Adaptive Output-feedback Stabilization for a Class of Uncertain Nonlinear Systems

SHANG Fang[1] LIU Yun-Gang[1]

Abstract This paper investigates the problem of global adaptive stabilization by output-feedback for a class of uncertain nonlinear systems. Due to the uncertain control coefficients and unknown linear growth rate, this problem is much complicated and quite difficult to solve. In this paper, a novel dynamic gain updated on-line is introduced, and based on this, high-gain K-filters are proposed to reconstruct the system states. Then, motivated by the universal control method, the backstepping design approach is developed for the adaptive output-feedback stabilizing controller. It is shown that the global stability of the closed-loop system can be guaranteed by the appropriate choice of the design parameters. A simulation example is also provided to illustrate the correctness of the theoretical results.

Key words Nonlinear systems, uncertain control coefficient, unknown linear growth rate, high-gain K-filters, output-feedback, adaptive control, backstepping

DOI 10.3724/SP.J.1004.2010.00092

Research on global stabilization via output-feedback for nonlinear systems has been accelerated over the recent two decades. With the help of the celebrated backstepping design methodology[1], numerous results have been obtained mainly on the lower triangular systems[2−11].

In this paper, we consider the global stabilization problem by output-feedback for a class of single-input single-output (SISO) uncertain nonlinear systems:

$$\begin{cases}\dot{\zeta}_i &=& g_i\zeta_{i+1}+\phi_i(t,\boldsymbol{\zeta},u), \quad i=1,\cdots,n-1\\ \dot{\zeta}_n &=& g_n u+\phi_n(t,\boldsymbol{\zeta},u)\\ y &=& \zeta_1\end{cases} \tag{1}$$

where $\boldsymbol{\zeta}=[\zeta_1,\cdots,\zeta_n]^{\mathrm{T}}\in\mathbf{R}^n$ is the system state with the initial condition $\boldsymbol{\zeta}(0)=\boldsymbol{\zeta}_0$; $u\in\mathbf{R}$ and $y\in\mathbf{R}$ are the control input and output, respectively; $g_i\neq 0$, $i=1,\cdots,n$ are unknown constants, called uncertain control coefficients; functions $\phi_i:\mathbf{R}^+\times\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}, i=1,\cdots,n$ are continuous in the first argument and locally Lipschitz in the rest arguments. In the following, we suppose only the system output is measurable.

Equation (1) represents a vast class of nonlinear systems that have been extensively investigated during the last two decades, but most works were devoted to the case where all state information is available and nonlinearities are dominated by functions in a lower triangular form[1−2, 4, 12−14], and another case where only the system output is measurable and nonlinearities are dominated by output-dependent functions[2, 7, 13]. The far more general case of output-feedback that has recently attracted a lot of attention is the systems [5−6, 15−20] with unmeasured states dependent nonlinearities. However, up to now, only several special classes of such systems have been considered and many problems remain open, partly because it is very hard to construct an implementable observer for the general case.

In this paper, the following assumptions are imposed on system (1):

Manuscript received August 7, 2008; accepted February 19, 2009
Supported by National Natural Science Foundation of China (60674036, 60974003), the Natural Science Foundation for Distinguished Young Scholar of Shandong Province of China (JQ200919), the Program for New Century Excellent Talents in University of China (NCET-07-0513), the Key Science and Technique Foundation of Ministry of Education of China (108079), the Excellent Young and Middle-Aged Scientist Award Grant of Shandong Province of China (2007BS01010)
1. School of Control Science and Engineering, Shandong University, Jinan 250061, P. R. China

Assumption 1. There exists an unknown positive constant c (usually called linear growth rate) such that

$$|\phi_i(t,\boldsymbol{\zeta},u)|\leq c(|\zeta_1|+\cdots+|\zeta_i|),\ i=1,\cdots,n$$

Assumption 2. The signs of g_i, $i=1,\cdots,n$ are known, and there exist known positive constants $\underline{g}_i$ and $\overline{g}_i$ satisfying $\underline{g}_i\leq|g_i|\leq\overline{g}_i$.

From Assumptions 1 and 2, it is easy to see that system (1) has uncertain control coefficients and unmeasured states dependent nonlinearities. In contrast with the previous literature, the major difference is the presence of uncertain control coefficients in the system under investigation. If g_i, $i=1,\cdots,n$ are exactly known, the output-feedback control design was investigated for systems with unmeasured states dependent growth in [5−6, 15, 17-20]. More specifically, the cases of known linear growth rate and unknown one were considered in [5] and [15, 18], respectively. Moreover, in [6, 17, 19], the output-feedback stabilization was investigated for systems with output dependent growth rate, and an extension was obtained to the systems with unmeasured states dependent growth rate in [20]. As pointed out in [21], unknown control coefficients will cause invalidity of the existing methods, mainly because the commonly used high-gain Luenberger-type observer becomes inapplicable[15, 17−18]. Besides, the existing techniques cannot be straightforwardly extended to be stable. Therefore, how to design an output-feedback stabilizing control law for system (1) under Assumptions 1 and 2 is a very meaningful problem.

This paper continues the investigation started in [21−22] and extends the existing results in [15, 18, 21−22] to globally stabilize system (1) via output-feedback. On the one hand, system (1) has uncertain control coefficients, and hence it is different from those studied in [15, 18]. On the other hand, the linear growth rate of system (1) is unknown, rather than known in [21−22]. Mainly due to the differences, the methods in [15, 18, 21−22] cannot be straightforwardly extended to output-feedback stabilize system (1). This motivates us to construct a new adaptive output-feedback controller of system (1), which is accomplished by flexibly combining the idea of universal control and the backstepping methodology. First, a novel dynamic gain that is updated online is introduced, and based on this, high-gain K-filters are proposed to reconstruct the system states. Then, the backstepping design approach is successfully developed for the adaptive output-feedback stabilizing controller in the spirit of [15, 21−22]. It is shown that the global stability of the closed-loop system can be guaranteed by appropriate choice of the design parameters. Finally, a simulation example is provided to illustrate the correctness of the theoretical results.

The remainder of this paper is organized as follows. Section 1 gives the main results of this paper. Specifically, in Subsection 1.1, the dynamic high-gain K-filters-based observer is constructed for the transformed new system; in Subsection 1.2, the output-feedback control design is then given using backstepping method; in Subsection 1.3, the main results are summarized, and the global stability of the closed-loop system is guaranteed when the design parameters are appropriately chosen. Section 2 gives a simulation example to demonstrate the correctness of the theoretical results. Section 3 presents some concluding remarks. The paper ends with an appendix that provides rigorous proofs of a proposition and a lemma.

The preliminary version of this paper, i.e., [23], was presented at the 27th China Control Conference, Kunming, China, July 16-18, 2008.

1 Adaptive output-feedback stabilizing control

Throughout the paper, for any real numbers ρ_i, $i = 1, \cdots, m$, we use $\rho_{j\sim k}$, $1 \le j \le k \le m$ to denote $\prod_{p=j}^{k} \rho_p$, and I the identity matrix with a suitable dimension.

According to Assumption 2, there exist known positive constants $g_N = \min\{\underline{g}_1, \underline{g}_1\underline{g}_2, \cdots, \underline{g}_{1\sim n}\}$ and $g_M = \max\{\overline{g}_1, \overline{g}_1\overline{g}_2, \cdots, \overline{g}_{1\sim n}\}$ such that $g_N \le |g_{1\sim i}| \le g_M$, $i = 1, \cdots, n$.

1.1 High-gain K-filters and state estimation

As system (1) is not convenient for control design, we first introduce the following linear state transformation:

$$x_1 = \zeta_1, \quad x_i = g_{1\sim(i-1)}\zeta_i, \quad i = 2, \cdots, n \tag{2}$$

Then, the dynamics of $\boldsymbol{x} = [x_1, \cdots, x_n]^{\rm T}$ are given by

$$\begin{cases} \dot{\boldsymbol{x}} &= A_n\boldsymbol{x} + g\boldsymbol{e}_n u + \boldsymbol{\varphi}(t, \boldsymbol{x}, u) \\ y &= x_1 \end{cases} \tag{3}$$

where $\boldsymbol{e}_n = [0, \cdots, 0, 1]^{\rm T} \in \mathbf{R}^n$, and $\boldsymbol{x} \in \mathbf{R}^n$ is the state of the new system with the initial condition depending on $\boldsymbol{\zeta}_0$ and (2), $g = g_{1\sim n}$, $\boldsymbol{\varphi} = [\varphi_1, \varphi_2, \cdots, \varphi_n]^{\rm T} = [\phi_1, g_1\phi_2, \cdots, g_{1\sim(n-1)}\phi_n]^{\rm T}|_{\zeta_1 = x_1, \zeta_i = \frac{1}{g_{1\sim(i-1)}}x_i, i=2,\cdots,n}$; and $A_n = \begin{bmatrix} 0 & & \\ \vdots & & I \\ 0\,0 & \cdots & 0 \end{bmatrix}$.

The objective now is to design an adaptive output-feedback stabilizing controller for system (3) as well as for system (1).

For system (3), as g is uncertain, it is quite difficult to find an appropriate state observer. As a special case, if g is known, a high-gain Luenberger-type observer has been developed to solve the output-feedback control problem[15, 18]. However, if a Luenberger-type observer is adopted for system (3), the resulting state estimation error dynamics will depend on the control input u, and this will make the output-feedback control design very hard, even impossible. On the other hand, for systems with uncertain control coefficients and output dependent nonlinearities, Chapter 8 of [2] investigated the output-feedback control design by proposing K-filters. Motivated by this, we introduce the following dynamic high-gain K-filters:

$$\begin{cases} \dot{\boldsymbol{\xi}} &= A_L\boldsymbol{\xi} + D_L\boldsymbol{l}y \\ \dot{\boldsymbol{\lambda}} &= A_L\boldsymbol{\lambda} + \boldsymbol{e}_n u \end{cases} \tag{4}$$

with time-varying high-gain L updated by

$$\dot{L} = \frac{y^2}{L^2} + \frac{l_1\lambda_1^2}{2L^3}, \quad L(0) = 1 \tag{5}$$

where $\boldsymbol{\xi} = [\xi_1, \cdots, \xi_n]^{\rm T}$, $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_n]^{\rm T}$, $D_L = {\rm diag}\{L, \cdots, L^n\}$, $\boldsymbol{l} = [l_1, \cdots, l_n]^{\rm T}$, and $A_L = A_n - D_L\boldsymbol{l}\boldsymbol{e}_1^{\rm T}$ with $\boldsymbol{e}_1 = [1, 0, \cdots, 0]^{\rm T} \in \mathbf{R}^n$.

Remark 1. Although the updated law of L given earlier is more complicated than that in [15], it is necessary for the backstepping control design, as will be seen later. Moreover, the complexity of (5) causes various difficulties in the stability analysis of the closed-loop system, which will be shown in the proof of Theorem 1 in Subsection 1.3.

Define $D = {\rm diag}\{1, \cdots, n\}$. The constant vector $\boldsymbol{l}$ is chosen such that matrix $A_{\boldsymbol{l}} = A_n - \boldsymbol{l}\boldsymbol{e}_1^{\rm T}$ is Hurwitz (this clearly implies $l_1 > 0$), and such that there exist a positive constant h and a symmetric positive definite matrix $P_{\boldsymbol{l}}$ satisfying

$$A_{\boldsymbol{l}}^{\rm T}P_{\boldsymbol{l}} + P_{\boldsymbol{l}}A_{\boldsymbol{l}} \le -I, \quad DP_{\boldsymbol{l}} + P_{\boldsymbol{l}}D \ge hI \tag{6}$$

It is necessary to point out that the aforementioned choice can always be carried out according to Lemma 1 in [17]. It is easily deduced from (5) and $l_1 > 0$ that $L \ge 1$.

