Turk J Math
33 (2009) , 131 – 142.
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doi:10.3906/mat-0802-10

The Existence of Triple Positive Solutions of Nonlinear Four-point Boundary Value Problem with p-Laplacian*

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Abstract

This paper deals with the multiplicity results of positive solutions of one-dimensional singular p-Laplace equation

$$(\varphi_p(u'(t)))' + a(t)f(t, u(t), u'(t)) = 0, \qquad 0 < t < 1$$

subject to the nonlinear boundary conditions

$$\alpha\varphi_p(u(0)) - \beta\varphi_p(u'(\xi)) = 0, \quad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(\eta)) = 0,$$

where $\varphi_p(x) = |x|^{p-2}x, p > 1$. By using the Avery-Peterson fixed point theorem, sufficient conditions for the existence of at least three positive solutions to the boundary value problem mentioned above are obtained.

Key Words: p-Laplacian; Avery-Peterson fixed-point theorem; positive solution; boundary value problem.

1. Introduction

In recent years, existence and multiplicity of positive solution for two-point boundary value problems involving p-Laplacian have been broadly investigated; see [2, 7, 8, 11-13] and references therein. There is much current attention focused on the study of nonlinear multi-point (at least three-point) boundary value problems. We refer the reader to [3-6, 9, 10] for details.

In this paper, we consider the following nonlinear four-point singular boundary value problem with p-Laplacian

$$(\varphi_p(u'))' + a(t)f(t, u(t), u'(t)) = 0 \qquad 0 < t < 1, \tag{1.1}$$

$$\alpha\varphi_p(u(0)) - \beta\varphi_p(u'(\xi)) = 0, \qquad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(\eta)) = 0, \tag{1.2}$$

where $\varphi_p(x) = |x|^{p-2}x, p > 1; \alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0, \xi, \eta \in (0, 1)$ are prescribed and $\xi < \eta$. Let $\varphi_q(x) = |x|^{q-2}x$ be the inverse function to φ_p; then we have $\frac{1}{p} + \frac{1}{q} = 1$. And $a(t)$ may be singular at $t = 0$ and/or $t = 1$.

2000 *AMS Mathematics Subject Classification:* 34B15, 34B16, 34B18.

*This work is sponsored by the Tutorial Scientific Research Program Foundation of Education Department of Gansu Province P.R. China (0710-04; 0810-03).

In [11, 12], Bai et al. studied the existence of at least three positive solutions of Eq. (1.1) together with the two-point boundary value conditions

$$u(0) = u(1) = 0, \quad \text{or} \quad u(1) = u'(1) = 0, \tag{1.3}$$

and

$$\alpha\varphi_p(u(0)) - \beta\varphi_p(u'(0)) = 0, \qquad \gamma\varphi_p(u(1)) + \delta\varphi_p(u'(1)) = 0, \tag{1.4}$$

and

$$u(0) - g_1(u'(0)) = 0, u(1)) + g_2(u'(1)) = 0, \tag{1.5}$$

by using a recent three-function fixed point theory and the Avery-Peterson fixed point theorem, respectively. In [2], Sun et al. obtained the existence of positive solutions of Eq. (1.1)–(1.4) and (1.1)–(1.5) and established the iterative schemes for the approximate solutions.

More recently, when the nonlinear term f does not depend on the first-order derivative, Eq. (1.1) together with some multi-point boundary conditions have been studied in several papers, for example, see [5, 7, 8, 10-13]. In [8], Xiong studied the existence of positive solution of Eq. (1.1)–(1.3) by means of the variational method. In [13], Li et al. considered the existence of at least three positive solutions of Eq. (1.1)–(1.5) by using a three-functional fixed point theorem. Zhao et al. considered the existence of three positive solutions of Eq. (1.1)–(1.3), the main tool is the Leggett-Williams fixed point theorem [7]. While in [5], Ji et al. studied the existence of multiple positive solutions of Eq. (1.1) in the case of $a(t) = 1$, subject to the nonlinear four-point boundary value conditions

$$u(0) - \alpha(u'(\xi)) = 0, \qquad u(1) + \beta(u'(\eta)) = 0. \tag{1.6}$$

The main tool is the fixed-point theorem in cones.

