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Comparison of Different Toll Policies in the Dynamic Second-best Optimal Toll Design Problem: Case study on a Three-link network

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In this paper, the dynamic optimal toll design problem is considered as a one leader-many followers hierarchical non-cooperative game. On a given network the road authority as the leader tolls some links in order to reach its objective, while travelers as followers minimize their perceived travel costs. So far toll has always been considered either as constant or as time-varying. Inspired by the San Diego's Interstate 15 congestion pricing project, in which heuristics with toll proportional to traffic flow are applied on a real two-link highway network, we consider toll as proportional to traffic flows in the network. On a three-link network we investigate various toll schemes and their influence on the outcome of the game for the road authority.
We show that the use of alternative toll schemes may improve system performance remarkably.

Keywords: Road pricing; dynamic optimal toll design problem; (inverse) Stackelberg games

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1 Introduction

Road pricing is one of the most efficient methods to avoid congestion problems on road networks (May, A.D. and Milne, D.S. (2000), Verhoef, E.T. (2002b)). With the use of appropriate tolls the road authority can influence travelers to behave so as to improve the performance of a given traffic system. This led to the introduction of the so-called optimal toll design problem (Joksimovič, D., Bliemer, M.C.J. and Bovy, P.H.L. (2004), Zhang, L. and Levinson, D. (2005), Larson, T. and Patriksson, M. (1997)).

Some researchers have attempted to solve the optimal toll design problem by means of a game-theoretic approach, in which the game is of a Stackelberg type (Chan, K.S. and Lam, W.H.K. (2005), Joksimovič, D., Bliemer, M.C.J. and Bovy, P.H.L. (2004), Verhoef, E.T. (2002a)).

We introduced (Staňková, K., Olsder, G.J. and Bliemer, M.C.J. (2006)) the static optimal toll design problem in which the link tolls are set in a different manner, as functions of link or route traffic flows in the network (We called the underlying game theoretic concept inverse Stackelberg game). We considered travelers driven by the Wardrop equilibrium and solved the problem analytically. With certain assumptions on network topology this approach led to better outcomes than the "conventional" toll. In Staňková, K., Bliemer, M.C.J. and Olsder, G.J. (2006) we extended the presented outcomes to the dynamic networks.

The approach proposed in this paper is inspired by San Diego's Interstate 15 congestion pricing project (Supernak, J., Golob, T., Kaschade, C., Kazimi, C., Schreffler, E. and Steffey, D. (2002)), in which the two-link highway network was studied. One of the links was tolled according to network occupancy and toll was established in a heuristic manner. We consider case studies with a three-link network, the problem being dynamic. The aim of the road authority is to minimize the total travel time of the system or to maximize the total toll revenue of the system by tolling a proper subset of the links, whereas each traveler decides which link to use so as to minimize his or her travel costs. As a reference case a situation with toll being constant or time-varying is considered. The outcome of this game is compared with outcomes of games in which the road authority uses alternative toll policies. As we will see, the traffic-flow dependent toll brings a better outcome for the road authority.

This paper is organized as follows: In Section 2 the dynamic optimal toll design problem is defined. In Section 3 we analytically solve case studies on a 3-link network. In the considered toll spaces the best possible tolls for the road authority are found and the results of these games are compared with the results of a traditional Stackelberg game with constant or time-varying toll. The results obtained and possibilities for future research are discussed in Section 4.

2 Dynamic Optimal Toll Design Problem

In this section the dynamic optimal toll design problem will be formulated as a leader-followers game. The dynamic optimal toll design problem was studied as a Stackelberg game in, e.g., Joksimovič, D., Bliemer, M.C.J. and Bovy, P.H.L. (2004), while its static version was studied in, e.g., Verhoef, E.T. (2002b). The static variant of the optimal toll design problem was introduced in Staňková, K., Olsder, G.J. and Bliemer, M.C.J. (2006).

2.1 Preliminaries

Let $\mathsf{K} = \{1,2,\ldots,|\mathsf{K}|\}$ ($|\mathsf{K}| \in \mathsf{N}$) be a time interval index set, i.e., $k \in \mathsf{K}$ identifies the k-th time interval, let Δ [h] be the time interval size. The time intervals are supposed to be equal and set to one hour. Let $\mathsf{G} = (\mathsf{N}, \mathsf{A})$ be a strongly connected road network with a finite nonempty node set N and a finite nonempty set $\mathsf{A} = \{l_1,\ldots,l_{|\mathsf{A}|}\}$ ($|\mathsf{A}| \in \mathsf{N}$) of directed links (arcs). Let $\mathsf{T} \subseteq \mathsf{A}$ be a set of tollable links. Let $\mathsf{OD} \subseteq \mathsf{N} \times \mathsf{N}$ be a set of origin-destination pairs. We will denote the nonempty set of simple routes from origin o to destination d by $\mathsf{R}^{(o,d)}$, and the set of all simple routes in the network by R. Let $D^{(o,d),(k)}$ [veh/h] be the average departure rate of travelers departing during k-th time interval from o to d. For the sake of simplicity, $D^{(o,d),(k)}$ is assumed to be inelastic and given. The link flow rate of travelers entering link $l_j \in \mathsf{A}$ during the k-th time interval will be denoted by $q_{l_j}^{(k)}$ [veh/h], the route flow of travelers departing during k-th time interval along route $r_i \in \mathsf{R}$ will be denoted by $f_{r_i}^{(k)}$ [veh/h]. In this paper the link travel time on link l_j for travelers entering link l_j during k-th time interval will be denoted by $\tau_{l_j}^{(k)}$ and will be defined as

