_\xi/I_\eta)\big). \\ q & : & X^{[n-1,n]} \to X \text{ with } q(\xi,\eta) = \mathrm{Supp}(I_\xi/I_\eta). \end{array}$$

Convention: Let V be an n-dimensional vector space. We use $\mathbb{P}(V)$ to denote the set of 1-dimensional quotients of the vector space V.

Theorem 2.2. *(see* [E-S]*) Adopt the above notations.*

(i) *The morphism* $\psi_n\colon X^{[n-1,n]} \to \mathcal{Z}_n$ *is canonically isomorphic to the projectification* $\mathbb{P}(\omega_{\mathcal{Z}_n}) \to \mathcal{Z}_n$ *where* $\omega_{\mathcal{Z}_n}$ *is the dualizing sheaf of* $\mathcal{Z}_n$*;*

(ii) *The morphism* $(f_n, q)\colon X^{[n-1,n]} \to X^{[n-1]} \times X$ *is canonically isomorphic to the blowing-up of* $X^{[n-1]} \times X$ *along* $\mathcal{Z}_{n-1}$*. The exceptional locus is*

$$E_n = \{(\xi,\eta) \in X^{[n-1,n]} \mid \mathrm{Supp}(\xi) = \mathrm{Supp}(\eta) \text{ and } \xi \subset \eta\}; \tag{14}$$

Let $\xi \in X^{[n-k]}$ and $\eta \in X^{[k]}$. If $\mathrm{Supp}(\xi) \cap \mathrm{Supp}(\eta) = \emptyset$, then we use $\xi + \eta$ to represent the closed subscheme $\xi \cup \eta$ in $X^{[n]}$. Similarly, given a subvariety Y of $X^{[n-k]}$ and a point $\eta \in X^{[k]}$ such that $\left(\bigcup_{\xi \in Y} \mathrm{Supp}(\xi)\right) \cap \mathrm{Supp}(\eta) = \emptyset$, we use $Y + \eta$ to represent the subvariety in $X^{[n]}$ consisting of all the points $\xi + \eta$ with $\xi \in Y$.

Next, we review some results on homology groups of the Hilbert scheme $X^{[n]}$ due to Göttsche [Got], Grojnowski [Gro], and Nakajima [Nak]. Their results say that the space $\mathbb{H} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} \bigoplus_{k=0}^{4n} H_k(X^{[n]}, \mathbb{C})$ is an irreducible highest weight representation of the Heisenberg algebra generated by $\mathfrak{a}_{-n}(\alpha), n \in \mathbb{Z}, \alpha \in H_*(X,\mathbb{C}) \stackrel{\text{def}}{=} \bigoplus_{k=0}^{4} H_k(X,\mathbb{C})$. Moreover, $|0\rangle \stackrel{\text{def}}{=} 1 \in H_0(X^{[0]}, \mathbb{C}) = \mathbb{C}$ is a highest weight vector. It follows that the space $\mathbb{H}$ is a linear span of elements of the form $\mathfrak{a}_{-n_1}(\alpha_1) \ldots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle$ where $k \ge 0$, $n_1, \ldots, n_k > 0$, and $\alpha_1, \ldots, \alpha_k \in H_*(X,\mathbb{C})$. The geometric interpretation of $\mathfrak{a}_{-n_1}(\alpha_1) \ldots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle$ for homogeneous classes $\alpha_1, \ldots, \alpha_k \in H_*(X,\mathbb{C})$ can be understood as follows. For $i = 1, \ldots, k$, let $\alpha_i \in H_{|\alpha_i|}(X,\mathbb{C})$ be represented by a cycle A_i such that $A_1, \ldots, A_k$ are in general position. Then,

$$\mathfrak{a}_{-n_1}(\alpha_1) \ldots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle \in H_m(X^{[n]}, \mathbb{C}) \tag{15}$$

where $n = \sum_{i=1}^{k} n_i$ and $m = \sum_{i=1}^{k}(2n_i - 2 + |\alpha_i|)$. In addition, up to a scalar, $\mathfrak{a}_{-n_1}(\alpha_1)\ldots\mathfrak{a}_{-n_k}(\alpha_k)|0\rangle$ is represented by the closure of the real-$\sum_{i=1}^{k}(2n_i - 2 + |\alpha_i|)$-dimensional subset:

$$\{\xi_1 + \ldots + \xi_k \in X^{[n]} | \xi_i \in M_{n_i}(A_i), \mathrm{Supp}(\xi_i) \cap \mathrm{Supp}(\xi_j) = \emptyset \text{ for } i \neq j\} \tag{16}$$

where $M_{n_i}(A_i)$ is the subset of $X^{[n_i]}$ defined by (13).

We shall write down the bases of the homology groups $H_2(X^{[n]}, \mathbb{C})$ and $H_4(X^{[n]}, \mathbb{C})$ in terms of the Heisenberg operators. The following definition introduces some special homology classes in $H_2(X^{[n]}, \mathbb{C})$ and $H_4(X^{[n]}, \mathbb{C})$.

Definition 2.1. Let $x \in X$, and C and $\widetilde{C}$ be real-2-dimensional submanifolds of X. Then, we define the following homology classes:

$$\begin{aligned}
\beta_C &= \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle \\
\beta_n &= \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle \\
\mathfrak{s}_{n,1} &= \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle \\
\mathfrak{s}_{n,2} &= \mathfrak{a}_{-2}(x)\mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-4}|0\rangle \\
\mathfrak{s}_{n,3} &= \mathfrak{a}_{-3}(x)\mathfrak{a}_{-1}(x)^{n-3}|0\rangle \\
\mathfrak{s}_{C,1} &= \mathfrak{a}_{-1}(C)\mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-3}|0\rangle \\
\mathfrak{s}_{C,2} &= \mathfrak{a}_{-2}(C)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle \\
\mathfrak{s}_{C,\widetilde{C}} &= \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(\widetilde{C})\mathfrak{a}_{-1}(x)^{n-2}|0\rangle.
\end{aligned}$$

Next, we discuss geometric representations of the above homology classes. First of all, we note from (15) that $\beta_C, \beta_n \in H_2(X^{[n]}, \mathbb{C})$ and $\mathfrak{s}_{n,1}$, $\mathfrak{s}_{n,2}$, $\mathfrak{s}_{n,3}$, $\mathfrak{s}_{C,1}$, $\mathfrak{s}_{C,2}$, $\mathfrak{s}_{C,\widetilde{C}} \in H_4(X^{[n]}, \mathbb{C})$. For $\eta \in X^{[n-1]}$ with $\mathrm{Supp}(\eta) \cap C = \emptyset$, we see from (16) that

$$\beta_C \sim C + \eta$$

where the symbol "$A_1 \sim A_2$" means that A_1 and A_2 are homologous as homology classes. Similarly, for $x \in X$ and $\eta \in X^{[n-2]}$ with $x \notin \mathrm{Supp}(\eta)$, we have

$$\beta_n \sim M_2(x) + \eta. \tag{17}$$

For $x_1, x_2 \in X$ and $\eta \in X^{[n-4]}$ satisfying $x_1 \neq x_2$ and $x_1, x_2 \notin \mathrm{Supp}(\eta)$,

