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Induced Mappings on Boolean Algebras of Clopen Sets and on Projections of the C^*-Algebra $C(X)$

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Abstract

For a compact space X, any group automorphism φ of $C(X,\mathbb{S}^1)$ induces a mapping Θ on the Boolean algebra of the clopen subsets of X. We prove that the disjointness of Θ equivalent to θ_φ is an orthoisomorphism on the sets of projections of the C^*-algebra $C(X)$, when $\varphi(-1) = -1$. Indeed, Θ is a Boolean isomorphism iff θ_φ preserves the product of projections. If X is equipped with a probability measure μ, on a certain $\sigma-$algebra of X, we show (under some condition) that Θ preserves the disjoint of clopen subsets, up to sets of measure zero, or equivalently, the mapping θ_φ is $\mu-$orthoisomorphism on the projections of the C^*-algebra $C(X)$.

Key Words: Unitary, Projections, Almost Isomorphisms, Boolean Algebra, Clopen Subset.

1. Introduction

For a unital C^*-algebra $\mathfrak{A}$, $\mathcal{P}(\mathfrak{A})$, $\mathcal{U}(\mathfrak{A})$ denotes the sets of projections, unitaries of $\mathfrak{A}$ respectively. If X is a compact Hausdorff space, then $C(X)$ denotes the C^*-algebra of continuous complex-valued functions on X, and $C(X,\mathbb{S}^1)$ is the set of all continuous functions on X with values in the unit circle $\mathbb{S}^1$. The set of all closed and open subsets of X (the clopen subsets) is denoted by $CO(X)$. For any set B, by B^c we mean the complement of B.

Let A and B be two unital C^*-algebras. A projection orthoisomorphism mapping is defined by H. Dye in [5], as a one-to-one correspondense θ between the projections of A

and B, which preserves the orthogonality, i.e. if p and q are projections of A such that $pq = 0$, then $\theta(p)\theta(q) = 0$.

If φ is a group isomorphism between the unitary groups of A and B, then φ maps the self-adjoint unitaries of A onto the self-adjoint unitaries of B, and therefore H. Dye in [5] defines a natural mapping θ_φ between the sets of projections of A and B via

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p).$$

Then a natural question arises here; for which C^*-algebra have we that the induced map θ_φ is an orthoisomorphism [1, Introd. Q.3]? This problem was already solved in the case of von Neumann factors by H. Dye, when he proved the following main lemma.

Lemma 1.1 [5, Lemma 13] *Let M and N be two factors. If φ is an isomorphism between $\mathcal{U}(M)$, $\mathcal{U}(N)$ and M is not of type I_{2n} ($n \geq 1$), then θ_φ is an orthoisomorphism.*

Another positive answer is given in [1, Chapt. 5], where the author shows that for a large class of simple, unital C^*-algebras, the map θ_φ is always an orthoisomorphism. This class includes in particular, the Cuntz algebras $\mathcal{O}_n$, $2 \leq n \leq \infty$, and the simple unital AF-algebras having 2-divisible K_0-group.

In this paper, we study the case of the non-simple, commutative C^*-algebra $C(X)$ of continuous complex-valued functions on a compact subset X of $\mathbb{R}$.

Remark 1.2 *Recall that if A is a simple, unital C^*-algebra, and φ is a group automorphism of the unitary group of A, then by the simplicity of A, we must have $\varphi(-1) = -1$, as also same holds in the case of factors discussed (see [5]). As we are working on the non-simple C^*-algebra $C(X)$, we shall only consider automorphisms φ of $C(X, \mathbb{S}^1)$, which satisfy $\varphi(-1) = -1$.*

Let φ be an automorphism of the unitary group $\mathcal{U}(C(X)) = C(X, \mathbb{S}^1)$ of $C(X)$. Suppose $\varphi(-1) = -1$, as in Remark (1.2). This automorphism φ induces a bijective map θ_φ on the sets of projections $\mathcal{P}(C(X))$ of the C^*-algebra $C(X)$. If $p \in \mathcal{P}(C(X))$, then $p = \chi_o$, where o is a clopen subset of X (see [4, IX.3]). Therefore, we define a map Θ^φ (induced by θ_φ) on the Boolean algebras (for more details about Boolean algebras, see [6]) of the clopen subsets $CO(X)$ of X. This gives the link between the C^*-algebra $C(X)$ and the Boolean algebra $CO(X)$, which is studied in the first part of this paper. We prove that θ_φ is an orthoisomorphism if and only if the mapping Θ preserves the disjointness of clopen subsets of X. Moreover, that Θ is a Boolean isomorphism is equivalent to θ_φ preserving the product of projection $\mathcal{P}(C(X))$.