$$\tau_{l_j}^{(k)} = \beta_{l_j} x_{l_j}^{(k)} + \gamma_{l_j}. \qquad (1)$$

Here β_{l_j} and γ_{l_j} are positive constants and $x_{l_j}^{(k)}$ [veh] is the number of travelers on link l_j (the link traffic volume) at the beginning of k-th time interval, defined as the cumulative inflow minus the cumulative outflow, i.e.,

$$x_{l_j}^{(k)} = \sum_{\tau=1}^{k} q_{l_j}^{(\tau)} - \sum_{\zeta \in \mathsf{W}_j^k} q_{l_j}^{(\zeta)}, \qquad (2)$$

where $\mathsf{W}_j^{(k)} = \{w \mid w + \tau_{l_j}^{(w)} \leq k\}$. This means that the number of drivers on link l_j in the k-th time interval is computed as the number of all drivers which entered the network till the k-th time interval minus the number of drivers who left this link during before time k [h]. The drivers entering the link in time interval $[k-1,k)$ are assigned to the traffic volume for the k-th time interval.

Generally, the link travel time can be any function increasing with the link volume on the same link. Initially, the network is considered to be empty.

The feasibility and nonnegativity conditions on the route flow rates have to be satisfied:

$$\sum_{i=1}^{\left|\mathsf{R}^{(o,d)}\right|} f_{r_i}^{(k)} = D^{(o,d),(k)}, \quad (o,d) \in \mathsf{OD}, \quad k \in \mathsf{K}, \quad (3)$$

$$f_{r_i}^{(k)} \geq 0, \quad r_i \in \mathsf{R}^{(o,d)}, \quad (o,d) \in \mathsf{OD}, \quad k \in \mathsf{K}. \quad (4)$$

Let $\left\lfloor \delta_{r_i,l_j}^{(k),(k')} \right\rfloor_{i \in \{1,\ldots,|\mathsf{R}|\},\, j \in \{1,\ldots,|\mathsf{A}|\},\, k,k' \in \mathsf{K}}$ be a dynamic link-route incidence identifier for G with

$$\delta_{r_i,l_j}^{(k),(k')} = \begin{cases} 1, & \textit{if travelers entering } r_i \in \mathsf{R}^{(o,d)} \textit{ during } k-th \textit{ time interval} \\ & \quad \textit{enter } l_j \in \mathsf{A} \textit{ during } k'-th \textit{ time interval}; \\ 0, & \textit{otherwise}. \end{cases}$$

Note that $[\delta_{r_i,l_j}^{(k),(k')}]_{r_i\in\mathsf{R},l_j\in\mathsf{A},k,k'\in\mathsf{K}}$ depends on the link travel times. Link flow rate $q_{l_j}^{(k')}$ [veh/h] is defined through the route flows as follows:

$$q_{l_j}^{(k')} = \sum_{k\in\mathsf{K}}\sum_{i=1}^{|\mathsf{R}|}\delta_{r_i,l_j}^{(k),(k')} f_{r_i}^{(k)}, \quad l_j \in \mathsf{A}.$$

With each $l_j \in \{1,\ldots,|\mathsf{A}|\}$ we associate the link travel cost $c_{l_j}^{(k)}$ [€] for travelers entering l_j during the k -th time interval. This cost is defined as

$$c_{l_j}^{(k)} = \alpha\tau_{l_j}^{(k)} + \theta_{l_j}^{(k)},$$

where $\tau_{l_j}^{(k)}$ [h] is the link travel time on l_j, α [/h] is the travelers' value of time (VOT), and $\theta_{l_j}^{(k)}$ [€] is the link toll paid by travelers entering link l_j during k -th time interval. The route travel times and the route travel costs are supposed to be additive, i.e.,

$$\tau_{r_i}^{(k)} = \sum_{k'\in\mathsf{K}}\sum_{j=1}^{|\mathsf{A}|}\delta_{r_i,l_j}^{(k),(k')}\tau_{l_j}^{(k')}, \quad c_{r_i}^{(k)} = \sum_{k'\in\mathsf{K}}\sum_{j=1}^{|\mathsf{A}|}\delta_{r_i,l_j}^{(k),(k')}c_{l_j}^{(k')}.$$

Let $\boldsymbol{\theta}^{(k)}$ be the $|\mathsf{T}|-$vector of nonnegative tolls on all tollable links during the k -th time interval, i.e., $\boldsymbol{\theta}^{(k)} = \left(\theta_{l_1}^{(k)},\ldots,\theta_{l_{|\mathsf{A}|}}^{(k)}\right)$, and let $\boldsymbol{\Theta}$ be a matrix of all tolls for all time intervals, i.e.,

$$\boldsymbol{\Theta} = \begin{pmatrix} \boldsymbol{\theta}^{(1)}, \\ \vdots \\ \boldsymbol{\theta}^{(|\mathsf{K}|)} \end{pmatrix} = \begin{pmatrix} \theta_{l_1}^{(1)}, & \ldots & \theta_{l_{|\mathsf{A}|}}^{(1)} \\ \vdots & \ddots & \vdots \\ \theta_{l_1}^{(|\mathsf{K}|)}, & \ldots & \theta_{l_{|\mathsf{A}|}}^{(|\mathsf{K}|)} \end{pmatrix}. \qquad (5)$$

The travelers are driven by the *dynamic route choice equilibrium assignment model* (Bliemer, M.C.J. (2001)), which is based on the assumptions that all road users have complete and accurate information about the traffic conditions, and that they choose the shortest routes available. In an equilibrium state, for each origin-destination pair and for each departure time interval, the actual route costs on all used routes are equal.