$$\mathfrak{s}_{n,2} \sim M_2(x_1) + M_2(x_2) + \eta. \tag{18}$$

For $x \in X$ and $\eta \in X^{[n-3]}$ with $x \notin C \cup \mathrm{Supp}(\eta)$ and $\mathrm{Supp}(\eta) \cap C = \emptyset$, we get

$$\mathfrak{s}_{n,3} \sim M_3(x) + \eta, \tag{19}$$

$$\mathfrak{s}_{C,1} \sim C + M_2(x) + \eta. \tag{20}$$

For a fixed $\eta \in X^{[n-2]}$ satisfying $\mathrm{Supp}(\eta) \cap C = \emptyset$, we have

$$\mathfrak{s}_{C,2} \sim M_2(C) + \eta. \tag{21}$$

For $\eta = x_1 + \ldots + x_{n-1} \in X^{[n-1]}$ where $x_1, \ldots, x_{n-1}$ are distinct, we obtain

$$\mathfrak{s}_{n,1} \sim \text{“the closure of } (X \backslash \mathrm{Supp}(\eta)) + \eta \text{ in } X^{[n]}\text{”}. \tag{22}$$

Alternatively, consider the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{X}_\eta & \subset & X^{[n-1,n]} & \xrightarrow{g_n} & X^{[n]} \\ \downarrow & & \downarrow (f_n, q) & & \\ \eta \times X & \subset & X^{[n-1]} \times X & & \end{array} \tag{23}$$

where $\widetilde{X}_\eta$ stands for the strict transform of $\eta \times X$. By Theorem 2.2 (ii), (f_n, q) is the blowup of $X^{[n-1]} \times X$ along $\mathcal{Z}_{n-1}$. So $\widetilde{X}_\eta$ is isomorphic to the blowup of X at the $(n-1)$ distinct points $x_1, \ldots, x_{n-1}$. Moreover, $g_n|_{\widetilde{X}_\eta} : \widetilde{X}_\eta \to g_n(\widetilde{X}_\eta)$ is an isomorphism and $g_n(\widetilde{X}_\eta)$ is precisely the closure of $(X \backslash \mathrm{Supp}(\eta)) + \eta$ in the Hilbert scheme $X^{[n]}$. So in view of (22), we conclude that

$$\mathfrak{s}_{n,1} \sim g_n(\widetilde{X}_\eta). \tag{24}$$

Note that the $(n-1)$ exceptional curves in the surface $g_n(\widetilde{X}_\eta)$ are

$$M_2(x_i) + (\eta \backslash \{x_i\}), \qquad i = 1, \ldots, n-1. \tag{25}$$

Finally, choose $\eta \in X^{[n-2]}$ such that $\mathrm{Supp}(\eta) \cap (C \cup \widetilde{C}) = \emptyset$. Then according to (16), when C and $\widetilde{C}$ are in general position, $\mathfrak{s}_{C,\widetilde{C}}$ is the closure of the subset

$$\{x + \tilde{x} + \eta |\ x \in C, \tilde{x} \in \widetilde{C}, \text{ and } x \neq \tilde{x}\} \subset X^{[n]}. \tag{26}$$

Lemma 2.3. *Assume that $n \geq 2$ and X is simply-connected. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a basis of $H_2(X, \mathbb{C})$ represented by real surfaces $\{C_1, \ldots, C_s\}$ respectively. Then,*

(i) *a basis of $H_2(X^{[n]}, \mathbb{C})$ consists of the homology classes $\beta_n, \beta_{C_1}, \ldots, \beta_{C_s}$;*

(ii) *a basis of $H_4(X^{[n]}, \mathbb{C})$ consists of the homology classes $\mathfrak{s}_{n,1}, \mathfrak{s}_{n,2}, \mathfrak{s}_{n,3}$, $\mathfrak{s}_{C_i,1}$ $(i = 1, \ldots, s)$, $\mathfrak{s}_{C_i,2}$ $(i = 1, \ldots, s)$, and $\mathfrak{s}_{C_i,C_j}$ $(i, j = 1, \ldots, s)$.*

Proof. We shall only prove (ii) since similar argument works for (i).

Fix a point $x \in X$. Expand the basis $\{\alpha_1, \ldots, \alpha_s\}$ of $H_2(X, \mathbb{C})$ to the basis $\{\alpha_0 = x, \alpha_1, \ldots, \alpha_s, \alpha_{s+1} = X\}$ of $H_*(X, \mathbb{C}) = H_0(X, \mathbb{C}) \oplus H_2(X, \mathbb{C}) \oplus H_4(X, \mathbb{C})$. By (15), a basis of $H_4(X^{[n]}, \mathbb{C})$ consists of

$$\mathfrak{a}_{-n_1}(\alpha_{m_1}) \ldots \mathfrak{a}_{-n_k}(\alpha_{m_k})|0\rangle \tag{27}$$

satisfying $n_i \geq 1$, $\sum_{i=1}^{k} n_i = n$, and $\sum_{i=1}^{k} (2n_i - 2 + |\alpha_{m_i}|) = 4$. Note that since X is simply-connected, $|\alpha_{m_i}| \in \{0, 2, 4\}$ for every i. Also, $n_i \leq 3$ for every i.

First of all, suppose that $n_i = 3$ for some i. From $\sum_{i=1}^{k}(2n_i - 2 + |\alpha_{m_i}|) = 4$, we see that such an i is unique and $n_j = 1$ for $j \neq i$. Moreover, $|\alpha_{m_j}| = 0$ for every j, i.e., $\alpha_{m_j} = \alpha_0 = x$ for every j. Since $\sum_{i=1}^{k} n_i = n$, we have $k = (n-2)$. So in view of Definition 2.1, the homology class (27) is $\mathfrak{s}_{n,3}$.

In the following, we assume that $n_i \le 2$ for every i. Then, $n_i = 2$ for at most two i's. Suppose $n_i = 2$ for two i's, say, $n_1 = n_2 = 2$. Then, $n_j = 1$ for $j \neq 1, 2$, $k = (n-2)$, and $|\alpha_{m_j}| = 0$ for every j. So the homology class (27) is $\mathfrak{s}_{n,2}$.

Next, suppose $n_i = 2$ for exactly one i (and $n_j = 1$ for $j \neq i$), say, $n_1 = 2$ (and $n_j = 1$ for $j \neq 1$). Then, $|\alpha_{m_{i_0}}| = 2$ for some i_0 and $|\alpha_{m_j}| = 0$ for $j \neq i_0$. Thus, the homology class (27) is $\mathfrak{s}_{C_{m_1},2}$ if $i_0 = 1$, and $\mathfrak{s}_{C_{m_1},1}$ if $i_0 > 1$.