However, for Eq. (1.1)–(1.2), there are currently few papers dealing with the existence of positive solution. Motivated by the papers mentioned above, in this paper we consider the existence of three positive solutions for nonlinear singular boundary value problem (1.1)–(1.2) by using the Avery-Peterson fixed point theorem. The purpose of this paper is to essentially improve and generalize the results in the above mentioned literatures.

In the rest of the paper, we make the following assumptions:

(H_1) $f \in C([0,1] \times (0,+\infty) \times (-\infty,+\infty),(0,+\infty))$;

(H_2) $a(t) \in C((0,1),[0,\infty))$, and $\int_0^1 a(t)\mathrm{d}t < \infty$. Furthermore, $a(t)$ is not identical zero on any compact subinterval of $(0,1)$. And $a(t)$ may be singular at $t = 0$ and/or $t = 1$.

2. Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces and a lemma which transform the problem (1.1)–(1.2) into an integral equation, and we state a three fixed points theorem duo to Avery and Peterson [1] for multiple fixed-points of a cone-preserving operator on ordered Banach space.

Definition 2.1 *Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is called a cone if it satisfies:*

(i) $u \in P, \lambda \geq 0,$ *implies* $\lambda u \in P$;

(ii) $u \in P, -u \in P,$ *implies* $u = 0$.

Definition 2.2 *A map α is said to be a nonnegative continuous concave functional on cone P of real Banach space E if*

$$\alpha : P \to [0, \infty)$$

is continuous and

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in P$ and $t \in [0,1]$. Similarly, we say the map β is a nonnegative continuous convex functional on cone P of real Banach space E if

$$\beta : P \to [0, \infty)$$

is continuous and

$$\beta(tu + (1-t)v) \leq t\beta(u) + (1-t)\beta(v)$$

for all $u, v \in P$ and $t \in [0,1]$.

Let γ and θ be nonnegative continuous convex functional on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$P(\gamma, d) = \{u \in P : \gamma(u) < d\},$$

$$\overline{P(\gamma, d)} = \{u \in P : \gamma(u) \leq d\},$$

$$P(\gamma, \alpha, b, d) = \{u \in P : b \leq \alpha(u), \gamma(u) \leq d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{u \in P : b \leq \alpha(u), \theta(u) \leq c, \gamma(u) \leq d\},$$

and a convex closed set

$$R(\gamma, \psi, a, d) = \{u \in P : a \leq \psi(u), \gamma(u) \leq d\}.$$

Lemma 2.3 *Suppose that conditions (H_1),(H_2) hold, then $u(t) \in C[0,1] \bigcap C^2(0,1)$ is a solution of boundary value problem (1.1)–(1.2), if and only if $u(t)$ is a solution of the following integral equation:*

$$u(t) = \begin{cases} \varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) + \int_0^t \varphi_q\left(\int_s^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\,\mathrm{d}s, 0 \leq t \leq \sigma \\ \varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) + \int_t^1 \varphi_q\left(\int_\sigma^s a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\,\mathrm{d}s, \sigma \leq t \leq 1. \end{cases} \tag{2.1}$$

where $\sigma \in [\xi, \eta] \subset (0,1)$ and $u'(\sigma) = 0$.

Proof. Firstly, assume that(2.1)holds, thus, we have

$$u'(t) = \begin{cases} \varphi_q\left(\int_t^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) \geq 0, & 0 \leq t \leq \sigma \\ -\varphi_q\left(\int_\sigma^t a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) \leq 0, & \sigma \leq t \leq 1. \end{cases} \tag{2.2}$$

Hence, thanks to (2.2), we have $(\varphi_p(u'(t)))' + a(t)f(t,u(t),u'(t)) = 0, \quad 0 < t < 1$, i.e., equation (1.1) holds. Furthermore, let $t = 0$ and $t = 1$ in (2.1) and let $t = \xi,\ t = \eta$ in (2.2), we can show the boundary conditions (1.2) is satisfied. Consequently, sufficiency is proved.

