Nilspaces, nilmanifolds and their morphisms

O.Antolin Camarena Balázs Szegedy

September 21, 2010

Abstract

Recent developments in ergodic theory, additive combinatorics, higher order Fourier analysis and number theory give a central role to a class of algebraic structures called *nilmanifolds*. In the present paper we continue a program started by Host and Kra. We introduce *nilspaces* as structures satisfying a variant of the Host-Kra axiom system for parallelepiped structures. We give a detailed structural analysis of abstract and compact topological nilspaces. Among various results it will be proved that compact nilspaces are inverse limits of finite dimensional ones. Then we show that finite dimensional compact connected nilspaces are nilmanifolds. The theory of compact nilspaces is a generalization of the theory of compact abelian groups. This paper is the main algebraic tool in the second authors approach to Gowers's uniformity norms and higher order Fourier analysis.

Contents

1 Introduction

We start with the formal definition of k-step nilmanifolds.

Definition 1.1. *Let L be a k-nilpotent Lie group. This means that the k-fold iterated commutator*

$$[\dots[[L,L],L],L\dots]$$

is trivial. Let Γ be a co-compact subgroup in L. The left coset space $N = L/\Gamma$ is a compact topological space which is called a k-step **nilmanifold***.*

Nilmanifolds were first introduced and studied by Mal'cev [10] in 1951. He proved many crucial facts which can be also found in the book [12]. Nilmanifolds are interesting from a purely geometric point of view [7],[11]. However recent development [5],[6],[8],[18] shows their important role in ergodic theory and additive combinatorics.

The main motivation for this paper comes from higher order Fourier analysis. Let f be a bounded measurable function on a compact abelian group A. We denote by $\Delta_t f$ the function $x \mapsto f(x)\overline{f(x+t)}$. The U_k uniformity norm of f introduced by Gowers [3],[4] is defined by

$$\|f\|_{U_k} = \left(\mathbb{E}_{t_1,t_2,\dots,t_k}\Delta_{t_1,t_2,\dots,t_k}(f)\right)^{2^{-k}}.$$

In particular it can be computed that

$$\|f\|_{U_2} = \Big(\sum_{\chi\in\hat{A}} |(f,\chi)|^4\Big)^{1/4}$$

where $\hat{A}$ is the dual group of A. This formula explains the behavior of the U_2 norm through ordinary Fourier analysis.

Based on results in ergodic theory [8],[18] it is expected that the behavior of the U_k norm is in some sense connected to $k-1$ step nilmanifolds. However to clarify the precise connection (at least in the second author's interpretation) one needs a generalization of k-step nilmanifolds that we call k-step nilspaces. (Another independent approach to this problem is announced in [6] which deals with Gowers norms on cyclic groups)

Before giving the precise definition of k-step nilspaces we give a list of motivations and reasons to generalize nilmanifolds.

1. The structures which naturally arise in ergodic theory are not nilmanifolds but inverse limits of them.
2. A k-step nilspace (even if it is a nilmanifold topologically) has an extra algebraic structure which seems to be needed in Higher order Fourier analysis.
3. In higher order Fourier analysis it will be convenient to study morphisms between nilmanifolds and nilspaces. It turns out that nilspaces are more natural for this purpose than nilmanifolds.
4. To study Gowers norms of functions on abelian groups with many bounded order elements nilmanifolds are not enough.
5. Nilspaces are directly defined through a simple set of axioms. This helps to separate the algebraic and analytic difficulty in Higher order Fourier analysis.
6. Gowers norms can be naturally defined for functions on compact nilspaces. This means that the notion of Higher order Fourier analysis naturally extends to them.
7. Related to the so-called limit theory for graphs and hypergraphs, interesting limit notions can be defined for functions on abelian groups. The limit objects are functions on nilspaces.

The axiom system of nilspaces is a variant of Host-Kra's axiom system [9] for parallelepiped structures. Roughly speaking, a nilspace is a structure in which cubes of every dimension are defined and they behave in a very similar way as in an abelian group. An abstract n-dimensional cube is the set $\{0,1\}^n$. A cube morphism $\phi : \{0,1\}^n \to \{0,1\}^m$ is a map which extends to an affine homomorphism (a homomorphism plus a shift) from $\mathbb{Z}^n$ to $\mathbb{Z}^m$. A nilspace is a set N together with sets $C^n(N) \subseteq N^{\{0,1\}^n}$ of n dimensional cubes $f : \{0,1\}^n \to N$ for every integer $n \geq 0$ which satisfy the following three axioms:

1. **(composition)**: If $\phi : \{0,1\}^n \to \{0,1\}^m$ is a morphism and $f \in C^m(N)$ then $\phi \circ f \in C^n(N)$
2. **(ergodicity)**: $C^1(N) = N^{\{0,1\}}$.
3. **(gluing)**: Let $f : \{0,1\}^n \setminus 1^n$ be a map whose restrictions to $n-1$ dimensional faces containing 0^n are all cubes. Then f extends to $\{0,1\}^n$ as an element in $C^n(N)$.