Finally, assume $n_i = 1$ for every i. Then, $k = n$ and $\sum_{i=1}^{k} |\alpha_{m_i}| = 4$. If $|\alpha_{m_{i_0}}| = 4$ for some i_0 and $|\alpha_{m_j}| = 0$ for $j \neq i_0$, then the homology class (27) is $\mathfrak{s}_{n,1}$. The remaining case is when $|\alpha_{m_{i_0}}| = |\alpha_{m_{i_1}}| = 2$ for some i_0 and i_1 with $i_0 \neq i_1$, and $|\alpha_{m_j}| = 0$ for $j \neq i_0, i_1$. In this case, the homology class (27) is $\mathfrak{s}_{C_{m_{i_0}}, C_{m_{i_1}}}$. □

Next, we recall certain results proved in section 4 of [LQZ].

Theorem 2.4. *(see* [LQZ]*) Let $n \ge 2$, and X be simply-connected.*

(i) *A curve γ in $X^{[n]}$ is homologous to β_n if and only if $\gamma = f_{n+1}(C)$ where C is a line in the projective space $(\psi_{n+1})^{-1}(\eta, x)$ for some $(\eta, x) \in \mathcal{Z}_{n+1}$. Moreover, in this case, the point (η, x) and the line C are uniquely determined by γ;*

(ii) *Let $\mathfrak{M}(\beta_n)$ be the moduli space of all the curves in the Hilbert scheme $X^{[n]}$ homologous to β_n. Then, $\mathfrak{M}(\beta_n)$ has dimension $(2n-2)$, and its top stratum consists of all the points corresponding to curves of the form (2);*

(iii) *Let γ be the curve of the form (2). Then, its normal bundle in $X^{[n]}$ is*

$$N_{\gamma \subset X^{[n]}} \cong \mathcal{O}_\gamma^{\oplus(2n-2)} \oplus \mathcal{O}_\gamma(-2). \tag{28}$$

3. The 1-point Gromov-Witten invariants $\langle \alpha \rangle_{0,d\beta_n}$ of $X^{[n]}$

In this section, we shall compute all the 1-point Gromov-Witten invariants $\langle \alpha \rangle_{0,d\beta_n}$ of $X^{[n]}$ for $n \ge 2$ and $d \ge 1$. One of the key steps is to determine the obstruction bundle over a Zariski open subset of the moduli space $\overline{\mathfrak{M}}_{0,0}(X^{[n]}, d\beta_n)$.

3.1. The obstruction bundle

We start with some notations. Let S_n be the symmetric group of n letters, and $|\operatorname{Supp}(\xi)|$ be the number of points in $\operatorname{Supp}(\xi)$. Recall from (1) the Hilbert-Chow map $\rho\colon X^{[n]} \to X^{(n)} = X^n/S_n$, where X^n is the Cartesian product of n copies of X. Let $\sigma : X^n \to X^{(n)}$ be the natural quotient map.

Notation. Put $X_*^{[n]} = \{\xi \in X^{[n]} \,|\, |\mathrm{Supp}(\xi)| \ge n-1\}$ and

$$\begin{aligned} X_*^{(n)} &= \rho(X_*^{[n]}), \\ X_*^n &= \sigma^{-1}(X_*^{(n)}), \\ B &= \{\xi \in X^{[n]} \,|\, |\mathrm{Supp}(\xi)| < n\}, \\ B_* &= \{\xi \in X^{[n]} \,|\, |\mathrm{Supp}(\xi)| = n-1\}, \\ X_{s*}^{(n)} &= \rho(B_*), \\ \Delta_{n*} &= \sigma^{-1}(\rho(B)) \cap X_*^n = \coprod_{1 \le i<j \le n} \Delta_{n*}^{i,j} \end{aligned}$$

where $\Delta_{n*}^{i,j} = \{(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \in X_*^n \,|\, x_i = x_j\}$ for $1 \le i < j \le n$.

When we compute the 1-point Gromov-Witten invariants $\langle \alpha \rangle_{0,d\beta_n}$, only $X_*^{[n]}$ is involved in most of the cases. Even though $X^{[n]}$ is very complicated, the open subset $X_*^{[n]}$ has a very simple description given below (see [Fo2]). Let $\widetilde{X_*^n}$ be the blow up of X_*^n along the big diagonal Δ_{n*}. The action of S_n on X_*^n lifts to an action on $\widetilde{X_*^n}$ and $X_*^{[n]} = \widetilde{X_*^n}/S_n$. Let $\tilde{\sigma} \colon \widetilde{X_*^n} \to X_*^{[n]}$ be the quotient map. Let $E_*^{i,j} \subset \widetilde{X_*^n}$ be the exceptional locus over $\Delta_{n*}^{i,j}$. Consider the following morphisms:

$$p_{1,2} \quad : \Delta_{n*}^{1,2} \longrightarrow X, \quad (x, x, x_3, \ldots, x_n) \to x, \tag{29}$$

$$j_2 \quad : X_{s*}^{(n)} \longrightarrow X, \quad 2x + x_3 + \ldots + x_n \to x. \tag{30}$$

Since the normal bundle of $\Delta_{n*}^{1,2}$ in X_*^n is isomorphic to $p_{1,2}^* T_X$, we have $E_*^{1,2} \cong \mathbb{P}(p_{1,2}^* T_X^*)$. The subgroup $S_2 \times S_{n-2} \subset S_n$ acts on $\Delta_{n*}^{1,2}$ with the S_2-factor acting trivially on $\Delta_{n*}^{1,2}$. The action of $S_2 \times S_{n-2}$ on $\Delta_{n*}^{1,2}$ lifts to an action on $E_*^{1,2}$. It is easy to see that $X_{s*}^{(n)} = \Delta_{n*}^{1,2}/(S_2 \times S_{n-2})$ and $B_* = E_*^{1,2}/(S_2 \times S_{n-2})$. Regard $p_{1,2} \colon \Delta_{n*}^{1,2} \to X$ as an $S_2 \times S_{n-2}$-equivariant morphism where $S_2 \times S_{n-2}$ acts on X trivially. Then, $S_2 \times S_{n-2}$ acts on $p_{1,2}^* T_X^*$, and the isomorphism $E_*^{1,2} \cong \mathbb{P}(p_{1,2}^* T_X^*)$ is $S_2 \times S_{n-2}$-equivariant. So we get an isomorphism

$$j_1 \colon B_* = E_*^{1,2}/(S_2 \times S_{n-2}) \cong \mathbb{P}(p_{1,2}^* T_X^*)/(S_2 \times S_{n-2}) \cong \mathbb{P}(j_2^* T_X^*).$$

where the last isomorphism is due to the fact that the S_2-factor acts trivially on $p_{1,2}^* T_X$ and the S_{n-2}-factor commutes with the morphism $p_{1,2}$.