Define $\hat{\boldsymbol{x}} = \boldsymbol{\xi} + g\boldsymbol{l}$ as the state estimate for system (3), and $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ the state estimation error which satisfies

$$\dot{\tilde{\boldsymbol{x}}} = A_L\tilde{\boldsymbol{x}} + \boldsymbol{\varphi} \tag{7}$$

To be simple and convenient in presentation and analysis, let us introduce the following scaling transformations as:

$$\begin{cases} \epsilon_i &= \dfrac{\tilde{x}_i}{L^i}, \quad \eta_i = \dfrac{\xi_i}{L^i}, \quad i = 1, \cdots, n \\ \varepsilon_1 &= \dfrac{y}{L}, \quad \varepsilon_i = \dfrac{\lambda_i}{L^i}, \quad i = 2, \cdots, n \end{cases} \tag{8}$$

Then, noting

$$\lambda_1 = \frac{1}{g}(y - \tilde{x}_1 - \xi_1) = \frac{L}{g}(\varepsilon_1 - \epsilon_1 - \eta_1) \tag{9}$$

we have

$$\begin{cases} \dot{\boldsymbol{\epsilon}} &= LA_{\boldsymbol{l}}\boldsymbol{\epsilon} + \boldsymbol{f} - \dfrac{\dot{L}}{L}D\boldsymbol{\epsilon} \\ \dot{\boldsymbol{\eta}} &= LA_{\boldsymbol{l}}\boldsymbol{\eta} + L\boldsymbol{l}\varepsilon_1 - \dfrac{\dot{L}}{L}D\boldsymbol{\eta} \\ \dot{\varepsilon}_1 &= gL\varepsilon_2 + L\epsilon_2 + L\eta_2 + f_1 - \dfrac{\dot{L}}{L}\varepsilon_1 \\ \dot{\varepsilon}_i &= -\dfrac{l_iL}{g}\varepsilon_1 + \dfrac{l_iL}{g}\epsilon_1 + \dfrac{l_iL}{g}\eta_1 + L\varepsilon_{i+1} - \dfrac{i\dot{L}}{L}\varepsilon_i, \\ & \quad i = 2, \cdots, n-1 \\ \dot{\varepsilon}_n &= -\dfrac{l_nL}{g}\varepsilon_1 + \dfrac{l_nL}{g}\epsilon_1 + \dfrac{l_nL}{g}\eta_1 + \dfrac{u}{L^n} - \dfrac{n\dot{L}}{L}\varepsilon_n \end{cases} \tag{10}$$

where $\boldsymbol{\epsilon} = [\epsilon_1, \cdots, \epsilon_n]^{\rm T}$, $\boldsymbol{\eta} = [\eta_1, \cdots, \eta_n]^{\rm T}$, $\boldsymbol{\varepsilon} = [\varepsilon_1, \cdots, \varepsilon_n]^{\rm T}$, and $\boldsymbol{f} = [f_1, f_2, \cdots, f_n]^{\rm T} = [\frac{\phi_1}{L}, \frac{g_1\phi_2}{L^2}, \cdots, \frac{g_{1\sim(n-1)}\phi_n}{L^n}]^{\rm T}$.

It should be stressed that under the new dynamic high-gain K-filters (4), the dynamics of the state estimation error (7) are independent of the control input u, and this will play a key role in the subsequent control design. As a commonly used technique[15], the state scaling transformations (8) are introduced to achieve system (10). It can be seen that the study of stabilization (by partial state-feedback) for the whole system (10) and (5) is equivalent to that (by output-feedback) for system (1). Noting that the $\boldsymbol{\varepsilon}$-dynamic system is in the strict-feedback form, some existing methods, e.g., backstepping approach, may be available for the control design of the whole system. However, because uncertain constant g exists in the dynamics of $\boldsymbol{\varepsilon}$, and L is a time-varying variable, there are still some obstacles to be overcome to explicitly construct the controller.

To prepare for the backstepping design procedure, we give the following two propositions that play an important role in the later control design and performance analysis. Specifically, Propositions 1 and 2 characterize the ISS-like properties of $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ of system (10), respectively. Besides, for the sake of compactness, the proof of Proposition 1 is provided in Appendix A.

Proposition 1. For the subsystem $\boldsymbol{\epsilon}$ of system (10), let $V_{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}^{\rm T} P_{\boldsymbol{l}} \boldsymbol{\epsilon}$. Then, in the maximal interval of existence for the solution of system (10), there is an unknown positive constant c_1 (depending on c), such that

$$\dot{V}_{\boldsymbol{\epsilon}} \le -\big(L - c_1\big)\|\boldsymbol{\epsilon}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\boldsymbol{\varepsilon}\|^2 \qquad (11)$$

Proposition 2. For the subsystem $\boldsymbol{\eta}$ of system (10), let $V_{\boldsymbol{\eta}} = \boldsymbol{\eta}^{\rm T} P_{\boldsymbol{l}} \boldsymbol{\eta}$. Then in the maximal interval of existence for the solution to system (10),

$$\dot{V}_{\boldsymbol{\eta}} \le -\frac{L}{2}\|\boldsymbol{\eta}\|^2 + 2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 L\varepsilon_1^2 - \frac{h\dot{L}}{L}\|\boldsymbol{\eta}\|^2$$

Proof. By (5) and (6), the time derivative of $V_{\boldsymbol{\eta}}$ along the trajectories of (10) satisfies

$$\begin{aligned}\dot{V}_{\boldsymbol{\eta}} \le\, & -L\|\boldsymbol{\eta}\|^2 + 2\boldsymbol{\eta}^{\rm T} P_{\boldsymbol{l}} L \boldsymbol{l}\varepsilon_1 - \frac{\dot{L}}{L}\boldsymbol{\eta}^{\rm T}(DP_{\boldsymbol{l}} + P_{\boldsymbol{l}}D)\boldsymbol{\eta} \le \\ & -L\|\boldsymbol{\eta}\|^2 + \frac{L}{2}\|\boldsymbol{\eta}\|^2 + 2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 L\varepsilon_1^2 - \frac{h\dot{L}}{L}\|\boldsymbol{\eta}\|^2 = \\ & -\frac{L}{2}\|\boldsymbol{\eta}\|^2 + 2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 L\varepsilon_1^2 - \frac{h\dot{L}}{L}\|\boldsymbol{\eta}\|^2 \end{aligned} \qquad \square$$

1.2 Output-feedback control design

This subsection is devoted to the constructive design of output-feedback control for system (10) by the traditional backstepping method, which is presented in a step-by-step manner. More specifically, Step 1 is the beginning of the design process from which the main techniques applied can be exhibited; Step 2 gives the initial assignment for the recursive steps k, $k = 2, \cdots, n$.

Step 1. Let $V_1 = V_{\boldsymbol{\epsilon}} + V_{\boldsymbol{\eta}} + \frac{1}{2L^4}\lambda_1^2 + \frac{1}{2}\varepsilon_1^2$ be the Lyapunov function candidate for this step, where $V_{\boldsymbol{\epsilon}}$ and $V_{\boldsymbol{\eta}}$ are defined in Propositions 1 and 2, respectively. Then, from Propositions 1, 2 and the dynamics of λ_1, ε_1, it follows that

$$\begin{aligned}\dot{V}_1 \le\, & -\big(L - c_1\big)\|\boldsymbol{\epsilon}\|^2 - \Big(\frac{L}{2} - 1\Big)\|\boldsymbol{\eta}\|^2 - \frac{h\dot{L}}{L}\|\boldsymbol{\eta}\|^2 - \frac{\dot{L}}{L}\varepsilon_1^2 - \\ & \frac{2\dot{L}}{L^5}\lambda_1^2 + \sum_{i=2}^{n}\varepsilon_i^2 - \frac{l_1}{L^3}\lambda_1^2 + \big(2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 L + 1\big)\varepsilon_1^2 + \\ & gL\varepsilon_1\varepsilon_2 + \frac{1}{L^2}\lambda_1\varepsilon_2 + L\varepsilon_1\eta_2 + L\varepsilon_1\epsilon_2 + f_1\varepsilon_1 \end{aligned} \qquad (12)$$

By the method of completing square and the fact $l_1 > 0$ pointed out earlier, we have the following estimation:

$$\begin{cases} L\varepsilon_1\epsilon_2 & \le \dfrac{L}{2}\epsilon_2^2 + \dfrac{L}{2}\varepsilon_1^2 \le \dfrac{L}{2}\|\boldsymbol{\epsilon}\|^2 + \dfrac{L}{2}\varepsilon_1^2 \\ \dfrac{1}{L^2}\lambda_1\varepsilon_2 & \le \dfrac{l_1}{2L^3}\lambda_1^2 + \dfrac{1}{2l_1L}\varepsilon_2^2 \le \dfrac{l_1}{2L^3}\lambda_1^2 + \dfrac{1}{2l_1}\varepsilon_2^2 \end{cases}$$

Substituting this and $f_1\varepsilon_1 \le c\varepsilon_1^2$ into (12) results in

$$\begin{aligned}\dot{V}_1 \le\, & -\Big(\frac{L}{2} - c_1\Big)\|\boldsymbol{\epsilon}\|^2 - \Big(\frac{L}{2} - 1\Big)\|\boldsymbol{\eta}\|^2 - \frac{h\dot{L}}{L}\|\boldsymbol{\eta}\|^2 - \\ & \frac{\dot{L}}{L}\varepsilon_1^2 - \frac{2\dot{L}}{L^5}\lambda_1^2 - \frac{l_1}{2L^3}\lambda_1^2 + \sum_{i=2}^{n}\varepsilon_i^2 + \frac{1}{2l_1}\varepsilon_2^2 + \\ & c\varepsilon_1^2 + \Big(\big(2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 + \frac{3}{2}\big)\varepsilon_1 + \eta_2\Big)L\varepsilon_1 + gL\varepsilon_1\varepsilon_2 \end{aligned} \qquad (13)$$

Choose the virtual controller

$$\varepsilon_2^* = -\alpha_1\varepsilon_1 - \gamma_1^{\eta_1}\eta_1 - \gamma_1^{\eta_2}\eta_2 \qquad (14)$$

where $\alpha_1 = \frac{{\rm sgn}(g)}{g_N}\big(2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 + \frac{3}{2} + b_1\big)$, $\gamma_1^{\eta_1} = 0$, $\gamma_1^{\eta_2} = \frac{{\rm sgn}(g)}{g_N}$, and $b_1 > 0$ is a constant to be determined later.

Define $z_1 = \varepsilon_1 - \varepsilon_1^*$ with $\varepsilon_1^* = 0$ and $z_2 = \varepsilon_2 - \varepsilon_2^*$. Then, $\big(1 + \frac{1}{2l_1}\big)\varepsilon_2^2 \le 4\big(1 + \frac{1}{2l_1}\big)z_2^2 + \mu_1^{z_1}z_1^2 + \mu_1^{\boldsymbol{\eta}}\|\boldsymbol{\eta}\|^2$, where $\mu_1^{z_1} = 4\big(1 + \frac{1}{2l_1}\big)\alpha_1^2$, $\mu_1^{\boldsymbol{\eta}} = 4\big(1 + \frac{1}{2l_1}\big)\max_{i=1,2}(\gamma_1^{\eta_i})^2$ are known positive constants. Substituting this and (14) into (13), we have

$$\begin{aligned}\dot{V}_1 \le\, & -d_1^{\boldsymbol{\epsilon}}\|\boldsymbol{\epsilon}\|^2 - d_1^{\boldsymbol{\eta}}\|\boldsymbol{\eta}\|^2 - \frac{h\dot{L}}{L}\|\boldsymbol{\eta}\|^2 - \frac{l_1}{2L^3}\lambda_1^2 - d_1^{z_1}z_1^2 - \\ & \frac{\dot{L}}{L}z_1^2 + \sum_{i=3}^{n}\varepsilon_i^2 + 4\Big(1 + \frac{1}{2l_1}\Big)z_2^2 + gLz_1z_2 \end{aligned} \qquad (15)$$

where $d_1^{\boldsymbol{\epsilon}} = \frac{L}{2} - c_1$, $d_1^{\boldsymbol{\eta}} = \frac{L}{2} - 1 - \mu_1^{\boldsymbol{\eta}}$, and $d_1^{z_1} = b_1L - c - \mu_1^{z_1}$. It should be pointed out that $d_1^{\boldsymbol{\epsilon}}$ and $d_1^{z_1}$ depend on L and c while $d_1^{\boldsymbol{\eta}}$ depends only on L.