We don't always assume the ergodicity axiom. If N is not ergodic then it can be decomposed into a disjoint union of ergodic nilspaces. We say that

N is a **k-step nilspace** if in the gluing axiom the extension is unique for $n = k + 1$. It is not hard to see that 1-step nilspaces are affine abelian groups with the usual notion of cubes. A cube $f : \{0,1\}^n \to A$ in an abelian group A is a map which extends to an affine homomorphism from $\mathbb{Z}^n \to A$.

If a set N satisfies the first axiom (but not necessarily the others) then we say that N is a **cubespace**. A **morphism** $h : N \to M$ between two cubespaces N and M is a cube preserving map. We require that for every $f \in C^n(N)$ the composition $f \circ h$ is in $C^n(M)$. We denote by $\mathrm{Hom}(N, M)$ the set of morphisms between N and M. In particular $C^n(N) = \mathrm{Hom}(\{0,1\}^n, N)$. With this notion we can introduce the categories of cubespaces and nilspaces.

Every morphism $\phi : \{0,1\}^n \to \{0,1\}^m$ induces a map $\hat{\phi} : C^m(N) \to C^n(N)$ such that $\hat{\phi}(f) = \phi \circ f$. We say that N is a compact nilspace if all the sets $C^n(N)$ are compact, Hausdorff, second countable topological spaces and the maps $\hat{\phi}$ are all continuous. Morphisms between compact nilspaces are required to be continuous.

The present paper consists of two parts. In the first part we study abstract nilspaces and in the second part we study compact nilspaces. The main topics in abstract nilspaces are the following:

1. For every natural number k and nilspace N we introduce a factor of N which is a k-step nilspace. Then we prove basic properties of these factors.
2. We give a structure theorem for k-step nilspaces in terms of iterated abelian bundles.
3. We introduce a cohomology theory for extensions of nilspaces.
4. We study a sequence of groups $\mathrm{Trans}_i(N)$ acting on a k-step nilspace N. They form a central series in the k-nilpotent group $\mathrm{Trans}_1(N)$.

The main topics in compact nilspaces are the following:

1. We generalize the concept of Haar measure for k-step compact nilspaces.
2. We prove a rigidity result for morphisms. This means that almost morphisms into finite dimensional nilspaces can be corrected into precise morphisms.
3. We show that a k-step compact nilspace is the inverse limit of finite dimensional ones.
4. We show that a finite dimensional compact nilspace consists of connected components that are nilmanifolds. In particular connected finite dimensional nilspaces are nilmanifolds.

To complete the picture about nilspaces we put in a chapter about category theoretic aspects of nilspaces. This is important for further generalizations in the subject. However proofs in the paper don't use the category theoretic terminology.

1.1 The role of nilspaces in Higher order Fourier analysis

This chapter is a short announcement of the upcoming paper [16]. The main goal in [16] is to give structure theorems for functions on compact abelian groups in terms of Gowers's uniformity norms. To be more precise let $f : A \to \mathbb{C}$ be a measurable function on the compact abelian group A such that $|f| \leq 1$. The goal is to decompose f as

$$f = f_s + f_e + f_r$$

where f_s is a structured part of bounded complexity, f_e is an error with small L^2 norm and f_r is quasi random with very small U_k norm. We will refer to this decomposition as the U_k**-regularity lemma.** (We omit here the precise statement) Note that the quadratic case ($k = 3$) was settled for arbitrary finite abelian groups in [15]. Of course the main question is the following.

What kind of structure is encoded in f_s?

It will turn out in [16] that f_s is the composition of two functions $\psi : A \to N$ and $g : N \to \mathbb{C}$ where N is a compact finite dimensional $k-1$-step nilspace of bounded complexity, ψ is a nilspace morphism and g is Lipschitz with bounded constant. (We omit here the definition of the complexity of a nilspace.)