Next, we study $\mathcal{O}_{B_*}(B_*)$. Since $\tilde{\sigma}^* \mathcal{O}_{X_*^{[n]}}(B_*) \cong \mathcal{O}_{\widetilde{X_*^n}}(2 \sum_{1 \le i<j \le n} E_*^{i,j})$ and $E_*^{i,j} \cap E_*^{1,2} \ne \emptyset$ if and only $i = 1$ and $j = 2$, we conclude that

$$(\tilde{\sigma}|_{E_*^{1,2}})^* \mathcal{O}_{B_*}(B_*) \cong \tilde{\sigma}^* \mathcal{O}_{X_*^{[n]}}(B_*)|_{E_*^{1,2}} \cong \mathcal{O}_{E_*^{1,2}}(2E_*^{1,2}) \cong \mathcal{O}_{\mathbb{P}(p_{1,2}^* T_X^*)}(-2) \tag{31}$$

where we have used the fact that $\mathcal{O}_{E_*^{1,2}}(E_*^{1,2}) \cong \mathcal{O}_{\mathbb{P}(p_{1,2}^* T_X^*)}(-1)$ via the isomorphism $E_*^{1,2} \cong \mathbb{P}(p_{1,2}^* T_X^*)$. Note that $\mathcal{O}_{\mathbb{P}(p_{1,2}^* T_X^*)}(-2) = \tau^* \big(\mathcal{O}_{\mathbb{P}(j_2^* T_X^*)}(-2)\big)$ where $\tau : \mathbb{P}(p_{1,2}^* T_X^*) \to \mathbb{P}(j_2^* T_X^*)$ is the natural morphism. Moreover, $j_1 \circ (\tilde{\sigma}|_{E_*^{1,2}}) = \tau$ via the isomorphism $E_*^{1,2} \cong$

$\mathbb{P}(p_{1,2}^*T_X^*)$. Combining with (31), we obtain $(\tilde{\sigma}|_{E_*^{1,2}})^*\mathcal{O}_{B_*}(B_*) \cong (\tilde{\sigma}|_{E_*^{1,2}})^* \Big(j_1^*\mathcal{O}_{\mathbb{P}(j_2^*T_X^*)}(-2)\Big)$. Since $\mathrm{Pic}(B_*)$ has no torsion, we have

$$\mathcal{O}_{B_*}(B_*) \cong j_1^*\mathcal{O}_{\mathbb{P}(j_2^*T_X^*)}(-2). \tag{32}$$

Consider the open subset $\mathfrak{U}_0$ of $\overline{\mathfrak{M}}_{0,0}(X^{[n]}, d\beta_n)$ consisting of stable maps $[\mu\colon D \to X^{[n]}]$ such that $\mu(D) \subset X_*^{[n]}$. Similarly, take the open subset $\mathfrak{U}_1$ of $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ consisting of stable maps $[\mu\colon (D;p) \to X^{[n]}]$ such that $\mu(D) \subset X_*^{[n]}$. Clearly $\mathfrak{U}_1 = f_{1,0}^{-1}(\mathfrak{U}_0)$. Let $[\mu\colon (D;p) \to X^{[n]}] \in \mathfrak{U}_1$. Since $\mu_*(D) \sim d\beta_n$, we must have $\mu(D) = M_2(x_2)+x_3+\ldots+x_n$ for some distinct points $x_2, \ldots, x_n \in X$. Hence $\mu(D) \subset B_*$. Moreover, the composite $\rho \circ ev_1$ sends the stable map $[\mu\colon (D;p) \to X^{[n]}]$ to the point $2x_2 + x_3 + \ldots + x_n$, which is independent of the marked point p on D. Hence ev_1 induces a morphism ϕ from $\mathfrak{U}_0$ to $\rho(B_*)$. Putting $\widetilde{ev}_1 = ev_1|_{\mathfrak{U}_1}$ and $\tilde{f}_{1,0} = f_{1,0}|_{\mathfrak{U}_1}$, we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathfrak{U}_1 & \stackrel{\widetilde{ev}_1}{\rightarrow} & B_* & \stackrel{j_1}{\cong} & \mathbb{P}(j_2^*T_X^*) & & \\ \downarrow^{\tilde{f}_{1,0}} & & \downarrow^{\rho} & & \downarrow^{\pi} & & \\ \mathfrak{U}_0 & \stackrel{\phi}{\rightarrow} & \rho(B_*) & = & X_{s*}^{(n)} & \stackrel{j_2}{\rightarrow} & X \end{array} \tag{33}$$

where $\pi\colon \mathbb{P}(j_2^*T_X^*) \to X_{s*}^{(n)}$ is the natural projection of the $\mathbb{P}^1$-bundle.

Note that the fiber $\phi^{-1}(2x_2+x_3+\ldots+x_n)$ over a fixed point $2x_2+x_3+\ldots+x_n \in \rho(B_*)$ is simply $\overline{\mathfrak{M}}_{0,0}(M_2(x_2) + x_3 + \ldots + x_n, d[M_2(x_2) + x_3 + \ldots + x_n])$ which is isomorphic to the moduli space $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$ via the isomorphism $M_2(x_2) + x_3 + \ldots + x_n \cong \mathbb{P}^1$. Hence the complex dimension of $\mathfrak{U}_0$ is equal to

$$\dim \overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1]) + 2(n-1) = 2n - 3 + 2d - 1.$$

The expected dimension of $\mathfrak{M}_{0,0}(X^{[n]}, d\beta_n)$ is $2n-3$ according to the formula (7) where we used $K_{X^{[n]}} \cdot d\beta_n = 0$. Hence the excess dimension of $\mathfrak{U}_0$ is $e = (2d-1)$.

Lemma 3.1. *With notations as above, the restriction of $R^1(f_{1,0})_*(ev_1^*T_{X^{[n]}})$ to $\mathfrak{U}_0$ is a locally free sheaf of rank $(2d-1)$.*

Proof. Take a stable map $u = [\mu\colon D \to X^{[n]}]$ in $\mathfrak{U}_0$, and consider

$$H^1(f_{1,0}^{-1}(u), (ev_1^*T_{X^{[n]}})|_{f_{1,0}^{-1}(u)}) \cong H^1(D, \mu^*T_{X^{[n]}}).$$

Since $\mu(D) = M_2(x_2) + x_3 + \ldots + x_n \cong \mathbb{P}^1$ for some distinct points $x_2, \ldots, x_n$, we have $T_{X^{[n]}}|_{\mu(D)} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{(2n-2)}$ by Theorem 2.4 (iii). Thus

$$H^1(D, \mu^*T_{X^{[n]}}) \cong H^1(D, \mu^*\mathcal{O}_{\mathbb{P}^1}(-2))$$

which has dimension equal to the excess dimension $e = (2d-1)$. Hence the direct image sheaf $R^1(f_{1,0})_*(ev_1^*T_{X^{[n]}})$ over $\mathfrak{U}_0$ is locally free of rank $(2d-1)$. □

Suppose that $\mathfrak{M}_1$ is a closed subset of $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ contained in $\mathfrak{U}_1$ and $\mathfrak{M}_0 = f_{1,0}(\mathfrak{M}_1) \subset \mathfrak{U}_0 \subset \overline{\mathfrak{M}}_{0,0}(X^{[n]}, d\beta_n)$. By Proposition 2.1 (i) and (iii), we have

$$[\overline{\mathfrak{M}}_{0,1}(Y,\beta)]^{\text{vir}}|_{\mathfrak{M}_1} = (\tilde{f}_{1,0})^* c_{2d-1}\big((R^1(f_{1,0})_*(ev_1)^*T_{X^{[n]}})|_{\mathfrak{M}_0}\big). \tag{34}$$

Hence it is crucial to determine the sheaf $R^1(f_{1,0})_*(ev_1)^*T_{X^{[n]}}$ over $\mathfrak{U}_0$.