Remark 2.1 So far we did not mention, for the sake of simplicity, independent variables in the description of traffic flows, times, volumes or costs. To stress the dependent variables, in the following we will add these variables to the descriptions of these functions. For example, we will use the notation $q_{l_j}^{(k)}(\xi)$ if the link flow rate function is in the k -th time interval dependent on ξ.

2.2 *The Dynamic Optimal Toll Design Problem*

from a Game-theoretic Viewpoint

Let $\mathbf{\Theta}(\cdot)$ denote the toll matrix (5), where the toll on each link and for each time interval is defined as a function of link flows in the network. Here $\theta_{l_j}^{(k)}(\cdot)$ belongs to the set Ω, which is defined as a set of all twice continuously differentiable mappings from $\left(q_{l_1}^{(k)},\ldots,q_{l_{|\mathsf{A}|}}^{(k)}\right)$ into R_+^0 for each k.

Two problems for the leader will be dealt with:

- Minimize the total travel time of the system by tolling tollable links, which can be symbolically written as

$$\mathbf{\Theta}^*(\cdot) = \arg \min_{\mathbf{\Theta}(\cdot)\in\Omega} \sum_{k\in\mathsf{K}} \sum_{i=1}^{|\mathsf{R}|} \tau_{r_i}^{(k)}(\cdot)\, f_{r_i}^{(k)}(\cdot). \qquad (6)$$

- Maximize the total toll revenue of the system by tolling tollable links, which can be symbolically written as

$$\mathbf{\Theta}^*(\cdot) = \arg \max_{\mathbf{\Theta}(\cdot)\in\Omega} \sum_{k\in\mathsf{K}} \sum_{j=1}^{|\mathsf{A}|} q_{l_j}^{(k)}(\cdot)\theta_{l_j}^{(k)}(\cdot). \qquad (7)$$

In both problems the drivers minimize their dynamic travel costs according to the dynamic route choice model and the traffic dynamics is defined in Section 2.1.

Followers $\mathsf{F}_1,\ldots\mathsf{F}_m$ are the drivers on the road network. The decision variables of the drivers are their route choices, i.e., $u_{\mathsf{F}_i}^{(k)} \in \mathsf{R}^{(o,d)}$ if F_i travels from o to d starting during k-th time interval. The decisions made by all the travelers determine the link volumes and flow rates in the network. In equilibrium state, the dynamic route choice equilibrium (Bliemer, M.C.J. (2001)) takes place.

In this paper we will attempt to answer the following question: ``How will the outcome of the game change with traffic-flow dependent tolls?'' As a reference case we take a situation with traffic-flow invariant, but possibly time-varying tolls. In Staňková, K., Bliemer, M.C.J. and Olsder, G.J. (2006) we showed that for a two-link static network the outcome can be remarkably improved by such a toll choice. In the following section we will answer this question for the dynamic case with a three-link network.

Remark 2.2 It can be shown that the problems P1 and P2 are NP-hard (Staňková, K. (2009)).

3 Case Studies

In this section problems P1 and P2 introduced in Section 2.2, played on the network depicted in Figure 1, will be dealt with.

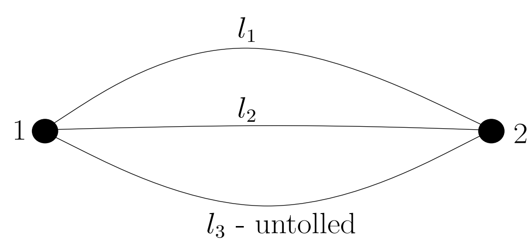

Figure 1. One origin-destination pair network with 3 links.

Here $K = \{1,\ldots,7\}$. Other initial parameters are given as follows: $d^{(1)} = 2000$ [veh/h], $d^{(2)} = 2000$ [veh/h], $d^{(3)} = 3000$ [veh/h], $d^{(4)} = 3000$ [veh/h], $d^{(5)} = 2500$ [veh/h], $d^{(6)} = 2000$ [veh/h], $d^{(7)} = 2000$ [veh/h], $\alpha = 8$ [/h], $\delta_1 = \frac{1}{5}$, $\delta_2 = \frac{1}{4}$, $\delta_3 = \frac{1}{3}$, $\beta_1 = \frac{1}{3000}$, $\beta_2 = \frac{1}{2000}$, $\beta_3 = \frac{1}{2500}$. The outcomes of the games will be found analytically. For each of the two problems, a few games, differing in chosen toll strategies, will be solved and compared.

3.1 *Total travel time minimization*

Let us consider problem P1 with objective to minimize the total travel time of the network. In the following four games we will consider different toll variants. We will compare a traffic-flow invariant and a traffic-flow dependent tolls. We will restrict ourselves to toll functions having the same number of unknown parameters as the corresponding standard toll cases. The aim is to find a toll strategy which does not increase complexity of the problem[4] and which will still provide a better outcome.

3.1.1 Game 1

Let only link l_1 be tolled. Two problems will be compared:

- The problem of total travel time minimization with uniform toll, i.e., $\theta_{l_1}^{(k)} \overset{\text{def}}{=} \theta_{l_1} \in \mathrm{R}_+^0$.
- The problem of total travel time minimization with toll defined as a ξ_1-multiple of an actual link traffic flow on link l_1, i.e., $\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right) \overset{\text{def}}{=} \xi_1 q_{l_1}^{(k)}$, $\xi_1 \in \mathrm{R}_+^0$.

The optimal toll for the first problem is $\frac{52}{135} \approx 0.39$ [€] and yields a total travel time of 9590.79 [h]. A slightly better outcome, 9583.12 [h], can be reached in the second game with an optimal value of ξ_1 equal to $\frac{3809}{10455525} \approx 0.36 \cdot 10^{-3}$.

