JUMPS AND MONODROMY OF ABELIAN VARIETIES

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Abstract. We prove a strong form of the motivic monodromy conjecture for abelian varieties, by showing that the order of the unique pole of the motivic zeta function is equal to the size of the maximal Jordan block of the corresponding monodromy eigenvalue. Moreover, we give a Hodge-theoretic interpretation of the fundamental invariants appearing in the proof.

1. Introduction

Let K be a henselian discretely valued field with algebraically closed residue field k, and let A be a tamely ramified abelian K-variety of dimension g. In [9], we introduced the *motivic zeta function* $Z_A(T)$ of A. It is a formal power series over the localized Grothendieck ring of k-varieties $\mathcal{M}_k$, and it measures the behaviour of the Néron model of A under tame base change. We showed that $Z_A(\mathbb{L}^{-s})$ has a unique pole, which coincides with Chai's *base change conductor* $c(A)$ of A, and that the order of this pole equals $1 + t_{\mathrm{pot}}(A)$, where $t_{\mathrm{pot}}(A)$ denotes the *potential toric rank* of A. Moreover, we proved that for every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the value $\exp(2\pi c(A)i)$ is an eigenvalue of the tame monodromy transformation on the ℓ-adic cohomology of A in degree g. The main ingredient of the proof is Edixhoven's filtration on the special fiber of the Néron model of A [8].

As we've explained in [9], this result is a global version of Denef and Loeser's motivic monodromy conjecture for hypersurface singularities in characteristic zero. Denef and Loeser's conjecture relates the poles of the motivic zeta function of the singularity to monodromy eigenvalues on the nearby cohomology. It is a motivic generalization of a conjecture of Igusa's for the p-adic zeta function, which relates certain arithmetic properties of polynomials f in $\mathbb{Z}[x_1, \ldots, x_n]$ (namely, the asymptotic behaviour of the number of solutions of the congruence $f \equiv 0$ modulo powers of a prime p) to the structure of the singularities of the complex hypersurface H_f defined by f. The conjectures of Igusa and Denef-Loeser have been solved, for instance, in the case $n = 2$ [11][17], but the general case remains wide open. We refer to [16] for a survey.

There also exists a stronger form of these conjectures, which says that the poles of the respective zeta functions are roots of the Bernstein polynomial $b_f(s)$ of f, and that the order of the pole is at most the multiplicity of the corresponding root of $b_f(s)$. It is well-known that, for every root α of $b_f(s)$, the value $\alpha' = \exp(2\pi i\alpha)$ is a monodromy eigenvalue on the nearby cohomology $R\psi_f(\mathbb{C})_x$ of f at some point x of $H_f(\mathbb{C})$ [10][12]. Moreover, if H_f has an isolated singularity at x, then the

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multiplicity m_α of α as a root of the local Bernstein polynomial $b_{f,x}(s)$ of f at x is closely related to the maximal size $m_{\alpha'}$ of the Jordan blocks with eigenvalue α' of the monodromy transformation on $R^{n-1}\psi_f(\mathbb{C})_x$. In particular, $m_\alpha \leq m_{\alpha'}$ if $\alpha \notin \mathbb{Z}$ [12, 7.1].

The aim of the present paper is twofold. First, we prove a strong form of the motivic monodromy conjecture for abelian varieties. There is no good notion of Bernstein polynomial in our setting, but we can look at the size of the Jordan blocks. We show that the order $1 + t_{\mathrm{pot}}(A)$ of the unique pole $s = c(A)$ of the motivic zeta function $Z_A(\mathbb{L}^{-s})$ is equal to the size of the maximal Jordan block of the corresponding monodromy eigenvalue on the degree g cohomology of A (Theorem 5.3). Next, we use the theory of Néron models of variations of Hodge structures to give a Hodge-theoretic interpretation of the jumps in Edixhoven's filtration. This is done in Theorems 6.2 and 6.3. In [9, 2.7], we speculated on a generalization of the monodromy conjecture to Calabi-Yau varieties over $\mathbb{C}((t))$ (i.e., smooth, proper, geometrically connected $\mathbb{C}((t))$-varieties with trivial canonical sheaf); we hope that the translation of Edixhoven's invariants to Hodge theory will help to extend the proof of the monodromy conjecture to that setting.

In order to obtain these results, we divide Edixhoven's jumps into three types: toric, abelian, and dual abelian. The properties of these types are discussed in Section 3, and they are related to the monodromy transformation on the Tate module of A in Section 4. Toric jumps correspond to monodromy eigenvalues with Jordan block of size two, and the abelian and dual abelian jumps to monodromy eigenvalues with Jordan block of size one (Theorem 4.3).

2. Preliminaries and notation

We denote by R a Henselian discrete valuation ring, with quotient field K and algebraically closed residue field k. We denote by K^a an algebraic closure of K, by K^s the separable closure of K in K^a, and by K^t the tame closure of K in K^s. We fix a topological generator σ of the tame monodromy group $G(K^t/K)$. We denote by p the characteristic exponent of k, and by $\mathbb{N}'$ the set of integers $d > 0$ prime to p. We denote by

$$(\cdot)_s : (R-\text{Schemes}) \to (k-\text{Schemes}) : \mathcal{X} \mapsto \mathcal{X}_s = \mathcal{X} \times_R k$$

the special fiber functor.

For every abelian variety B over a field F, we denote its dual abelian variety by $B^\vee$. For every abelian K-variety A with Néron model $\mathcal{A}$, we denote by $t(A)$, $u(A)$ and $a(A)$ the reductive, resp. unipotent, resp. abelian rank of $\mathcal{A}_s^o$. We call these values the *toric*, resp. *unipotent*, resp. *abelian* rank of A.

By Grothendieck's semi-stable reduction theorem, there exists a finite extension K' of K in K^s such that $A \times_K K'$ has semi-abelian reduction [2, IX.3.6] (i.e., the identity component of the special fiber of its Néron model is a semi-abelian k-variety). The value $t_{\mathrm{pot}}(A) = t(A \times_K K')$ is called the *potential toric rank* of A, and the value $a_{\mathrm{pot}}(A) = a(A \times_K K')$ the *potential abelian rank*. It follows from [2, IX.3.9] that these values are independent of the choice of K'. We say that A is *tamely ramified* if we can take for K' a *tame* finite extension of K (this means that the degree $[K' : K]$ is prime to p) .

For every scheme S, every S-group scheme G and every integer $n > 0$, we denote by $n_G : G \to G$ the multiplication by n, and by ${}_nG$ its kernel.

If $\mathcal{S}$ is a set, and $g : \mathcal{S} \to \mathbb{R}$ a function with finite support, we set

$$\|g\| = \sum_{s \in \mathcal{S}} g(s).$$

We denote the support of g by $\mathrm{Supp}(g)$.

Definition 2.1. *For every function*

$$f : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$$

we define its reflection

$$f^* : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$$

by

$$f^*(x) = f(-x).$$

We call f complete *if for every $x \in \mathbb{Q}/\mathbb{Z}$, the value $f(x)$ only depends on the order of x in the group $\mathbb{Q}/\mathbb{Z}$. We say that f is* semi-complete *if $f + f^*$ is complete.*

Consider a function

$$f : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$$

and assume that there exists an element e of $\mathbb{Z}_{>0}$ such that $\mathrm{Supp}(f)$ is contained in $((1/e)\mathbb{Z})/\mathbb{Z}$. Let F be any algebraically closed field such that e is prime to the characteristic exponent p' of F. For each generator ζ of $\mu_e(F)$, we put

$$P_{f,\zeta}(t) = \prod_{i \in ((1/e)\mathbb{Z})/\mathbb{Z}} (t - \zeta^{i \cdot e})^{f(i)}$$

in $F[t]$. For each integer $d > 0$, we denote by $\Phi_d(t)$ the cyclotomic polynomial whose roots are the primitive d-th roots of unity. We say that $\Phi_d(t)$ is F-tame if d is prime to p'.

Lemma 2.2. *The function f is complete iff for some generator ζ of $\mu_e(F)$, the polynomial $P_{f,\zeta}(t)$ is the image of a product $Q_f(t)$ of F-tame cyclotomic polynomials under the unique ring morphism*

$$\rho : \mathbb{Z}[t] \to F[t]$$

mapping t to t. If f is complete, then $P_{f,\zeta}(t)$ is independent of ζ and e, and $Q_f(t)$ is unique. In that case, if we choose a primitive e-th root of unity ξ in an algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$, then

$$Q_f(t) = \prod_{i \in ((1/e)\mathbb{Z})/\mathbb{Z}} (t - \xi^{i \cdot e})^{f(i)}.$$

Proof. First, assume that f is complete, and put

$$Q_f(t) = \prod_{i \in ((1/e)\mathbb{Z})/\mathbb{Z}} (t - \xi^{i \cdot e})^{f(i)}$$

for some primitive e-th root of unity ξ in $\mathbb{Q}^a$. Then $Q_f(t)$ is a product of F-tame cyclotomic polynomials, because f is complete and e is prime to p'. There is a unique ring morphism

$$\widetilde{\rho} : \mathbb{Z}[\xi][t] \to F[t]$$

that maps ξ to ζ and t to t. We clearly have $\widetilde{\rho}(Q_f(t)) = P_{f,\zeta}(T)$. Since $Q_f(t)$ belongs to $\mathbb{Z}[t]$, it follows that $\rho(Q_f(t)) = P_{f,\zeta}(T)$ so that $P_{f,\zeta}(t)$ does not depend on ζ. Uniqueness of $Q_f(t)$ follows from [9, 5.10].

Conversely, if $P_{f,\zeta}(t)$ is the image under ρ of a product $Q(t)$ of F-tame cyclotomic polynomials, then it is easily seen that f is complete. $\square$

3. Toric and abelian multiplicity

3.1. **Galois action on Néron models.** Let A be a tamely ramified abelian K-variety of dimension g, and let K' be a finite extension of K in K^t such that $A' = A \times_K K'$ has semi-abelian reduction. We denote by R' the normalization of R in K', and by $\mathfrak{m}'$ the maximal ideal of R'. We put $d = [K' : K]$.