Lemma 3.2. *Let $\mathcal{V}$ denote the restriction of $R^1(f_{1,0})_*(ev_1)^*T_{X^{[n]}}$ to $\mathfrak{U}_0$. Then,*
(i) $\mathcal{V} \cong R^1(\tilde{f}_{1,0})_*(\widetilde{ev}_1)^*\mathcal{O}_{B_*}(B_*) \cong R^1(\tilde{f}_{1,0})_*(j_1 \circ \widetilde{ev}_1)^*\mathcal{O}_{\mathbb{P}(j_2^*T_X^*)}(-2)$.
(ii) *the locally free sheaf $\mathcal{V}$ sits in the exact sequence*

$$\begin{aligned} 0 \quad \to \quad & (j_2 \circ \phi)^*\mathcal{O}_X(-K_X) \to \mathcal{V} \\ \to \quad & R^1(\tilde{f}_{1,0})_*(j_1 \circ \widetilde{ev}_1)^*((j_2 \circ \pi)^*T_X \otimes \mathcal{O}_{\mathbb{P}(j_2^*T_X^*)}(-1)) \to 0. \end{aligned} \tag{35}$$

Proof. (i) Since $ev_1(\mathfrak{U}_1) \subset B_*$, we have $((ev_1)^*T_{X^{[n]}})|_{\mathfrak{U}_1} = (\widetilde{ev}_1)^*(T_{X_*^{[n]}}|_{B_*})$ and $\mathcal{V} = (R^1(f_{1,0})_*(ev_1)^*T_{X^{[n]}})|_{\mathfrak{U}_0} = R^1(\tilde{f}_{1,0})_*\big(((ev_1)^*T_{X^{[n]}})|_{\mathfrak{U}_1}\big) = R^1(\tilde{f}_{1,0})_*(\widetilde{ev}_1)^*(T_{X^{[n]}}|_{B_*})$. Since B_* is a smooth codimension-1 subvariety of $X^{[n]}$, we obtain the exact sequence

$$0 \to T_{B_*} \to T_{X^{[n]}}|_{B_*} \to \mathcal{O}_{B_*}(B_*) \to 0. \tag{36}$$

Applying $(\widetilde{ev}_1)^*$ and $(\tilde{f}_{1,0})_*$ to the exact sequence (36), we get

$$R^1(\tilde{f}_{1,0})_*(\widetilde{ev}_1)^*T_{B_*} \to \mathcal{V} \to R^1(\tilde{f}_{1,0})_*(\widetilde{ev}_1)^*\mathcal{O}_{B_*}(B_*) \to 0.$$

where we have used $R^2(\tilde{f}_{1,0})_*(\widetilde{ev}_1)^*T_{B_*} = 0$ since $\tilde{f}_{1,0}$ is of relative dimension 1.

If $[\mu\colon D \to X^{[n]}]$ is a stable map in $\mathfrak{U}_0$, then $\mu(D) = M_2(x_2) + x_3 + \ldots + x_n$. Hence the normal bundle of $\mu(D)$ in B_* is trivial since $\mu(D)$ is a fiber of the $\mathbb{P}^1$-bundle $\mathbb{P}(j_2^*T_X^*)$ over $X_{s*}^{(n)}$. Thus $T_{B_*}|_{\mu(D)} \cong \mathcal{O}_{\mu(D)}(2) \oplus \mathcal{O}_{\mu(D)}^{\oplus(2n-2)}$. Therefore, $H^1(D, \mu^*T_{B_*}) \cong H^1(D, \mu^*(\mathcal{O}_{\mu(D)}(2) \oplus \mathcal{O}_{\mu(D)}^{\oplus(2n-2)})) = 0$, and $R^1(\tilde{f}_{1,0})_*(\widetilde{ev}_1)^*T_{B_*} = 0$. So in view of (32), we have

$$\mathcal{V} \cong R^1(\tilde{f}_{1,0})_*(\widetilde{ev}_1)^*\mathcal{O}_{B_*}(B_*) \cong R^1(\tilde{f}_{1,0})_*(j_1 \circ \widetilde{ev}_1)^*\mathcal{O}_{\mathbb{P}(j_2^*T_X^*)}(-2).$$

(ii) For simplicity, we denote $\mathbb{P}(j_2^*T_X^*)$ by $\mathbb{P}$. Consider the natural surjection $\pi^*(j_2^*T_X^*) \to \mathcal{O}_{\mathbb{P}}(1) \to 0$. The kernel of this surjection is a line bundle. By comparing the first Chern classes, we get the following exact sequence:

$$0 \to \pi^*\mathcal{O}_{X_{s*}^{(n)}}(j_2^*K_X) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \pi^*(j_2^*T_X^*) \to \mathcal{O}_{\mathbb{P}}(1) \to 0. \tag{37}$$

Tensoring (37) with $\pi^*\mathcal{O}_{X_{s*}^{(n)}}(-j_2^*K_X) \otimes \mathcal{O}_{\mathbb{P}}(-1)$, we get

$$\begin{aligned} 0 \quad \to \quad & \mathcal{O}_{\mathbb{P}}(-2) \to (j_2 \circ \pi)^*(T_X^* \otimes \mathcal{O}_X(-K_X)) \otimes \mathcal{O}_{\mathbb{P}}(-1) \\ & \to (j_2 \circ \pi)^*\mathcal{O}_X(-K_X) \to 0. \end{aligned} \tag{38}$$

Note that $T_X^* \otimes \mathcal{O}_X(-K_X) \cong T_X$. Applying $(j_1 \circ \widetilde{ev}_1)^*$ to (38) yields

$$\begin{aligned} 0 \quad \to \quad & (j_1 \circ \widetilde{ev}_1)^*\mathcal{O}_{\mathbb{P}}(-2) \to (j_1 \circ \widetilde{ev}_1)^*((j_2 \circ \pi)^*T_X \otimes \mathcal{O}_{\mathbb{P}}(-1)) \\ & \to (j_2 \circ \pi \circ j_1 \circ \widetilde{ev}_1)^*\mathcal{O}_X(-K_X) \to 0. \end{aligned} \tag{39}$$

By (33), we have $(j_2 \circ \pi \circ j_1 \circ \widetilde{ev}_1)^* = (j_2 \circ \phi \circ \tilde{f}_{1,0})^* = (\tilde{f}_{1,0})^* \circ (j_2 \circ \phi)^*$. So rewriting the 3rd term in the exact sequence (39), we obtain

$$\begin{aligned} 0 \quad \to \quad & (j_1 \circ \widetilde{ev}_1)^* \mathcal{O}_{\mathbb{P}}(-2) \to (j_1 \circ \widetilde{ev}_1)^*((j_2 \circ \pi)^* T_X \otimes \mathcal{O}_{\mathbb{P}}(-1)) \\ & \to (\tilde{f}_{1,0})^*((j_2 \circ \phi)^* \mathcal{O}_X(-K_X)) \to 0. \end{aligned} \tag{40}$$