Next, by the boundary conditions (1.2), we have $u'(\xi) \geq 0, \quad u'(\eta) \leq 0$. Then there exists a constant $\sigma \in [\xi, \eta]$, such that $u'(\sigma) = 0$. Thus, for $t \in (0, \sigma)$, integrating the two side of the equation (1.1) over $(0, \sigma)$, and notice that $u'(\sigma) = 0$, we have

$$\varphi_p(u'(\sigma)) - \varphi_p(u'(t)) = -\int_t^\sigma a(s)f(s, u(s), u'(s))\,\mathrm{d}s. \tag{2.3}$$

Then $u'(t) = \varphi_q\left(\int_t^\sigma a(s)f(s, u(s), u'(s))\,\mathrm{d}s\right)$. Hence,

$$u(\sigma) - u(t) = \int_t^\sigma \varphi_q\left(\int_s^\sigma a(r)f(r, u(r), u'(r))\,\mathrm{d}r\right)\mathrm{d}s. \tag{2.4}$$

Let $t = \xi$ in (2.3) and notice that $u'(\sigma) = 0$, we have

$$\varphi_p(u'(\xi)) = \int_\xi^\sigma a(s)f(s, u(s), u'(s))\,\mathrm{d}s.$$

With the boundary condition (1.2), we have

$$\varphi_p(u(0)) = \frac{\beta}{\alpha}\varphi_p(u'(\xi)).$$

Then

$$u(0) = \varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(s)f(s, u(s), u'(s))\,\mathrm{d}s\right). \tag{2.5}$$

Let $t = 0$ in (2.4), by (2.4) and (2.5), we can obtain that for any $t \in (0, \sigma)$

$$u(t) = \varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r, u(r), u'(r))\,\mathrm{d}r\right) + \int_0^t \varphi_q\left(\int_s^\sigma a(r)f(r, u(r), u'(r))\,\mathrm{d}r\right)\mathrm{d}s.$$

Similarly, for $t \in (\sigma, 1)$, by integrating the two sides of equation (1.1) over $(\sigma, 1)$, we can obtain that for any $t \in (\sigma, 1)$

$$u(t) = \varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r, u(r), u'(r))\,\mathrm{d}r\right) + \int_t^1 \varphi_q\left(\int_\sigma^s a(r)f(r, u(r), u'(r))\,\mathrm{d}r\right)\mathrm{d}s.$$

This ends the proof of Lemma 2.1. □

In order to fulfil the proof of the main result, the following fixed point theorem, due to Avery and Peterson, will be fundamental.

Theorem A [1]. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functional on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,*

$$\alpha(u) \leq \psi(u), \quad \text{and} \quad \|u\| \leq M\gamma(u), \quad \text{for all} \quad u \in \overline{P(\gamma, d)}. \tag{2.6}$$

Suppose $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$ is completely continuous and there exist positive numbers a,b and c with $a<b$ and such that

$(C_1)\{u\in P(\gamma,\theta,\alpha,b,c,d):\alpha(u)>b\}\neq\emptyset$, and $\alpha(Tu)>b$, for $u\in P(\gamma,\theta,\alpha,b,c,d)$;

$(C_2)\alpha(Tu)>b$, for $u\in P(\gamma,\alpha,b,d)$ with $\theta(Tu)>c$;

$(C_3)0\notin R(\gamma,\psi,a,d)$ and $\psi(Tu)<a$ for $u\in R(\gamma,\psi,a,d)$ with $\psi(u)=a$.

Then T has at least three fixed points $u_1,u_2,u_3\in\overline{P(\gamma,d)}$ such that

$$\gamma(u_i)\le d,\quad \text{for}\quad i=1,2,3,$$

$$b<\alpha(u_1),$$

$$a<\psi(u_2),\quad \text{with}\quad \alpha(u_2)<b,$$

and

$$\psi(u_3)<a.$$

3. Existence of three positive solutions of problem(1.1)–(1.2)

In this section, we define two appropriate Banach spaces and the two cones, and provide two technical lemmas which need in the proof of our main result, then present our main result and its proof.