We denote by μ the Galois group $G(K'/K)$, and we let μ act on K' from the left. The action of $\zeta \in \mu$ on $\mathfrak{m}'/(\mathfrak{m}')^2$ is multiplication by $\iota(\zeta)$, for some $\iota(\zeta)$ in the group $\mu_d(k)$ of d-th roots of unity in k, and the map

$$\mu \to \mu_d(k) : \zeta \mapsto \iota(\zeta)$$

is an isomorphism.

We denote by $\mathcal{A}$ and $\mathcal{A}'$ the Néron models of A, resp. A'. By the universal property of the Néron model, there exists a unique morphism of R'-group schemes

$$h : \mathcal{A} \times_R R' \to \mathcal{A}'$$

that extends the canonical isomorphism between the generic fibers. It induces an injective morphism of free rank g R'-modules

$$\mathrm{Lie}(h) : \mathrm{Lie}(\mathcal{A} \times_R R') \to \mathrm{Lie}(\mathcal{A}').$$

Definition 3.1 (Chai [4])**.** *The* base change conductor $c(A)$ *of* A *is* $[K' : K]^{-1}$ *times the length of the cokernel of* $\mathrm{Lie}(h)$*.*

The definition does not require that A is tamely ramified. The base change conductor is a positive rational number, independent of the choice of K'. It vanishes iff A has semi-abelian reduction.

The right μ-action on A' extends uniquely to a right μ-action on $\mathcal{A}'$ such that the structural morphism

$$\mathcal{A}' \to \mathrm{Spec}\, R'$$

is μ-equivariant. We denote by

$$0 \to T \to (\mathcal{A}'_s)^o \to B \to 0 \tag{3.1}$$

the Chevalley decomposition of $(\mathcal{A}'_s)^o$, with T a k-torus and B an abelian k-variety. There exist unique right μ-actions on T, resp. B, such that (3.1) is μ-equivariant. The right μ-action on B induces a left μ-action on the dual abelian variety $B^\vee$.

Lemma 3.2. *The sequence*

$$0 \to T^\mu \to ((\mathcal{A}'_s)^o)^\mu \to B^\mu \to 0 \tag{3.2}$$

obtained from (3.1) by taking μ-invariants, is exact.

Taking identity components, we get a sequence

$$0 \to (T^\mu)^o \to ((\mathcal{A}'_s)^\mu)^o \to (B^\mu)^o \to 0 \tag{3.3}$$

which is exact at the left and at the right. The quotient

$$B' = ((\mathcal{A}'_s)^\mu)^o/(T^\mu)^o$$

is an abelian k-variety, and the natural morphism $f : B' \to (B^\mu)^o$ is a separable isogeny.

Moreover, if we denote by h the unique morphism

$$h : \mathcal{A} \times_R R' \to \mathcal{A}'$$

extending the natural isomorphism between the generic fibers, then the k-morphism $h_s : \mathcal{A}_s \to \mathcal{A}'_s$ factors through a morphism

$$g : \mathcal{A}_s \to (\mathcal{A}'_s)^{\mu}$$

The morphism g is smooth and surjective, and its kernel is a connected unipotent smooth algebraic k-group. The identity component $((\mathcal{A}'_s)^{\mu})^o$ is semi-abelian.

Proof. Left exactness of (3.2) is clear. To prove that (3.2) is exact, it suffices to show that $((\mathcal{A}'_s)^o)^{\mu} \to B^{\mu}$ is smooth and surjective. For any commutative k-group scheme Z endowed with a right μ-action, consider the morphism $N_Z : Z \to Z^{\mu}$ defined by

$$N_Z(S) : Z(S) \to Z^{\mu}(S) : s \mapsto \sum_{\zeta \in \mu} s * \zeta$$

for all k-schemes S. If we denote by ι_Z the tautological closed immersion $Z^{\mu} \to Z$, then $N_Z \circ \iota_Z$ equals $d_{Z^{\mu}}$ (=multiplication by d on Z^{μ}).

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{\alpha} & (\mathcal{A}'_s)^o & \xrightarrow{\beta} & B & \longrightarrow & 0 \\ & & \downarrow N_T & & \downarrow N_{(\mathcal{A}'_s)^o} & & \downarrow N_B & & \\ 0 & \longrightarrow & T^{\mu} & \longrightarrow & ((\mathcal{A}'_s)^o)^{\mu} & \xrightarrow{\beta^{\mu}} & B^{\mu} & & \end{array}$$

Since d_B is surjective [14, II.4, p. 42], the morphism N_B is surjective. By surjectivity of β, this implies that β^{μ} is surjective. But β is smooth since T is smooth over k [1, $\mathrm{VI_B}$.9.2], so that β^{μ} is smooth and surjective [8, 3.5].

Taking identity components in (3.2), we get a sequence

$$(T^{\mu})^o \xrightarrow{\alpha^o} ((\mathcal{A}'_s)^{\mu})^o \xrightarrow{\beta^o} (B^{\mu})^o$$

By [8, 3.5], all members of this sequence are smooth algebraic k-groups. Injectivity of α^o is obvious, and surjectivity of β^o follows from [1, $\mathrm{IV_B}$.3.11]. We put

$$B' = ((\mathcal{A}'_s)^{\mu})^o/(T^{\mu})^o$$

This is a connected smooth algebraic k-group, by [1, $\mathrm{VI_B}$.9.2]. The kernel of the natural morphism $f : B' \to (B^{\mu})^o$ is canonically isomorphic to

$$\ker(\beta^o)/(T^{\mu})^o$$

By smoothness of β, we know that $\ker(\beta^o)$ is smooth over k, so that $\ker(f)$ is smooth over k, by [1, $\mathrm{VI_B}$.9.2]. Surjectivity of β^o implies that f is a surjection between algebraic k-groups of the same dimension, so that $\ker(f)$ is finite and étale. Hence, B' is an abelian variety, and f a separable isogeny.

Since h is μ-equivariant, and μ acts trivially on the special fiber $\mathcal{A}_s$ of $\mathcal{A} \times_R R'$, the morphism h_s factors through a morphism $g : \mathcal{A}_s \to (\mathcal{A}'_s)^{\mu}$. By [8, 5.3], the morphism g is smooth and surjective, and its kernel is a connected unipotent smooth algebraic k-group. By [8, 3.5], $((\mathcal{A}'_s)^{\mu})^o$ is a connected smooth closed k-subgroup scheme of the semi-abelian k-group scheme $(\mathcal{A}'_s)^o$, so that $((\mathcal{A}'_s)^{\mu})^o$ is semi-abelian by [9, 5.2]. □

3.2. **Multiplicity functions.** Fix an element $e \in \mathbb{N}'$. For every finite dimensional k-vector space V with a right $\mu_e(k)$-action

$$* : V \times \mu_e(k) \to V : (v, \zeta) \mapsto v * \zeta$$

and for every integer i in $\{0, \dots, e-1\}$, we denote by $V[i]$ the maximal subspace of V such that

$$v * \zeta = \zeta^i \cdot v$$

for all $\zeta \in \mu_e(k)$ and all $v \in V[i]$. We define the *multiplicity function*

$$m_{V,\mu_e(k)} : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$$

by

$$\begin{cases} m_{V,\mu_e(k)}(i/e) = \dim(V[i]) & \text{for } i \in \{0, \dots, e-1\} \\ m_{V,\mu_e(k)}(x) = 0 & \text{if } x \notin ((1/e)\mathbb{Z})/\mathbb{Z} \end{cases}$$

Note that $m_{V,\mu_e(k)}$ determines the $k[\mu_e(k)]$-module V up to isomorphism, since the order of $\mu_e(k)$ is invertible in k.

In an analogous way, we define the multiplicity function $m_{\mu_e(k),W}$ for a finite dimensional k-vector space W with left $\mu_e(k)$-action. The inverse of the right $\mu_e(k)$-action on V is the left action

$$\mu_e(k) \times V \to V : (\zeta, v) \mapsto v * \zeta^{-1}.$$

Its multiplicity function $m_{\mu_e(k),V}$ is equal to the reflection $(m_{V,\mu_e(k)})^*$ of the multiplicity function $m_{V,\mu_e(k)}$.

Let A be a tamely ramified abelian K-variety. We adopt the notations of Section 3.1. In the set-up of (3.1), the group $\mu \cong \mu_d(k)$ acts on the k-vector spaces $\mathrm{Lie}(T)$, $\mathrm{Lie}(\mathcal{A}'_s)$ and $\mathrm{Lie}(B)$ from the right, and on $\mathrm{Lie}(B^\vee)$ from the left (via the dual action of μ on $B^\vee$). Hence, we can state the following definitions.

Definition 3.3. *We define the toric multiplicity function m_A^{tor} of A by*

$$m_A^{\mathrm{tor}} = m_{\mathrm{Lie}(T),\mu}.$$

We define the abelian multiplicity function m_A^{ab} of A by

$$m_A^{\mathrm{ab}} = m_{\mathrm{Lie}(B),\mu}.$$

We define the dual abelian multiplicity function $\breve{m}_A^{\mathrm{ab}}$ of A by

$$\breve{m}_A^{\mathrm{ab}} = m_{\mu,\mathrm{Lie}(B^\vee)}.$$

Finally, we define the multiplicity function m_A of A by

$$m_A = m_A^{\mathrm{tor}} + m_A^{\mathrm{ab}} = m_{\mathrm{Lie}(\mathcal{A}'_s),\mu}.$$

Using [2, IX.3.9], it is easily checked that these definitions only depend on A, and not on the choice of K'.

Proposition 3.4. *For every tamely ramified abelian K-variety A, we have*

$$\breve{m}_A^{\mathrm{ab}} = (m_{A^\vee}^{\mathrm{ab}})^*.$$

Proof. We adopt the notations of Section 3.1. We set $(A^\vee)' = A^\vee \times_K K'$ and we denote its Néron model by $(\mathcal{A}^\vee)'$. The identity component of $(\mathcal{A}^\vee)'_s$ is a semi-abelian k-variety [2, IX.2.2.7]. We denote by C its abelian part.