The proof of the decomposition theorem is based on a decomposition theorem on ultra product groups. Let $\mathbf{A}$ be the ultra product of finite (or more generally compact) abelian groups. One can introduce a natural measure space structure on $\mathbf{A}$ and a σ-topology (like a topology but only countable unions of open sets need be open). A topological factor of $\mathbf{A}$ is given by a surjective continuous map $f : \mathbf{A} \to T$ (called factor map) where T is a separable compact Hausdorff space. (Such a factor can also be viewed as an equivalence relation on $\mathbf{A}$ whose classes are the fibres of f.) Every such factor inherits a cubespace structure from $\mathbf{A}$ by composing the cubes in $\mathbf{A}$ with the factor map f. A **nilspace factor** of $\mathbf{A}$ is a topological factor of $\mathbf{A}$ whose inherited cubespace structure satisfies the nilspace axioms.

The non-standard U_k-regularity lemma is much simpler and cleaner than the standard one. It says the following.

Non-standard U_k-regularity lemma: *Every measurable function $f : \mathbf{A} \to \mathbb{C}$ with $\|f\|_\infty \leq \infty$ can be (uniquely) decomposed as $f = f_n + f_r$ where*

$\|f_r\|_{U_k} = 0$ *and* f_s *is Borel measurable in a compact* $k-1$ *step nilspace factor.*

Note that U_k is only a semi-norm on $\mathbf{A}$ so it is possible that f_r is not 0 but $\|f_r\|_{U_k}$ is 0. The non-standard U_k-regularity lemma implies the ordinary one using the rigidity theorem for morphisms proved in the present paper. One can give restrictions on the structure of the nilspace factors if the abelian groups (that we take the ultra product of) are chosen from a special families (for example exponent 2 groups). We don't discuss the details of this here.

Limits of functions on abelian groups: Quite interestingly the non-standard U_k-regularity lemma can also be used for a different purpose. For every fixed natural number k, one can introduce limit objects for functions on finite (or compact) abelian groups which are measurable functions on compact k-step nilspaces. We demonstrate the convergence notion in a simplified version where the functions are $\{0,1\}$ valued. This means that they can be viewed as subsets in abelian groups. For a subset $S \subseteq A$ in an abelian group A we introduce a $k+1$ uniform hypergraph on the vertex set A whose edges are the $k+1$ tuples $(x_1, x_2, \dots, x_{k+1})$ in A^{k+1} satisfying $\sum_{i=1}^{k+1} x_i \in S$. We say that a sequence of subsets $\{S_i\}_{i=1}^{\infty}$ in abelian groups $\{A_i\}_{i=1}^{\infty}$ is k-convergent if the corresponding $k+1$-uniform hypergraphs converge in the sense of [2]. The non-standard U_{k+1}-regularity lemma implies that the appropriate limit objects for this convergence notion are measurable functions on k-step compact nilspaces. Note that as a special case we get limit objects for Cayley graphs in commutative groups. **Limits of Cayley graphs** in general (not necessarily commutative) groups are analyzed in [17]. It is proved that the limit of Cayley graphs is a Cayley graphon of a compact topological group.

1.2 Nilmanifolds as nilspaces

Let G be an at most k-nilpotent group. Let $\{G_i\}_{i=1}^{k+1}$ be a central series with $G_{k+1} = \{1\}$, $G_1 = G$ and $[G_i, G_j] \subseteq G_{i+j}$. We define a cubic structure on G which depends on the given central series. The set of n dimensional cubes $f : \{0,1\}^n \to G$ is the smallest set satisfying the following properties.

1. *The constant* 1 *map is a cube,*
2. *If* $f : \{0,1\}^n \to G$ *is a cube and* $g \in G_i$ *then the function* f' *obtained from* f *by multiplying the values on some* $(n-i)$*-dimensional face from the left by* g *is a cube.*

This definition builds up cubes by a generating system. However there is another way of describing them through equations. For every n we introduce an ordering $g_n : \{0,1\}^n \to \{1, 2, \dots, 2^n\}$ in the following way. If $n = 1$ then $g_1(0) = 1, g_1(1) = 2$. If $n > 1$ then

$$g_n(a_1, a_2, \dots, a_n) = g_{n-1}(a_1, a_2, \dots, a_{n-1})$$

if $a_n = 0$ and

$$g_n(a_1, a_2, \dots, a_n) = 2^n + 1 - g_{n-1}(a_1, a_2, \dots, a_{n-1})$$

if $a_n = 1$. It is clear that (a cyclic version of) this ordering defines a Hamiltonian cycle of the one dimensional skeleton of $\{0,1\}^n$.

Definition 1.2. *Let G be a group and $f : \{0,1\}^n \to G$. We say that f satisfy the* **Gray code property** *if*

$$\prod_{i=1}^{2^n} f(g_n^{-1}(i))^{(-1)^i} = 1.$$

A function $f : \{0,1\}^n \to G$ is a cube if for every $i \in \mathbb{N}$ and i-dimensional face F the restriction of f to F satisfies the Gray code property modulo G_i. If $i \geq k+1$ then we define G_1 to be trivial. An easy induction shows that cubes in G defined as above are symmetric under the automorphisms of $\{0,1\}^n$.