Remark 2. If the order of system (1) is 1, then $z_2 = 0$, and from (15) one can show that the controller $u = L^2\varepsilon_2^*$ with ε_2^* designed in (14) does stabilize the closed-loop system (see Theorem 1 later). If system (1) is of order $n > 1$, then the unstabilized terms containing ε_i and z_2 in (15) can be handled in the following steps.

Step 2. Let $V_2 = \sigma_1V_1 + \frac{1}{2}z_2^2$ be the Lyapunov function candidate, where $\sigma_1 > 0$ is a constant to be determined later. Then, the time derivative of V_2 satisfies

$$\dot{V}_2 = \sigma_1\dot{V}_1 + z_2\dot{z}_2 \qquad (16)$$

Noting $z_2 = \varepsilon_2 + \alpha_1\varepsilon_1 + \frac{{\rm sgn}(g)}{g_N}\eta_2$, we have

$$\begin{aligned}\dot{z}_2 =\, & L\varepsilon_3 - \frac{2\dot{L}}{L}z_2 + \frac{\dot{L}}{L}\big(\gamma_{2,1}^{z_1}z_1 + \gamma_{2,1}^{\eta_1}\eta_1\big) + \sum_{i=1}^{2}\gamma_2^{z_i}Lz_i + \\ & \sum_{i=1}^{3}\gamma_2^{\eta_i}L\eta_i + \sum_{i=1}^{2}L(\delta_2^{z_i}z_i + \delta_2^{\epsilon_i}\epsilon_i + \delta_2^{\eta_i}\eta_i) + \delta_2^{f_1}f_1 \end{aligned} \qquad (17)$$

where $\gamma_{2,1}^{z_1} = \alpha_1$, $\gamma_{2,1}^{\eta_1} = 0$, $\gamma_2^{z_1} = \frac{{\rm sgn}(g)}{g_N}l_2$, $\gamma_2^{z_2} = 0$, $\gamma_2^{\eta_1} = -\frac{{\rm sgn}(g)}{g_N}l_2$, $\gamma_2^{\eta_2} = \alpha_1$, $\gamma_2^{\eta_3} = \frac{{\rm sgn}(g)}{g_N}$, $\delta_2^{z_1} = -\frac{l_2}{g} - g\alpha_1^2$, $\delta_2^{z_2} = g\alpha_1$, $\delta_2^{f_1} = \alpha_1$, $\delta_2^{\epsilon_1} = \frac{l_2}{g}$, $\delta_2^{\epsilon_2} = \alpha_1$, $\delta_2^{\eta_1} = \frac{l_2}{g}$, and $\delta_2^{\eta_2} = -\frac{|g|}{g_N}\alpha_1$. It is easy to see that $\gamma_{2,1}^{z_1}$, $\gamma_{2,1}^{\eta_1}$, $\gamma_2^{z_i}$, $i = 1, 2$, and $\gamma_2^{\eta_i}$, $i = 1, 2, 3$ are known, and $\delta_2^{f_1}$, $\delta_2^{z_i}$, $\delta_2^{\epsilon_i}$, and $\delta_2^{\eta_i}$, $i = 1, 2$ may be unknown but have known upper bounds by Assumption 2 and (5). For simplicity, we denote their upper bounds as $\bar{\delta}_2^{f_1}$, $\bar{\delta}_2^{z_i}$, $\bar{\delta}_2^{\epsilon_i}$, and $\bar{\delta}_2^{\eta_i}$, respectively.

Before deriving the virtual controller ε_3^*, we should eliminate the "undesired" effect of z_2 in (16). For this purpose, by the method of completing square, we obtain

$$\begin{cases}(\gamma_{2,1}^{z_1}z_1+\gamma_{2,1}^{\eta_1}\eta_1)z_2 \ \le \dfrac{1}{2}(\gamma_{2,1}^{z_1})^2z_1^2+\dfrac{1}{2}(\gamma_{2,1}^{\eta_1})^2\|\boldsymbol{\eta}\|^2+z_2^2\\ \delta_2^{z_1}Lz_1z_2 \ \le \ \sigma_1Lz_1^2+\dfrac{1}{4\sigma_1}(\bar{\delta}_2^{z_1})^2Lz_2^2\\ \delta_2^{f_1}f_1z_2 \ \le \ \sigma_1c^2z_1^2+\dfrac{1}{4\sigma_1}(\bar{\delta}_2^{f_1})^2z_2^2\\ \displaystyle\sum_{i=1}^{2}\delta_2^{\epsilon_i}L\epsilon_iz_2 \ \le \ \frac{\sigma_1}{4}L\|\boldsymbol{\epsilon}\|^2+\frac{1}{\sigma_1}\sum_{i=1}^{2}(\bar{\delta}_2^{\epsilon_i})^2Lz_2^2\\ \displaystyle\sum_{i=1}^{2}\delta_2^{\eta_i}L\eta_iz_2 \ \le \ \frac{\sigma_1}{4}L\|\boldsymbol{\eta}\|^2+\frac{1}{\sigma_1}\sum_{i=1}^{2}(\bar{\delta}_2^{\eta_i})^2Lz_2^2\end{cases}$$

Substituting this and (17) into (16) and noting $\delta_2^{z_2}Lz_2^2\le\bar{\delta}_2^{z_2}Lz_2^2$ and $gLz_1z_2\le Lz_1^2+\frac{1}{4}\max\{g_M^2,1\}Lz_2^2$, we have

$$\begin{aligned}\dot{V}_2\le&-\sigma_1\Big(d_1^{\boldsymbol{\epsilon}}-\frac{1}{4}L\Big)\|\boldsymbol{\epsilon}\|^2-\sigma_1\Big(d_1^{\boldsymbol{\eta}}-\frac{1}{4}L\Big)\|\boldsymbol{\eta}\|^2-\Big(\sigma_1h-\\&\frac{1}{2}(\gamma_{2,1}^{\eta_1})^2\Big)\frac{\dot{L}}{L}\|\boldsymbol{\eta}\|^2-\frac{\sigma_1l_1}{2L^3}\lambda_1^2-\sigma_1\big(d_1^{z_1}-2L-c^2\big)z_1^2-\\&\Big(\sigma_1-\frac{1}{2}(\gamma_{2,1}^{z_1})^2\Big)\frac{\dot{L}}{L}z_1^2-\frac{\dot{L}}{L}z_2^2+\sigma_1\sum_{i=3}^{n}\varepsilon_i^2+\\&Lz_2\bigg(\gamma_2^{z_1}z_1+\sum_{i=1}^{3}\gamma_2^{\eta_i}\eta_i+\bar{\alpha}_2z_2\bigg)+Lz_2\varepsilon_3\end{aligned}\tag{18}$$

where $\bar{\alpha}_2=4\sigma_1\big(1+\frac{1}{2l_1}\big)+\bar{\delta}_2^{z_2}+\frac{1}{4\sigma_1}\big((\bar{\delta}_2^{f_1})^2+(\bar{\delta}_2^{z_1})^2\big)+\frac{1}{\sigma_1}\sum_{i=1}^{2}\big((\bar{\delta}_2^{\epsilon_i})^2+(\bar{\delta}_2^{\eta_i})^2\big)+\frac{1}{4}\sigma_1\max\{g_M^2,1\}+\gamma_2^{z_2}$.

Choose the virtual controller

$$\varepsilon_3^*=-\alpha_2z_2-\gamma_2^{z_1}z_1-\sum_{i=1}^{3}\gamma_2^{\eta_i}\eta_i\tag{19}$$

where $\alpha_2=b_2+\bar{\alpha}_2$ and $b_2>0$ is a constant to be determined later.

Define $z_3=\varepsilon_3-\varepsilon_3^*$. Then, we have $\varepsilon_3^2\le6z_3^2+\sum_{i=1}^{2}\mu_2^{z_i}z_i^2+\mu_2^{\boldsymbol{\eta}}\|\boldsymbol{\eta}\|^2$, where $\mu_2^{z_1}=6(\gamma_2^{z_1})^2$, $\mu_2^{z_2}=6\alpha_2^2$, and $\mu_2^{\boldsymbol{\eta}}=6\max_{i=1,2,3}(\gamma_2^{\eta_i})^2$. Substituting this and (19) into (18), we have

$$\begin{aligned}\dot{V}_2\le&-d_2^{\boldsymbol{\epsilon}}\|\boldsymbol{\epsilon}\|^2-d_2^{\boldsymbol{\eta}}\|\boldsymbol{\eta}\|^2-\frac{d_{2,1}^{\boldsymbol{\eta}}\dot{L}}{L}\|\boldsymbol{\eta}\|^2-\frac{\sigma_1l_1}{2L^3}\lambda_1^2-\sum_{i=1}^{2}d_2^{z_i}z_i^2-\\&\frac{d_{2,1}^{z_1}\dot{L}}{L}z_1^2-\frac{\dot{L}}{L}z_2^2+\sigma_1\sum_{i=4}^{n}\varepsilon_i^2+6\sigma_1z_3^2+Lz_2z_3\end{aligned}$$

where $d_2^{\boldsymbol{\epsilon}}=\sigma_1\big(d_1^{\boldsymbol{\epsilon}}-\frac{1}{4}L\big)$, $d_2^{\boldsymbol{\eta}}=\sigma_1\big(d_1^{\boldsymbol{\eta}}-\frac{1}{4}L-\mu_2^{\boldsymbol{\eta}}\big)$, $d_2^{z_1}=\sigma_1\big(d_1^{z_1}-2L-c^2-\mu_2^{z_1}\big)$, $d_2^{z_2}=b_2L-\sigma_1\mu_2^{z_2}$, $d_{2,1}^{z_1}=\sigma_1-\frac{1}{2}(\gamma_{2,1}^{z_1})^2$, and $d_{2,1}^{\boldsymbol{\eta}}=\sigma_1h-\frac{1}{2}(\gamma_{2,1}^{\eta_1})^2$.