3.1.2 Game 2

Let both links l_1 and l_2 be tolled. Two problems will be compared:

- A problem of total travel time minimization, where toll is uniform, i.e., $\theta_{l_j}^{(k)} \overset{\text{def}}{=} \theta_{l_j} \in \mathrm{R}_+^0$, $j \in \{1,2\}$.
- A problem of total travel time minimization with toll on link l_j ($j \in \{1,2\}$) defined as a ξ_j-multiple of actual link traffic flow on link l_j, i.e., $\theta_{l_j}^{(k)}\left(q_{l_j}^{(k)}\right) \overset{\text{def}}{=} \xi_1 \cdot q_{l_j}^{(k)}$, $\xi_j \in \mathrm{R}_+^0$, $j \in \{1,2\}$.

[4] This may be important for possible real-time applications.

For the first problem the optimal tolls on links l_1 and l_2 are $\frac{8}{15}$ [€] and $\frac{1}{3}$ [€], respectively, and yield a total travel time of 9590.79 [h] (the same outcome as in the previous case). The optimal values of ξ_1 and ξ_2 for the second problem are $0.50 \cdot 10^{-3}$ and $0.51 \cdot 10^{-3}$, respectively, and yield the outcome 9578.36 [h].

3.1.3 Game 3

Let only link l_1 be tolled. Two problems to be compared are:

- Find $\theta_{l_1}^{(k)}$ minimizing the total travel time of the system, where

$$\theta_{l_1}^{(k)} \stackrel{\text{def}}{=} \begin{cases} \theta_{l_1}, & k \in \{1,2,6,7\}, \\ \tilde{\theta}_{l_1}, & k \in \{3,4,5\}. \end{cases}$$

- Find $\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right)$ minimizing the total travel time of the system, where

$$\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right) \stackrel{\text{def}}{=} \begin{cases} \xi_1 q_{l_1}^{(k)}, & k \in \{1,2,6,7\}, \\ \tilde{\xi}_1 q_{l_1}^{(k)}, & k \in \{3,4,5\}. \end{cases}$$

The optimal values of θ_{l_1} and $\tilde{\theta}_{l_1}$ in the first game are $\frac{52}{135} \approx 0.39$ [€] and $\frac{52}{135} \approx 0.39$ [€], respectively, and yield the total travel time 9590.79 [h]. The optimal values of ξ_1 and $\tilde{\xi}_1$ are $\frac{13}{29925} \approx 0.43 \cdot 10^{-3}$ and $\frac{1}{225} \approx 0.44 \cdot 10^{-2}$, respectively, and yield a total travel time of 9582.68 [h].

3.1.4 Game 4

Let links l_1 and l_2 be tolled. Two problems to be solved are:

- Find $\theta_{l_1}^{(k)}$, $\theta_{l_2}^{(k)}$, minimizing the total travel time of the system, where

$$\theta_{l_j}^{(k)} \stackrel{\text{def}}{=} \begin{cases} \theta_{l_j} \in \mathrm{R}_+^{(0)}, & k \in \{1,2,6,7\}, \quad j \in \{1,2\}, \\ \tilde{\theta}_{l_j} \in \mathrm{R}_+^{(0)}, & k \in \{3,4,5\}, \quad j \in \{1,2\}. \end{cases}$$

- Find $\theta_{l_1}^{(k)}$, $\theta_{l_2}^{(k)}$, minimizing the total travel time of the system, where

$$\theta_{l_j}^{(k)}\left(q_{l_j}^{(k)}\right) \stackrel{\text{def}}{=} \begin{cases} \xi_j q_{l_j}^{(k)}, & k \in \{1,2,6,7\}, \quad j \in \{1,2\} \\ \tilde{\xi}_j q_{l_j}^{(k)} & k \in \{3,4,5\}, \quad j \in \{1,2\}. \end{cases}$$

The optimal values of θ_{l_1}, θ_{l_2}, $\tilde{\theta}_{l_1}$, and $\tilde{\theta}_{l_2}$ for the first problem are $\frac{8}{15}$ [€], $\frac{8}{15}$ [€], $\frac{1}{3}$ [€], and $\frac{1}{3}$ [€], respectively, and yield a total travel time of 9649.51 [h]. The optimal values of ξ_1, ξ_2, $\tilde{\xi}_1$, and $\tilde{\xi}_2$ for the second problem are $0.77 \cdot 10^{-2}$, $1.29 \cdot 10^{-2}$, $0.78 \cdot 10^{-2}$, and $1.26 \cdot 10^{-2}$, respectively, and yield a total travel time of 9577.38 [h].

3.1.5 General outcome

Minimization of the total travel time function with respect to the traffic flows yields the link traffic flows and the link travel times as depicted in Table 1. If these traffic flows and travel times are the travelers' response to the tolls, minimal total travel time $\frac{1034347}{108} \approx 9577.29$ [h] will be obtained. This means that the second strategy from Game 4 yields a total travel time close to the optimal outcome. To reach the (first-best) optimal outcome 9577.29 [h] more parameters in toll functions should be included. In Table 2 you can see the optimal linear toll strategy ($\theta_{l_j}^{(k)} \overset{\text{def}}{=} a_j^{(k)} q_{l_j}^{(k)} + b_j^{(k)}$) and the optimal standard Stackelberg strategy, when minimizing the total travel time of the system. Since for linear toll strategy parameters $a_j^{(k)}$ are free (and therefore the solution of the game with linear tolls is nonunique), it can be seen that 7 parameters in toll function are needed to obtain the optimal outcome. Obviously, with setting $a_j^{(k)}$ to 0 the optimal standard Stackelberg strategy will be reached. Therefore, with enough toll parameters the outcomes of the two strategies would be the same. However, in practical applications we cannot change toll on each link and for each time period.