As explained in Section 3.1, the left Galois action of μ on K' induces a right action of μ on C. The canonical divisorial correspondence on $A \times_K A^\vee$ induces a divisorial correspondence on $B \times_k C$ that identifies C with the dual abelian variety of B [2, IX.5.4]. It suffices to show that the right μ-action on C is the inverse of the dual of the right μ-action on B. To this end, we take a closer look at the construction of the divisorial correspondence on $B\times_k C$. Here we need the language of biextensions [2]. We note that the following proof does not use the assumption that A is tamely ramified and that K' is a tame extension of K.

The canonical divisorial correspondence on $A\times_K A^\vee$ can be interpreted as a *Poincaré biextension* $\mathscr{P}$ of $(A,A^\vee)$ by $\mathbb{G}_{m,K}$ [2, VII.2.9.5], which is defined up to isomorphism. It induces a biextension $\mathscr{P}'$ of $(A',(A^\vee)')$ by $\mathbb{G}_{m,K'}$ by base change. By [2, VIII.7.1], the biextension $\mathscr{P}'$ extends uniquely to a biextension of $((\mathcal{A}')^o,((\mathcal{A}^\vee)')^o)$ by $\mathbb{G}_{m,R'}$, which restricts to a biextension $\mathscr{P}'_s$ of $((\mathcal{A}')^o_s,((\mathcal{A}^\vee)')^o_s)$ by $\mathbb{G}_{m,k}$. By [2, VIII.4.8], $\mathscr{P}'_s$ induces a biextension $\mathscr{Q}$ of (B,C) by $\mathbb{G}_{m,k}$ that is characterized (up to isomorphism) by the fact that its pullback to $((\mathcal{A}')^o_s,((\mathcal{A}^\vee)')^o_s)$ is isomorphic to $\mathscr{P}'_s$. The theorem in [2, IX.5.4] asserts that $\mathscr{Q}$ is a Poincaré biextension.

Since $\mathscr{P}'$ is obtained from the biextension $\mathscr{P}$ over K by base change to K', it follows easily from the construction that, for every element ζ of μ, the pullback of the biextension $\mathscr{Q}$ through the k-morphisms

$$\begin{aligned} r_\zeta &: B\to B \\ r_\zeta &: C\to C \end{aligned}$$

is isomorphic to $\mathscr{Q}$, where r_ζ stands for the right multiplication by ζ on B and C. Interpreting $\mathscr{Q}$ as an isomorphism

$$i : B\to C^\vee$$

in the way of [2, VIII.3.2.2], this means that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & C^\vee \\ {\scriptstyle r_\zeta}\downarrow & & \uparrow{\scriptstyle (r_\zeta)^\vee} \\ B & \xrightarrow{\ i\ } & C^\vee \end{array}$$

commutes, which is what we wanted to show. □

In the following proposition, we see how the multiplicity functions of a tamely ramified abelian K-variety A are related to Edixhoven's jumps and Chai's elementary divisors of A. These jumps and elementary divisors are rational numbers in $[0,1[$ that measure the behaviour of the Néron model of A under tame ramification of the base field K. For the definition of Edixhoven's jumps, we refer to [8, 5.4.5]. The terminology we use is the one from [9, 4.12]. For Chai's elementary divisors, we refer to [4, 2.4]. By definition, the base change conductor $c(A)$ of A is equal to the sum of the elementary divisors.

Proposition 3.5. *Let A be a tamely ramified abelian K-variety. The functions m_A, m_A^{tor}, m_A^{ab} and $\breve{m}_A^{\mathrm{ab}}$ are supported on*

$$((1/e)\mathbb{Z})/\mathbb{Z},$$

with e the degree of the minimal extension of K where A acquires semi-abelian reduction.

If we identify $[0,1[\cap\mathbb{Q}$ with $\mathbb{Q}/\mathbb{Z}$ via the bijection

$$[0,1[\cap\mathbb{Q}\to\mathbb{Q}/\mathbb{Z} : x\mapsto x \bmod \mathbb{Z}$$

then for every $x \in [0,1[\cap\mathbb{Q}$, the value $m_A(x)$ is equal to the multiplicity of x as a jump in Edixhoven's filtration for A. In particular, the support of m_A is the set of jumps in Edixhoven's filtration. The value $m_A(x)$ is also equal to the number of Chai's elementary divisors of A that are equal to x, and the base change conductor $c(A)$ of A is given by

$$c(A) = \sum_{[0,1[\cap\mathbb{Q}} (m_A(x)\cdot x).$$

Proof. See [8, 5.3 & 5.4.5] and [9, 4.8 & 4.13 & 4.18]. □

Proposition 3.6. *We have the following equalities:*

$$\begin{array}{rclcrcl} \|m_A\| &=& \dim(A), && m_A^{\mathrm{tor}}(0) &=& t(A), \\ \|m_A^{\mathrm{tor}}\| &=& t_{\mathrm{pot}}(A), && m_A^{\mathrm{ab}}(0) &=& a(A), \\ \|m_A^{\mathrm{ab}}\| &=& a_{\mathrm{pot}}(A), && \|\breve{m}_A^{\mathrm{ab}}\| &=& a_{\mathrm{pot}}(A). \end{array}$$

Moreover, we have

$$\sum_{x\in(\mathbb{Q}/\mathbb{Z})\setminus\{0\}} m_A(x) = u(A).$$

Proof. We adopt the notations of Section 3.1. It follows immediately from the definitions that

$$\begin{array}{rcrclcl} && \|m_A\| &=& \dim(\mathrm{Lie}(\mathcal{A}'_s)) &=& \dim(A), \\ && \|m_A^{\mathrm{tor}}\| &=& \dim(\mathrm{Lie}(T)) &=& t_{\mathrm{pot}}(A), \\ \|\breve{m}_A^{\mathrm{ab}}\| &=& \|m_A^{\mathrm{ab}}\| &=& \dim(\mathrm{Lie}(B)) &=& a_{\mathrm{pot}}(A). \end{array}$$

By Lemma 3.2, the abelian, resp., reductive rank of $\mathcal{A}_s^o$ is equal to the abelian, resp., reductive rank of the semi-abelian k-variety $((\mathcal{A}'_s)^\mu)^o$. In the notation of Lemma 3.2, the Chevalley decomposition of $((\mathcal{A}'_s)^\mu)^o$ is given by

$$0 \to (T^\mu)^o \to ((\mathcal{A}'_s)^\mu)^o \to B' \to 0$$

and there exists a natural separable isogeny $f : B' \to (B^\mu)^o$. By [8, 3.2], the natural morphisms

$$\begin{array}{c} \mathrm{Lie}(T^\mu) \to \mathrm{Lie}(T)^\mu = \mathrm{Lie}(T)[0] \\ \mathrm{Lie}(B^\mu) \to \mathrm{Lie}(B)^\mu = \mathrm{Lie}(B)[0] \end{array}$$

are isomorphisms. Since $\mathrm{Lie}(f)$ is also an isomorphism, we find

$$\begin{array}{rcl} m_A^{\mathrm{tor}}(0) &=& t(A), \\ m_A^{\mathrm{ab}}(0) &=& a(A). \end{array}$$

It follows that

$$\begin{array}{rcl} \displaystyle\sum_{x\in(\mathbb{Q}/\mathbb{Z})\setminus\{0\}} m_A(x) &=& \|m_A\| - m_A^{\mathrm{tor}}(0) - m_A^{\mathrm{ab}}(0) \\ &=& \dim(A) - t(A) - a(A) \\ &=& u(A). \end{array}$$

□

Lemma 3.7. *If A_1 and A_2 are tamely ramified abelian K-varieties, then*

$$\begin{aligned} m^{\mathrm{tor}}_{A_1\times_K A_2} &= m^{\mathrm{tor}}_{A_1} + m^{\mathrm{tor}}_{A_2}, \\ m^{\mathrm{ab}}_{A_1\times_K A_2} &= m^{\mathrm{ab}}_{A_1} + m^{\mathrm{ab}}_{A_2}, \\ \breve{m}^{\mathrm{ab}}_{A_1\times_K A_2} &= \breve{m}^{\mathrm{ab}}_{A_1} + \breve{m}^{\mathrm{ab}}_{A_2}. \end{aligned}$$

Proof. If we denote by $\mathcal{A}_1$ and $\mathcal{A}_2$ the Néron models of A_1, resp. A_2, then it follows immediately from the universal property of the Néron model that $\mathcal{A}_1 \times_R \mathcal{A}_2$ is a Néron model for $A_1 \times_K A_2$. Since the Chevalley decomposition of a connected smooth algebraic k-group commutes with finite fibered products over k and $\mathrm{Lie}(G_1 \times_k G_2)$ is canonically isomorphic to $\mathrm{Lie}(G_1) \oplus \mathrm{Lie}(G_2)$ for any pair of algebraic k-groups G_1, G_2, the result follows. □

Proposition 3.8. *Let A be a tamely ramified abelian K-variety. Let L be a finite tame extension of K of degree e, and put $A_L = A \times_K L$. Then for each $x \in \mathbb{Q}/\mathbb{Z}$, we have*

$$\begin{aligned} m^{\mathrm{tor}}_{A_L}(x) &= \sum_{y\in\mathbb{Q}/\mathbb{Z},\, e\cdot y = x} m^{\mathrm{tor}}_{A}(y) \\ m^{\mathrm{ab}}_{A_L}(x) &= \sum_{y\in\mathbb{Q}/\mathbb{Z},\, e\cdot y = x} m^{\mathrm{ab}}_{A}(y) \\ \breve{m}^{\mathrm{ab}}_{A_L}(x) &= \sum_{y\in\mathbb{Q}/\mathbb{Z},\, e\cdot y = x} \breve{m}^{\mathrm{ab}}_{A}(y) \end{aligned}$$

Proof. We adopt the notations of Section 3.1. Since the multiplicity functions do not depend on the choice of the field K' where A acquires semi-abelian reduction, we may assume that L is contained in K'. If ζ is a generator of μ, then the Galois group $G(K'/L)$ is generated by ζ^e. Now the result easily follows from the definition of the multiplicity functions. □

Proposition 3.9. *If $f : A_1 \to A_2$ is an isogeny of tamely ramified abelian K-varieties, and the degree $\deg(f)$ of f is prime to p, then*

$$m^{\mathrm{ab}}_{A_1} = m^{\mathrm{ab}}_{A_2} \quad \textit{and} \quad \breve{m}^{\mathrm{ab}}_{A_1} = \breve{m}^{\mathrm{ab}}_{A_2}.$$

Proof. We put $n = \deg(f)$. The kernel of f is a finite étale K-group scheme of rank n, so it is contained in ${}_n(A_1)$. Hence, there exists an isogeny $g : A_2 \to A_1$ such that $g \circ f = n_{A_1}$.