If X is a connected space (i.e. $C(X)$ has only the trivial projections 0 and 1), then Θ preserves the disjoint of the clopen subsets. Therefore, we consider the case where X is not a connected space. Recall that, if X is the Cantor ternary set, then $C(X)$ is an AF-algebra; more generally, if X is a second countable space, then $C(X)$ is an AF-algebra iff X is a totally disconnected space [7, 5.B]. Moreover, let us recall Stone's representation theorem:

Theorem 1.3 *[8] If B is a Boolean algebra, then there exists a totally disconnected compact Hausdorff space X such that B is isomorphic to the Boolean algebra of clopen subsets of X.*

In the second part of this paper (Section 3), we consider X to be a compact subset of $\mathbb{R}$, a $\sigma-$algebra $\mathcal{A}$ of subsets of X, which contains the Borel subsets of X, and μ to be any probability measure on $\mathcal{A}$. We define the $\mu-$orthoisomorphism map (or almost orthoisomorphism) on $\mathcal{P}(C(X))$; we find a condition on φ, in order that θ_φ becomes $\mu-$orthoisomorphism, by imitating the technique used in proving the following theorem proved by the author.

Theorem 1.4 *[1, Theorem (5.1.0.2)] Let A, B be simple, unital C^*-algebras, and φ be an isomorphism from $\mathcal{U}(A)$ to $\mathcal{U}(B)$. If there exist faithful, normalized traces τ_A and τ_B on A and B, respectively, such that*

$$\tau_A(u) = \tau_B(\varphi(u)), \textit{ for every self-adjoint} \, u \in \mathcal{U}(A),$$

then θ_φ is a projection orthoisomorphism.

So finally, we prove (under some conditions on φ) that if o_1 and o_2 are disjoint clopen subsets of X, then $\Theta(o_1)\Theta(o_2) = \chi_o$, such that $\mu(o) = 0$, i.e. Θ preserves disjoint of clopen subsets of X, up to sets of measure zero.

2. On The Boolean Algebra $CO(X)$

By direct computation, one can easily deduce the following: If o, o_1 and o_2 are clopen subsets of X, then $\chi_{o_1 \cap o_2} = \chi_{o_1}\chi_{o_2}$, $\chi_{o^c} = 1 - \chi_o$, $\chi_{o_1 \cup o_2} = \chi_{o_1} + \chi_{o_2} - \chi_{o_1 \cap o_2}$; and hence if o_1 and o_2 are disjoint clopen sets, then $\chi_{o_1 \cup o_2} = \chi_{o_1} + \chi_{o_2}$.

Let φ be a group automorphism of $\mathcal{U}(C(X))$. Therefore φ induces a natural mapping θ_φ on the set of projections $\mathcal{P}(C(X))$ via

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p).$$

Then θ_φ induces a mapping Θ on the Boolean algebra of the clopen subsets $CO(X)$ of X, which is defined as follows.

Definition 2.1 *If φ is a group automorphism of $\mathcal{U}(C(X))$, then the following commutative diagram defines the map Θ^φ (or simply, Θ) on the Boolean algebra $CO(X)$:*

$$\begin{array}{ccc} \mathcal{P}(C(X)) & \xrightarrow{\theta_\varphi} & \mathcal{P}(C(X)) \\ \uparrow & & \downarrow \\ CO(X) & \xrightarrow{\Theta} & CO(X) \end{array}$$

i.e. $o \in CO(X) \Longleftrightarrow \chi_o \in \mathcal{P}(C(X))$, and $\chi_{\Theta(o)} = \theta_\varphi(\chi_o)$.