Applying $(\tilde{f}_{1,0})_*$ to the above exact sequence and using part (i), we have

$$\begin{aligned} 0 \quad \to \quad & (\tilde{f}_{1,0})_*(\tilde{f}_{1,0})^*((j_2 \circ \phi)^* \mathcal{O}_X(-K_X)) \to \mathcal{V} \\ \to \quad & R^1(\tilde{f}_{1,0})_*(j_1 \circ \widetilde{ev}_1)^*((j_2 \circ \pi)^* T_X \otimes \mathcal{O}_{\mathbb{P}}(-1)) \\ \to \quad & R^1(\tilde{f}_{1,0})_*(\tilde{f}_{1,0})^*((j_2 \circ \phi)^* \mathcal{O}_X(-K_X)). \end{aligned}$$

where we have used $(\tilde{f}_{1,0})_*(j_1 \circ \widetilde{ev}_1)^*((j_2 \circ \pi)^* T_X \otimes \mathcal{O}_{\mathbb{P}}(-1)) = 0$. Note that $(\tilde{f}_{1,0})_* \mathcal{O}_{\mathfrak{U}_1} \cong \mathcal{O}_{\mathfrak{U}_0}$ and $R^1(\tilde{f}_{1,0})_* \mathcal{O}_{\mathfrak{U}_1} = 0$. So we get

$$\begin{aligned} (\tilde{f}_{1,0})_*(\tilde{f}_{1,0})^*((j_2 \circ \phi)^* \mathcal{O}_X(-K_X)) \quad & \cong \quad (j_2 \circ \phi)^* \mathcal{O}_X(-K_X) \otimes (\tilde{f}_{1,0})_* \mathcal{O}_{\mathfrak{U}_1} \\ & \cong \quad (j_2 \circ \phi)^* \mathcal{O}_X(-K_X) \end{aligned}$$

by the projection formula. Similarly, $R^1(\tilde{f}_{1,0})_*(\tilde{f}_{1,0})^*((j_2 \circ \phi)^* \mathcal{O}_X(-K_X)) = 0$. Therefore, the locally free sheaf $\mathcal{V}$ sits in the exact sequence (35). □

Remark 3.1. Fix distinct points $x_2, \ldots, x_n$ on X. Via the isomorphism $\phi^{-1}(2x_2 + x_3 + \ldots + x_n) \cong \overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$, the restriction of $R^1(\tilde{f}_{1,0})_*(j_1 \circ \widetilde{ev}_1)^*((j_2 \circ \pi)^* T_X \otimes \mathcal{O}_{\mathbb{P}}(-1))$ to $\phi^{-1}(2x_2 + x_3 + \ldots + x_n)$ is isomorphic to

$$R^1(f_{1,0})_*(ev_1)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$$

where by abusing notations, we still use $f_{1,0}$ and ev_1 to denote the forgetful map and the evaluation map from $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^1, d[\mathbb{P}^1])$ to $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$ and $\mathbb{P}^1$ respectively.

3.2. The 1-point Gromov-Witten invariants $\langle \alpha \rangle_{0,d\beta_n}$

In this subsection, we compute all the 1-point Gromov-Witten invariants $\langle \alpha \rangle_{0,d\beta_n}$ for the Hilbert schemes $X^{[n]}$. Recall from (8) and (7) that $|\alpha| = 4n - 4$. In view of Lemma 2.3 (ii), we need only to compute $\langle \alpha \rangle_{0,d\beta_n}$ when α is the Poincaré duals of $\mathfrak{s}_{n,1}$, $\mathfrak{s}_{n,2}$, $\mathfrak{s}_{n,3}$, $\mathfrak{s}_{C_1,1}$, $\mathfrak{s}_{C_1,2}$, and $\mathfrak{s}_{C_1,C_2}$ where C_1 and C_2 are two smooth real surfaces in X. These six cases will be divided into two lemmas.

Lemma 3.3. *Let $d \geq 1$, and C_1 and C_2 be smooth real surfaces in X.*
(i) *If α is the Poincaré dual of $\mathfrak{s}_{n,1}$, $\mathfrak{s}_{C_1,C_2}$, or $\mathfrak{s}_{C_1,1}$, then $\langle \alpha \rangle_{0,d\beta_n} = 0$.*
(ii) *If α is the Poincaré dual of $\mathfrak{s}_{C_1,2}$, then $\langle \alpha \rangle_{0,d\beta_n} = 2(K_X \cdot C_1)/d^2$.*

Proof. (i) Suppose that α is Poincaré dual to $\mathfrak{s}_{n,1}$. Fix distinct points $x_1, \ldots, x_{n-1} \in X$ which are not contained in $C_1 \cup C_2$. By (24), $\mathfrak{s}_{n,1} \sim g_n(\widetilde{X_\eta}) \cong \widetilde{X_\eta}$ where $\widetilde{X_\eta}$ is the blow up of X along $\eta = x_1 + \ldots + x_{n-1}$. Moreover, the exceptional curves in $g_n(\widetilde{X_\eta})$ are $\rho^{-1}(x_1 + \ldots + x_{i-1} + 2x_i + x_{i+1} + \ldots + x_{n-1})$ for $1 \leq i \leq n-1$. Let $\mathfrak{M}_1$ be the subset of $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ consisting of all the stable maps $[\mu\colon (D;p) \to X^{[n]}]$ such that $\mu(p) \in g_n(\widetilde{X_\eta})$. In this case, $\mu(D)$ is one of the exceptional curves in $g_n(\widetilde{X_\eta}) \subset B_*$. In particular, the stable maps $[\mu\colon (D;p) \to X^{[n]}]$ are contained in $\mathfrak{U}_1$, and $\mathfrak{M}_1 = \coprod_{1\le i\le n-1} (\tilde{f}_{1,0})^{-1}(\phi^{-1}(x_1+\ldots+x_{i-1}+2x_i+x_{i+1}+\ldots+x_{n-1}))$. So as algebraic cycles, we have $[\mathfrak{M}_1] = \sum_{i=1}^{n-1} (\tilde{f}_{1,0})^*\phi^*[x_1+\ldots+x_{i-1}+2x_i+x_{i+1}+\ldots+x_{n-1}]$. By Lemma 3.2 (ii), we get $c_{2d-1}(\mathcal{V}) = -(j_2\circ\phi)^*K_X \cdot c_{2d-2}(\mathcal{E})$ where $\mathcal{E} = R^1(\tilde{f}_{1,0})_*(j_1\circ \widetilde{ev}_1)^*((j_2\circ\pi)^*T_X \otimes \mathcal{O}_{\mathbb{P}(j_2^*T_X^*)}(-1))$. In view of (9) and (34),

$$\begin{aligned}
\langle\alpha\rangle_{0,d\beta_n} &= \int_{[\overline{\mathfrak{M}}_{0,1}(Y,\beta)]^{\mathrm{vir}}} (ev_1)^*\alpha = [\mathfrak{M}_1]\cdot[\overline{\mathfrak{M}}_{0,1}(Y,\beta)]^{\mathrm{vir}} \\
&= [\mathfrak{M}_1]\cdot[\overline{\mathfrak{M}}_{0,1}(Y,\beta)]^{\mathrm{vir}}|_{\mathfrak{M}_1} = [\mathfrak{M}_1]\cdot(\tilde{f}_{1,0})^*(c_{2d-1}(\mathcal{V})) \\
&= -\sum_{i=1}^{n-1}(\tilde{f}_{1,0})^*\Big(\phi^*([x_1+\ldots+2x_i+\ldots+x_{n-1}]\cdot j_2^*K_X)\cdot c_{2d-2}(\mathcal{E})\Big) = 0.
\end{aligned}$$