Let K' be a tame finite extension of K such that A_1 and A_2 acquire semi-abelian reduction over K', and denote by R' the normalization of R in K'. We denote the Néron model of $(A_i) \times_K K'$ by $\mathcal{A}'_i$, for $i = 1, 2$. The morphisms $f \times_K K'$ and $g \times_K K'$ extend uniquely to morphisms of R'-group schemes

$$\begin{aligned} f' &: \mathcal{A}'_1 \to \mathcal{A}'_2 \\ g' &: \mathcal{A}'_2 \to \mathcal{A}'_1. \end{aligned}$$

For $i = 1, 2$, we denote by B_i the abelian part of the semi-abelian k-variety $(\mathcal{A}'_i)^o_s$. By functoriality of the Chevalley decomposition, f'_s induces a morphism of k-group schemes $f'_B : B_1 \to B_2$. Likewise, g'_s induces a morphism of k-group schemes $g'_B : B_2 \to B_1$. Since $g' \circ f'$ is multiplication by n, the same holds for $g'_B \circ f'_B$. In particular, the degree of f'_B is prime to p. It follows that f'_B is a μ-equivariant separable isogeny, so that $\mathrm{Lie}(f'_B) : \mathrm{Lie}(B_1) \to \mathrm{Lie}(B_2)$ is a μ-equivariant isomorphism, and $m^{\mathrm{ab}}_{A_1} = m^{\mathrm{ab}}_{A_2}$.

By [14, p. 143], the dual morphism $(f'_B)^\vee$ is again an isogeny, and its kernel is the Cartier dual of the kernel of f'_B. In particular, f'_B and $(f'_B)^\vee$ have the same degree, so that $(f'_B)^\vee$ is separable. Since it is also equivariant for the left μ-action on $B^\vee$, we find that $\breve{m}^{\mathrm{ab}}_{A_1} = \breve{m}^{\mathrm{ab}}_{A_2}$. □

Remark 3.10. The same proof shows that m_A^{tor} is invariant under isogenies of degree prime to p. We'll see in Corollary 4.4 that, more generally, the functions m_A^{tor} and $m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}$ are invariant under *all* isogenies.

Corollary 3.11. *Let A be a tamely ramified abelian K-variety. If k has characteristic zero, or A is principally polarized, then*

$$m_A^{\mathrm{ab}} = m_{A^\vee}^{\mathrm{ab}}$$

and

$$\breve{m}_A^{\mathrm{ab}} = (m_A^{\mathrm{ab}})^*.$$

Proof. The first equality follows from Proposition 3.9. Together with Proposition 3.4, it implies the second equality. □

We will see in Theorem 6.3 that, when R is the ring of germs of holomorphic functions at the origin of $\mathbb{C}$, the equality

$$\breve{m}_A^{\mathrm{ab}} = (m_A^{\mathrm{ab}})^*$$

expresses that the monodromy eigenvalues on the $(-1,0)$-component of a certain limit mixed Hodge structure associated to A are the complex conjugates of the monodromy eigenvalues on the $(0,-1)$-component. Corollary 3.11 generalizes this Hodge symmetry.

Question 3.12. Is it true that

$$\breve{m}_A^{\mathrm{ab}} = (m_A^{\mathrm{ab}})^*$$

for every tamely ramified abelian K-variety A?

4. Jumps and monodromy

Proposition 4.1. *Let B be an abelian k-variety, and T an algebraic k-torus. Fix an element $e \in \mathbb{N}'$, and assume that $\mu_e(k)$ acts on B, resp. T from the right. We consider the dual left action of $\mu_e(k)$ on $B^\vee$. The functions $m_1 := m_{\mathrm{Lie}(T),\mu_e(k)}$ and*

$$m_2 := m_{\mathrm{Lie}(B),\mu_e(k)} + m_{\mu_e(k),\mathrm{Lie}(B^\vee)}$$

are complete.

Moreover, for each generator ζ of $\mu_e(k)$, the characteristic polynomial $P_1(t)$ of ζ on the ℓ-adic Tate module

$$\mathscr{V}_\ell T \cong \mathscr{T}_\ell T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is equal to $Q_{m_1}(t)$ (in the notation of Lemma 2.2). Likewise, the characteristic polynomial $P_2(t)$ of ζ on $\mathscr{V}_\ell B$ is equal to $Q_{m_2}(t)$

Proof. We denote by

$$\rho : \mathbb{Z}[t] \to k[t]$$

the unique ring morphism that maps t to t. It is well-known that the characteristic polynomials $P_1(t)$ and $P_2(t)$ belong to $\mathbb{Z}[t]$. For $P_1(t)$, this follows from the canonical isomorphism

$$\mathscr{V}_\ell T \cong \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Q}_\ell(1))$$

where $X(T)$ denotes the character group of T. For $P_2(t)$, it follows from [14, § 19, Thm.4].

Since e is invertible in k, $P_1(t)$ and $P_2(t)$ are products of k-tame cyclotomic polynomials. In the notation of Lemma 2.2, the characteristic polynomial of the automorphism induced by ζ on $\mathrm{Lie}(T)$ is equal to $P_{m_1,\zeta}(T)$. Likewise, $P_{m_2,\zeta}(T)$ equals the product of the characteristic polynomials of the automorphism induced by ζ on $\mathrm{Lie}(B)$ and the dual automorphism on $\mathrm{Lie}(B^\vee)$.

By Lemma 2.2, we only have to show that the image of $P_1(t)$ under ρ equals $P_{m_1,\zeta}(T)$ and the image of $P_2(t)$ equals $P_{m_2,\zeta}(t)$. This follows from the proof of [9, 5.12], by the canonical isomorphism

$$H^1(B, \mathcal{O}_B) \cong \mathrm{Lie}(B^\vee)$$

(see [14, § 13, Cor.3]). □

For every $n \in \mathbb{Z}_{>0}$ and every $a \in \mathbb{C}$, we denote by $\mathrm{Diag}_n(a)$ the rank n diagonal matrix with diagonal $(a, \ldots, a)$, and by $\mathrm{Jord}_n(a)$ the rank n Jordan matrix with diagonal $(a, \ldots, a)$ and subdiagonal $(1, \ldots, 1)$. For any two complex square matrices M and N, of rank m, resp. n, we denote by $M \oplus N$ the rank $m + n$ square matrix

$$M \oplus N = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

For every integer $q > 0$, we put

$$\oplus^q M = \underbrace{M \oplus \cdots \oplus M}_{q \text{ times}}.$$

Definition 4.2. *For $i = 1, 2$, let*

$$m_i : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$$

be a function with finite support. The Jordan matrix $\mathrm{Jord}(m_1, m_2)$ associated to the couple (m_1, m_2) is the complex square matrix of rank $\|m_1\| + 2 \cdot \|m_2\|$ given by

$$\begin{aligned} \mathrm{Jord}(m_1, m_2) \quad &= \bigoplus_{x \in \mathrm{Supp}(m_1)} \Big(\mathrm{Diag}_{m_1(x)}(\exp(2\pi i x))\Big) \\ &\oplus \bigoplus_{y \in \mathrm{Supp}(m_2)} \Big(\oplus^{m_2(y)} \mathrm{Jord}_2(\exp(2\pi i y))\Big) \end{aligned}$$

where we ordered the set $\mathbb{Q}/\mathbb{Z}$ using the bijection $\mathbb{Q} \cap [0, 1[\to \mathbb{Q}/\mathbb{Z}$ and the usual ordering on $[0, 1[$.

Theorem 4.3. *We fix an embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. If A is a tamely ramified abelian K-variety, then the monodromy action of σ on $H^1(A \times_K K^t, \mathbb{Q}_\ell)$ has Jordan form*

$$\mathrm{Jord}(m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}, m_A^{\mathrm{tor}}).$$

Moreover, the functions m_A^{tor} and $m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}$ are complete.

Proof. We adopt the notations of Section 3.1. We denote by $\mathscr{T}_\ell A$ the ℓ-adic Tate module of A. We put $I = G(K^s/K)$ and $I' = G(K^s/K')$. Recall that there exists a canonical I-equivariant isomorphism

$$H^1(A \times_K K^s, \mathbb{Q}_\ell) \cong Hom_{\mathbb{Z}_\ell}(\mathscr{T}_\ell A, \mathbb{Q}_\ell) \tag{4.1}$$

(see [13, 15.1]). Since A is tamely ramified, the wild inertia subgroup $P \subset I$ acts trivially on $H^1(A \times_K K^s, \mathbb{Q}_\ell)$ and $\mathscr{T}_\ell A$, so that the I-action on these modules factors through an action of $I/P = G(K^t/K)$.

Since P is a p-group and p is prime to ℓ, there exists for every K-variety X and every integer $i \geq 0$ a canonical $G(K^t/K)$-equivariant isomorphism

$$H^i(X \times_K K^t, \mathbb{Q}_\ell) \cong H^i(X \times_K K^s, \mathbb{Q}_\ell)^P$$

(see [2, I.2.7.1]). In our case, this yields a canonical $G(K^t/K)$-equivariant isomorphism

$$H^1(A \times_K K^s, \mathbb{Q}_\ell) = H^1(A \times_K K^s, \mathbb{Q}_\ell)^P \cong H^1(A \times_K K^t, \mathbb{Q}_\ell). \tag{4.2}$$

By (4.1) and (4.2), it suffices to show that the action of σ on

$$\mathscr{V}_\ell A = \mathscr{T}_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

has Jordan form

$$\mathrm{Jord}(m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}, m_A^{\mathrm{tor}})$$

and that $m_A^{\mathrm{ab}} + \breve{m}_A^{\mathrm{ab}}$ and m_A^{tor} are complete.