Assume that G has a transitive action on a set N. Then we say that $f : \{0,1\}^n \to N$ is a cube if $f(v) = x^{f'(v)}$ where $f' : \{0,1\}^n \to G$ is a cube and $x \in N$ is a fixed element.

1.3 The category of nilspaces

The definition of nilspace presented in the introduction implicitly makes use of what we call the *category of discrete cubes*: the category whose objects are the sets of the form $\{0,1\}^n$ and whose morphisms $\{0,1\}^m \to \{0,1\}^n$ are those functions which are restrictions of some affine homomorphism $\mathbb{Z}^m \to \mathbb{Z}^n$. There are two other descriptions of the morphisms of the category which are easily seen to be equivalent to this description:

1. $f : \{0,1\}^m \to \{0,1\}^n$ is a morphism of discrete cubes iff it can be written as $f(x_1, \dots, x_m) = (y_1, \dots, y_n)$ where each y_i is either 0, 1, x_j or $1 - x_j$ for some j (depending on i).
2. We can think of $\{0,1\}^n$ as the set of all characteristic vectors of subsets of $\{1, \dots, n\}$. A function $f : \{0,1\}^m \to \{0,1\}^n$ is a morphism sending $(0, \dots, 0)$ to $(0, \dots, 0)$ iff, regarded as a function from subsets of $\{1, \dots, m\}$ to subsets of $\{1, \dots, n\}$, it sends disjoint unions to disjoint unions. A general morphism (one not necessarily sending 0 to 0) is of the form $f(A) = g(A) \oplus S$ where g is a morphism sending 0 to 0, S is some subset of $\{1, \dots, m\}$, and $\oplus$ denotes the symmetric difference of sets.

We can rephrase the definition of nilspace in category theoretical terms using this category of discrete cubes. To begin, the condition that cubes in a nilspace be closed under composition with morphisms of discrete cubes says that given a nilspace N, the assignment

$$\{0,1\}^n \mapsto C^n(N) = \text{the set of } n\text{-cubes in } N,$$

is the object part of a contravariant functor from the category $\mathbf{Cubes}$ of discrete cubes to the category of sets (on a morphism β the functor gives the function of composition with β). In category theory, contravariant functors from a category $\mathcal{C}$ to the category of sets are often called *presheaves on* $\mathcal{C}$. The collection of all presheaves on $\mathcal{C}$ forms a category in which the morphisms are just the natural transformations of functors, and after recasting nilspaces as presheaves satisfying certain conditions we will indeed organize them into a category by simply taking all natural transformations between them as morphisms. From this point of view it also makes sense to talk about morphisms from arbitrary presheaves on $\mathbf{Cubes}$ into nilspaces, and we will occasionally do so.

To say which presheaves on $\mathbf{Cubes}$ arise from nilspaces, consider an abstract presheaf $F : \mathbf{Cubes} \to \mathbf{Sets}$. If it did come from a nilspace, we could recover the set of points as $N := F(\{0,1\}^0)$. (Well almost: the definition does not require every point of the underlying set of the nilspace to actually appear as a vertex in any cube. Of course, those points that are not vertices of cubes are totally irrelevant and can be ignored, or, say, assumed to not exist.) Also, given any abstract n-cube $c \in F(\{0,1\}^n)$, we could recover the function $\{0,1\}^n \to N$ that corresponds to c by using all the 2^n morphisms from the 0-dimensional cube to the n-dimensional one: namely, if for any $p \in \{0,1\}^n$, we denote by ι_p the morphism in the category of discrete cubes sending $\{0,1\}^0$ to $p \in \{0,1\}^n$, then the cube c corresponds to the function $\{0,1\}^n \to N$ given by $p \mapsto F(\iota_p)(c)$.

Definition 1.3. *We say that a presheaf $F : \mathbf{Cubes}^{\mathrm{op}} \to \mathbf{Sets}$ is* determined by its points *if for any n, the function*

$$F(\{0,1\}^n) \to \big(F(\{0,1\}^0)\big)^{2^n}$$

whose coordinates are the $F(\iota_p)$ for $p \in \{0,1\}^n$ is injective.

A morphism between presheaves determined by their points is just a function between their point sets that sends cubes to cubes. So these presheaves are exactly the cubespaces from the introduction.