Next let α be the Poincaré dual of $\mathfrak{s}_{C_1,C_2}$. We may assume that C_1 and C_2 intersect transversally at the points $y_1,\ldots,y_m$. By (26), $\mathfrak{s}_{C_1,C_2}$ is the closure of

$$\{x+\tilde{x}+x_1+\ldots+x_{n-2}|\ x\in C_1, \tilde{x}\in C_2, \text{ and } x\ne\tilde{x}\} \subset X^{[n]}.$$

Let $\mathfrak{M}_1' \subset \overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ consist of all the stable maps $[\mu\colon (D;p)\to X^{[n]}]$ such that $\mu(p)\in\mathfrak{s}_{C_1,C_2}$. In this case, $\rho(\mu(D)) = 2y_k + x_1+\ldots+x_{n-2}$ for some k with $1\le k\le m$. Therefore, $\mu(D) = \rho^{-1}(2y_k+x_1+\ldots+x_{n-2})$. Hence the stable map $[\mu\colon (D;p)\to X^{[n]}]$ is contained in $\mathfrak{U}_1$. So $\mathfrak{M}_1'\subset \mathfrak{U}_1$ is the disjoint union of $(\tilde{f}_{1,0})^{-1}(\phi^{-1}(2y_k+x_1+\ldots+x_{n-2}))$, $1\le k\le m$, with $\pm$ orientations. By the same computations as in the previous paragraph, we obtain $\langle\alpha\rangle_{0,d\beta_n}=0$.

For the case of $\mathfrak{s}_{C_1,1}$, the proof is similar to the cases of $\mathfrak{s}_{n,1}$ and $\mathfrak{s}_{C_1,C_2}$.

(ii) Let $\tilde{\eta} = x_1+\ldots+x_{n-2}$. By (21), $\mathfrak{s}_{C_1,2}\sim M_2(C_1)+\tilde{\eta} = \rho^{-1}(2C_1+\tilde{\eta})$. Thus, we have $\alpha = \mathrm{PD}(\rho^{-1}(2C_1+\tilde{\eta}))$ where PD stands for the Poincaré dual. So we see from (34) and Lemma 3.2 (ii) that

$$\begin{aligned}
\langle\alpha\rangle_{0,d\beta_n} &= \int_{[\overline{\mathfrak{M}}_{0,1}(Y,\beta)]^{\mathrm{vir}}}(ev_1)^*\alpha = \int_{-(\tilde{f}_{1,0})^*(j_2\circ\phi)^*K_X\cdot(\tilde{f}_{1,0})^*c_{2d-2}(\mathcal{E})}(ev_1)^*\alpha \\
&= -\int_{(\widetilde{ev}_1)^*(\rho^*j_2^*K_X)\cdot(\tilde{f}_{1,0})^*c_{2d-2}(\mathcal{E})}(ev_1)^*\alpha \\
&= -\int_{(\widetilde{ev}_1)^*(\rho^*j_2^*K_X)\cdot(\tilde{f}_{1,0})^*c_{2d-2}(\mathcal{E})}(\widetilde{ev}_1)^*\mathrm{PD}(\rho^{-1}(2C_1+\tilde{\eta})\cdot c_1(\mathcal{O}_{B_*}(B_*))) \\
&= -\int_{(\tilde{f}_{1,0})^*c_{2d-2}(\mathcal{E})}(\widetilde{ev}_1)^*\mathrm{PD}\left(\rho^{-1}\Big((j_2^*K_X)\cdot(2C_1+\tilde{\eta})\Big)\cdot c_1(\mathcal{O}_{B_*}(B_*))\right) \\
&= 2(K_X\cdot C_1)\cdot\int_{(\tilde{f}_{1,0})^*c_{2d-2}(\mathcal{E})}(\widetilde{ev}_1)^*\mathrm{PD}(\xi) \qquad (41)
\end{aligned}$$

where $\xi \in \rho^{-1}(2x + \tilde{\eta}) = \rho^{-1}(2x + x_1 + \ldots + x_{n-2})$ is a fixed point for some fixed point $x \in C_1$. Also, we have used the isomorphism (32) in the last step.

Let $\mathfrak{M}_1'' \subset \overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ be the subset consisting of all stable maps $[\mu\colon (D; p) \to X^{[n]}]$ with $\mu(p) = \xi$. If $[\mu\colon (D; p) \to X^{[n]}] \in \mathfrak{M}_1''$, then $\rho(\mu(D)) = \rho(\mu(p)) = 2x + x_1 + \ldots + x_{n-2}$. So $\mu(D) = \rho^{-1}(2x + x_1 + \ldots + x_{n-2})$. Thus the restriction of the forgetful map $\tilde{f}_{1,0}$ to $\mathfrak{M}_1''$ gives a degree-d morphism from $\mathfrak{M}_1''$ to $\mathfrak{M}_0'' \stackrel{\text{def}}{=} \phi^{-1}(2x + x_1 + \ldots + x_{n-2})$. Hence, as algebraic cycles, we have $(\tilde{f}_{1,0})_*[\mathfrak{M}_1''] = d[\mathfrak{M}_0''] = d \cdot \phi^*[2x + x_1 + \ldots + x_{n-2}]$. By (41), we obtain

$$\begin{aligned} \langle \alpha \rangle_{0,d\beta_2} &= 2(K_X \cdot C_1) \cdot [\mathfrak{M}_1''] \cdot (\tilde{f}_{1,0})^* c_{2d-2}(\mathcal{E}) \\ &= 2(K_X \cdot C_1) \cdot (\tilde{f}_{1,0})_*[\mathfrak{M}_1''] \cdot c_{2d-2}(\mathcal{E}) \\ &= 2d(K_X \cdot C_1) \cdot \phi^*[2x + x_1 + \ldots + x_{n-2}] \cdot c_{2d-2}(\mathcal{E}) \\ &= 2d(K_X \cdot C_1) \cdot c_{2d-2}(\mathcal{E}|_{\phi^{-1}(2x+x_1+\ldots+x_{n-2})}). \end{aligned} \tag{42}$$

By Remark 3.1, $\mathcal{E}|_{\phi^{-1}(2x+x_1+\ldots+x_{n-2})} \cong R^1(f_{1,0})_*(ev_1)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ where $f_{1,0}$ and ev_1 denote the forgetful map and the evaluation map from the moduli space $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^1, d[\mathbb{P}^1])$ to $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$ and $\mathbb{P}^1$ respectively. By the Theorem 9.2.3 in [C-K], $c_{2d-2}(R^1(f_{1,0})_*(ev_1)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) = 1/d^3$. So we have

$$c_{2d-2}(\mathcal{E}|_{\phi^{-1}(2x+x_1+\ldots+x_{n-2})}) = c_{2d-2}(R^1(f_{1,0})_*(ev_1)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = 1/d^3.$$