Let $E_1=C^1[0,1]$, endowed with the norm

$$\|u\|_1=\max\left\{\max_{0\le t\le 1}|u(t)|,\quad \max_{0\le t\le 1}|u'(t)|\right\},$$

and $E_2=C[0,1]$, endowed with the maximum norm

$$\|u\|_2=\max_{0\le t\le 1}|u(t)|.$$

It is easy to check that if $u(t)$ satisfy

$$(\varphi_p(u'(t)))'=-a(t)f(t,u(t),u'(t))\le 0$$

then u is concave on $[0,1]$. Thus, we can define cone $P_1\subset E_1$ by

$$P_1=\{u\in E_1: u(t) \text{ is nonnegative and concave on } [0,1], \text{ and satisfying}$$

$$\alpha\varphi_p(u(0))-\beta\varphi_p(u'(\xi))=0,\quad \gamma\varphi_p(u(1))+\delta\varphi_p(u'(\eta))=0\}$$

and define cone $P_2\subset E_2$ by

$$P_2=\{u\in E_2: u(t) \text{ is nonnegative and concave on } [0,1]\}.$$

Obviously, $E_1\subset E_2, P_1\subset P_2$.

Let the nonnegative continuous concave functional α_1, the nonnegative continuous convex functional θ_1, γ_1, and the nonnegative continuous functional ψ_1 be defined on the cone P_1 given by

$$\alpha_1(u) = \min_{\omega \le t \le (1-\omega)} u(t), \text{ for } \quad \omega \in (0, \frac{1}{2}),$$

$$\gamma_1(u) = \max_{0 \le t \le 1} |u'(t)|, \qquad \psi_1(u) = \theta_1(u) = \max_{0 \le t \le 1} u(t).$$

In order to prove our main result, we will make use of the following technical lemmas.

Lemma 3.1 *[4, **Lemma 2.1**]. Let $u \in P_2$ and $\omega \in (0, \frac{1}{2})$, then*

$$u(t) \ge \omega \|u\|_2, \quad t \in [\omega, 1-\omega].$$

Apparently, the conclusion is also true for $u \in P_1$.

Lemma 3.2 *Let $u \in P_1$, then there exists a positive constant L, such that*

$$\max_{0 \le t \le 1} |u(t)| \le L \max_{0 \le t \le 1} |u'(t)|.$$

Proof. By $u(t) - u(0) = \int_0^t u'(s)\,\mathrm{d}s$, we have

$$\max_{0 \le t \le 1} |u(t)| \le |u(0)| + \max_{0 \le t \le 1} |u'(t)|.$$

On the other hand, by the first equation of (1.2), we have

$$|u(0)| = \varphi_q(\frac{\beta}{\alpha})|u'(\xi)| \le \varphi_q(\frac{\beta}{\alpha}) \max_{0 \le t \le 1} |u'(t)|.$$

Thus, we have

$$\max_{0 \le t \le 1} |u(t)| \le \left(1 + \varphi_q(\frac{\beta}{\alpha})\right) \max_{0 \le t \le 1} |u'(t)|.$$

Similarly, by the second equation of (1.2), we have

$$\max_{0 \le t \le 1} |u(t)| \le \left(1 + \varphi_q(\frac{\delta}{\gamma})\right) \max_{0 \le t \le 1} |u'(t)|.$$

Therefore, setting

$$L = \min\left\{1 + \varphi_q(\frac{\beta}{\alpha}), 1 + \varphi_q(\frac{\delta}{\gamma})\right\},$$

the proof of Lemma 3.2 ends. □

For convenience, we introduce the following constants:

$$N = \max\left\{\varphi_q\left(\int_0^\eta a(r)\,\mathrm{d}r\right), \varphi_q\left(\int_\xi^1 a(r)\,\mathrm{d}r\right)\right\},$$

$$M=\max\left\{\varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)\,\mathrm{d}r\right)+\int_0^\eta\varphi_q\left(\int_s^\eta a(r)\,\mathrm{d}r\right)\,\mathrm{d}s,\varphi_q\left(\frac{\delta}{\gamma}\int_\xi^\eta a(r)\,\mathrm{d}r\right)+\int_\xi^1\varphi_q\left(\int_\xi^s a(r)\,\mathrm{d}r\right)\,\mathrm{d}s\right\},$$

$$m=\min\left\{\int_0^\xi\varphi_q\left(\int_s^\xi a(r)\,\mathrm{d}r\right)\,\mathrm{d}s,\int_\eta^1\varphi_q\left(\int_\eta^s a(r)\,\mathrm{d}r\right)\,\mathrm{d}s\right\}.$$

Our main result is as follows.