Table 1. The optimal link traffic flows [veh/h] and link travel times [h]-total travel time minimization

k	$q_{l_1}^{(k)}$	$q_{l_2}^{(k)}$	$q_{l_3}^{(k)}$
1	$\frac{2660}{3}$	$\frac{4870}{9}$	$\frac{5150}{9}$
2	$\frac{2660}{3}$	$\frac{4870}{9}$	$\frac{5150}{9}$
3	$\frac{3860}{3}$	$\frac{7270}{9}$	$\frac{8150}{9}$
4	$\frac{3860}{3}$	$\frac{7270}{9}$	$\frac{8150}{9}$
5	$\frac{3260}{3}$	$\frac{6070}{9}$	$\frac{6650}{9}$
6	$\frac{2660}{3}$	$\frac{4870}{9}$	$\frac{5150}{9}$
7	$\frac{2660}{3}$	$\frac{4870}{9}$	$\frac{5150}{9}$

Continued Table 1.

k	$\tau_{l_1}^{(k)}$	$\tau_{l_2}^{(k)}$	$\tau_{l_3}^{(k)}$
1	$\frac{223}{450}$	$\frac{937}{1800}$	$\frac{253}{450}$
2	$\frac{223}{450}$	$\frac{937}{1800}$	$\frac{253}{450}$
3	$\frac{283}{450}$	$\frac{1177}{1800}$	$\frac{313}{450}$
4	$\frac{283}{450}$	$\frac{1177}{1800}$	$\frac{313}{450}$
5	$\frac{253}{450}$	$\frac{1057}{1800}$	$\frac{283}{450}$
6	$\frac{223}{450}$	$\frac{937}{1800}$	$\frac{253}{450}$
7	$\frac{223}{450}$	$\frac{937}{1800}$	$\frac{253}{450}$

Table 2. The optimal link toll function coefficients and optimal tolls (standard Stackelberg game) [€]: Total travel time minimization

k	$b_1^{(k)}$	$b_2^{(k)}$
1	$\frac{8}{15}-\frac{2660}{3}a_1^{(1)}$	$\frac{1}{3}-\frac{4870}{9}a_2^{(1)}$
2	$\frac{8}{15}-\frac{2660}{3}a_1^{(2)}$	$\frac{1}{3}-\frac{4870}{9}a_2^{(2)}$
3	$\frac{8}{15}-\frac{3860}{3}a_1^{(3)}$	$\frac{1}{3}-\frac{7270}{9}a_2^{(3)}$
4	$\frac{8}{15}-\frac{3860}{3}a_1^{(4)}$	$\frac{1}{3}-\frac{7270}{9}a_2^{(4)}$
5	$\frac{8}{15}-\frac{3260}{3}a_1^{(5)}$	$\frac{1}{3}-\frac{6070}{9}a_2^{(5)}$
6	$\frac{8}{15}-\frac{2660}{3}a_1^{(6)}$	$\frac{1}{3}-\frac{4870}{9}a_2^{(6)}$
7	$\frac{8}{15}-\frac{2660}{3}a_1^{(7)}$	$\frac{1}{3}-\frac{4870}{9}a_2^{(7)}$

Continued Table 2.

k	$\theta_{l_1}^{(k)}$	$\theta_{l_2}^{(k)}$
1	$\frac{8}{15}$	$\frac{1}{3}$
2	$\frac{8}{15}$	$\frac{1}{3}$
3	$\frac{8}{15}$	$\frac{1}{3}$
4	$\frac{8}{15}$	$\frac{1}{3}$
5	$\frac{8}{15}$	$\frac{1}{3}$
6	$\frac{8}{15}$	$\frac{1}{3}$
7	$\frac{8}{15}$	$\frac{1}{3}$

3.2 *Total toll revenue maximization*

Let us deal with problem P2 to maximize the total toll revenue of the network depicted in Figure 1. The total toll revenue function is the sum of all tolls that the drivers have to pay when traveling in the network during the observed time interval. To obtain the first-best tolls, one has to know the derivative of the objective function with respect to the traffic flows. But in this case, the objective function changes according to our choice of the toll function. Therefore, the first-best tolls are not explicitly known in advance.

Please note that leaving link l_3 untolled prevents the situation in which the tolls on links l_1 and l_2 would be set infinitely high in order to obtain higher profit. In such a case all the drivers would use link l_3, as it follows from the dynamic route choice equilibrium assignment model (Bliemer, M.C.J. (2001)).

3.2.1 Game 1

We will first assume that only link l_1 is tolled. Two problems will be compared:

- The problem of total toll revenue maximization, where the toll is uniform, i.e., $\theta_{l_1}^{(k)} \stackrel{\text{def}}{=} \theta_{l_1} \in \mathrm{R}_+^0$.

- The problem of total toll revenue maximization with toll defined as a ξ_1-multiple of actual link traffic flow on link l_1, i.e., $\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right) \stackrel{\text{def}}{=} \xi_1 \cdot q_{l_1}^{(k)},\ \ \xi_1 \in \mathrm{R}_+^0$.

The optimal toll for the first problem is $\frac{2344}{945} \approx 2.48$ [€] and yields a total toll revenue of 9690.19 [€]. The solution of the second problem is $\xi_j = \frac{1}{225}$ and yields a total toll revenue of 9931.46 [€].