Consider the filtration

$$(\mathscr{T}_\ell A)^{\mathrm{et}} \subset (\mathscr{T}_\ell A)^{\mathrm{ef}} \subset \mathscr{T}_\ell A \tag{4.3}$$

from [2, IX.4.1.1], with $(\mathscr{T}_\ell A)^{\mathrm{ef}}$ the *essentially fixed part* of the Tate module $\mathscr{T}_\ell A$, and $(\mathscr{T}_\ell A)^{\mathrm{et}}$ the *essentially toric part*. By definition,

$$(\mathscr{T}_\ell A)^{\mathrm{ef}} = (\mathscr{T}_\ell A)^{I'}$$

and $(\mathscr{T}_\ell A)^{\mathrm{et}}$ is stable under the action of I on $\mathscr{T}_\ell A$. We denote by

$$(\mathscr{V}_\ell A)^{\mathrm{et}} \subset (\mathscr{V}_\ell A)^{\mathrm{ef}} = (\mathscr{V}_\ell A)^{I'} \subset \mathscr{V}_\ell A \tag{4.4}$$

the filtration obtained from (4.3) by tensoring with $\mathbb{Q}_\ell$. By [2, IX.4.1.2] there exists an I-equivariant isomorphism

$$\mathscr{V}_\ell A/(\mathscr{V}_\ell A)^{\mathrm{ef}} \cong ((\mathscr{V}_\ell A)^{\mathrm{et}})^\vee. \tag{4.5}$$

In particular, I' acts trivially on $\mathscr{V}_\ell A/(\mathscr{V}_\ell A)^{\mathrm{ef}}$. It follows that the I'-action on $\mathscr{V}_\ell A$ is unipotent of level ≤ 2, and that the I-action on $(\mathscr{V}_\ell A)^{\mathrm{et}}$ and $\mathscr{V}_\ell A/(\mathscr{V}_\ell A)^{\mathrm{ef}}$ factors through an action of $I/I' \cong \mu = \mu_d(k)$, where $d = [K' : K]$. We denote by $\overline{\sigma}$ the image of σ under the projection $G(K^t/K) \to \mu$.

The element σ^d belongs to I', so that the action of σ^d on $\mathscr{V}_\ell A$ is unipotent of level ≤ 2. Combining (4.4) and (4.5) and using some elementary linear algebra, we see that the action of σ on $\mathscr{V}_\ell A$ has the following Jordan form: for every eigenvalue α of $\overline{\sigma}$ on $(\mathscr{V}_\ell A)^{\mathrm{et}}$ there is a Jordan block of size two with eigenvalue α, and for every eigenvalue β of $\overline{\sigma}$ on $(\mathscr{V}_\ell A)^{\mathrm{ef}}/(\mathscr{V}_\ell A)^{\mathrm{et}}$ there is a Jordan block of size one with eigenvalue β.

Hence, in order to prove the theorem, it suffices to prove the following claim:

(1) the $\overline{\sigma}$-action on $(\mathscr{V}_\ell A)^{\mathrm{et}}$ has Jordan form $\mathrm{Jord}(m_A^{\mathrm{tor}}, 0)$, and m_A^{tor} is complete,

(2) the $\overline{\sigma}$-action on $(\mathscr{V}_\ell A)^{\mathrm{ef}}/(\mathscr{V}_\ell A)^{\mathrm{et}}$ has Jordan form $\mathrm{Jord}(m_A^{\mathrm{ab}}+\breve{m}_A^{\mathrm{ab}},0)$, and $m_A^{\mathrm{ab}}+\breve{m}_A^{\mathrm{ab}}$ is complete.

By [2, IX.4.2.7 & IX.4.2.9] there exist μ-equivariant isomorphisms

$$\begin{aligned}(\mathscr{V}_\ell A)^{\mathrm{et}} &\cong \mathscr{V}_\ell T\\ (\mathscr{V}_\ell A)^{\mathrm{ef}}/(\mathscr{V}_\ell A)^{\mathrm{et}} &\cong \mathscr{V}_\ell B\end{aligned}$$

so that the claim follows from Proposition 4.1. □

Corollary 4.4. *The functions $m_A^{\mathrm{ab}}+\breve{m}_A^{\mathrm{ab}}$ and m_A^{tor} are invariant under isogeny. In particular, $m_A^{\mathrm{tor}}=m_{A^\vee}^{\mathrm{tor}}$, and*

$$m_A^{\mathrm{ab}}+\breve{m}_A^{\mathrm{ab}}=m_{A^\vee}^{\mathrm{ab}}+\breve{m}_{A^\vee}^{\mathrm{ab}}.$$

Corollary 4.5. *Let A be a tamely ramified abelian K-variety. If we assume either that k has characteristic zero, or that A is principally polarized, then m_A^{ab} is semi-complete.*

Proof. This follows from Corollary 3.11. □

Corollary 4.6. *Let A be a tamely ramified abelian K-variety, and let e be the degree of the minimal extension of K where A acquires semi-abelian reduction. Fix a primitive e-th root of unity ξ in an algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$. The characteristic polynomial*

$$P_\sigma(t)=det(t\cdot\mathrm{Id}-\sigma\,|\,H^1(A\times_K K^t,\mathbb{Q}_\ell))$$

of σ on $H^1(A\times_K K^t,\mathbb{Q}_\ell)$ is given by

$$P_\sigma(t)=\prod_{i\in((1/e)\mathbb{Z})/\mathbb{Z}}(t-\xi^{e\cdot i})^{m_A^{\mathrm{ab}}(i)+\breve{m}_A^{\mathrm{ab}}(i)+2m_A^{\mathrm{tor}}(i)}\qquad\in\mathbb{Z}[t].$$

Corollary 4.7. *Let A be a tamely ramified abelian K-variety. Assume either that k has characteristic zero, or that A is principally polarized. Then the monodromy action of σ on $H^1(A\times_K K^t,\mathbb{Q}_\ell)$ has Jordan form*

$$\mathrm{Jord}(m_A^{\mathrm{ab}}+(m_A^{\mathrm{ab}})^*,m_A^{\mathrm{tor}}).$$

In the notation of Corollary 4.6, we have

$$P_\sigma(t)=\prod_{i\in((1/e)\mathbb{Z})/\mathbb{Z}}(t-\xi^{e\cdot i})^{m_A^{\mathrm{ab}}(i)+m_A^{\mathrm{ab}}(-i)+2m_A^{\mathrm{tor}}(i)}\qquad\in\mathbb{Z}[t].$$

Proof. Apply Corollary 3.11. □

5. Potential toric rank and Jordan blocks

Lemma 5.1. *Let F be an algebraically closed field of characteristic zero, and let $V\neq\{0\}$ be a vector space over F. Let α be an endomorphism of V, with Jordan form*

$$\mathrm{Jord}_m(\xi)$$

where $m\in\mathbb{Z}_{>0}$ and $\xi\in F^\times$. For each integer i in $\{0,\ldots,m\}$, the endomorphism

$$\beta:=(\wedge^i\alpha)-\xi^i\cdot\mathrm{Id}$$

of $\wedge^i V$ is nilpotent of order $1+i(m-i)$.

Proof. We choose a basis $v_1, \ldots, v_m$ of V such that

$$(\alpha - \xi \cdot \mathrm{Id})v_\ell = v_{\ell-1}$$

for each $\ell \in \{1, \ldots, m\}$, where we put $v_0 = 0$.

We may assume that i belongs to $\{1, \ldots, m\}$, because the result is obvious for $i = 0$. We denote by $\mathscr{J}$ the set of strictly increasing maps

$$j : \{1, \ldots, i\} \to \{1, \ldots, m\}.$$

We define a partial ordering on $\mathscr{J}$ by putting $j_1 \leq j_2$ iff $j_1(x) \leq j_2(x)$ for all $x \in \{1, \ldots, i\}$.

For each $j \in \mathscr{J}$, we define the *weight* of j by

$$\omega(j) = \sum_{a=1}^{i} (j(a) - a)$$

and we put

$$v_j := v_{j(1)} \wedge \cdots \wedge v_{j(i)} \ \in \wedge^i V.$$

Note that $\omega(j)$ is contained in the set $\{0, \ldots, i(m-i)\}$. For every $n \in \mathbb{N}$ we denote by $\mathscr{J}_n$ the subset of $\mathscr{J}$ consisting of the elements j of weight $\omega(j) = n$. We denote by $j_{\min}$ the unique element in $\mathscr{J}_0$, i.e., the inclusion

$$j_{\min} : \{1, \ldots, i\} \to \{1, \ldots, m\}.$$

For every $n \in \mathbb{N}$, we denote by $W_{<n}$ the linear subspace of $\wedge^i V$ generated by the elements v_j with $j \in \mathscr{J}$ and $\omega(j) < n$.

Since $\{v_j \,|\, j \in \mathscr{J}\}$ is a basis of $\wedge^i V$, the lemma follows immediately from the following claim.

Claim. For each element j in $\mathscr{J}$, and every $n \in \mathbb{N}$, we have $\beta^n(v_j) = 0$ if $n > \omega(j)$. If $n = \omega(j)$, then

$$\beta^n(v_j) = c\xi^{n(i-1)} v_{j_{\min}}$$

with $c \in \mathbb{Z}_{>0}$.