Remark 1.1. *The property of being determined by points is analogous to (part of) the difference between a simplicial complex and a Δ-complex: in a simplicial complex each simplex is determined by the set of its vertices, while in a general Δ-complex this is not the case, and, for example, two simplices can share their boundary.*

Remark 1.2. *Presheaves on $\mathbf{Cubes}$ are closely related to what are called* cubical sets *in the algebraic topology literature (see e.g., the work of Brown* *and Higgins on strict ω-groupoids, or the recent book [1]). Our category of discrete cubes has a subcategory $\mathbf{Cubes}_0$ with the same objects but whose only morphisms are those given by formulas of the form $(x_1, \ldots, x_m) \mapsto (y_1, \ldots, y_n)$ where each y_i is either 0, 1 or x_j for some j depending on i such that the sequence of j's used is strictly increasing. This subcategory is generated by projections and embeddings of cubes into higher dimensional ones as faces. A cubical set is precisely a presheaf on this category $\mathbf{Cubes}_0$, which means that each presheaf F on $\mathbf{Cubes}$ gives rise to a cubical set by composition:*

$$\mathbf{Cubes}_0^{\mathrm{op}} \hookrightarrow \mathbf{Cubes}^{\mathrm{op}} \xrightarrow{F} \mathbf{Sets}.$$

Now we will restate the glueing property from the definition of nilspaces in a more geometrical language. This will be immediately clear to readers familiar with either cubical complexes or simplicial sets in algebraic topology.

Presheaves on any base category can be thought of in a geometrical way as some sort of complexes built out of objects of the base category by gluing along morphisms. So, just like cubical complexes, presheaves on $\mathbf{Cubes}$ are geometric objects that are built out of cubes. As a very simple example, each cube can be regarded as a representable presheaf: the n-cube is the contravariant functor $\{0,1\}^m \mapsto \mathrm{Hom}_{\mathbf{Cubes}}(\{0,1\}^m, \{0,1\}^n)$. The Yoneda lemma says that given an arbitrary presheaf F on $\mathbf{Cubes}$, the set of morphisms from this n-cube to F is in bijection with $F(\{0,1\}^n)$, which is what we previously called the set of n-cubes in F.

Now we will define a presheaf that corresponds to a *corner of a cube*, i.e., a cube minus one point: $\{0,1\}^n \setminus \{(1,1,\ldots,1)\}$. Such an object can be obtained by glueing together n different $(n-1)$-cubes along $(n-2)$ dimensional faces. Categorically, this means the n-corner is the colimit of the relevant diagram of $(n-2)$- and $(n-1)$-cubes. A simple alternative explicit description is as follows: the corner as a subset of $\{0,1\}^n$ is the union of the images of the face embeddings $\alpha_{i,n} : \{0,1\}^{n-1} \to \{0,1\}^n$, $\alpha_{i,n}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$, so we can define the n-corner as the presheaf of maps that factor through one of these:

$$\{0,1\}^m \mapsto \{\gamma : \{0,1\}^m \to \{0,1\}^n \mid \gamma \text{ factors through some } \alpha_{i,n}\}.$$

One can easily check that this has the desired property that morphisms of presheaves from the n-corner to an arbitrary presheaf F are in bijection with n-tuples $(c_1, \ldots, c_n)$ of $(n-1)$-cubes of F (that is, each $c_i \in F(\{0,1\}^{n-1})$) that fit together to form a corner, i.e., tuples such that $F(\alpha_{i,n-1})(c_j) = F(\alpha_{j+1,n-1})(c_i)$ for all $i \leq j$. Also notice that if F is a presheaf determined by its points, a morphism from the n-corner into F is simply a function from $\{0,1\}^n \setminus \{(1,1,\ldots,1)\}$ to the set of points of F, such that the restrictions to all $(n-1)$-dimensional faces of the corner are $(n-1)$-cubes of F.

The glueing condition in term of presheaves simply says that any morphism of presheaves (that is, any natural transformation) from the corner of an n-cube to a nilspace, can be extended to a morphism from the whole n-cube.

Remark 1.3. *The glueing property is reminiscent of the Kan condition for simplicial sets in algebraic topology. For those familiar with cubical sets it should be pointed out that this extension condition is not the same as the condition for a cubical set to be fibrant: that every morphism from a cube without the interior and without one face can be extended to the whole cube.*

Finally, note that erodicity is also an extension condition. One can define a presheaf that corresponds to two disjoint points (namely, this is the non-erogodic nilspace with two points and only constant maps as cubes). This embeds into the representable presheaf given by $\{0,1\}^1$ and ergodicity simply means that any morphism from the pair of points extends to the 1-dimensional cube.