3.2.2 Game 2

Let both links l_1 and l_2 be tolled. We will compare two problems:

- The problem of total toll revenue maximization, where the toll is uniform, i.e., $\theta_{l_j}^{(k)} \stackrel{def}{=} \theta_{l_j} \in \mathrm{R}_{+}^{0}, \quad j \in \{1,2\}$.
- The problem of total toll revenue maximization, with toll defined as a ξ_j-multiple of actual link traffic flow on link l_j, i.e., $\theta_{l_j}^{(k)}\left(q_{l_1}^{(k)}\right) \stackrel{def}{=} \xi_1 \cdot q_{l_j}^{(k)}, \quad \xi_j \in \mathrm{R}_{+}^{0}, \quad j \in \{1,2\}$.

The optimal tolls for the first problem are $\frac{452}{105} \approx 4.30$ [€] and $\frac{431}{105} \approx 4.10$ [€] for links l_1 and l_2, respectively, yielding the total toll revenue 26071.23 [€]. For the second problem the optimal values of ξ_1 and ξ_2 are $0.77 \cdot 10^{-2}$ and $1.27 \cdot 10^{-2}$, respectively. The resulting total toll revenue is 26794.74 [€].

3.2.3 Game 3

Let only link l_1 be tolled. We will compare two problems:

- Find $\theta_{l_1}^{(k)}$ maximizing the total toll revenue of the system, with

$$\theta_{l_1}^{(k)} \stackrel{def}{=} \begin{cases} \theta_{l_1}, & k \in \{1,2,6,7\} \\ \tilde{\theta}_{l_j}, & k \in \{3,4,5\}. \end{cases}$$

- Find $\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right)$ maximizing the total toll revenue of the system, with

$$\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right) \stackrel{def}{=} \begin{cases} \xi_1 q_{l_1}^{(k)}, & k \in \{1,2,6,7\} \\ \tilde{\xi}_1 q_{l_1}^{(k)}, & k \in \{3,4,5\}. \end{cases}$$

The optimal values of θ_{l_1} and $\tilde{\theta}_{l_1}$ for the first problem are $\frac{292}{135} \approx 2.16$ [€] and $\frac{392}{135} \approx 2.90$ [€], respectively, and yield the total toll revenue 9901.83 [€]. The optimal values of ξ_1 and $\tilde{\xi}_1$ for the second problem are $\frac{1}{225}$ and $\frac{1}{225}$, respectively, and yield a total toll revenue of 9931.46 [€].

3.2.4 Game 4

Let both link l_1 and l_2 be tolled. We will compare two problems:

- Find $\theta_{l_1}^{(k)}$ and $\theta_{l_2}^{(k)}$ maximizing total toll revenue of the system, with

$$\theta_{l_j}^{(k)} \stackrel{\text{def}}{=} \begin{cases} \theta_{l_j} \in \mathrm{R}_{+}^{(0)}, & k \in \{1,2,6,7\}, \quad j \in \{1,2\}, \\ \tilde{\theta}_{l_j} \in \mathrm{R}_{+}^{(0)}, & k \in \{3,4,5\}, \quad j \in \{1,2\}. \end{cases}$$

- Find $\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right)$, $\theta_{l_2}^{(k)}\left(q_{l_2}^{(k)}\right)$, maximizing the total toll revenue of the system, with

$$\theta_{l_j}^{(k)}\left(q_{l_j}^{(k)}\right) \stackrel{\text{def}}{=} \begin{cases} \xi_j q_{l_j}^{(k)}, & k \in \{1,2,6,7\}, \quad j \in \{1,2\}, \\ \tilde{\xi}_j q_{l_j}^{(k)} & k \in \{3,4,5\}, \quad j \in \{1,2\}. \end{cases}$$

The optimal values of θ_{l_1}, θ_{l_2}, $\tilde{\theta}_{l_1}$, and $\tilde{\theta}_{l_2}$ for the first problem are $\frac{56}{15} \approx 3.73$ [€], $\frac{73}{15} \approx 5.07$ [€], $\frac{53}{15} \approx 3.53$ [€], and $\frac{73}{15} \approx 4.87$ [€], respectively, and yield a total toll revenue of 26706.15 [€]. The optimal values of ξ_1, ξ_2, $\tilde{\xi}_1$, and $\tilde{\xi}_2$ for the second problem are $0.77 \cdot 10^{(-2)}$, $1.29 \cdot 10^{-2}$, $0.78 \cdot 10^{-2}$, and $1.26 \cdot 10^{-2}$, respectively, and yield a total toll revenue of 26795.01 [€].

Since the total toll revenue function will vary depending on the chosen structure of the toll functions, it is impossible to know the maximal total toll revenue before knowing the toll structure used. In the following game the optimal value of the total toll revenue with linear tolls will be computed, as this toll brought the best possible outcome when various polynomial toll functions were tested.

3.2.5 Game 5

We will consider the situation, where the road authority maximizes the total toll revenue of the system by setting tolls defined as follows:

$$\theta_{l_1}^{(k)}\left(q_{l_1}^{(k)}\right) \stackrel{\text{def}}{=} a_1^{(k)} q_{l_1}^{(k)} + b_1^{(k)}, \quad \theta_{l_2}^{(k)}\left(q_{l_1}^{(k)}\right) \stackrel{\text{def}}{=} a_2^{(k)} q_{l_2}^{(k)} + b_2^{(k)}. \tag{8}$$

Provided that coefficients $a_j^{(k)}$, $j = 1,2$, $k \in \{1,\ldots,7\}$ are negative, local maxima of the total toll revenue function with respect to the link traffic flows will be reached with flows depicted in Table 3. These traffic flows are dependent on $a_j^{(k)}$ and $b_j^{(k)}$ ($j = 1,2,3$, $k = 1,\ldots,7$).