Theorem 3.3 *Assume that (H_1), (H_2) hold, and suppose that there exist positive constants a,b,d such that $0<a<b<\omega Ld$. Also assume that f satisfies the following conditions:*

(H_3) $f(t,u,u')\le\varphi_p(\frac{d}{N})$, *for* $(t,u,u')\in[0,1]\times[0,Ld]\times[-d,d]$;

(H_4) $f(t,u,u')>\varphi_p(\frac{b}{\omega m})$, *for* $(t,u,u')\in[\omega,1-\omega]\times[b,\frac{b}{\omega}]\times[-d,d]$;

(H_5) $f(t,u,u')<\varphi_p(\frac{a}{M})$, *for* $(t,u,u')\in[0,1]\times[0,a]\times[-d,d]$.

Then the boundary value problem (1.1)–(1.2) has at least three positive solutions u_1,u_2, and u_3 such that

$$\max_{0\le t\le1}|u_i'(t)|\le d,\quad \textit{for}\quad i=1,2,3.$$

$$b<\min_{\omega\le t\le1-\omega}|u_1(t)|,\quad \textit{with}\quad \max_{0\le t\le1}|u_1(t)|\le Ld,$$

$$1<\max_{0\le t\le1}|u_2(t)|<\frac{b}{\omega},\quad \textit{with}\quad \min_{\omega\le t\le1-\omega}|u_2(t)|<b,$$

$$\max_{0\le t\le1}|u_3|<a.$$

Proof. We define the operator $T:P_1\to E_1$ given by

$$(Tu)(t)=\begin{cases}\varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)+\int_0^t\varphi_q\left(\int_s^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\,\mathrm{d}s,0\le t\le\sigma\\ \varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)+\int_t^1\varphi_q\left(\int_\sigma^s a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\,\mathrm{d}s,\sigma\le t\le1.\end{cases}\tag{3.1}$$

First, it follows from Lemma 2.1 that operator T is well-defined. Next, because

$$(Tu)'(t)=\begin{cases}\varphi_q\left(\int_t^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\ge0,\quad 0\le t\le\sigma\\ -\varphi_q\left(\int_\sigma^t a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\le0,\quad \sigma\le t\le1\end{cases},\tag{3.2}$$

it is obvious that the operator $(Tu)'$ is continuous monotone decreasing on $[0,1]$, and $(Tu)'(\sigma)=0$. Meanwhile, it follows from the definition of operator T that for each $u\in P_1,Tu\in E_1$ is nonnegative continuous, and with (3.1), (3.2), we can obtain

$$\alpha\varphi_p((Tu)(0))-\beta\varphi_p((Tu)'(\xi))=0,\quad \gamma\varphi_p((Tu)(1))+\delta\varphi_p((Tu)'(\eta))=0$$

These show that $Tu\in P_1$, *i.e.*, $T(P_1)\subset P_1$, and we can obtain $(Tu)(\sigma)=\|Tu\|_2$ with

$$\varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)+\int_0^\sigma\varphi_q\left(\int_s^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\,\mathrm{d}s$$

$$= \varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) + \int_\sigma^1 \varphi_q\left(\int_\sigma^s a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s.$$