Now let us establish some basic properties of the induced map Θ^φ on the Boolean algebras $CO(X)$, which are similar to those results established for θ_φ in [3]. As the induced map θ_φ is a bijective map from its definition, consequently, we have the following result.

Proposition 2.2 *The induced Θ is a bijective map on the Boolean algebra $CO(X)$.*

Proof Is obvious.

In the following proposition, we establish a functorial property of the map $\varphi \mapsto \Theta$.

Proposition 2.3 *(i) If φ and ψ are two automorphisms of $C(X, \mathbb{S}^1)$, then $\Theta^{\psi\varphi} = \Theta^\psi\Theta^\varphi$.*
(ii) If ι is the identity map of $C(X, \mathbb{S}^1)$, then Θ^ι is the identity map on $CO(X)$.
(iii) If φ is an automorphisms of $C(X, \mathbb{S}^1)$, then $(\Theta^\varphi)^{-1} = \Theta^{\varphi^{-1}}$.

Proof (i) Let φ and ψ be two automorphisms of $C(X,\mathbb{S}^1)$. It is enough to prove $\chi_{\Theta^{\psi\varphi}} = \chi_{\Theta^{\psi}\Theta^{\varphi}}$. If $o \in CO(X)$, then

$$\begin{aligned}
\chi_{\Theta^{\psi\varphi}(o)} &= \theta_{\psi\varphi}(\chi_o) \\
&= \theta_{\psi}(\theta_{\varphi}(\chi_o)) \quad \text{by [3, Prop. (3.1.0.4)(i)]} \\
&= \theta_{\psi}(\chi_{\Theta^{\varphi}(o)}) \\
&= \chi_{\Theta^{\psi}\Theta^{\varphi}(o)}.
\end{aligned}$$

(ii) If $o \in CO(X)$, then $\chi_{\Theta^{\iota}(o)} = \theta_{\iota}(\chi_o) = \chi_o$ as θ_ι is the identity map on the sets of projections by [3, Prop. (3.1.0.4); (ii)], therefore $\Theta^{\iota}(o) = o$.
(iii) If $o \in CO(X)$, then also by [3, Prop. (3.1.0.4); (iii)] we have

$$\chi_{(\Theta^{\varphi})^{-1}(o)} = (\theta_{\varphi})^{-1}(\chi_o) = \theta_{\varphi^{-1}}(\chi_o) = \chi_{\Theta^{\varphi^{-1}}(o)}.$$

Hence the proposition has been checked. $\square$

Consequently, we have proved the following corollary.

Corollary 2.4 *The group of all automorphisms φ of $C(X,\mathbb{S}^1)$ induces a group of mappings θ^{φ} of the Boolean algebra $CO(X)$.*

Remark 2.5 *For the rest of this paper, if φ is an automorphism of the unitary group $C(X,\mathbb{S}^1)$ of $C(X)$, then as mentioned in Remark (1.2), φ is assumed to satisfy $\varphi(-1) = -1$. Therefore, from [2] or [3, Lemma (3.1.0.3)(6)] we have that $\theta_\varphi(1) = 1$ and $\theta_\varphi(1-p) = 1-\theta_\varphi(p)$, for any projection p of $C(X)$. In the case of factors which is discussed in [5], $\varphi(-1) = -1$ is already satisfied, as well as in the case of simple C^*-algebras.*

Now we discuss the question whether the map Θ is a Boolean isomorphism, or under what conditions it becomes so? Also we give a characterization of Θ being a Boolean isomorphism in term of the induced map θ_φ on the projections of $C(X)$. Also a characterization of Θ preserving the disjointness of the clopen sets.

As the induced map θ_φ on the set of projections preserves the partition of the unity (i.e. if $pq = 0$ and $p + q = 1$, then $\theta_\varphi(p)\theta_\varphi(q) = 0$), then consequently, we have the following proposition.