Let us prove the claim. We proceed by induction on $\omega(j)$. It is easily seen that $\beta(v_{j_{\min}}) = 0$, so the claim holds for $\omega(j) = 0$. Assume that $\omega(j) > 0$ and that the claim holds for elements of $\mathscr{J}$ of weight strictly smaller than $\omega(j)$. Direct computation shows that

$$\beta(v_j) = \sum_{j' \in \mathscr{J}_{\omega(j)-1},\, j' \leq j} \xi^{i-1} v_{j'} + w$$

with $w \in W_{<\omega(j)-1}$. By our induction hypothesis, we know that $\beta^{\omega(j)-1}(w) = 0$, and that for each $j' \in \mathscr{J}_{\omega(j)-1}$,

$$\beta^{\omega(j)-1}(v_{j'}) = c_{j'} \xi^{(\omega(j)-1)(i-1)} v_{j_{\min}}$$

with $c_{j'} \in \mathbb{Z}_{>0}$. It follows that

$$\beta^{\omega(j)}(v_j) = \left(\sum_{j' \in \mathscr{J}_{\omega(j)-1},\, j' \leq j} c_{j'} \right) \xi^{\omega(j)(i-1)} v_{j_{\min}}$$

and $\beta^n(v_j) = 0$ for $n > \omega(j)$. $\square$

The following proposition will allow us to compute the maximal size of certain Jordan blocks of monodromy on the cohomology of a tamely ramified abelian K-variety (Theorem 5.3). The proof of the proposition consists of some elementary, but quite tedious, linear algebra.

Proposition 5.2. *Let F be an algebraically closed field of characteristic zero, and let $V \neq \{0\}$ be a vector space over F. Let α be an endomorphism of V, with Jordan form*

$$\mathrm{Jord}_{m_1}(\xi_1) \oplus \cdots \oplus \mathrm{Jord}_{m_q}(\xi_q)$$

where $q \in \mathbb{Z}_{>0}$, $m \in (\mathbb{Z}_{>0})^q$ and $\xi_j \in F^\times$ for $j = 1, \ldots, q$.

We fix an integer $i > 0$. For every element ζ of F, we denote by M_ζ the size of the largest Jordan block of $\wedge^i\alpha$ on $\wedge^i V$ with eigenvalue ζ. If we denote by $\mathscr{S}$ the set of tuples $s \in \mathbb{N}^q$ such that $\|s\| = i$ and $s_j \leq m_j$ for each $j \in \{1, \ldots, q\}$, then

$$M_\zeta = \max\{1 + \sum_{j \in \mathrm{Supp}(s)} (s_j(m_j - s_j)) \,|\, s \in \mathscr{S},\ \prod_{j=1}^{q} (\xi_j)^{s_j} = \zeta\}$$

for every $\zeta \in F$, with the convention that $\max \emptyset = 0$.

Proof. We fix $\zeta \in F$, and we put

$$M = \max\{1 + \sum_{j \in \mathrm{Supp}(s)} (s_j(m_j - s_j)) \,|\, s \in \mathscr{S},\ \prod_{j=1}^{q} (\xi_j)^{s_j} = \zeta\}.$$

We have to show that $M = M_\zeta$.

We can write

$$V = V_1 \oplus \cdots \oplus V_q$$

such that $\alpha(V_j) \subset V_j$ for each j and the restriction α_j of α to V_j has Jordan form $\mathrm{Jord}_{m_j}(\xi_j)$. If we put

$$V_s = (\wedge^{s_1} V_1) \otimes \cdots \otimes (\wedge^{s_q} V_q)$$

for each $s \in \mathscr{S}$, then we have a canonical isomorphism

$$\wedge^i V \cong \bigoplus_{s \in \mathscr{S}} V_s$$

such that every summand V_s is stable under $\wedge^i\alpha$ and the restriction of $\wedge^i\alpha$ to V_s equals

$$(\wedge^{s_1}\alpha_1) \otimes \cdots \otimes (\wedge^{s_q}\alpha_q).$$

The endomorphism $\wedge^i\alpha$ has a unique eigenvalue on V_s, which is equal to

$$\xi_s := \prod_{j=1}^{q} (\xi_j)^{s_j}.$$

It suffices to prove the following claims.

Claim 1. For each $s \in \mathscr{S}$ such that $\xi_s = \zeta$, we have $(\wedge^i\alpha - \zeta \cdot \mathrm{Id})^M = 0$ on V_s.

Assume that $s \in \mathscr{S}$ such that $\xi_s = \zeta$. For each subset S of $\{1, \ldots, q\}$, and each $j \in \{1, \ldots, q\}$, we denote by $\alpha_{S,j}$ the endomorphism $(\wedge^{s_j}\alpha_j - (\xi_j)^{s_j} \cdot \mathrm{Id})$ of $\wedge^{s_j} V_j$ if $j \in S$, and the endomorphism $(\xi_j)^{s_j} \cdot \mathrm{Id}$ of $\wedge^{s_j} V_j$ else. We denote by α_S the endomorphism

$$\alpha_{S,1} \otimes \cdots \otimes \alpha_{S,q}$$

of V_s. Then we have, for every $w \in V_s$,

$$\begin{aligned}(\wedge^i \alpha - \zeta \cdot \mathrm{Id})^M(w) &= (\otimes_{j=1}^q (\wedge^{s_j}\alpha_j) - \otimes_{j=1}^q ((\xi_j)^{s_j} \cdot \mathrm{Id}))^M(w) \\ &= (\sum_{\emptyset \neq S \subset \{1,\ldots,q\}} \alpha_S)^M(w)\end{aligned}$$

because

$$\otimes_{j=1}^q (\wedge^{s_j}\alpha_j) = \otimes_{j=1}^q ((\wedge^{s_j}\alpha_j - (\xi_j)^{s_j} \cdot \mathrm{Id}) + (\xi_j)^{s_j} \cdot \mathrm{Id}) = \sum_{S \subset \{1,\ldots,q\}} \alpha_S.$$

Let $S_1, \ldots, S_M$ be (not necessarily distinct) non-empty subsets of $\{1, \ldots, q\}$. It is enough to show that

$$(\alpha_{S_1} \circ \cdots \circ \alpha_{S_M})(w) = 0$$

for all $w \in V_s$. For every $j \in \{1, \ldots, q\}$, we denote by ν_j the cardinality of the set

$$\{r \in \{1, \ldots, M\} \mid j \in S_r\}.$$

Since

$$\sum_{j=1}^q \nu_j = \sum_{r=1}^M |S_r| \geq M > \sum_{j=1}^q s_j(m_j - s_j),$$

there exists an element j' of $\{1, \ldots, q\}$ such that

$$\nu_{j'} > s_{j'}(m_{j'} - s_{j'}).$$

By Lemma 5.1, this implies that

$$(\wedge^{s_{j'}}\alpha_{j'} - (\xi_{j'})^{s_{j'}} \cdot \mathrm{Id})^{\nu_{j'}}$$

vanishes on $\wedge^{s_{j'}} V_{j'}$, so that

$$\alpha_{S_1} \circ \cdots \circ \alpha_{S_M}$$

vanishes on V_s.

Claim 2. If $M > 0$, then there exists an element $s \in \mathscr{S}$ such that $\xi_s = \zeta$ and such that $(\wedge^i \alpha - \zeta \cdot \mathrm{Id})^{M-1} \neq 0$ on V_s.

Since $M > 0$, there exists an element $s \in \mathscr{S}$ such that $\prod_{j=1}^q (\xi_j)^{s_j} = \zeta$ and

$$M = 1 + \sum_{j \in \mathrm{Supp}(s)} (s_j(m_j - s_j)).$$

By Lemma 5.1, we can choose, for each $j \in \mathrm{Supp}(s)$, an element w_j in $\wedge^{s_j} V_j$ such that

$$(\wedge^{s_j}\alpha_j - (\xi_j)^{s_j} \cdot \mathrm{Id})^{s_j(m_j - s_j)}(w_j) \neq 0.$$

We put

$$w = \otimes_{j \in \mathrm{Supp}(s)} w_j \quad \in V_s.$$

It suffices to show that

$$(\wedge^i \alpha - \zeta \cdot \mathrm{Id})^{M-1}(w) \neq 0.$$

With the notations of the proof of Claim 1, we have that

$$\begin{aligned}(\wedge^i \alpha - \zeta \cdot \mathrm{Id})^{M-1}(w) &= (\sum_{\emptyset \neq S \subset \{1,\ldots,q\}} \alpha_S)^{M-1}(w) \\ &= (\sum_{\emptyset \neq S_1, \ldots, S_{M-1} \subset \{1,\ldots,q\}} \alpha_{S_1} \circ \cdots \circ \alpha_{S_{M-1}})(w).\end{aligned}$$

Consider (not necessarily distinct) non-empty subsets $S_1, \dots, S_{M-1}$ of $\{1, \dots, q\}$. As we've seen above,

$$(\alpha_{S_1} \circ \cdots \circ \alpha_{S_{M-1}})(w)$$

vanishes unless

$$\nu'_j := |\{r \in \{1, \dots, M-1\} \mid j \in S_r\}| \leq s_j(m_j - s_j)$$

for each $j \in \{1, \dots, q\}$. Since

$$M - 1 \leq \sum_{r=1}^{M-1} |S_r| = \sum_{j=1}^{q} \nu'_j$$

this happens iff $\nu'_j = s_j(m_j - s_j)$ for each j. Note that this situation occurs, i.e., there exist non-empty subsets $S_1, \dots, S_{M-1}$ of $\{1, \dots, q\}$ such that $\nu'_j = s_j(m_j - s_j)$ for each j. It suffices to take

$$S_r = \{j(r)\}$$

for each $r \in \{1, \dots, M-1\}$, with $j(r)$ the unique element of $\{1, \dots, q\}$ such that

$$\sum_{a=1}^{j-1} (s_a(m_a - s_a)) < r \leq \sum_{a=1}^{j} (s_a(m_a - s_a)).$$

This means that

$$(\alpha_{S_1} \circ \cdots \circ \alpha_{S_{M-1}})(w)$$

is non-zero for certain non-empty subsets $S_1, \dots, S_{M-1}$ of $\{1, \dots, q\}$, and that, in this case, it is equal to

$$\bigotimes_{j=1}^{q} \Big((\wedge^{s_j} \alpha_j - (\xi_j)^{s_j} \cdot \mathrm{Id})^{s_j(m_j - s_j)} ((\xi_j)^{s_j(M-1-s_j(m_j-s_j))} \cdot w_j) \Big) \quad \in V_s.$$