In what follows, we will prove that $T: P_1 \to P_1$ is completely continuous operator. The continuity of T is obvious because of the the continuity of f and a. Now, we prove T is compact. Let $D \subset P_1$ be a bounded set, then, there exists $R > 0$, such that $D \subset \{u \in P_1 | \|u\| \le R\}$. For any $u \in D$, we have $0 \le \int_0^1 a(s)f(s,u(s),u'(s))\,\mathrm{d}s \le \sup_{s\in[0,1]} a(s)f(s,u(s),u'(s)) = K$. Then, we have

$$\|Tu\| \le \varphi_q(K)\max\{\varphi_q(\frac{\beta}{\alpha}) + 1, \varphi_q(\frac{\delta}{\gamma}) + 1\},$$

$$\|(Tu)'\| \le \varphi_q(K),$$

$$\|(\varphi_p(Tu)')'\| \le K.$$

Thus, the Arzela-Ascoli theorem implies that $T: P_1 \to P_1$ is completely continuous. and it follows from Lemma 2.1 that each fixed point of T in P_1 is a positive solution of (1.1)–(1.2).

We now verify that the other conditions of Theorem A are satisfied.

Firstly, we choose $u \in \overline{P_1(\gamma_1, d)}$, then $\gamma_1(u) = \max_{0\le t\le 1}|u'(t)| \le d$. By Lemma 3.2, there is $\max_{0\le t\le 1}|u(t)| \le Ld$, thus, assumption (H_3) yields $f(t,u(t),u'(t)) \le \varphi_p(\frac{d}{N})$, for arbitrary $0 \le t \le 1$. Note that $\max_{0\le t\le 1}|(Tu)'(t)| = \max\{(Tu)'(0), |(Tu)'(1)|\}$, and from (3.2),we have

$$\begin{aligned}
&\gamma_1(Tu) = \max_{0\le t\le 1}|(Tu)'(t)| \\
&= \max\left\{\varphi_q\left(\int_0^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right), \varphi_q(\int_\sigma^1 a(r)f(r,u(r),u'(r))\,\mathrm{d}r)\right\} \\
&\le \max\left\{\varphi_q\left(\int_0^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right), \varphi_q(\int_\xi^1 a(r)f(r,u(r),u'(r))\,\mathrm{d}r)\right\} \\
&\le \tfrac{d}{N}\max\left\{\varphi_q\left(\int_0^\eta a(r)\,\mathrm{d}r\right), \varphi_q(\int_\xi^1 a(r)\,\mathrm{d}r)\right\} = d.
\end{aligned}$$

This shows that $T: \overline{P_1(\gamma_1, d)} \to \overline{P_1(\gamma_1, d)}$, and by the previous proof, we know that $T: \overline{P_1(\gamma_1, d)} \to \overline{P_1(\gamma_1, d)}$ is completely continuous.

In addition, thanks to Lemma3.1, Lemma3.2 and the definition of $\alpha_1, \gamma_1, \theta_1, \psi_1$, we have

$$\omega\theta_1(u) \le \alpha_1(u) \le \theta_1(u) = \psi_1(u), \tag{3.3}$$

$$\|u\|_1 = \max\{\theta_1(u), \gamma_1(u)\} \le L\gamma_1(u), \quad \text{for arbitrary} \quad u \in P_1, \tag{3.4}$$

and

$$\psi_1(\lambda u) = \max_{0\le t\le 1}|\lambda u(t)| = \lambda \max_{0\le t\le 1}|u(t)| = \lambda\psi_1(u), \quad 0 \le \lambda \le 1.$$

Therefore, the condition(2.6) of Theorem A is satisfied.

Secondly, we take $u(t) \equiv \frac{b}{2\omega}$, for $t \in [0,1]$, we have

$$\alpha_1(u(t)) = \min_{\omega\le t\le 1-\omega} u(t) = \frac{b}{2\omega} > b,$$

$$\gamma_1(u(t)) = \max_{0\le t\le 1} |u'(t)| = 0 < d,$$

$$\theta_1(u(t)) = \max_{0\le t\le 1} u(t) = \frac{b}{2\omega} < \frac{b}{\omega},$$

hence $u(t) \equiv \frac{b}{2\omega} \in P_1(\gamma_1, \theta_1, \alpha_1, b, \frac{b}{\omega}, d)$ and $\alpha_1(u(t)) > b$.