Proposition 2.6 *Let φ be as in Remark (2.5). If o_1 and o_2 are two clopen subsets of X which form a partition of X, then $\Theta(o_1) \cap \Theta(o_2) = \emptyset$.*

Proof Let $p = \chi_{o_1}$ and $q = \chi_{o_2}$. Then p and q are projections of $C(X)$ such that $p + q = 1$ and $pq = 0$. Therefore,

$$\theta_\varphi(p)\theta_\varphi(q) = \theta_\varphi(p)\theta_\varphi(1-p) = \theta_\varphi(p)(1-\theta_\varphi(p)) = 0.$$

Then we have

$$\chi_{\Theta(o_1)\cap\Theta(o_2)} = \chi_{\Theta(o_1)}\chi_{\Theta(o_2)} = 0,$$

and hence $\Theta(o_1) \cap \Theta(o_2) = \emptyset$, which ends the proof. □

Proposition 2.7 *Let φ be as in Remark (2.5). If o is a clopen subset of X, then $\Theta(o^c) = (\Theta(o))^c$.*

Proof Let $o \in CO(X)$. It's enough to prove $\chi_{\Theta(o^c)} = \chi_{(\Theta(o))^c}$. This is equivalent to prove $\theta_\varphi(\chi_{o^c}) = 1 - \chi_{\Theta(o)}$. But

$$\theta_\varphi(\chi_{o^c}) = \theta_\varphi(1-\chi_o) = 1 - \theta_\varphi(\chi_o).$$

Hence the proposition has been checked. □

Now in the following lemma, we characterize the concept of orthoisomorphism on the sets of projections of the C^*-algebra $C(X)$. The characterization is valid for any automorphism φ of $C(X, \mathbb{S}^1)$, without any restrictions as in Remark (2.5).

Lemma 2.8 *Let φ be any automorphism of $C(X, \mathbb{S}^1)$. Then θ_φ is an orthoisomorphism on the projections of $C(X)$ iff Θ preserves the disjoint of the clopen sets in the Boolean algebra $CO(X)$.*

Proof Suppose that θ_φ is an orthoisomorphism. Let o_1 and o_2 be two disjoint clopen subsets of X. Then

$$\begin{aligned}
o_1 \cap o_1 = \emptyset \quad &\Leftrightarrow \quad \chi_{o_1}\chi_{o_2} = 0 \\
&\Leftrightarrow \quad \theta_\varphi(\chi_{o_1})\theta_\varphi(\chi_{o_2}) = 0 \\
&\Leftrightarrow \quad \chi_{\Theta(o_1)}\chi_{\Theta(o_2)} = 0 \\
&\Leftrightarrow \quad \chi_{\Theta(o_1)\cap\Theta(o_2)} = 0 \\
&\Leftrightarrow \quad \Theta(o_1) \cap \Theta(o_2) = \emptyset.
\end{aligned}$$

For the converse, suppose that Θ preserves the disjoint of clopen subsets of X. Let $p, q \in \mathcal{P}(C(X))$, such that $pq = 0$. Then $p = \chi_{o_1}$ and $q = \chi_{o_2}$, for some $o_1, o_2 \in CO(X)$. As $\chi_{o_1 \cap o_2} = 0$, then $o_1 \cap o_2 = \emptyset$; therefore, by assumption, we have $\Theta(o_1) \cap \Theta(o_2) = \emptyset$, and then

$$0 = \chi_{\Theta(o_1) \cap \Theta(o_2)} = \chi_{\Theta(o_1)} \chi_{\Theta(o_2)};$$

thus we have $\theta_\varphi(p)\theta_\varphi(q) = 0$. $\square$

Now recall that if p and q are commuting projections of a unital C^*-algebra A, then the symmetric difference projection of p and q, which is denoted by $p\Delta q$ (the same notation for symmetric difference of two sets), is defined by $p\Delta q = p + q - 2pq$. It is easy to prove that if o_1 and o_2 are clopen subsets of X, then $\chi_{o_1} \Delta \chi_{o_2} = \chi_{o_1 \Delta o_2}$. As the induced map θ_φ between the sets of projections preserves the symmetric difference ([5, Lemma 9], [3, Lemma (3.1.0.3)]), then we have the following lemma.