Inductive step. Suppose at Step $k-1\,(k=3,\cdots,n)$, there exists a smooth, positive definite and proper function $V_{k-1}(\boldsymbol{\eta},\boldsymbol{\epsilon},\lambda_1,z_1,\cdots,z_{k-1})$ whose time derivative satisfies

$$\begin{aligned}\dot{V}_{k-1}\le&-d_{k-1}^{\boldsymbol{\epsilon}}\|\boldsymbol{\epsilon}\|^2-d_{k-1}^{\boldsymbol{\eta}}\|\boldsymbol{\eta}\|^2-\frac{d_{k-1,1}^{\boldsymbol{\eta}}\dot{L}}{L}\|\boldsymbol{\eta}\|^2-\sum_{i=1}^{k-1}d_{k-1}^{z_i}z_i^2-\\&\frac{\sigma_{1\sim(k-2)}l_1}{2L^3}\lambda_1^2-\sum_{i=1}^{k-2}\frac{d_{k-1,1}^{z_i}\dot{L}}{L}z_i^2-\frac{\dot{L}}{L}z_{k-1}^2+\\&\sigma_{1\sim(k-2)}\sum_{i=k+1}^{n}\varepsilon_i^2+2k\sigma_{1\sim(k-2)}z_k^2+Lz_{k-1}z_k\end{aligned}$$

where $z_1=\varepsilon_1$, $z_i=\varepsilon_i-\varepsilon_i^*$, $i=2,\cdots,k$, and the virtual controller ε_i^* satisfies

$$\varepsilon_i^*=-\alpha_{i-1}z_{i-1}-\sum_{j=1}^{i-2}\gamma_{i-1}^{z_j}z_j-\sum_{j=1}^{i}\gamma_{i-1}^{\eta_j}\eta_j,\ i=2,\cdots,k\tag{20}$$

The dynamics of $z_i\,(i=2,\cdots,k-1)$ can be immediately computed from (20) as

$$\begin{aligned}\dot{z}_i=&L\varepsilon_{i+1}-\frac{i\dot{L}}{L}z_i+\frac{\dot{L}}{L}\sum_{j=1}^{i-1}\big(\gamma_{i,1}^{z_j}z_j+\gamma_{i,1}^{\eta_j}\eta_j\big)+\sum_{j=1}^{i}\gamma_i^{z_j}Lz_j+\\&\sum_{j=1}^{i+1}\gamma_i^{\eta_j}L\eta_j+\sum_{j=1}^{i-1}\delta_i^{z_j}Lz_j+\delta_i^{f_1}f_1+\sum_{j=1}^{2}\delta_i^{\epsilon_j}L\epsilon_j+\sum_{j=1}^{i}\delta_i^{\eta_j}L\eta_j\end{aligned}$$

In what follows, we will show that the aforementioned statements still hold at Step k. For this aim, choose $V_k=\sigma_{k-1}V_{k-1}+\frac{1}{2}z_k^2$ as the Lyapunov function candidate for Step k with $\sigma_{k-1}>0$, a constant to be determined, where $z_k=\varepsilon_k-\varepsilon_k^*$ and the virtual controller ε_k^* are smooth functions. For notational convenience and consistency, let $L\varepsilon_{n+1}=\frac{u}{L^n}$. Then, computing the time derivative of V_k, we have

$$\dot{V}_k=\sigma_{k-1}\dot{V}_{k-1}+z_k\dot{z}_k\tag{21}$$

where

$$\begin{aligned}\dot{z}_k=&L\varepsilon_{k+1}-\frac{k\dot{L}}{L}z_k+\frac{\dot{L}}{L}\sum_{i=1}^{k-1}\big(\gamma_{k,1}^{z_i}z_i+\gamma_{k,1}^{\eta_i}\eta_i\big)+\sum_{i=1}^{k}\gamma_k^{z_i}Lz_i+\\&\sum_{i=1}^{k+1}\gamma_k^{\eta_i}L\eta_i+\sum_{i=1}^{k-1}\delta_k^{z_i}Lz_i+\delta_k^{f_1}f_1+\sum_{i=1}^{2}\delta_k^{\epsilon_i}L\epsilon_i+\sum_{i=1}^{k}\delta_k^{\eta_i}L\eta_i\end{aligned}\tag{22}$$

This can be easily obtained after dull computation. As before, $\gamma_{k,1}^{z_i}$, $\gamma_{k,1}^{\eta_i}$, $\gamma_k^{z_i}$, and $\gamma_k^{\eta_i}$ are known, and $\delta_k^{f_1}$, $\delta_k^{z_i}$, $\delta_k^{\epsilon_i}$, and $\delta_k^{\eta_i}$ may be unknown but have known upper bounds $\bar{\delta}_k^{f_1}$, $\bar{\delta}_k^{z_i}$, $\bar{\delta}_k^{\epsilon_i}$, and $\bar{\delta}_k^{\eta_i}$, respectively.

Similarly, before deriving the virtual controller ε_{k+1}^*, we should eliminate the "undesired" effect of z_k in (21). For this purpose, by the method of completing square, we have

$$\begin{cases}\displaystyle\sum_{i=1}^{k-1}(\gamma_{k,1}^{z_i}z_i+\gamma_{k,1}^{\eta_i}\eta_i)z_k \ \le \ \frac{1}{2}\sum_{i=1}^{k-1}(\gamma_{k,1}^{z_i})^2z_i^2+(k-1)z_k^2+\\ \qquad\qquad\qquad\qquad\qquad\displaystyle\frac{1}{2}\max_{i=1,\cdots,k-1}(\gamma_{k,1}^{\eta_i})^2\|\boldsymbol{\eta}\|^2\\ \delta_k^{z_i}Lz_iz_k \ \le \ \sigma_{1\sim(k-1)}Lz_i^2+\dfrac{(\bar{\delta}_k^{z_i})^2}{4\sigma_{1\sim(k-1)}}Lz_k^2\\ \delta_k^{f_1}f_1z_k \ \le \ \sigma_{1\sim(k-1)}c^2z_1^2+\dfrac{(\bar{\delta}_k^{f_1})^2}{4\sigma_{1\sim(k-1)}}z_k^2\\ \displaystyle\sum_{i=1}^{2}\delta_k^{\epsilon_i}L\epsilon_iz_k \ \le \ \frac{\sigma_{1\sim(k-1)}}{2^k}L\|\boldsymbol{\epsilon}\|^2+\sum_{i=1}^{2}\frac{2^{k-2}(\bar{\delta}_k^{\epsilon_i})^2}{\sigma_{1\sim(k-1)}}Lz_k^2\\ \displaystyle\sum_{i=1}^{k}\delta_k^{\eta_i}L\eta_iz_k \ \le \ \frac{\sigma_{1\sim(k-1)}}{2^k}L\|\boldsymbol{\eta}\|^2+\sum_{i=1}^{k}\frac{2^{k-2}(\bar{\delta}_k^{\eta_i})^2}{\sigma_{1\sim(k-1)}}Lz_k^2\end{cases}$$

Substituting this and (22) into (21), and noting that $Lz_{k-1}z_k\le Lz_{k-1}^2+\frac{1}{4}\max\{g_M^2,1\}Lz_k^2$, we have

$$\begin{aligned}\dot{V}_k\le&-\sigma_{k-1}\Big(d_{k-1}^{\boldsymbol{\epsilon}}-\frac{\sigma_{1\sim(k-2)}}{2^k}L\Big)\|\boldsymbol{\epsilon}\|^2-\sigma_{k-1}\Big(d_{k-1}^{\boldsymbol{\eta}}-\\&\frac{\sigma_{1\sim(k-2)}}{2^k}L\Big)\|\boldsymbol{\eta}\|^2-\frac{l_1}{2L^3}\sigma_{1\sim(k-1)}\lambda_1^2-\frac{\dot{L}}{L}\Big(\sigma_{k-1}d_{k-1,1}^{\boldsymbol{\eta}}-\end{aligned}$$

$$\frac{1}{2}\max_{i=1,\cdots,k-1}(\gamma_{k,1}^{\eta_i})^2\Big)\|\boldsymbol{\eta}\|^2-\sigma_{k-1}\big(d_{k-1}^{z_1}-\sigma_{1\sim(k-2)}L-\sigma_{1\sim(k-2)}c^2\big)z_1^2-\sum_{i=2}^{k-2}\sigma_{k-1}\big(d_{k-1}^{z_i}-\sigma_{i\sim(k-2)}L\big)z_i^2-\sigma_{k-1}\times \big(d_{k-1}^{z_{k-1}}-2L\big)z_{k-1}^2-\frac{\dot L}{L}\sum_{i=1}^{k-2}\Big(\sigma_{k-1}d_{k-1,1}^{z_i}-\frac{1}{2}(\gamma_{k,1}^{z_i})^2\Big)z_i^2-\frac{\dot L}{L}\Big(\sigma_{k-1}-\frac{1}{2}(\gamma_{k,1}^{z_{k-1}})^2\Big)z_{k-1}^2-\frac{\dot L}{L}z_k^2+Lz_k\Big(\sum_{i=1}^{k-1}\gamma_k^{z_i}z_i+\sum_{i=1}^{k+1}\gamma_k^{\eta_i}\eta_i+\bar\alpha_kz_k\Big)+\sigma_{1\sim(k-1)}\sum_{i=k+1}^{n}\varepsilon_i^2+Lz_k\varepsilon_{k+1} \quad (23)$$

where $\bar\alpha_k=\gamma_k^{z_k}+\frac{\sigma_{k-1}}{4}\max\{g_M^2,1\}+\sum_{i=1}^{k-1}\frac{(\bar\delta_k^{z_i})^2}{4\sigma_{i\sim(k-1)}}+\sum_{i=1}^{2}\frac{2^{k-2}(\bar\delta_k^{\epsilon_i})^2}{\sigma_{1\sim(k-1)}}+\sum_{i=1}^{k}\frac{2^{k-2}(\bar\delta_k^{\eta_i})^2}{\sigma_{1\sim(k-1)}}+\frac{(\bar\delta_k^{f_1})^2}{4\sigma_{1\sim(k-1)}}+2k\sigma_{1\sim(k-1)}$.

Thus, we can choose the virtual controller

$$\varepsilon_{k+1}^*=-\alpha_kz_k-\sum_{i=1}^{k-1}\gamma_k^{z_i}z_i-\sum_{i=1}^{k+1}\gamma_k^{\eta_i}\eta_i \quad (24)$$

where $\eta_{n+1}=0$, $\alpha_k=b_k+\bar\alpha_k$, and $b_k>0$ is a constant to be determined later.

Define $z_{k+1}=\varepsilon_{k+1}-\varepsilon_{k+1}^*$. Then, $\varepsilon_{k+1}^2\le(2k+2)z_{k+1}^2+\sum_{i=1}^{k}\mu_k^{z_i}z_i^2+\mu_k^{\boldsymbol{\eta}}\|\boldsymbol{\eta}\|^2$, where $\mu_k^{z_i}=(2k+2)(\gamma_k^{z_i})^2$, $i=1,\cdots,k-1$, $\mu_k^{z_k}=(2k+2)\alpha_k^2$, and $\mu_k^{\boldsymbol{\eta}}=(2k+2)\max_{i=1,\cdots,k+1}(\gamma_k^{\eta_i})^2$, independent of L. Substituting this and (24) into (23) results in

$$\dot V_k\le-d_k^{\boldsymbol{\epsilon}}\|\boldsymbol{\epsilon}\|^2-d_k^{\boldsymbol{\eta}}\|\boldsymbol{\eta}\|^2-\frac{d_{k,1}^{\boldsymbol{\eta}}\dot L}{L}\|\boldsymbol{\eta}\|^2-\frac{l_1}{2L^3}\sigma_{1\sim(k-1)}\lambda_1^2-\sum_{i=1}^{k}d_k^{z_i}z_i^2-\sum_{i=1}^{k-1}\frac{d_{k,1}^{z_i}\dot L}{L}z_i^2-\frac{\dot L}{L}z_k^2+\sigma_{1\sim(k-1)}\sum_{i=k+2}^{n}\varepsilon_i^2+2(k+1)\sigma_{1\sim(k-1)}z_{k+1}^2+Lz_kz_{k+1}$$

where $k=3,\cdots,n$ and

$$\begin{cases}d_k^{\boldsymbol{\epsilon}}=\sigma_{k-1}\Big(d_{k-1}^{\boldsymbol{\epsilon}}-\dfrac{\sigma_{1\sim(k-2)}}{2^k}L\Big)\\ d_k^{\boldsymbol{\eta}}=\sigma_{k-1}\Big(d_{k-1}^{\boldsymbol{\eta}}-\dfrac{\sigma_{1\sim(k-2)}}{2^k}L-\sigma_{1\sim(k-2)}\mu_k^{\boldsymbol{\eta}}\Big)\\ d_{k,1}^{\boldsymbol{\eta}}=\sigma_{k-1}d_{k-1,1}^{\boldsymbol{\eta}}-\dfrac{1}{2}\max\limits_{i=1,\cdots,k-1}(\gamma_{k,1}^{\eta_i})^2\\ d_{k,1}^{z_i}=\sigma_{k-1}d_{k-1,1}^{z_i}-\dfrac{1}{2}(\gamma_{k,1}^{z_i})^2,\ i=1,\cdots,k-2\\ d_{k,1}^{z_{k-1}}=\sigma_{k-1}-\dfrac{1}{2}(\gamma_{k,1}^{z_{k-1}})^2\\ d_k^{z_1}=\sigma_{k-1}\big(d_{k-1}^{z_1}-\sigma_{1\sim(k-2)}(L+c^2)-\sigma_{1\sim(k-2)}\mu_k^{z_1}\big)\\ d_k^{z_i}=\sigma_{k-1}\big(d_{k-1}^{z_i}-\sigma_{i\sim(k-2)}L-\sigma_{1\sim(k-2)}\mu_k^{z_i}\big),\\ \qquad i=2,\cdots,k-2\\ d_k^{z_{k-1}}=\sigma_{k-1}\big(d_{k-1}^{z_{k-1}}-2L-\sigma_{1\sim(k-2)}\mu_k^{z_{k-1}}\big)\\ d_k^{z_k}=b_kL-\sigma_{1\sim(k-1)}\mu_k^{z_k}\end{cases}$$