2 Abstract nilspaces

2.1 Notation and basics

When composing two functions f and g we will use the notation $(f \circ g)(x)$ for $g(f(x))$.

Let N be a nilspace. For a natural number k we denote the set of k-dimensional cubes in N by $C^k(N)$. In $\mathbb{Z}^k$ we denote by 0^k and 1^k the everywhere 0 and everywhere 1 vectors. If S is a finite set and h is a subset of S then we denote by $\{0,1\}^S_h$ the set of vectors supported on h which can be regarded as the discrete cube of dimension $|h|$ is the obvious way.

Definition 2.1. *Let S be a finite set and H be an arbitrary set system in S. The collection of all cube morphisms*

$$\{f : \{0,1\}^n \to \{0,1\}^S_h \mid n \in \mathbb{N},\ h \in H\}$$

defines a presheaf structure on $\cup_{h\in H}\{0,1\}^S_h$. Cubic presheaves arising this way will be called **simplicial***.*

Note that without loss of generality we can assume that H is downwards closed. This means that if $h \in H$ then every subset of h is also in H. Such set systems are called simplicial complexes.

The above construction produces a cubespace for every simplicial complex. It is not quite a functor from simplicial complexes to cubespaces: any dimension-preserving simplicial map between simplicial complexes produces a morphism between the corresponding cubespaces, but maps that identify two vertices of a single simplex do not induce a morphism.

Lemma 2.1 (Simplicial gluing)**.** *Let N be a nilspace, S a finite set and P be a simplicial presheaf corresponding to a set system H. Then any morphism $f : P \to N$ extends to a morphism $f_2 : \{0,1\}^S \to N$ of the full cube $\{0,1\}^S$.*

Proof. We assume that H is a simplicial complex. If H is the full complex of subsets in S then there is nothing to prove. If H is not the full complex then there is a set h' which is not in H but every subset of h' is contained in H. Let $H' = H \cup \{h'\}$ be a new simplicial complex. The gluing axiom guarantees that we can extend f to $\cup_{h \in H'} \{0,1\}^S_h$ with the presheaf structure corresponding to H'. By iterating this step we can extend f to the full cube. □

2.2 Operations with nilspaces

If N_1 and N_2 are nilspaces then we define their **direct product** as the nilspace $N_1 \times N_2$ whose cubes are functions $f : \{0,1\}^n \to N_1 \times N_2$ such that the projections f_1 and f_2 to the direct components are both cubes. (Ergodicity and the gluing axiom hold automatically for $N_1 \times N_2$.

If N is a nilspace then the previous construction yields a nilspace structure on $N \times N$. However there is another interesting cubic structure on $N \times N$ and we will refer to it as the **arrow space**. In this construction a map $f_1 \times f_2 : \{0,1\}^n \to N \times N$ is a cube if the function $f' : \{0,1\}^{n+1} \to N$ defined by $f'(v,0) = f_1(v), f'(v,1) = f_2(v)$ is a cube in N. The arrow space has fewer cubes than the direct product. The arrow space is not necessarily ergodic but lemma 2.1 implies that it satisfies the gluing axiom and so all its ergodic components are nilspaces.

We will also need the following variants of arrow spaces. The i-th arrow space is a (not necessarily ergodic) nilspace on $N \times N$. Let $f_1, f_2 : \{0,1\}^n \to N$ be two maps. We denote by $(f_1, f_2)_i$ the map $g : \{0,1\}^{n+i} \to N$ such that $g(v,w) = f_1(v)$ if $w \in \{0,1\}^i \setminus \{1^i\}$ and $g(v,w) = f_2(v)$ if $w = 1^i$. If $f : \{0,1\}^n \to N \times N$ is a single map with components f_1, f_2 then we denote by $(f)_i$ the map $(f_1, f_2)_i$. A map $f : \{0,1\}^n \to N \times N$ is a cube in the i-th arrow space if $(f)_i$ is a cube in N.

2.3 The 3-cubes

In this section we define a class of cube spaces which will be useful in many calculations. These will simply be n-cubes of side length two, divided into unit cubes. (They are called 3-cubes because they have 3 vertices on each side). We will typically use them to form new cubes in a nilspace by glueing together other cubes into a 3-cube and taking the outer vertices, as justified in Lemma 2.2 below.

Let $T_n = \{-1,0,1\}^n$. For every vector $v = (v_1, v_2, \ldots, v_n) \in \{-1,1\}^n$ we define the cube

$$\Psi(v) = \prod_{i=1}^{n} \{0, v_i\}$$

in T_n. The cubes of the form $\Psi(v)$ span a cubespace structure on T_n (this just means that the N-cubes of T_n are taken to be the maps $\{0,1\}^N \to T_n$

that factor through the inclusion of some $\Psi(v)$). Note that in terms of the direct product introduced above, T_n is just $(T_1)^n$.