Lemma 2.9 *Let φ be as in Remark (2.5). If o_1 and o_2 are two clopen subsets of X, then $\Theta(o_1 \Delta o_2) = \Theta(o_1)\Delta\Theta(o_2)$, i.e. Θ preserves the symmetric difference of the clopen subsets.*

Proof It is enough to prove $\chi_{\Theta(o_1 \Delta o_2)} = \chi_{\Theta(o_1)\Delta\Theta(o_2)}$. As

$$\begin{aligned}
\chi_{\Theta(o_1 \Delta o_2)} &= \theta_\varphi(\chi_{o_1 \Delta o_2}) \\
&= \theta_\varphi(\chi_{o_1} \Delta \chi_{o_2}) \\
&= \theta_\varphi(\chi_{o_1}) \Delta \theta_\varphi(\chi_{o_2}) \\
&= \chi_{\Theta(o_1)} \Delta \chi_{\Theta(o_2)},
\end{aligned}$$

the lemma is checked. $\square$

As a consequence result of Lemma (2.9) and Lemma (2.8) , we can easily prove.

Corollary 2.10 *Let φ be as in Remark (2.5). If the map θ_φ is an orthoisomorphism between the sets of projections of $C(X)$, then the map Θ preserves the disjoint union of the clopen subsets of X.*

Proof Let o_1 and o_2 be two disjoint clopen subsets of X. As $o_1 \cup o_2 = o_1 \Delta o_2$, then by applying Lemma (2.9) together with Lemma (2.8), we prove $\Theta(o_1 \cup o_2) = \Theta(o_1) \cup \Theta(o_2)$. □

Now let's prove the following result, which shows that, if θ_φ is an orthoisomorphism between the sets of projections, then saying that the map Θ preserves the union of clopen subsets of X is equivalent to saying that the map Θ preserves the intersection of the clopen sets.

Lemma 2.11 *Let φ be as in Remark (2.5) such that the induced map θ_φ is an orthoisomorphism. Let o_1 and o_2 be two clopen subsets of X. Then*

$$\Theta(o_1 \cap o_2) = \Theta(o_1) \cap \Theta(o_2) \quad \text{iff} \quad \Theta(o_1 \cup o_2) = \Theta(o_1) \cup \Theta(o_2).$$

Proof As $o_1 \cup o_2 = (o_1 \Delta o_2) \cup (o_1 \cap o_2)$ where the union in the right hand side is a disjoint union, then by Corollary (2.10)

$$\Theta(o_1 \cup o_2) = \Theta(o_1 \Delta o_2) \cup \Theta(o_1 \cap o_2). \tag{1}$$

If Θ preserves the union of the clopen sets, then by Lemma (2.9), we have $\Theta(o_1) \cup \Theta(o_2) = (\Theta(o_1) \Delta \Theta(o_2)) \cup \Theta(o_1 \cap o_2)$ therefore,

$$\Theta(o_1 \cap o_2) = (\Theta(o_1) \cup \Theta(o_2)) \backslash (\Theta(o_1) \Delta \Theta(o_2)) = \Theta(o_1) \cap \Theta(o_2).$$

On the other hand, if Θ preserves the intersection of clopen sets, then again by (1) and Lemma (2.9), we have

$$\Theta(o_1 \cup o_2) = (\Theta(o_1) \Delta \Theta(o_2)) \cup (\Theta(o_1) \cap \Theta(o_2)) = \Theta(o_1) \cup \Theta(o_2).$$

Hence the lemma is checked. □

Now let's give a characterization of Θ being a Boolean isomorphism on $CO(X)$ in terms of the induced map θ_φ on the set of projections $\mathcal{P}(C(X))$.

Theorem 2.12 *Let φ be as in Remark (2.5). Then the map Θ on the Boolean algebra $CO(X)$ is a Boolean isomorphism iff the induced map θ_φ on the projections of the C^*-algebra $C(X)$ preserves the product of projections, and hence an orthoisomorphism.*

Proof Suppose that Θ is a Boolean isomorphism on $CO(X)$. If $p, q \in \mathcal{P}(C(X))$, then $p = \chi_{o_1}$ and $q = \chi_{o_2}$. Moreover,

$$\begin{aligned}
\theta_\varphi(pq) &= \theta_\varphi(\chi_{o_1}\chi_{o_2}) \\
&= \theta_\varphi(\chi_{o_1 \cap o_2}) \\
&= \chi_{\Theta(o_1 \cap o_2)} \\
&= \chi_{\Theta(o_1) \cap \Theta(o_2)} \\
&= \chi_{\Theta(o_1)}\chi_{\Theta(o_2)} \\
&= \theta_\varphi(p)\theta_\varphi(q).
\end{aligned}$$