Therefore, $\{u \in P_1(\gamma_1, \theta_1, \alpha_1, b, \frac{b}{\omega}, d) : \alpha_1(u) > b\} \neq \emptyset$, and for any $u \in P_1(\gamma_1, \theta_1, \alpha_1, b, \frac{b}{\omega}, d)$, there is $b \le u(t) \le \frac{b}{\omega}, |u'(t)| \le d$, for $t \in [\omega, 1-\omega]$.

Thus, by condition (H_4) and Lemma 3.1, we have

$$\alpha_1(Tu) = \min_{\omega\le t\le 1-\omega} |(Tu)(t)| \ge \omega\theta_1((Tu)(t)) = \omega(Tu)(\sigma)$$

$$= \omega\left(\varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) + \int_0^\sigma \varphi_q\left(\int_s^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s\right)$$

$$= \omega\left(\varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) + \int_\sigma^1 \varphi_q\left(\int_\sigma^s a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s\right)$$

$$\ge \omega \min\left\{\int_0^\xi \varphi_q\left(\int_s^\xi a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s, \int_\eta^1 \varphi_q\left(\int_\eta^s a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s\right\}$$

$$> \omega\frac{b}{\omega m}\min\left\{\int_0^\xi \varphi_q\left(\int_s^\xi a(r)\,\mathrm{d}r\right)\mathrm{d}s, \int_\eta^1 \varphi_q\left(\int_\eta^s a(r)\,\mathrm{d}r\right)\mathrm{d}s\right\} = b.$$

Thereupon, $\alpha_1(Tu) > b$, for arbitrary $u \in P_1(\gamma_1, \theta_1, \alpha_1, b, \frac{b}{\omega}, d)$.
Consequently, condition (C_1) of Theorem A is satisfied.

Thirdly, we claim that condition(C_2) of Theorem A is satisfied. For this, we choose $u \in P_1(\gamma_1, \alpha_1, b, d)$ with $\theta_1(Tu) > \frac{b}{\omega}$, it follows from Lemma3.1 that

$$\alpha_1(Tu) \ge \omega\theta_1(Tu) > \omega\frac{b}{\omega} = b.$$

Thus, condition (C_2) of Theorem A is satisfied.

Finally, we show that condition(C_3) of Theorem A is also satisfied. It is easy to see that $\psi_1(0) = 0 < a$, hence $0 \notin R(\gamma_1, \psi_1, a, d)$. Now assume that $u \in R(\gamma_1, \psi_1, a, d)$ with $\psi_1(u) = a$, thus, by the condition (H_5), we have

$$\psi_1(Tu) = \max_{0\le t\le 1} |(Tu)(t)| = (Tu)(\sigma)$$

$$= \varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) + \int_0^\sigma \varphi_q\left(\int_s^\sigma a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s$$

$$= \varphi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right) + \int_\sigma^1 \varphi_q\left(\int_\sigma^s a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s$$

$$\leq \max\left\{ \int_0^\eta \varphi_q\left(\int_s^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s + \varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right),\right.$$

$$\left.\int_\xi^1 \varphi_q\left(\int_\xi^s a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\mathrm{d}s + \varphi_q\left(\frac{\delta}{\gamma}\int_\xi^\eta a(r)f(r,u(r),u'(r))\,\mathrm{d}r\right)\right\}$$

$$< \frac{a}{M}\max\left\{\int_0^\eta \varphi_q\left(\int_s^\eta a(r)\,\mathrm{d}r\right)\mathrm{d}s+\varphi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)\,\mathrm{d}r\right), \int_\xi^1 \varphi_q\left(\int_\xi^s a(r)\,\mathrm{d}r\right)\mathrm{d}s+\varphi_q\left(\frac{\delta}{\gamma}\int_\xi^\eta a(r)\,\mathrm{d}r\right)\right\} = a$$

Consequently, condition (C_3) of Theorem A is satisfied.