Let $f : \{-1,0,1\} \to \{0,1\}$ be a function such that $f(0) = 0$. Then $f^n : \{-1,0,1\}^n \to \{0,1\}^n$ is a morphism.

Similarly, let f be the function $f(1) = (1,0), f(0) = (0,0), f(-1) = (0,1)$. Then $q = f^n$ is an embedding of T_n into the $2n$ dimensional cube $\{0,1\}^{2n}$. By abusing the notation we will identify T_n with the subset $q(T_n)$ in $\{0,1\}^{2n}$.

Finally, let $\omega : \{0,1\}^n \to T_n$ be equal to f^n where $f(0) = -1$ and $f(1) = 1$. Since T_n is a subset of $\{0,1\}^{2n}$ we can regard ω as a map from $\{0,1\}^n$ to $\{0,1\}^{2n}$.

Lemma 2.2. *Let $m : T_n \to N$ be a morphism into a nilspace N. Then the composition $\omega \circ m$ is in $C^n(N)$.*

Proof. It is clear that T_n is simplicial in $\{0,1\}^{2n}$ so by lemma 2.1 the map m extends to $\{0,1\}^{2n}$. On the other hand ω is a cube morphism of $\{0,1\}^n$ into $\{0,1\}^{2n}$. □

2.4 Characteristic factors

Let N be a nilspace. A congruence of a N is an equivalence relation $\sim$ on N such that the cube space on $N/\sim$ induced by the map $N \to N/\sim$ satisfies the nilspace axioms. The nilspace $N/\sim$ obtained this way will be called a factor of N. In this section we introduce factors of nil-spaces that are crucial building blocks of them.

Definition 2.2. *Let $\sim_k$ be the relation defined through the property that $x \sim_k y$ if and only if there are two cubes $c_1, c_2 \in C^{k+1}(N)$ such that $c_1(1^{k+1}) = x, c_2(1^{k+1}) = y$ and $c_1(v) = c_2(v)$ for every element $v \in \{0,1\}^{k+1} \setminus \{1^{k+1}\}$.*

The relation $\sim_k$ is obviously reflexive and symmetric. The next lemma will imply transitivity.

Lemma 2.3. *Two elements $x, y \in N$ satisfy $x \sim_k y$ if and only if there is a cube $c \in C^{k+1}(N)$ such that $c(1^{k+1}) = y$ and $c(v) = x$ for all $v \in \{0,1\}^{k+1} \setminus \{1^{k+1}\}$.*

Proof. Let c_1, c_2 be two cubes satisfying the condition in definition 2.2. Let us define the map $\phi = f^{k+1} : T_{k+1} \to \{0,1\}^{k+1}$ on the 3-cube T_{k+1} where $f(-1) = 1, f(0) = 0, f(1) = 1$. We denote by $g : T_{k+1} \to N$ the function which is obtained from $\phi \circ c_1$ by modifying the value on 1^{k+1} from x to y. The condition on c_1 and c_2 guarantees that g is a morphism. Using lemma 2.2 we get that $\omega \circ g$ is in $C^{k+1}(N)$. □

Corollary 2.1. *The relation $\sim_k$ is an equivalence relation for every $k \in \mathbb{N}$ and nilspace N.*

Proof. Assume that in N three elements satisfy $x \sim_k y$ and $y \sim_k z$. then by symmetry and lemma 2.3 we obtain that there are two cubes $c_1, c_2 \in C^{k+1}(N)$ such that $c_1(1^{k+1}) = x, c_2(1^{k+1}) = z$ and $c_1(v) = c_2(v) = y$ for every $v \neq 1^{k+1}$. This means that $x \sim_k z$. □

Lemma 2.4. *Two elements $x, y \in N$ satisfy $x \sim_k y$ if and only if for every cube $c_1 \in C^{k+1}(N)$ with $c_1(0^{k+1}) = x$ the map $c_2 : \{0,1\}^{k+1} \to N$ satisfying*

$$c_2(0^{k+1}) = y \text{ and } c_2(v) = c_1(v) \ \forall\, v \in \{0,1\}^{k+1} \setminus \{0^{k+1}\}$$

is in $C^{k+1}(N)$.