From this and the expressions of $d_i^{\boldsymbol{\epsilon}}$, $d_i^{\boldsymbol{\eta}}$, $d_i^{z_1}$, $i=1,2$ and $d_2^{z_2}$ given before, we have

$$\begin{cases}d_k^{\boldsymbol{\epsilon}}=\sigma_{1\sim(k-1)}\Big(\dfrac{L}{2^k}-c_1\Big)\\ d_k^{\boldsymbol{\eta}}=\sigma_{1\sim(k-1)}\Big(\dfrac{L}{2^k}-1-\sum\limits_{i=1}^{k}\mu_i^{\boldsymbol{\eta}}\Big)\\ d_{k,1}^{\boldsymbol{\eta}}=\sigma_{1\sim(k-1)}h-\sum\limits_{i=2}^{k}\dfrac{\sigma_{i\sim(k-1)}}{2}\max\limits_{p=1,\cdots,i-1}(\gamma_{i,1}^{\eta_p})^2\\ d_{k,1}^{z_i}=\sigma_{i\sim(k-1)}-\sum\limits_{j=i+1}^{k}\dfrac{\sigma_{j\sim(k-1)}}{2}(\gamma_{j,1}^{z_i})^2,\ i=1,\cdots,k-1\\ d_k^{z_1}=\sigma_{1\sim(k-1)}\Big((b_1-k)L-c-(k-1)c^2-\sum\limits_{j=1}^{k}\mu_j^{z_1}\Big)\\ d_k^{z_i}=\sigma_{i\sim(k-1)}\Big((b_i-k+i-1)L-\sigma_{1\sim(i-1)}\sum\limits_{q=i}^{k}\mu_q^{z_i}\Big),\\ \qquad i=2,\cdots,k-1\\ d_k^{z_k}=b_kL-\sigma_{1\sim(k-1)}\mu_k^{z_k}\end{cases}$$

with $\sigma_{i\sim j}=1$ if $i>j$.

At the last step, using the inductive procedure, we can design the actual controller

$$u=-L^{n+1}\alpha_nz_n-L^{n+1}\sum_{i=1}^{n-1}\gamma_n^{z_i}z_i-L^{n+1}\sum_{i=1}^{n}\gamma_n^{\eta_i}\eta_i \quad (25)$$

Noting that $\sum_{i=n+2}^{n}\varepsilon_i^2=0$ and $z_{n+1}=0$, we have

$$\dot V_n\le-\sigma_{1\sim(n-1)}\Big(\Big(\frac{L}{2^n}-c_1\Big)\|\boldsymbol{\epsilon}\|^2+\Big(\frac{L}{2^n}-\sum_{i=1}^{n-1}\mu_i^{\boldsymbol{\eta}}-1\Big)\|\boldsymbol{\eta}\|^2+\frac{l_1}{2L^3}\lambda_1^2\Big)-\frac{\dot L}{L}\Big(\sigma_{1\sim(n-1)}h-\sum_{i=2}^{n}\frac{\sigma_{i\sim(n-1)}}{2}\max_{j=1,\cdots,i-1}(\gamma_{i,1}^{\eta_j})^2\Big)\times\|\boldsymbol{\eta}\|^2-\sigma_{1\sim(n-1)}\Big((b_1-n)L-c-(n-1)c^2-\sum_{j=1}^{n-1}\mu_j^{z_1}\Big)z_1^2-\sum_{i=2}^{n-1}\sigma_{i\sim(n-1)}\Big((b_i-n+i-1)L-\sigma_{1\sim(i-1)}\sum_{j=i}^{n-1}\mu_j^{z_i}\Big)z_i^2-b_nLz_n^2-\frac{\dot L}{L}\sum_{i=1}^{n-1}\Big(\sigma_{i\sim(n-1)}-\sum_{j=i+1}^{n}\frac{\sigma_{j\sim(n-1)}}{2}(\gamma_{j,1}^{z_i})^2\Big)z_i^2-\frac{\dot L}{L}z_n^2 \quad (26)$$

where $V_n(\boldsymbol{\eta},\boldsymbol{\epsilon},\lambda_1,\boldsymbol{z})$ is a positive definite and proper function defined by $V_n=\sigma_{1\sim(n-1)}\left(\boldsymbol{\epsilon}^{\rm T}P_l\boldsymbol{\epsilon}+\boldsymbol{\eta}^{\rm T}P_l\boldsymbol{\eta}+\frac{1}{2L^4}\lambda_1^2\right)+\frac{1}{2}\sum_{i=1}^{n}\sigma_{j\sim(n-1)}z_i^2$.

This completes the constructive design of the output-feedback controller.

From (26), if the controller (25) ensures the negative definiteness of the time derivative of V_n, the closed-loop system will be globally stable. In Subsection 1.3, we will show that if L is bounded on $[0,+\infty)$ and large enough, the global-asymptotic stability of the other closed-loop system states can be guaranteed by appropriate choice of the design parameters b_i, $i=1,\cdots,n$ and σ_i, $i=1,\cdots,n-1$.

1.3 Main results

From (26), to achieve the stabilization of the closed-loop system, we should first choose the design parameters b_i, $i=$

$1,\cdots,n$ and σ_i, $i=1,\cdots,n-1$ to satisfy

$$\begin{cases} b_i-n+i-1>0, \quad i=1,\cdots,n-1, \quad b_n>0\\ \sigma_{1\sim(n-1)}h-\sum\limits_{i=2}^{n}\dfrac{\sigma_{i\sim(n-1)}}{2}\max\limits_{j=1,\cdots,i-1}(\gamma_{i,1}^{\eta_j})^2>0\\ \sigma_{i\sim(n-1)}-\sum\limits_{j=i+1}^{n}\dfrac{\sigma_{j\sim(n-1)}}{2}(\gamma_{j,1}^{z_i})^2>0,\ i=1,\cdots,n-1 \end{cases} \tag{27}$$

The following lemma provides the appropriate choice of the design parameters satisfying (27).

Lemma 1. There always exist positive design parameters b_i, $i=1,\cdots,n$ and σ_i, $i=1,\cdots,n-1$, such that (27) holds.

Proof. It suffices to give the choice of b_i and σ_i satisfying (27).

1) Choice of b_i, $i=1,\cdots,n$.

Clearly, it is enough to select b_i to satisfy $b_i>n-i+1$, $i=1,\cdots,n-1$, and $b_n>0$.

2) Choice of σ_i, $i=1,\cdots,n-1$.

By careful observation of (27), we see that the last two lines of (27) can be rewritten as

$$\begin{cases} \sigma_{n-1}\Big(\cdots\Big(\sigma_2\big(\sigma_1 h-\dfrac{1}{2}(\gamma_{2,1}^{\eta_1})^2\big)-\dfrac{1}{2}\max\limits_{i=1,2}(\gamma_{3,1}^{\eta_i})^2\Big)\cdots\Big)-\\ \quad\dfrac{1}{2}\max\limits_{i=1,\cdots,n-1}(\gamma_{n,1}^{\eta_i})^2>0\\ \sigma_{n-1}\Big(\cdots\Big(\sigma_{i+1}\big(\sigma_i-\dfrac{1}{2}(\gamma_{i+1,1}^{z_i})^2\big)-\dfrac{1}{2}(\gamma_{i+2,1}^{z_i})^2\Big)\cdots\Big)-\\ \quad\dfrac{1}{2}(\gamma_{n,1}^{z_i})^2>0,\ \ i=1,\cdots,n-1 \end{cases}$$

which shows design parameters σ_i, $i=1,\cdots,n-1$ can be chosen by the rule: first σ_1, then σ_2 under selected σ_1, and till σ_{n-1} under selected $\sigma_1,\cdots,\sigma_{n-2}$. Explicitly, we first choose σ_1 to satisfy $\sigma_1 h-\frac{1}{2}(\gamma_{2,1}^{\eta_1})^2>0$ and $\sigma_1-\frac{1}{2}(\gamma_{2,1}^{z_1})^2>0$. Then, choose σ_2 to satisfy $\sigma_2\big(\sigma_1 h-\frac{1}{2}(\gamma_{2,1}^{\eta_1})^2\big)-\frac{1}{2}\max_{i=1,2}(\gamma_{3,1}^{\eta_i})^2>0$, $\sigma_2\big(\sigma_1-\frac{1}{2}(\gamma_{2,1}^{z_1})^2\big)-\frac{1}{2}(\gamma_{3,1}^{z_1})^2>0$, and $\sigma_2-\frac{1}{2}(\gamma_{3,1}^{z_2})^2>0$. In the same way, we select σ_i, $i=3,\cdots,n-1$.

The aforementioned choice of the design parameters b_i and σ_i obviously means that (27) holds. □

It is easy to verify that the right-hand side of the closed-loop system is locally Lipschitz in $(\boldsymbol{\zeta},\boldsymbol{\xi},\boldsymbol{\lambda},L)$ in a neighborhood of the initial condition, and hence the closed-loop system has a unique solution on a small interval $[0,\,t_f)$. Let $[0,\,T_f)$ be its maximal interval on which a unique solution exists, where $0<T_f\le+\infty$.

The following lemma is given which will play an important role in proving Theorem 1 below, and its proof is provided in Appendix B for compactness.

Lemma 2. If L is bounded on $[0,T_f)$, then the states $\boldsymbol{\eta}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\epsilon}$ are bounded on $[0,T_f)$, and moreover,

$$\int_0^{T_f}\big(\|\boldsymbol{\varepsilon}(t)\|^2+\|\boldsymbol{\epsilon}(t)\|^2+\|\boldsymbol{\eta}(t)\|^2\big)\mathrm{d}t<+\infty$$

The main results in the paper are summarized in the following theorem.

Theorem 1. Consider system (1) under Assumptions 1 and 2. If $\boldsymbol{l}=[l_1,\cdots,l_n]^{\mathrm{T}}$ is chosen such that matrix $A_{\boldsymbol{l}}=A_n-\boldsymbol{l}\boldsymbol{e}_1{}^{\mathrm{T}}$ is Hurwitz and the design parameters b_i, $i=1,\cdots,n$, and σ_i, $i=1,\cdots,n-1$ satisfy (27), then the adaptive output-feedback controller (25) based on the high-gain K-filters (4) and (5) guarantees that all the states of the closed-loop system are bounded on $[0,\,+\infty)$, and furthermore, $\lim_{t\to+\infty}(\boldsymbol{\zeta}(t),\boldsymbol{\xi}(t),\boldsymbol{\lambda}(t))=(0,0,0)$.

Proof. In the following, we will first prove that the closed-loop solution is well-defined and bounded on $[0,+\infty)$, and then prove that the system states $\boldsymbol{\zeta}(t)$, $\boldsymbol{\xi}(t)$, and $\boldsymbol{\lambda}(t)$ are asymptotically stable.