Table 3. Optimal link flows: Total toll revenue maximization with linear toll functions

k	$q_{l_1}^{(k)}$	$q_{l_2}^{(k)}$	$q_{l_3}^{(k)}$
1	$-\frac{b_1^{(1)}}{2a_1^{(1)}}$	$-\frac{b_2^{(1)}}{2a_2^{(1)}}$	$\frac{4000a_2^{(1)}a_1^{(1)}+a_2^{(1)}b_1^{(1)}+a_1^{(1)}b_2^{(1)}}{2a_2^{(1)}a_1^{(1)}}$
2	$-\frac{b_1^{2}}{2a_1^{(2)}}$	$-\frac{b_2^{(2)}}{2a_2^{(2)}}$	$\frac{4000a_2^{(2)}a_1^{(2)}+a_2^{(2)}b_1^{(2)}+a_1^{(2)}b_2^{(2)}}{2a_2^{(2)}a_1^{(2)}}$
3	$-\frac{b_1^{(3)}}{2a_1^{(3)}}$	$-\frac{b_2^{(3)}}{2a_2^{(3)}}$	$\frac{6000a_2^{(3)}a_1^{(3)}+a_2^{(3)}b_1^{(3)}+a_1^{(3)}b_2^{(3)}}{2a_2^{(3)}a_1^{(3)}}$
4	$-\frac{b_1^{(4)}}{2a_1^{(4)}}$	$-\frac{b_2^{(4)}}{2a_2^{(4)}}$	$\frac{6000a_2^{(4)}a_1^{(4)}+a_2{}^{(4)}b_1^{(4)}+a_1^{(4)}b_2^{(4)}}{2a_2^{(4)}a_1^{(4)}}$
5	$-\frac{b_1^{(5)}}{2a_1^{(5)}}$	$-\frac{b_2^{(5)}}{2a_2{}^{5}}$	$\frac{5000a_2^{(5)}a_1^{(5)}+a_2^{(5)}b_1^{(5)}+a_1^{(5)}b_2^{(5)}}{2a_2^{(5)}a_1^{(5)}}$
6	$-\frac{b_1^{(6)}}{2a_1^{(6)}}$	$-\frac{b_2^{(6)}}{2a_2^{(6)}}$	$\frac{4000a_2^{(6)}a_1^{(6)}+a_2^{(6)}b_1^{(6)}+a_1^{(6)}b_2^{(6)}}{2a_2^{(6)}a_1^{(6)}}$
7	$-\frac{b_1^{(7)}}{2q_1^{(7)}}$	$-\frac{b_2^{(7)}}{2a_2^{(7)}}$	$\frac{4000a_2^{(7)}a_1^{(7)}+a_2^{(7)}b_1^{(7)}+a_1^{(7)}b_2^{(7)}}{2a_2^{(7)}a_1^{(7)}}$

Table 4. Coefficient of linear toll functions yielding local maximum of the total toll revenue

k	$b_1^{(k)}$	$b_2^{(k)}$
1	$\frac{80a_1^{(1)}\left(-73+17500a_2^{(1)}\right)}{3-550a_2^{(1)}-675a_1^{(1)}+93750a_1^{(1)}a_2^{(1)}}$	$\frac{40a_2^{(1)}\left(-247+99375a_1^{(1)}\right)}{3\left(3-550a_2^{(1)}-675a_1^{(1)}+93750a_1^{(1)}a_2^{(1)}\right)}$
2	$\frac{80a_1^{(2)}\left(-73+17500a_2^{(2)}\right)}{3-550a_2^{(2)}-675a_1^{(2)}+93750a_1^{(2)}a_2^{(2)}}$	$\frac{40a_2^{(2)}\left(-247+99375a_1^{(2)}\right)}{3\left(3-550a_2^{(2)}-675a_1^{(2)}+93750a_1^{(2)}a_2^{(2)}\right)}$
3	$\frac{80a_1^{(3)}\left(-103+25000a_2^{(3)}\right)}{3-675a_1^{(3)}-550a_2^{(3)}+93750a_2^{(3)}a_1^{(3)}}$	$\frac{40a_2^{(3)}\left(-367+144375a_1^{(3)}\right)}{3\left(3-675a_1^{(3)}-550a_2^{(3)}+93750a_2^{(3)}a_1^{(3)}\right)}$
4	$\frac{80a_1^{(4)}\left(-103+25000a_2^{(4)}\right)}{3-675a_1^{(4)}-550a_2^{(4)}+93750a_2^{(4)}a_1^{(4)}}$	$\frac{40a_2^{(4)}\left(-367+144375a_1^{(4)}\right)}{3\left(3-675a_1^{(4)}-550a_2^{(4)}+93750a_2^{(4)}a_1^{(4)}\right)}$
5	$\frac{160a_1^{(5)}\left(10625a_2^{(5)}-44\right)}{3-675a_1^{(5)}-550a_2^{(5)}+93750a_1^{(5)}a_2^{(5)}}$	$\frac{40a_2^{(5)}\left(-307+121875a_1^{(5)}\right)}{3\left(3-675a_1^{(5)}-550a_2^{(5)}+93750a_1^{(5)}a_2^{(5)}\right)}$
6	$\frac{80a_1^{(6)}\left(-73+17500a_2^{(6)}\right)}{3-675a_1^{(6)}-550a_2^{(6)}+93750a_1^{(6)}a_2^{(6)}}$	$\frac{40a_2^{(6)}\left(-247+99375a_1^{(6)}\right)}{3\left(3-675a_1^{(6)}-550a_2^{(6)}+93750a_1^{(6)}a_2^{(6)}\right)}$
7	$\frac{80a_1^{(7)}\left(-73+17500a_2^{(7)}\right)}{3-675a_1^{(7)}-550a_2^{(7)}+93750a_2^{(7)}a_1^{(7)}}$	$\frac{40a_2^{(7)}\left(-247+99375a_1^{(7)}\right)}{3\left(3-675a_1^{(7)}-550a_2^{(7)}+93750a_2^{(7)}a_1^{(7)}\right)}$