Thus, it follows from Theorem A that the boundary value problem (1.1)–(1.2) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\max_{0\leq t\leq 1}|u_i'(t)| \leq d, \quad \text{for} \quad i = 1,2,3;$$

$$b < \min_{\omega\leq t\leq 1-\omega}|u_1(t)|, \max_{0\leq t\leq 1}|u_1(t)| \leq Ld;$$

$$a < \max_{0\leq t\leq 1}|u_2(t)| < \frac{b}{\omega} \quad \text{with} \quad \min_{\omega\leq t\leq 1-\omega}|u_2(t)| < b;$$

$$\max_{0\leq t\leq 1}|u_3| < a.$$

The proof of Theorem 3.1 is completed. □

4. An Example

In order to illustrate our result, we present an example as follows.

example. Consider the singular boundary value problem with p-Laplacian

$$\begin{cases} (\varphi_p(u'(t)))' + \frac{1}{2}t^{-\frac{1}{2}}f(t,u(t),u'(t)) = 0, \qquad 0 < t < 1, \\ \alpha\varphi_p(u(0)) - \beta\varphi_p\left(u'(\frac{1}{4})\right) = 0, \gamma\varphi_p(u(1)) + \delta\varphi_p\left(u'(\frac{1}{2})\right) = 0. \end{cases} \tag{4.1}$$

where

$$p = \frac{3}{2}, \alpha = \beta, \gamma = \delta, \xi = \frac{1}{4}, \eta = \frac{1}{2}, \omega = \frac{1}{4}, a(t) = \frac{1}{2}t^{-\frac{1}{2}}$$

and

$$f(t,u,u') = \begin{cases} \frac{t^2}{4} + \frac{5}{2}u + \frac{\sin u'}{100}, & (t,u,u') \in [0,1]\times[0,1]\times(-\infty,\infty), \\ \frac{t^2}{4} + \frac{5}{2}u^4 + \frac{\sin u'}{100}, & (t,u,u') \in [0,1]\times[1,2]\times(-\infty,\infty), \\ \frac{t^2}{4} + 40 + \frac{\sin u'}{100}, & (t,u,u') \in [0,1]\times[2,10^4]\times(-\infty,\infty), \\ \frac{t^2}{4} + 40 + \frac{u-10^4}{\sqrt{u}} + \frac{\sin u'}{100}, & (t,u,u') \in [0,1]\times[10^4,+\infty)\times(-\infty,+\infty). \end{cases}$$

Then (4.1) has at least three positive solutions.

Proof. It follows from a direct calculation that

$$L = 2, N = \frac{1}{2}, M = \frac{79-48\sqrt{2}}{96}, m = \frac{1}{96}.$$

Clearly, condition $(H_1), (H_2)$ hold.
Let $a = 1, b = 2, d = 5000$, thus, we have

$$f(t, u, u') < \varphi_p(\frac{d}{N}) = \varphi(2d) = 100, \text{for} \quad (t, u, u') \in [0, 1] \times [0, 10000] \times [-5000, 5000],$$

$$f(t, u, u') > \varphi_p(\frac{b}{\omega m}) = 16\sqrt{3}, \text{for} \quad (t, u, u') \in [\frac{1}{4}, \frac{3}{4}] \times [2, 8] \times [-5000, 5000],$$

and

$$f(t, u, u') < \varphi_p(\frac{96}{79 - 48\sqrt{3}}), \text{for} \quad (t, u, u') \in [0, 1] \times [0, 1] \times [-5000, 5000].$$

Consequently, conditions $(H_3), (H_4), (H_5)$ of Theorem 3.1 are satisfied. Thus, by Theorem 3.1, the singular boundary value problem with p-Laplacian (4.1) has at least three positive solutions, and such that

$$\max_{0 \le t \le 1} |u_i'(t)| \le d, \quad \text{for} \quad i = 1, 2, 3.$$

$$b < \min_{\omega \le t \le 1-\omega} |u_1(t)|, \quad \text{with} \quad \max_{0 \le t \le 1} |u_1(t)| \le Ld,$$

$$1 < \max_{0 \le t \le 1} |u_2(t)| < \frac{b}{\omega}, \quad \text{with} \quad \min_{\omega \le t \le 1-\omega} |u_2(t)| < b,$$

$$\max_{0 \le t \le 1} |u_3| < a.$$

Remark 4.1 Problem (4.1) is singular at $t = 0$.

Acknowledgement

The author is very grateful to the referee for her/his important comments and suggestions.

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Received 12.02.2008

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