We are now ready to show the boundedness of L on $[0,\,T_f)$. From (5) and the fact $l_1>0$, it follows that $L(t)\ge1$, $\forall t\in[0,\,T_f)$. Then, the boundedness of L on $[0,\,T_f)$ can be proven by a contradiction argument. Suppose L is not bounded on $[0,\,T_f)$. This means that $\lim_{t\to T_f}L(t)=+\infty$. Consequently, there exists a finite time $0<t_1<T_f$, such that for $\forall t\in[t_1,\,T_f)$,

$$L(t)\ge\max\Bigg\{2^n(c_1+1),\ 2^n\Big(2+\sum_{i=1}^{n-1}\mu_i^{\boldsymbol{\eta}}\Big),\ \frac{1}{b_1^*}\Big(c+(n-1)c^2+1+\sum_{i=1}^{n-1}\mu_i^{z_1}\Big),\ \frac{1}{b_i^*}\Big(\sigma_{1\sim(i-1)}\sum_{j=i}^{n-1}\mu_j^{z_i}+1\Big),\ i=2,\cdots,n-1,\ \frac{1}{b_n^*}\Bigg\}$$

where $b_i^*=b_i-n+i-1$, $i=1,\cdots,n-1$ and $b_n^*=b_n$, all positive numbers. This together with (26) results in

$$\dot V_n(\lambda_1(t),\boldsymbol{X}(t),\boldsymbol{z}(t))\le-\sigma_{1\sim(n-1)}\|\boldsymbol{X}(t)\|^2-\frac{\sigma_{1\sim(n-1)}l_1}{2L^3}\lambda_1^2-\sum_{i=1}^{n}\sigma_{i\sim(n-1)}z_i^2,\ \forall t\in[t_1,\,T_f)$$

where $\boldsymbol{X}=[\boldsymbol{\epsilon}^{\mathrm{T}},\boldsymbol{\eta}^{\mathrm{T}}]^{\mathrm{T}}$, and hence,

$$\int_{t_1}^{T_f}\Big(z_1^2(t)+\frac{l_1}{2L^3(t)}\lambda_1^2(t)\Big)\mathrm{d}t\le\frac{1}{\sigma_{1\sim(n-1)}}V_n(\lambda_1(t_1),\boldsymbol{X}(t_1),\boldsymbol{z}(t_1))<+\infty$$

By this and the fact $\dot L=z_1^2+\frac{l_1}{2L^3}\lambda_1^2$ (see (5)), we have

$$+\infty=L(T_f)-L(t_1)=\int_{t_1}^{T_f}\dot L(t)\mathrm{d}t=\int_{t_1}^{T_f}\Big(z_1^2(t)+\frac{l_1}{2L^3(t)}\lambda_1^2(t)\Big)\mathrm{d}t<+\infty$$

which is impossible and thus implies that L is bounded on $[0,\,T_f)$.

Up to now, we can verify that $T_f=+\infty$. From (26) and the boundedness of L, it follows that $\dot V_n\le\beta V_n$ for some positive constant β and hence $V_n(\lambda_1(t),\boldsymbol{X}(t),\boldsymbol{z}(t))\le \mathrm{e}^{\beta t}V_n(\lambda_1(0),\boldsymbol{X}(0),\boldsymbol{z}(0))$, $t\in[0,T_f)$. Suppose T_f is finite, we have $+\infty=V_n(\lambda_1(T_f),\boldsymbol{X}(T_f),\boldsymbol{z}(T_f))\le \mathrm{e}^{\beta T_f}V_n(\lambda_1(0),\boldsymbol{X}(0),\boldsymbol{z}(0))<+\infty$. This contradiction means that $T_f=+\infty$.

Noting the boundedness of L on $[0,\,T_f)$ and $T_f=+\infty$, from Lemma 2, we know that $\boldsymbol{\eta}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\epsilon}$ are bounded on $[0,\,+\infty)$. It is easily seen that z_i can be represented by a linear time-invariant function of ε_j and η_j, where $i=1,\cdots,n$, $j=1,\cdots,i$ and, consequently, $\boldsymbol{z}$ is bounded on $[0,\,+\infty)$. Till now, all the states of the closed-loop system have been well-defined and bounded on $[0,\,+\infty)$.

The rest is to prove the asymptotic stability of $(\boldsymbol{\zeta},\boldsymbol{\xi},\boldsymbol{\lambda})$. Using the boundedness of $(L,\boldsymbol{\varepsilon},\boldsymbol{\eta},\boldsymbol{\epsilon})$ on $[0,\,+\infty)$, it can be deduced that $\dot{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{\eta}}$, and $\dot{\boldsymbol{\epsilon}}$ are bounded on $[0,\,+\infty)$ as well. Then, by Lemma 2 and Barbalat's Lemma[24], we have

$$\lim_{t\to+\infty}\boldsymbol{\varepsilon}(t)=\boldsymbol{0},\quad \lim_{t\to+\infty}\boldsymbol{\eta}(t)=\boldsymbol{0},\quad \lim_{t\to+\infty}\boldsymbol{\epsilon}(t)=\boldsymbol{0}$$

which together with (8) and (9) results in

$$\lim_{t\to+\infty}\boldsymbol{\lambda}(t)=\boldsymbol{0},\quad \lim_{t\to+\infty}\boldsymbol{\xi}(t)=\boldsymbol{0},\quad \lim_{t\to+\infty}\tilde{\boldsymbol{x}}(t)=\boldsymbol{0}$$

Therefore, $\boldsymbol{x} = \boldsymbol{\xi} + g\boldsymbol{\lambda} + \tilde{\boldsymbol{x}}$ is bounded on $[0, +\infty)$ and $\lim_{t\to+\infty} \boldsymbol{x}(t) = 0$. This together with (2) yields the boundedness of $\boldsymbol{\zeta}$ on $[0, +\infty)$ and $\lim_{t\to+\infty} \boldsymbol{\zeta}(t) = 0$. □

2 Simulation example

Consider the following second-order nonlinear system:

$$\dot{\zeta}_1 = g_1\zeta_2 + \theta\zeta_1 \sin\zeta_2, \quad \dot{\zeta}_2 = g_2 u, \quad y = \zeta_1$$

where θ is an unknown constant. Suppose this system satisfies Assumptions 1 and 2 with $c = |\theta|$, $1 \le |g_1| \le 1.5$, $1 \le |g_2| \le 1.2$. Without loss of generality, the signs of g_1 and g_2 are assumed to be positive.

Choose $\boldsymbol{l} = [2, 1]^{\rm T}$. Then, by Theorem 1, we design an adaptive output-feedback controller of the form (4)-(5)-(25) with $n = 2$.

Let the initial value of the state be $\boldsymbol{\zeta}_0 = [1, 0]^{\rm T}$, $\theta = 0.1$ and $g_1 = g_2 = 1$. Choosing design parameters $b_1 = 2.1$, $b_2 = 0.5$, and $\sigma_1 = 10.6$, we can obtain Figs. 1 ∼ 4. From these figures, we know that L is bounded, and $\boldsymbol{\zeta}$, $\boldsymbol{\xi}$, and $\boldsymbol{\lambda}$ are asymptotically stable.

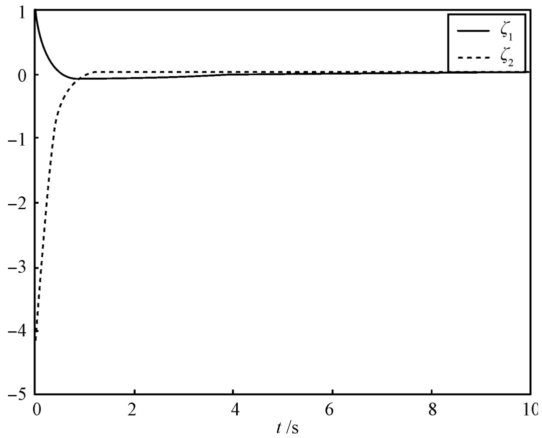

Fig. 1 System state $\boldsymbol{\zeta}$

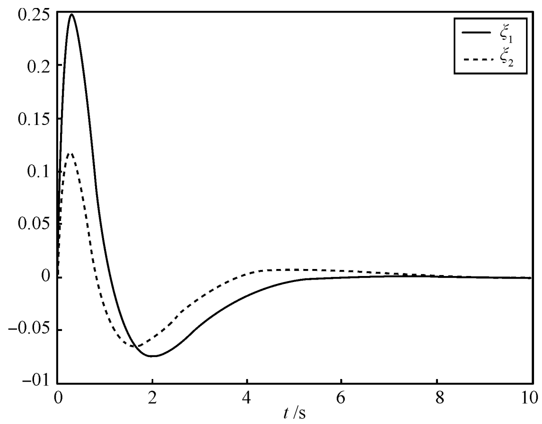

Fig. 2 State $\boldsymbol{\xi}$ of high-gain K-filters

3 Concluding remarks

In this paper, the output-feedback stabilization problem has been investigated for a class of uncertain nonlinear systems. After introducing linear state transformation, the control design becomes convenient since the converted system has known virtual control coefficients. However, the commonly used high-gain observer is inapplicable to the output-feedback control of the system under consideration. Therefore, an adaptive high-gain observer is proposed, which is the extension of the customary K-filters-based observer. Then, by the backstepping approach, the output-feedback controller is successfully designed. It has been shown that the global stability of the closed-loop system can be guaranteed by the dynamic gain and the appropriate choice of the design parameters.

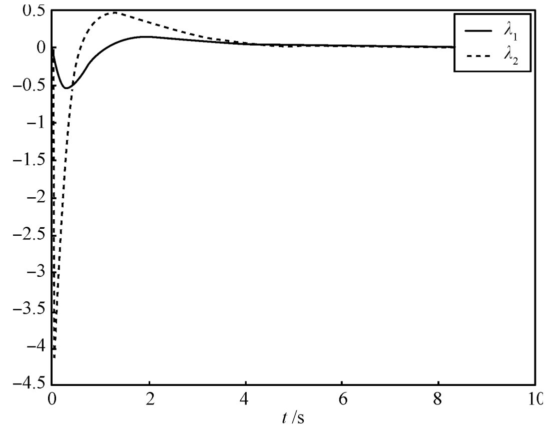

Fig. 3 State $\boldsymbol{\lambda}$ of high-gain K-filters

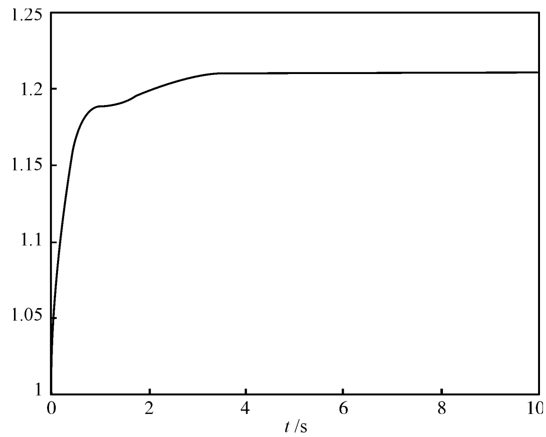

Fig. 4 Dynamic gain L of the K-filters

Appendix

In this appendix, the proofs of Proposition 1 and Lemma 2 are provided, respectively.