The road authority maximizing the total toll revenue, aiming to influence the travelers so that the traffic flows depicted in Table 3 will be obtained, has to take into account the dynamic deterministic user equilibrium conditions. If all three links are used, these conditions will yield coefficients $b_1^{(k)}$ and $b_2^{(k)}$ as depicted in Table 4.

Here $a_1^{(k)},\ a_2^{(k)},\ k=1,\ldots,7,$ are free. However, after substituting $b_1^{(k)},\ b_2^{(k)},\ k=1,\ldots,7,$ from Table 4 into the total toll revenue function and maximizing the obtained function with respect to $a_1^{(k)}$, $a_2^{(k)},\ k=1,\ldots,7,$ the values of the coefficients of the toll function can be obtained. These coefficients are depicted in Table 5 and yield the maximal toll revenue $2.6795\cdot 10^4$ [€].

Table 5. Optimal toll function coefficients and resulting tolls [€]: Total toll revenue maximization

k	$a_1^{(k)}$	$b_1^{(k)}$	$a_2^{(k)}$	$b_2^{(k)}$	$\theta_{l_1}^{(k)}$	$\theta_{l_2}^{(k)}$
1	-0.0077	7.4795	-0.0129	7.0757	3.7397	3.5379
2	-0.0077	7.4795	-0.0129	7.0757	3.7397	3.5379
3	-0.0078	10.6860	-0.0126	10.2769	5.3430	5.1384
4	-0.0078	10.6860	-0.0126	10.2768	5.3430	5.1384
5	-0.0077	9.0533	-0.0127	8.6608	4.5267	4.3304
6	-0.0077	7.4795	-0.0129	7.0757	3.7397	3.5379
7	-0.0077	7.4795	-0.0129	7.0757	3.7397	3.5379

Substituting the coefficients $a_1^{(k)},\ b_1^{(k)},\ a_2^{(k)},$ and $b_2^{(k)}$ from Table 5 into (8) will result in toll values $\theta_{l_1}^{(k)}$ and $\theta_{l_2}^{(k)}$ as depicted in the same table. We have also considered tolls defined as polynomial function (of the actual link flow) of degree higher than $1.$ This choice of toll did not lead to a better outcome, thus this is the best outcome that we could achieve. The second strategy for the road authority from Game 4 is the best strategy that we could find.

3.3 Discussion

The case studies suggest that the traffic-flow dependent toll is a very promising tool for improving the system performance for the second-best pricing problems. This follows also from the results obtained in the first-best pricing theory (following, e.g., Pigou, A.C. (1920) or Wardrop, J.G. (1952)). If all links could be tolled the toll minimizing a general objective function of the road authority would be traffic-flow dependent.[5]

More complicated toll functions may improve the system performance even further, if the first-best optimum cannot be reached by simple toll choice. However, the considered traffic model is very simple. In the next step of our research the departure time choice will be included and more realistic travel time functions will be considered. Although the authors believe that also in this case the traffic-flow dependent toll improves the system performance, this belief needs to be validated.

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[5] For example, the first-best toll minimizing the total travel costs is equal to the marginal external cost and is therefore traffic-flow dependent. The first-best toll maximizing the total toll revenue of the system is defined as the sum of the toll minimizing the total travel costs and of an origin-destination dependent surcharge based on marginal revenues. Also this toll is traffic-flow dependent.

Applicability of the traffic-flow dependent in real-time problems has to be discussed, too. Some discussion on this topic can be found in Staňková, K. (2009).

Our results suggest that this traffic-flow dependence applies also for the second-best pricing case.

4 Conclusions & Future Research

In this paper we dealt with the dynamic optimal toll design problem as a game of Stackelberg type, with travelers as followers driven by a dynamic route choice equilibrium and the road authority as leader minimizing the total travel time or maximizing the total toll revenue of the system. Alternative toll strategies, where the toll was set as a function of the traffic flow, were considered, and outcomes of the games with such strategies were compared to outcomes of the games with standard (traffic-flow invariant) toll strategies.

Moreover, on a benchmark network inspired by the San Diego experiment (Supernak, J., Golob, T., Kaschade, C., Kazimi, C., Schreffler, E. and Steffey, D. (2002)) we performed case studies with different toll strategies, and computed analytically their outcomes for the road authority. In this way we illustrated that the road authority choosing even a very simple alternative toll strategy may improve the system performance remarkably. It is quite clear that since the alternative tolls are defined as a generalization of standard toll strategies, with use of the same initial conditions these alternative tolls will never bring a worse outcome for the road authority than the standard tolls.

The use of the alternative tolls is one of the possible methods for avoiding congestion on the road networks. Further research on this topic may help to build more-efficient tolling systems in the future. Future research will focus on more complex toll schemes, too.

Also, additional research is needed to solve large problems of the same type. For these purposes a numerical model has been developed.

The problem considered in this paper was fully deterministic. Stochastic nature has to be included into the model, too, to provide more realistic view on the problem of congestion pricing.

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