Note that the latter value is independent of $S_1, \dots, S_{M-1}$. Summing over all non-empty subsets $S_1, \dots, S_{M-1}$ of $\{1, \dots, q\}$, we find that

$$(\wedge^i \alpha - \zeta \cdot \mathrm{Id})^{M-1}(w) \neq 0.$$

□

Theorem 5.3. *Let A be a tamely ramified abelian K-variety of dimension g. For every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the value $\alpha = \exp(2\pi i c(A))$ is an eigenvalue of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. Each Jordan block of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ has size at most $t_{\mathrm{pot}}(A) + 1$, and σ has a Jordan block with eigenvalue α on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ with size $t_{\mathrm{pot}}(A) + 1$.*

Proof. Since A is tamely ramified, we have a canonical $G(K^t/K)$-equivariant isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$H^g(A \times_K K^t, \mathbb{Q}_\ell) \cong \bigwedge^g H^1(A \times_K K^t, \mathbb{Q}_\ell).$$

By Theorem 4.3, the monodromy operator σ has precisely $\|m_A^{\mathrm{tor}}\|$ Jordan blocks of size 2 on $H^1(A \times_K K^t, \mathbb{Q}_\ell)$, and no larger Jordan blocks. It follows from Proposition 5.2 that the size of the Jordan blocks of σ on $H^g(A \times_K K^t, \mathbb{Q}_\ell)$ is bounded by $1 + \|m_A^{\mathrm{tor}}\|$. By Proposition 3.6, we know that $\|m_A^{\mathrm{tor}}\| = t_{\mathrm{pot}}(A)$.

By Proposition 3.5, the image in $\mathbb{Q}/\mathbb{Z}$ of the base change conductor $c(A)$ equals

$$\sum_{x \in \mathbb{Q}/\mathbb{Z}} ((m_A^{\mathrm{tor}}(x) + m_A^{\mathrm{ab}}(x)) \cdot x)$$

and by Proposition 3.6, we have

$$\sum_{x\in\mathbb{Q}/\mathbb{Z}}(m_A^{\mathrm{tor}}(x)+m_A^{\mathrm{ab}}(x))=g.$$

Hence, by Theorem 4.3 and Proposition 5.2, σ has a Jordan block of size $1+t_{\mathrm{pot}}(A)$ with eigenvalue α on $H^g(A\times_K K^t,\mathbb{Q}_\ell)$. □

6. Limit Mixed Hodge structure

Let A be a tamely ramified abelian K-variety of dimension g. Theorem 4.3 shows that the couple of functions $(m_A^{\mathrm{tor}},m_A^{\mathrm{ab}}+\breve{m}_A^{\mathrm{ab}})$ and the Jordan form of σ on the tame degree one cohomology of A determine each other. It does not tell us how to recover m_A^{ab} and $\breve{m}_A^{\mathrm{ab}}$ individually from the cohomology of A.

In this section, we assume that A is obtained by base change from a family of abelian varieties over a smooth complex curve. We will show how the functions m_A^{ab} and $\breve{m}_A^{\mathrm{ab}}$ can be read from the limit mixed Hodge structure on the cohomology of the family, and we obtain a Hodge-theoretic interpretation of the multiplicity functions m_A^{ab}, $\breve{m}_A^{\mathrm{ab}}$ and m_A^{tor}.

6.1. **Limit mixed Hodge structure of a family of abelian varieties.** Let $\overline{S}$ be a connected smooth complex algebraic curve, let s be a closed point on $\overline{S}$, and choose a local parameter t on $\overline{S}$ at s. We put $K=\mathbb{C}((t))$, $R=\mathbb{C}[[t]]$ and $S=\overline{S}\setminus\{s\}$. Let

$$f:X\to S$$

be a projective family of abelian varieties over S, of relative dimension g, and put

$$A=X\times_S\operatorname{Spec}K.$$

We choose an extension of f to a projective morphism

$$\overline{f}:\overline{X}\to\overline{S}$$

with $\overline{X}$ a smooth complex variety. We denote by $\overline{X}_s$ the fiber of $\overline{f}$ over the point s.

For every $i\in\mathbb{N}$, we consider the limit homology, resp. cohomology

$$\begin{aligned}H_i(X_\infty,\mathbb{Z}) &:= \mathbb{H}^{2g-i}(\overline{X}_s(\mathbb{C}),R\psi_{\overline{f}}(\mathbb{Z}))(g)\\ H^i(X_\infty,\mathbb{Z}) &:= \mathbb{H}^{i}(\overline{X}_s(\mathbb{C}),R\psi_{\overline{f}}(\mathbb{Z}))\end{aligned}$$

of $\overline{f}$ at s. Here

$$R\psi_{\overline{f}}(\mathbb{Z})\in D^b_c(\overline{X}_s(\mathbb{C}),\mathbb{Z})$$

denotes the complex analytic nearby cycles associated to $\overline{f}$. The limit homology and cohomology are independent of the chosen compactification $\overline{f}$, and they carry natural limit mixed Hodge structures [19]. We put

$$\begin{aligned}H_i(X_\infty,\mathbb{Q}) &:= H_i(X_\infty,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\\ H_i(X_\infty,\mathbb{C}) &:= H_i(X_\infty,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}.\end{aligned}$$

Poincaré duality yields a canonical isomorphism of mixed Hodge structures

$$H_i(X_\infty,\mathbb{Z})\cong H^i(X_\infty,\mathbb{Z})^\vee.$$

For all $i\in\mathbb{N}$, we denote by M the monodromy operator on $H_i(X_\infty,\mathbb{Q})$. The action of the semi-simple part M_s of M on

$$H_i(X_\infty,\mathbb{Q})$$

is a morphism of rational mixed Hodge structures, by [15, 15.13].

For every $i \in \mathbb{N}$, there exists an isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$H^i(X_\infty, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^i(A \times_K K^a, \mathbb{Q}_\ell) \tag{6.1}$$

such that the monodromy action on the left hand side corresponds to the action of the canonical topological generator of $G(K^a/K) \cong \widehat{\mathbb{Z}}(1)(\mathbb{C})$ on the right hand side. This follows from Deligne's comparison theorem for ℓ-adic vs. complex analytic nearby cycles [3, 2.8]. Thus, if A acquires semi-abelian reduction on an extension of K of degree d, then $(M_s)^d$ is the identity on $H_i(X_\infty, \mathbb{Q})$ for all $i \geq 0$. Identifying M_s with the canonical generator of $\mu_d(\mathbb{C})$, we obtain an action of $\mu_d(\mathbb{C})$ on the mixed Hodge structures $H_i(X_\infty, \mathbb{Q})$.

Proposition 6.1. *For each $i \in \mathbb{N}$, the morphism*

$$\wedge^i_{\mathbb{Z}} H^1(X_\infty, \mathbb{Z}) \to H^i(X_\infty, \mathbb{Z}) \tag{6.2}$$

induced by the cup product is an isomorphism of mixed Hodge structures.

Proof. The morphism (6.2) is an isomorphism, since this holds for every fiber of f [14, p. 3]. Hence, it is enough to show that (6.2) is a morphism of mixed Hodge structures. By functoriality of the construction of the limit of a variation of Hodge structures, this follows immediately from the fact that the cup product defines a morphism of pure Hodge structures on the cohomology of every fiber of f. $\square$

By Proposition 6.1, in order to describe the limit mixed Hodge structure on $H^i(X_\infty, \mathbb{Z})$ for $i \geq 0$, it suffices to consider the case where $i = 1$.

We denote by $(\cdot)^{\mathrm{an}}$ the complex analytic GAGA functor, and by

$$\mathscr{V} \to S^{\mathrm{an}}$$

the polarizable variation of Hodge structures

$$R^{2g-1}(f^{\mathrm{an}})_*(\mathbb{Z}_{X^{\mathrm{an}}})(g)$$

of type $\{(0,-1),(-1,0)\}$ [6, 4.4.3]. To simplify notation, we will omit the superscript "an" if no confusion can occur. We denote by $\mathscr{V}_{\mathbb{Z}}$, $\mathscr{V}_{\mathbb{Q}}$ and $\mathscr{V}_{\mathbb{C}}$ the integer, resp. rational, resp. complex component of $\mathscr{V}$. By Poincaré duality, the fiber of $\mathscr{V}_{\mathbb{Z}}$ over a point z of S^{an} is canonically isomorphic to $H_1(X_z(\mathbb{C}), \mathbb{Z})$, where X_z denotes the fiber of f over z. The limit of $\mathscr{V}$ at the point s is precisely the mixed Hodge structure $H_1(X_\infty, \mathbb{Z})$. It is of type

$$\{(0,0),\,(-1,0),\,(0,-1),\,(-1,-1)\}.$$

Moreover,

$$Gr^W_{-1} H_1(X_\infty, \mathbb{Z})$$

is polarizable, so that $H_1(X_\infty, \mathbb{Z})$ is a mixed Hodge 1-motive in the sense of [7, § 10].