A Proof of Proposition 1

Along the trajectories of the subsystem $\boldsymbol{\epsilon}$ in (10), the time derivative of $V_{\boldsymbol{\epsilon}}$ satisfies

$$\dot{V}_{\boldsymbol{\epsilon}} \le -L\|\boldsymbol{\epsilon}\|^2 + 2\boldsymbol{\epsilon}^{\rm T} P_{\boldsymbol{l}} \boldsymbol{f} - \frac{\dot{L}}{L}\boldsymbol{\epsilon}^{\rm T}(DP_{\boldsymbol{l}} + P_{\boldsymbol{l}}D)\boldsymbol{\epsilon} \qquad \text{(A1)}$$

We will first handle the second term on the right-hand side of the aforementioned inequality. Noting that $x_1 = y = L\varepsilon_1$, $x_i = \tilde{x}_i + \xi_i + g\lambda_i = L^i\epsilon_i + L^i\eta_i + gL^i\varepsilon_i$ for $i = 2, \cdots, n$ and by the fact $L \ge 1$, we have $|f_1| \le c|\varepsilon_1| \le c_2|\varepsilon_1|$ and

$$|f_i| = \left|\frac{g_{1\sim(i-1)}\phi_i}{L^i}\right| \le c|g_{1\sim(i-1)}|\Big(\frac{|y|}{L^i} + \sum_{j=2}^{i}\frac{|x_j|}{|g_{1\sim(j-1)}|L^i}\Big) \le$$
$$c_2\Big(\sum_{j=1}^{i}|\varepsilon_j| + \|\boldsymbol{\epsilon}\| + \|\boldsymbol{\eta}\|\Big)$$

where $c_2 = \frac{cg_M}{g_N}\max\{\sqrt{n-1}, g_M\}$ is an unknown positive constant depending on c. Then we have

$$2\|P_{\boldsymbol{l}}\| \cdot \|\boldsymbol{\epsilon}\| \cdot \|\boldsymbol{f}\|_1 \le c_1\|\boldsymbol{\epsilon}\|^2 + \|\boldsymbol{\varepsilon}\|^2 + \|\boldsymbol{\eta}\|^2$$

where $c_1 = (n+1)n^2c_2^2\|P_{\boldsymbol{l}}\|^2 + 2nc_2\|P_{\boldsymbol{l}}\|$ is an unknown positive constant depending on c. Substituting this into (A1), and noting (5) and (6), we obtain (11). $\square$

B Proof of Lemma 2

We first show that

$$\int_0^{T_f} \varepsilon_1^2(t)\mathrm{d}t < +\infty, \quad \int_0^{T_f} \lambda_1^2(t)\mathrm{d}t < +\infty \tag{B1}$$

Otherwise, we have $\int_0^{T_f}\left(z_1^2(t) + \frac{l_1\lambda_1^2(t)}{2L^3(t)}\right)\mathrm{d}t = +\infty$. Then, from $\dot{L} = z_1^2 + \frac{l_1\lambda_1^2}{2L^3}$, we conclude that L is unbounded on $[0,\ T_f)$, and this contradicts the boundedness of L on $[0,\ T_f)$.

Let us next prove the boundedness of $\boldsymbol{\eta}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\epsilon}$ on $[0,\ T_f)$. For the subsystem $\boldsymbol{\eta}$ of system (10), by Proposition 2, we have, for $\forall t \in [0,\ T_f)$,

$$\begin{aligned}&\lambda_{\min}(P_{\boldsymbol{l}})\|\boldsymbol{\eta}(t)\|^2 - \boldsymbol{\eta}(0)^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{\eta}(0) \leq \\ &V_{\boldsymbol{\eta}}(\boldsymbol{\eta}(t)) - V_{\boldsymbol{\eta}}(\boldsymbol{\eta}(0)) \leq \\ &2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2\int_0^t L(\tau)\varepsilon_1^2(\tau)\mathrm{d}\tau - \frac{1}{2}\int_0^t \|\boldsymbol{\eta}(\tau)\|^2\mathrm{d}\tau\end{aligned}$$

from which, (B1) and the boundedness of L, it follows that, for $\forall t \in [0,\ T_f)$,

$$\begin{aligned}\|\boldsymbol{\eta}(t)\|^2 \leq \frac{1}{\lambda_{\min}(P_{\boldsymbol{l}})}\Big(&\boldsymbol{\eta}(0)^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{\eta}(0) + \\ &2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2\int_0^t L(\tau)\varepsilon_1^2(\tau)\mathrm{d}\tau\Big) < +\infty\end{aligned}$$

namely, $\boldsymbol{\eta}$ is bounded on $[0,\ T_f)$, so is $\boldsymbol{\xi}$, and

$$\begin{aligned}\int_0^{T_f}\|\boldsymbol{\eta}(t)\|^2\mathrm{d}t \leq\ & 2\boldsymbol{\eta}(0)^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{\eta}(0) + \\ &4\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2\int_0^{T_f}L(t)\varepsilon_1^2(t)\mathrm{d}t < +\infty\end{aligned} \tag{B2}$$

According to (9), we have $\epsilon_1 = \varepsilon_1 - \eta_1 - \frac{g}{L}\lambda_1$. By $|g| \leq g_M$, (B1), (B2), and the boundedness of L on $[0, T_f)$, we obtain $\int_0^{T_f}\epsilon_1^2(t)\mathrm{d}t < +\infty$. In order to verify the boundedness of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\epsilon}$ on $[0, T_f)$, we first introduce the change of coordinates

$$\begin{cases}\bar{\epsilon}_i = \dfrac{\tilde{x}_i}{(L^*)^i}, & i = 1,\cdots,n\\ \bar{\varepsilon}_1 = \dfrac{y}{gL^*}, \quad \bar{\varepsilon}_i = \dfrac{\lambda_i}{(L^*)^i}, & i = 2,\cdots,n\end{cases} \tag{B3}$$

where L^* is a constant satisfying

$$\begin{aligned}L^* \geq \max\Big\{&10 + 4\sqrt{n}\max\{g_M, 1\}L(T_f)\max_{i=1,\cdots,n}|\bar{a}_i|\|P_{\boldsymbol{l}}\|, \\ &12 + 12n\bar{c}_1\|P_{\boldsymbol{l}}\| + 4n^2\bar{c}_1^2\|P_{\boldsymbol{l}}\|^2 + \frac{16}{g_N^2}\|P_{\boldsymbol{l}}\|^2\|B_1\|^2, L(T_f)\Big\}\end{aligned} \tag{B4}$$

with $\bar{a}_i$, $i = 1,\cdots,n$ and $\bar{c}_1$ constants to be determined.

Then, the dynamics of $\bar{\boldsymbol{\epsilon}} = [\bar{\epsilon}_1,\cdots,\bar{\epsilon}_n]^{\mathrm{T}}$ satisfy

$$\dot{\bar{\boldsymbol{\epsilon}}} = L^*A_{\boldsymbol{l}}\bar{\boldsymbol{\epsilon}} + L^*\boldsymbol{l}\bar{\epsilon}_1 - L\Gamma_1\boldsymbol{l}\bar{\epsilon}_1 + \boldsymbol{\Psi}$$

where $\Gamma_1 = \mathrm{diag}\Big\{1, \frac{L}{L^*},\cdots,\left(\frac{L}{L^*}\right)^{n-1}\Big\}$ and $\boldsymbol{\Psi} = [\frac{\varphi_1}{L^*},\cdots,\frac{\varphi_n}{(L^*)^n}]^{\mathrm{T}}$.

By $z_i = \varepsilon_i - \varepsilon_i^*$ aforementioned, (20) and (25), there exist known constants $\bar{a}_i$, a_i, $i = 1,\cdots,n$ such that

$$u = -|g|L^nL^*\bar{a}_1\bar{\varepsilon}_1 - \sum_{i=2}^n L^{n+1-i}(L^*)^i\bar{a}_i\bar{\varepsilon}_i - L^{n+1}\sum_{i=1}^n a_i\eta_i$$

By (9), the dynamics of $\bar{\boldsymbol{\varepsilon}} = [\bar{\varepsilon}_1,\cdots,\bar{\varepsilon}_n]^{\mathrm{T}}$ satisfy

$$\begin{aligned}\dot{\bar{\boldsymbol{\varepsilon}}} =\ & L^*A_{\boldsymbol{l}}\bar{\boldsymbol{\varepsilon}} + L^*\boldsymbol{l}\bar{\varepsilon}_1 - L\Gamma_2\boldsymbol{l}\bar{\varepsilon}_1 - L\boldsymbol{e}_n|g|\left(\frac{L}{L^*}\right)^{n-1}\bar{a}_1\bar{\varepsilon}_1 - \\ &L\boldsymbol{e}_n\sum_{i=2}^n\left(\frac{L}{L^*}\right)^{n-i}\bar{a}_i\bar{\varepsilon}_i + \frac{L^*}{g}B_1\bar{\boldsymbol{\epsilon}} + B_2\boldsymbol{\eta} + \frac{\boldsymbol{e}_1\varphi_1}{gL^*}\end{aligned}$$

where $\Gamma_2 = \mathrm{diag}\Big\{0, \frac{L}{L^*},\cdots,\left(\frac{L}{L^*}\right)^{n-1}\Big\}$, and

$$B_1 = \begin{bmatrix}0 & 1\,0\cdots 0\\ \left(\frac{L}{L^*}\right)^2 l_2 & 0\,0\cdots 0\\ \vdots & \vdots\,\vdots\,\ddots\,\vdots\\ \left(\frac{L}{L^*}\right)^n l_n & 0\,0\cdots 0\end{bmatrix}$$

$$B_2 = \frac{L}{g}B_1 + \frac{L(L-L^*)}{gL^*}\boldsymbol{e}_1\boldsymbol{e}_2{}^{\mathrm{T}} - \frac{L^{n+1}}{(L^*)^n}\boldsymbol{e}_n\boldsymbol{a}^{\mathrm{T}}$$

with $\boldsymbol{e}_2 = [0,1,0,\cdots,0]^{\mathrm{T}} \in \mathbf{R}^n$ and $\boldsymbol{a} = [a_1,\cdots,a_n]^{\mathrm{T}}$. Clearly, $\|B_1\|$ and $\|B_2\|$ are bounded.

Let $V_{\bar{\boldsymbol{\epsilon}}} = \bar{\boldsymbol{\epsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\bar{\boldsymbol{\epsilon}}$. Then, the time derivative of $V_{\bar{\boldsymbol{\epsilon}}}$ satisfies

$$\dot{V}_{\bar{\boldsymbol{\epsilon}}} \leq -L^*\|\bar{\boldsymbol{\epsilon}}\|^2 + 2L^*\bar{\boldsymbol{\epsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{l}\bar{\epsilon}_1 - 2L\bar{\boldsymbol{\epsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\Gamma_1\boldsymbol{l}\bar{\epsilon}_1 + 2\bar{\boldsymbol{\epsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{\Psi} \tag{B5}$$

We next handle the last three terms on the right-hand side of the aforementioned inequality. First, note that $\|\Gamma_1\|_1 = 1$ and

$$\begin{aligned}\left|\frac{\varphi_i}{(L^*)^i}\right| \leq\ & c|g_{1\sim(i-1)}|\Big(\frac{|y|}{(L^*)^i} + \sum_{j=2}^i\frac{|x_j|}{|g_{1\sim(j-1)}|(L^*)^i}\Big) \leq \\ &\frac{cg_M}{g_N}\Big(g_N|\bar{\varepsilon}_1| + \sum_{j=2}^i\Big(|\bar{\epsilon}_j| + \left(\frac{L}{L^*}\right)^j|\eta_j| + g_M|\bar{\varepsilon}_j|\Big)\Big) \leq \\ &\bar{c}_1\left(\|\bar{\boldsymbol{\varepsilon}}\| + \|\bar{\boldsymbol{\epsilon}}\| + \|\boldsymbol{\eta}\|\right)\end{aligned}$$

where $\bar{c}_1$ is an unknown positive constant depending on c. Then, we have

$$\begin{cases}2L^*\bar{\boldsymbol{\epsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{l}\bar{\epsilon}_1 \leq \|\bar{\boldsymbol{\epsilon}}\|^2 + (L^*)^2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2\bar{\epsilon}_1^2\\ -2L\bar{\boldsymbol{\epsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\Gamma_1\boldsymbol{l}\bar{\epsilon}_1 \leq \|\bar{\boldsymbol{\epsilon}}\|^2 + L^2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2\bar{\epsilon}_1^2\\ 2\bar{\boldsymbol{\epsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{\Psi} \leq 2\|P_{\boldsymbol{l}}\|\cdot\|\bar{\boldsymbol{\epsilon}}\|\cdot\|\boldsymbol{\Psi}\|_1\\ \qquad \leq \left(3n\bar{c}_1\|P_{\boldsymbol{l}}\| + n^2\bar{c}_1^2\|P_{\boldsymbol{l}}\|^2\right)\|\bar{\boldsymbol{\epsilon}}\|^2 + \|\bar{\boldsymbol{\varepsilon}}\|^2 + \\ \qquad n\bar{c}_1\|P_{\boldsymbol{l}}\|\cdot\|\boldsymbol{\eta}\|^2\end{cases}$$