Combining this with (42), we conclude that $\langle \alpha \rangle_{0,d\beta_n} = 2(K_X \cdot C_1)/d^2$. ☐

Lemma 3.4. *Let $d \geq 1$. If α is the Poincaré dual of $\mathfrak{s}_{n,2}$ or $\mathfrak{s}_{n,3}$, then $\langle \alpha \rangle_{0,d\beta_n} = 0$.*

Proof. Since similar argument works for $\mathfrak{s}_{n,2}$, we shall only prove the lemma for $\mathfrak{s}_{n,3}$. So assume that α is the Poincaré dual of $\mathfrak{s}_{n,3}$. Let $x_1, \ldots, x_{n-2} \in X$ be fixed distinct points on X contained in a small analytic open subset U of X. We may assume that U is independent of the smooth surface X. Let $\mathfrak{U}_1' \subset \overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ be the analytic open subset consisting of all stable maps $[\mu\colon (D; p) \to X^{[n]}]$ with $\mu(p) \in U^{[n]}$. Since $\mu_*(D) \sim d\beta_n$, we see that $\text{Supp}(\mu(D)) = \text{Supp}(\mu(p))$ for $[\mu\colon (D; p) \to X^{[n]}] \in \overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$. So $\mu(D) \subset U^{[n]}$, and $\mathfrak{U}_1'$ is independent of X.

Next, recall from (19) that $\mathfrak{s}_{n,3}$ is represented by $M_3(x_1) + x_2 + \ldots + x_{n-2}$. Let $\mathfrak{M}_1 \subset \overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$ be the closed subset consisting of all stable maps $[\mu\colon (D; p) \to X^{[n]}]$ with $\mu(p) \in M_3(x_1) + x_2 + \ldots + x_{n-2}$. Then, $\mathfrak{M}_1 \subset \mathfrak{U}_1'$ since $M_3(x_1) + x_2 + \ldots + x_{n-2} \subset U^{[n]}$. In addition, since $\text{Supp}(\mu(D)) = \text{Supp}(\mu(p))$, we must have $\mu(D) \subset M_3(x_1) + x_2 + \ldots + x_{n-2}$ for every $[\mu\colon (D; p) \to X^{[n]}] \in \mathfrak{M}_1$. So $\mathfrak{M}_1$ is independent of X. Thus the pull-back $ev_1^*(\alpha)$ is also independent of X.

In summary, $\mathfrak{M}_1 \subset \mathfrak{U}_1'$, $\mathfrak{U}_1'$ is analytic open in $\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)$, and $\mathfrak{M}_1$ and $\mathfrak{U}_1'$ are independent of X. It follows from the constructions of the virtual fundamental class (see [LT2, LT3, Ru1]) that the restriction $[\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)]^{\text{vir}}|_{\mathfrak{M}_1}$ is independent of the smooth surface X. So the 1-point Gromov-Witten invariant $\langle \alpha \rangle_{0,d\beta_n}$, which is defined to be $[\overline{\mathfrak{M}}_{0,1}(X^{[n]}, d\beta_n)]^{\text{vir}} \cdot ev_1^*(\alpha)$ with $ev_1^*(\alpha)$ being independent of X, is independent of X as well. Since all the Gromov-Witten invariants $\langle \alpha_1, \ldots, \alpha_k \rangle_{0,\beta}$ with $\beta \neq 0$ for a $K3$-surface are zero, we conclude that $\langle \alpha \rangle_{0,d\beta_n} = 0$ for $d \geq 1$. □

Summarizing Lemma 3.3 and Lemma 3.4, we obtain our main result.

Theorem 3.5. *Let X be a simply-connected smooth projective surface. Let $n \geq 2$, $d \geq 1$, and C_1 and C_2 be two smooth real surfaces in X.*

(i) *If α is the Poincaré dual of $\mathfrak{s}_{n,1}$, $\mathfrak{s}_{C_1,C_2}$, $\mathfrak{s}_{C_1,1}$, $\mathfrak{s}_{n,2}$ or $\mathfrak{s}_{n,3}$, then $\langle \alpha \rangle_{0,d\beta_n} = 0$.*

(ii) *If α is the Poincaré dual of $\mathfrak{s}_{C_1,2}$, then $\langle \alpha \rangle_{0,d\beta_n} = 2(K_X \cdot C_1)/d^2$.* □

Acknowledgments

The authors thank Dan Edidin, Sheldon Katz, Jun Li, Yongbin Ruan, Qi Zhang for stimulating discussions and valuable helps.

References

[Beh] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997) 601-617.

[B-F] K. Behrend, B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997) 45-88.

[C-K] D. Cox, S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**, Amer. Math. Soc., Providence, RI (1999).

[E-S] G. Ellingsrud, S.A. Stromme, *An intersection number for the punctual Hilbert scheme of a surface*, Trans. of A.M.S. **350** (1999) 2547-2552.

[Fo1] J. Fogarty, Algebraic families on an algebraic surface, Amer. J. Math. **90** (1968) 511-520.

[Fo2] J. Fogarty, *Algebraic families on an algebraic surface. II: The Picard scheme of the punctual Hilbert scheme*, Amer. J. Math. **95** (1973) 660-687.

[F-P] W. Fulton, R. Pandharipande, *Notes on stable maps and quantum cohomology*. Algebraic Geometry—Santa Cruz 1995, 45-96, Proc. Sympos. Pure Math. **62**, Amer. Math. Soc., Providence, RI (1997).

[Get] E. Getzler, *Intersection theory on $\overline{M}_{1,4}$ and elliptic Gromov-Witten invariants*, J. AMS **10** (1997) 973-998.

[Got] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990) 193–207.

[Gro] I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Lett. **3** (1996) 275–291.

[LiJ] J. Li, *Private communication*, 2000.

[LQZ] W.-P. Li, Z. Qin, Q. Zhang, *On the geometry of the Hilbert schemes of points in the projective plane*, preprint, math.AG/0105213.

[LT1] J. Li, G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. A.M.S. **11** (1998) 19-174.

[LT2] J. Li, G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, (1998) 47–83.

[LT3] J. Li, G. Tian, *Comparison of the algebraic and symplectic definitions of GW invariants*, Asian J. Math. **3** (1999) 689–728.

[Nak] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. Math. **145** (1997) 379–388.

[Ru1] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves*, Proceedings of 6th Gökova Geometry-Topology Conference. Turkish J. Math. **23** (1999) 161–231.

[Ru2] Y. Ruan, *Cohomology ring of crepant resolutions of orbifolds*, preprint, math.AG/0108195.

Department of Mathematics, HKUST, Clear Water Bay, Kowloon, Hong Kong
E-mail address: mawpli@ust.hk

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: zq@math.missouri.edu