Theorem 6.2. *We apply the terminology of Section 3.1 to the abelian K-variety A and define in this way the degree d extension K' of K, as well as the torus T and the abelian variety B over $\mathbb{C}$, endowed with a right action of $\mu \cong \mu_d(\mathbb{C})$. There exist canonical μ-equivariant isomorphisms of pure Hodge structures*

$$\begin{aligned} Gr^W_0(H_1(X_\infty, \mathbb{Z})) &\cong H_1(T(\mathbb{C}), \mathbb{Z})(-1) \\ Gr^W_{-1}(H_1(X_\infty, \mathbb{Z})) &\cong H_1(B(\mathbb{C}), \mathbb{Z}) \\ Gr^W_{-2}(H_1(X_\infty, \mathbb{Z})) &\cong H_1(T(\mathbb{C}), \mathbb{Z}). \end{aligned}$$

Proof. We denote by $\mathbb{C}(S')$ the algebraic closure of the function field $\mathbb{C}(S)$ in K', and we consider the normalization

$$\overline{S}' \to \overline{S}$$

of $\overline{S}$ in $\mathbb{C}(S')$. This is a ramified Galois covering, obtained by taking a d-th root of the local parameter t. Its Galois group is canonically isomorphic to μ. With abuse of notation, we denote again by s the unique point of the inverse image of s in $\overline{S}'$, and we put $S' = \overline{S}' \setminus \{s\}$. Then

$$f' : X' := X \times_S S' \to S'$$

is a projective family of abelian varieties, and we have a canonical isomorphism

$$A' := A \times_K K' \cong X' \times_{S'} \operatorname{Spec} K'.$$

Since A' has semi-abelian reduction, the variation of Hodge structures

$$\mathscr{V}' := \mathscr{V} \times_S S' \cong R^{2g-1} f'_*(\mathbb{Z}_{X'})(g)$$

has unipotent monodromy around s.

We denote by $\mathcal{X}'$ the Néron model of X' over $\overline{S}'$, and by $\mathcal{A}'$ the Néron model of A'. Note that there is a canonical isomorphism

$$\mathcal{A}' \cong \mathcal{X}' \times_{\overline{S}'} \operatorname{Spec} R'$$

where R' is the normalization of R in K'. The analytic family of abelian varieties

$$(f')^{\mathrm{an}} : (X')^{\mathrm{an}} \to (S')^{\mathrm{an}}$$

is canonically isomorphic to the Jacobian

$$J(\mathscr{V}') \to (S')^{\mathrm{an}}$$

of the variation of Hodge structures $\mathscr{V}'$ [18, 2.10.1]. We will now explain the relation between the complex semi-abelian variety $(\mathcal{A}')^o_s$ and the limit mixed Hodge structure $H_1(X_\infty, \mathbb{Z})$ of $\mathscr{V}'$ at the point s. To simplify notation, we put $H_C = H_1(X_\infty, C)$ for $C = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, and we denote by H the mixed Hodge structure

$$(H_{\mathbb{Z}}, W_\bullet H_{\mathbb{Q}}, F^\bullet H_{\mathbb{C}}).$$

By [18, 4.5(i)], $(\mathcal{X}')^{\mathrm{an}}$ is canonically isomorphic to Clemens's Néron model of $\mathscr{V}'$ over $\overline{S}'$ [5][18, 2.5]. It follows that

$$((\mathcal{X}')^o)^{\mathrm{an}}$$

is canonically isomorphic to the Zucker extension $J^Z_{\overline{S}'}(\mathscr{V}')$ of $\mathscr{V}'$ [20][18, 2.1]. This extension is given explicitly by

$$J^Z_{\overline{S}'}(\mathscr{V}') = j_*\mathscr{V}'_{\mathbb{Z}} \backslash \widehat{\mathscr{V}'_{\mathbb{C}}} / F^0 \widehat{\mathscr{V}'_{\mathbb{C}}}$$

where $\widehat{\mathscr{V}'}$ is the Deligne extension of $\mathscr{V}'_{\mathbb{C}}$ to $\overline{S}'$, j is the open immersion of S' into $\overline{S}'$, and $F^0\widehat{\mathscr{V}'_{\mathbb{C}}}$ is the extension of $F^0\mathscr{V}'_{\mathbb{C}}$ to a holomorphic subbundle of $\widehat{\mathscr{V}'_{\mathbb{C}}}$. We can describe the fiber

$$J^Z_{\overline{S}'}(\mathscr{V}')_s \cong ((\mathcal{X}')^o_s)^{\mathrm{an}} \cong ((\mathcal{A}')^o_s)^{\mathrm{an}}$$

of $J^Z_{\overline{S}'}(\mathscr{V}')$ at s in terms of the mixed Hodge structure H, as follows.

The fiber of $\widehat{\mathscr{V}'_{\mathbb{C}}}$ over s is canonically isomorphic to $H_{\mathbb{C}}$, and $F^0\widehat{\mathscr{V}'_{\mathbb{C}}}$ coincides with the degree zero part of the Hodge filtration on $H_{\mathbb{C}}$. Moreover, the fiber of $j_*\mathscr{V}'_{\mathbb{Z}}$ at s is the $\mathbb{Z}$-module of elements in $H_{\mathbb{Z}}$ that are invariant under the monodromy action of M^d. By definition, the weight filtration on $H_{\mathbb{Q}}$ is the filtration centered at -1

defined by the logarithm N of M^d. Since $(M^d - \mathrm{Id})^2 = 0$, we have $N = M^d - \mathrm{Id}$ and $N^2 = 0$, and we see that

$$(j_*\mathscr{V}'_{\mathbb{Z}})_s = \ker(N) = W_{-1}H_{\mathbb{Z}}.$$

Thus, we find canonical isomorphisms

$$\begin{aligned} ((\mathcal{A}')^o_s)^{\mathrm{an}} &\cong J^Z_{\overline{S}'}(\mathscr{V}')_s \\ &\cong W_{-1}H_{\mathbb{Z}}\backslash H_{\mathbb{C}}/F^0H_{\mathbb{C}} \\ &\cong W_{-1}H_{\mathbb{Z}}\backslash W_{-1}H_{\mathbb{C}}/(F^0H_{\mathbb{C}} \cap W_{-1}H_{\mathbb{C}}) \end{aligned}$$

where the last isomorphism follows from the fact that $Gr^W_0 H$ is purely of type $(0,0)$ so that $F^0 Gr^W_0 H = Gr^W_0 H$. By [7, 10.1], we have an extension

$$0 \to J(Gr^W_{-2}H) \to ((\mathcal{A}')^o_s)^{\mathrm{an}} \to J(Gr^W_{-1}H) \to 0 \tag{6.3}$$

where

$$J(Gr^W_{-2}H) = Gr^W_{-2}H_{\mathbb{C}}/Gr^W_{-2}H_{\mathbb{Z}}$$

is a torus, and

$$J(Gr^W_{-1}H) = H_{\mathbb{Z}}\backslash Gr^W_{-1}H_{\mathbb{C}}/F^0Gr^W_{-1}H_{\mathbb{C}}$$

an abelian variety, because the Hodge structure $Gr^W_{-1}H$ is polarizable. By [7, 10.1.1.3], the extension (6.3) is the analytification of the Chevalley decomposition

$$0 \to T \to (\mathcal{A}')^o_s \to B \to 0.$$

Hence, there exist canonical isomorphisms of pure Hodge structures

$$Gr^W_{-1}(H) \cong H_1(B(\mathbb{C}), \mathbb{Z}) \tag{6.4}$$

$$Gr^W_{-2}(H) \cong H_1(T(\mathbb{C}), \mathbb{Z}). \tag{6.5}$$

Moreover, by definition of the weight filtration on H, the operator N defines a μ-equivariant isomorphism of Hodge structures

$$Gr^W_0(H) \to Gr^W_{-2}(H)(-1).$$

It remains to show that the isomorphisms (6.4) and (6.5) are μ-equivariant. It is enough to prove that the Galois action of μ on

$$\mathscr{V}' \to S'$$

extends analytically to the Zucker extension

$$J^Z_{\overline{S}'}(\mathscr{V}') \to \overline{S}'$$

in such a way that the action of the canonical generator of $\mu = \mu_d(\mathbb{C})$ on

$$J^Z_{\overline{S}'}(\mathscr{V}')_s = W_{-1}H_{\mathbb{Z}}\backslash H_{\mathbb{C}}/F^0H_{\mathbb{C}}$$

coincides with the semi-simple part of the monodromy action. This follows easily from the constructions. □

6.2. **Multiplicity functions and limit mixed Hodge structure.**

Theorem 6.3. *We keep the notations of Section 6.1.*

(1) *The potential toric rank* $t_{\text{pot}}(A)$ *is equal to the largest integer* α *such that* $\exp(2\pi c(A)i)$ *is an eigenvalue of* M_s *on* $gr^W_{g+\alpha}H^g(X_\infty,\mathbb{Q})$.

(2) *The Jordan form of* M_s *on*

$$\begin{array}{lll} (gr^W_{-1}H_1(X_\infty,\mathbb{Q}))^{1,0} & \textit{is} & \mathrm{Jord}(m_A^{\mathrm{ab}},0), \\ (gr^W_{-1}H_1(X_\infty,\mathbb{Q}))^{0,1} & \textit{is} & \mathrm{Jord}(\check{m}_A^{\mathrm{ab}},0), \\ gr^W_{-2}H_1(X_\infty,\mathbb{Q}) & \textit{is} & \mathrm{Jord}(m_A^{\mathrm{tor}},0), \\ gr^W_{0}H_1(X_\infty,\mathbb{Q}) & \textit{is} & \mathrm{Jord}(m_A^{\mathrm{tor}},0). \end{array}$$

Proof. We denote by M_u the unipotent part of the monodromy, and by N its logarithm. By definition, the weight filtration on $H^g(X_\infty,\mathbb{Q})$ is the filtration with center g associated to the nilpotent operator N. Hence, in order to prove (1), it is enough to show that there exists an eigenvector v with eigenvalue $\exp(2\pi c(A)i)$ for the action of M on $H^g(X_\infty,\mathbb{Q})$ such that $N^{t_{\text{pot}}(A)}v\neq 0$, and that for any such v, we have $N^{t_{\text{pot}}(A)+1}v=0$. This follows from Theorem 5.3 and the isomorphism (6.1).

Point (2) follows from Theorem 6.2 and the canonical μ-equivariant isomorphisms

$$\begin{array}{lll} H_1(B(\mathbb{C}),\mathbb{C})^{1,0} & \cong & \mathrm{Lie}(B) \\ H_1(B(\mathbb{C}),\mathbb{C})^{0,1} & \cong & \mathrm{Lie}(B^\vee)^\vee \\ H_1(T(\mathbb{C}),\mathbb{C}) & \cong & \mathrm{Lie}(T) \end{array}$$

(see [14, pp. 4 and 86] for the dual isomorphisms on the level of cohomology). □

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