Substituting this into (B5), we have

$$\begin{aligned}\dot{V}_{\bar{\boldsymbol{\epsilon}}} \leq\ & -\Big(L^* - 2 - 3n\bar{c}_1\|P_{\boldsymbol{l}}\| - n^2\bar{c}_1^2\|P_{\boldsymbol{l}}\|^2\Big)\|\bar{\boldsymbol{\epsilon}}\|^2 + \|\bar{\boldsymbol{\varepsilon}}\|^2 + \\ &\Big((L^*)^2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 + L^2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2\Big)\bar{\epsilon}_1^2 + n\bar{c}_1\|P_{\boldsymbol{l}}\|\cdot\|\boldsymbol{\eta}\|^2\end{aligned} \tag{B6}$$

Let $V_{\bar{\boldsymbol{\varepsilon}}} = \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\bar{\boldsymbol{\varepsilon}}$. Then, the time derivative of $V_{\bar{\boldsymbol{\varepsilon}}}$ satisfies

$$\begin{aligned}\dot{V}_{\bar{\boldsymbol{\varepsilon}}} \leq\ & -L^*\|\bar{\boldsymbol{\varepsilon}}\|^2 + 2L^*\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{l}\bar{\varepsilon}_1 - 2L\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{e}_n|g|\left(\frac{L}{L^*}\right)^{n-1}\bar{a}_1\bar{\varepsilon}_1 - \\ &2L\bar{\boldsymbol{\varepsilon}}P_{\boldsymbol{l}}\Gamma_2\boldsymbol{l}\bar{\varepsilon}_1 - 2L\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{e}_n\sum_{i=2}^n\left(\frac{L}{L^*}\right)^{n-i}\bar{a}_i\bar{\varepsilon}_i + \\ &\frac{2L^*}{g}\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}B_1\bar{\boldsymbol{\epsilon}} + 2\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}B_2\boldsymbol{\eta} + \frac{2}{gL^*}\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{e}_1\varphi_1\end{aligned} \tag{B7}$$

Let us first handle the last seven terms on the right-hand side of the aforementioned inequality. Noting that $\|\Gamma_2\|_1 \leq 1$ and (B4), we have

$$\begin{cases}-2L\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{e}_n|g|\left(\frac{L}{L^*}\right)^{n-1}\bar{a}_1\bar{\varepsilon}_1 - 2L\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{e}_n\sum_{i=2}^n\left(\frac{L}{L^*}\right)^{n-i}\bar{a}_i\bar{\varepsilon}_i \leq \\ \qquad 2\sqrt{n}\max\{g_M,1\}L\max_{i=1,\cdots,n}|\bar{a}_i|\|P_{\boldsymbol{l}}\|\|\bar{\boldsymbol{\varepsilon}}\|^2\\ 2L^*\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{l}\bar{\varepsilon}_1 \leq \|\bar{\boldsymbol{\varepsilon}}\|^2 + (L^*)^2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2\bar{\varepsilon}_1^2\\ -2L\bar{\boldsymbol{\varepsilon}}P_{\boldsymbol{l}}\Gamma_2\boldsymbol{l}\bar{\varepsilon}_1 \leq \|\bar{\boldsymbol{\varepsilon}}\|^2 + L^2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2\bar{\varepsilon}_1^2\\ \dfrac{2L^*}{g}\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}B_1\bar{\boldsymbol{\epsilon}} \leq \dfrac{L^*}{4}\|\bar{\boldsymbol{\varepsilon}}\|^2 + \dfrac{4L^*}{g_N^2}\|P_{\boldsymbol{l}}\|^2\|B_1\|^2\|\bar{\boldsymbol{\epsilon}}\|^2\\ 2\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}B_2\boldsymbol{\eta} \leq \|\bar{\boldsymbol{\varepsilon}}\|^2 + \|P_{\boldsymbol{l}}\|^2\|B_2\|^2\|\boldsymbol{\eta}\|^2\\ \dfrac{2}{gL^*}\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}P_{\boldsymbol{l}}\boldsymbol{e}_1\varphi_1 \leq \|\bar{\boldsymbol{\varepsilon}}\|^2 + \dfrac{c^2}{g_N^2}\|P_{\boldsymbol{l}}\|^2\bar{\varepsilon}_1^2\end{cases}$$

Substituting these into (B7), we have

$$\begin{aligned}\dot{V}_{\bar{\boldsymbol{\varepsilon}}} \leq\ & -\Big(\frac{3}{4}L^* - 2\sqrt{n}\max\{g_M,1\}L\max_{i=1,\cdots,n}|\bar{a}_i|\|P_{\boldsymbol{l}}\| - 4\Big)\times \\ &\|\bar{\boldsymbol{\varepsilon}}\|^2 + \frac{4L^*}{g_N^2}\|P_{\boldsymbol{l}}\|^2\|B_1\|^2\|\bar{\boldsymbol{\epsilon}}\|^2 + \Big((L^*)^2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 +\end{aligned}$$

$$\frac{c^2}{g_N^2}\|P_{\boldsymbol{l}}\|^2 + L^2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2\Big)\bar{\varepsilon}_1^2 + \|P_{\boldsymbol{l}}\|^2\|B_2\|^2\|\boldsymbol{\eta}\|^2 \qquad \text{(B8)}$$

Define $V(\bar{\boldsymbol{\varepsilon}},\bar{\boldsymbol{\epsilon}}) = V_{\bar{\boldsymbol{\varepsilon}}} + \frac{L^*}{4}V_{\bar{\boldsymbol{\epsilon}}}$. Then, from (B6) and (B8), we have

$$\begin{aligned}\dot{V} \le & -\Big(\frac{1}{2}L^* - 2\sqrt{n}\max\{g_M, 1\}L\max_{i=1,\cdots,n}|\bar{a}_i|\|P_{\boldsymbol{l}}\| - 4\Big)\|\bar{\boldsymbol{\varepsilon}}\|^2 - \\ & \Big(\frac{L^*}{4} - \Big(2 + 3n\bar{c}_1\|P_{\boldsymbol{l}}\| + n^2\bar{c}_1^2\|P_{\boldsymbol{l}}\|^2 + \\ & \frac{4}{g_N^2}\|P_{\boldsymbol{l}}\|^2\|B_1\|^2\Big)\Big)L^*\|\bar{\boldsymbol{\epsilon}}\|^2 + \Big((L^*)^2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 + \\ & \frac{c^2}{g_N^2}\|P_{\boldsymbol{l}}\|^2 + L^2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2\Big)\bar{\varepsilon}_1^2 + \frac{L^*}{4}\Big((L^*)^2\|P_{\boldsymbol{l}}\boldsymbol{l}\|^2 + \\ & L^2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2\Big)\bar{\boldsymbol{\epsilon}}_1^2 + \Big(\frac{L^*}{4}n\bar{c}_1\|P_{\boldsymbol{l}}\| + \|P_{\boldsymbol{l}}\|^2\|B_2\|^2\Big)\|\boldsymbol{\eta}\|^2\end{aligned}$$

By (B4) and the boundedness of L, the inequality above becomes

$$\dot{V} \le -\|\bar{\boldsymbol{\varepsilon}}\|^2 - \|\bar{\boldsymbol{\epsilon}}\|^2 + c_3\bar{\varepsilon}_1^2 + c_3\bar{\epsilon}_1^2 + c_3\|\boldsymbol{\eta}\|^2 \qquad \text{(B9)}$$

where $c_3 > 0$ is a proper constant depending on the unknown constant c.

By (B3), $\int_0^{T_f}\varepsilon_1^2(t)\mathrm{d}t < +\infty$ and $\int_0^{T_f}\epsilon_1^2(t)\mathrm{d}t < +\infty$, it is easy to show that $\int_0^{T_f}\bar{\varepsilon}_1^2(t)\mathrm{d}t < +\infty$ and $\int_0^{T_f}\bar{\epsilon}_1^2(t)\mathrm{d}t < +\infty$. Then, from (B9), it follows that, for $\forall t \in [0, T_f)$,

$$\begin{aligned}\|\bar{\boldsymbol{\varepsilon}}(t)\|^2 \le & \frac{1}{\lambda_{\min}(P_{\boldsymbol{l}})}\Big(V(\bar{\boldsymbol{\varepsilon}}(0),\bar{\boldsymbol{\epsilon}}(0)) + \\ & c_3\int_0^t\Big(\bar{\varepsilon}_1^2(\tau) + \bar{\epsilon}_1^2(\tau) + \|\boldsymbol{\eta}(\tau)\|^2\Big)\mathrm{d}\tau\Big) < +\infty \\ \|\bar{\boldsymbol{\epsilon}}(t)\|^2 \le & \frac{4}{L^*\lambda_{\min}(P_{\boldsymbol{l}})}\Big(V(\bar{\boldsymbol{\varepsilon}}(0),\bar{\boldsymbol{\epsilon}}(0)) + \\ & c_3\int_0^t\Big(\bar{\varepsilon}_1^2(\tau) + \bar{\epsilon}_1^2(\tau) + \|\boldsymbol{\eta}(\tau)\|^2\Big)\mathrm{d}\tau\Big) < +\infty\end{aligned}$$

namely, $\bar{\boldsymbol{\varepsilon}}$ and $\bar{\boldsymbol{\epsilon}}$ are bounded on $[0, T_f)$, so is $\boldsymbol{\lambda}$, and

$$\begin{aligned}\int_0^{T_f}\|\bar{\boldsymbol{\varepsilon}}(t)\|^2\mathrm{d}t \le & V(\bar{\boldsymbol{\varepsilon}}(0),\bar{\boldsymbol{\epsilon}}(0)) + c_3\int_0^{T_f}\Big(\bar{\varepsilon}_1^2(t) + \\ & \bar{\epsilon}_1^2(t) + \|\boldsymbol{\eta}(t)\|^2\Big)\mathrm{d}t < +\infty \\ \int_0^{T_f}\|\bar{\boldsymbol{\epsilon}}(t)\|^2\mathrm{d}t \le & V(\bar{\boldsymbol{\varepsilon}}(0),\bar{\boldsymbol{\epsilon}}(0)) + c_3\int_0^{T_f}\Big(\bar{\varepsilon}_1^2(t) + \\ & \bar{\epsilon}_1^2(t) + \|\boldsymbol{\eta}(t)\|^2\Big)\mathrm{d}t < +\infty\end{aligned}$$

In view of (B3) and (8), the conclusion can be obtained that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\epsilon}$ are bounded on $[0, T_f)$. □

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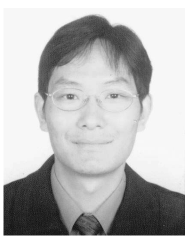
SHANG Fang Ph. D. candidate at the School of Control Science and Engineering, Shandong University. Her research interest covers nonlinear output feedback control, and adaptive control theory.
E-mail: shangfang929@163.com



LIU Yun-Gang Professor at the School of Control Science and Engineering, Shandong University. His research interest covers stochastic control, nonlinear system analysis, and adaptive control. Corresponding author of this paper.
E-mail: lygfr@sdu.edu.cn