Suppose that θ_φ preserves the product of projections in $\mathcal{P}(C(X))$, hence it is an orthoisomorphism. By Lemma (2.11) and Proposition (2.7), it is enough to prove that Θ preserves the intersection of the clopen sets, and this holds as

$$\chi_{\Theta(o_1 \cap o_2)} = \theta_\varphi(\chi_{o_1}\chi_{o_2}) = \theta_\varphi(\chi_{o_1})\theta_\varphi(\chi_{o_2}) = \chi_{\Theta(o_1)}\chi_{\Theta(o_2)}.$$

Hence the proof of the theorem is completed. □

It is clear that if X and Y are homeomorphic topological spaces, then their Boolean algebras of clopen subsets $CO(X)$ and $CO(Y)$ are Boolean isomorphic. Also the C^*-algebras $C(X)$ and $C(Y)$ are $*-$isomorphic.

3. Almost Orthoisomorphism On Projections Of $C(X)$

Let X be a compact space, with a $\sigma-$algebra $\mathcal{A}$ of subsets of X containing the Borel subsets of X, and μ be a probability measure on $\mathcal{A}$. Also let φ be an automorphism of $C(X, \mathbb{S}^1)$, as in Remark (2.5) i.e. $\varphi(-1) = -1$. In this section, we concentrate on some compact spaces X, such that under some conditions, we have that the induced map θ_φ is an orthoisomorphism (in some sense) on the sets of projections $\mathcal{P}(C(X))$; and as discussed in the previous part, we find a condition on φ in order that the map Θ preserves the disjointness of clopen subsets, up to sets of measure zero, i.e. if o_1 and o_2 are disjoint clopen subsets of X, then $\Theta(o_1)\Theta(o_2)$ equals to a clopen subset of measure zero.

Recall that a state (see [7, 6.3]) on a C^*-algebra A is a linear map $s : A \to \mathbb{C}$ which is positive i.e. $s(a^*a) \geq 0$ and $\|s\| = 1$. Moreover, by [7, Example 6.5], if μ is a probability measure on X, then the map $s_\mu : C(X) \to \mathbb{C}$ defined by $f \mapsto \int_X f\, d\mu$ gives a state on $C(X)$ and hence define a normalized trace.

Let φ be as in Remark (2.5). We discuss whether the induced map θ_φ on $\mathcal{P}(C(X))$ is an orthoisomorphism. We are going to define a normalized faithful (in some sense) trace on $C(X)$, where φ is invariant under τ, in order to use [1, Theorem 5.1.0.2].

Definition 3.1 *Let τ be a trace on $C(X)$. Then τ is called $\mu-$almost faithful ($\mu-$faithful), if $f \geq 0$ and $\tau(f) = 0$ implies $f = 0$, $\mu-$almost everywhere.*

Definition 3.2 *A mapping α on $\mathcal{P}(C(X))$ is called a $\mu-$orthoisomorphism if for all $p, q \in \mathcal{P}(C(X))$, $pq = 0$ implies $\alpha(p)\alpha(q) = 0$ $\mu-$almost everywhere.*

Let us prove the following.

Proposition 3.3 *The mapping $\tau_\mu : C(X) \to \mathbb{C}$ which is defined by $f \mapsto \int_X f\, d\mu$ is a normalized $\mu-$faithful trace on $C(X)$.*

Proof It is clear that τ_μ is a normalized trace on $C(X)$. To prove that τ_μ is $\mu-$faithful, let $f \in C(X)$, and suppose that $\tau_\mu(f^* f) = 0$. Therefore

$$\int_X |f|^2\, d\mu = 0,$$

and then $|f|^2 = 0$ almost everywhere on X, hence the proposition has been proved. $\square$

Recall that if u is a self-adjoint unitary of $C(X)$, then $u \in C(X, \{1, -1\})$, then let's define the following.

Definition 3.4 *Any self-adjoint unitary u of $C(X)$, associates the partition $\{X_u^+, X_u^-\}$ of X induced by u, where*

$$X_u^+ := \{x \in X|\ u(x) = 1\},$$

$$X_u^- := \{x \in X|\ u(x) = -1\}.$$

Notice that if u is any self-adjoint unitary of $C(X)$, then both X_u^+ and X_u^- are measurable subsets of X.

Remark 3.5 *The only purpose from assuming the $\sigma-$algebra $\mathcal{A}$ containing the Borel sets of X, is to ensure that X_u^+ and X_u^- are measurable subsets of X. So $\mathcal{A}$ can be replaced by any $\sigma-$algebra which satisfies this condition.*

In the proof of our result, we assume that φ satisfies the following condition: For any self-adjoint unitary u of $C(X)$

$$\mu(X_u^+) + \mu(X_{\varphi(u)}^-) = \mu(X_{\varphi(u)}^+) + \mu(X_u^-) \tag{2}$$

It is clear that the identity automorphism of $\mathcal{U}(C(X))$, and the automorphism $u \mapsto \bar{u}$ both satisfy condition (2).

Now let's prove the following lemma.

Lemma 3.6 *Let φ be as in Remark (2.5). If φ satisfies condition (2), then $\tau_\mu(\varphi(u)) = \tau_\mu(u)$, for all self-adjoint unitaries of $C(X)$.*

Proof If u is a self-adjoint unitary, then we have

$$\begin{aligned}
\tau_\mu(\varphi(u)) &= \int_X \varphi(u)\, d\mu \\
&= \int_X (\varphi(u))^+\, d\mu - \int_X (\varphi(u))^-\, d\mu \\
&= \mu(X_{\varphi(u)}^+) - \mu(X_{\varphi(u)}^-) \\
&= \mu(X_u^+) - \mu(X_u^-)\,, \quad \text{by condition (2)} \\
&= \int_X u\, d\mu \\
&= \tau_\mu(u).
\end{aligned}$$

Hence the lemma is checked. □

Now we are in the position to prove the following result.

Theorem 3.7 *Let X be a compact subset of $\mathbb{R}$, with a $\sigma-$algebra $\mathcal{A}$, containing the Borel subsets of X (or as in Remark (3.5)), and μ be a probability measure on $\mathcal{A}$. Let φ be an automorphism of $C(X, \mathbb{S}^1)$, such that $\varphi(-1) = -1$. If φ satisfies condition (2), then the induced map θ_φ is $\mu-$orthoisomorphism on $\mathcal{P}(C(X))$.*

Proof Let $p, q \in \mathcal{P}(C(X))$ such that $pq = 0$, and $u,\ v$ be the self-adjoint unitaries $1-2p,\ 1-2q$ (respectively). If $x = \theta_\varphi(p)\theta_\varphi(q)$, then

$$\begin{aligned} xx^* &= \frac{1}{4}[(1-\varphi(u))(1-\varphi(v))] \\ &= \frac{1}{4}[1-\varphi(u)-\varphi(v)+\varphi(uv)]. \end{aligned}$$

Therefore, by Lemma (3.6), we have

$$\begin{aligned} \tau_\mu(xx^*) &= \frac{1}{4}[1-\tau_\mu(u)-\tau_\mu(v)+\tau_\mu(uv)] \\ &= \tau_\mu(pq) \\ &= 0. \end{aligned}$$

Then by Proposition (3.3), we have that $x = \theta_\varphi(p)\theta_\varphi(q) = 0$, $\mu-$almost everywhere; hence the proof is completed. □

Consequently, we have the following result of the map Θ on the Boolean algebra $CO(X)$.

Corollary 3.8 *Let (X, μ) and φ be as in the previous theorem. If o_1 and o_2 are disjoint clopen subsets of X, then $\Theta(o_1)\Theta(o_2) = o$, such that $\mu(o) = 0$, i.e. Θ preserves the disjoint of the clopen subsets of X, up to sets of measure zero.*

Proof Direct from the previous theorem and Lemma (2.8). □

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