Proof. Let $\phi = f^{k+1} : T_{k+1} \to \{0,1\}^{k+1}$ where $f(-1) = 0, f(0) = 0, f(1) = 1$. Let $g : T_{k+1} \to N$ be the function obtained from $\phi \circ c_1$ by modifying the value on $(-1)^{k+1}$ from x to y. Lemma 2.3 guarantees that g is a morphism. According to lemma 2.2 the composition of ω and g is in $C^{k+1}(N)$. On the other hand $c_2 = \omega \circ g$. □

Corollary 2.2. *For every $k \in \mathbb{N}$ and cube $c \in C^{k+1}(N)$ we have that if a function $c_2 : \{0,1\}^{k+1} \to N$ satisfies $c(v) \sim_k c_2(v)$ for every $v \in \{0,1\}^{k+1}$ then $c_2 \in C^{k+1}(N)$.*

Proof. We get the statement by iterating lemma 2.4. Note that by the symmetries of cubes the vector 0^{k+1} can be replaced by any other vector in lemma 2.4. □

Corollary 2.3. *A cube $c \in C^n(N/\sim_k)$ is uniquely determined by the elements $c(v)$ where $v \in \{0,1\}^n$ contains at most k one's.*

Proof. For $n = k + 1$ it follows directly from corollary 2.2. If $n > k + 1$ then straightforward induction on the number of one's in v complete the proof. □

Lemma 2.5. *For every $k \in \mathbb{N}$ and nilspace N the equivalence relation $\sim_k$ is a congruence.*

Proof. Let $M = N/\sim_k$ with the induced cubespace structure. It is clear that M satisfies the ergodicity property. We need to check the gluing axiom. Let $f : \{0,1\}^n \setminus \{1^n\} \to M$ be a map which is a morphism of the corner of the n dimensional cube to the presheaf M. We need to show that f extends to the whole cube $\{0,1\}^n$ as a morphism. Let T be the subset in $\{0,1\}^n$ of vectors with at most $k+1$ one's in the coordinates. Corollary 2.2 shows that the restriction of f to T can be lifted from M to N as a morphism. Let $\bar{f}$ denote a lift. Lemma 2.1 implies that $\bar{f}$ extends to a morphism $\bar{f}_2$ of the whole cube $\{0,1\}^n$ to N. It is easy to see that the composition (call it f_2) of $\bar{f}_2$ with the factor map $\pi : N \to M$ is equal to f when restricted to

$\{0,1\}^n \setminus \{1^n\}$. Now corollary 2.3 shows that the restriction of f_2 to each face in $\{0,1\}^n$ of dimension $n-1$ and containing 0^n is equal to f. This completes the proof. $\square$

Definition 2.3. *For a nilspace N we denote by $\mathcal{F}_k(N)$ the factor $N/\sim_k$. We say that N is a k*-**step nilspace** *if $N = \mathcal{F}_k(N)$.*

Another way of formulating the previous definition is that N is a k-step nilspace if and only if every morphism of the corner of the $k+1$ cube to N extends in a unique way to a morphism of the $k+1$ dimensional cube. In other words the gluing axiom for $k+1$ dimensional cubes holds in a stronger form where uniqueness of the extension is guaranteed. Note that this **unique closing property** also appears in the Host-Kra theory of parallelepiped structures.

Definition 2.4. *We say that a cubespace P has the* **lifting property** *if for every nilspace N and natural number k we have that every morphism $\phi : P \to \mathcal{F}_k(N)$ has a lift $\phi' : P \to \mathcal{F}_{k+1}(N)$ such that $\phi = \phi'$ modulo $\sim_k$.*

Lemma 2.6. *Every simplicial cubespace P has the lifting property.*

Proof. Lemma 2.1 shows that if $\phi : P \to \mathcal{F}_k(N)$ is a morphism then it extend to a morphism of the corresponding cube. On the other hand cubes have the lifting property by the definition of the cubespace structure on $\mathcal{F}_k$. $\square$

Lemma 2.7. *Let N be a k-step nilspace and $n \geq k+2$. A function $c : \{0,1\}^n \to N$ is in $C^n(N)$ if and only if its restrictions to $k+1$ dimensional faces with at least one point with 0 in the last coordinate are all in $C^{k+1}(N)$.*

Proof. Let P be the set of elements in $\{0,1\}^n$ with at most k ones. Note that P is the union of the k-dimensional faces containing 0^n. The condition of the lemma implies that c restricted to such faces are cubes. Using lemma 2.1 and the fact that N is k-step we get that there is a unique element c' in $C^n(N)$ whose restriction to P is equal to the restriction of c to P. We claim that $c = c'$. Let t be the maximal integer such that $c = c'$ on every element $v \in \{0,1\}^n$ with at most t ones in its coordinates. By contradiction assume that $t < n$. Then there is an element $w \in \{0,1\}^n$ with $t+1$ ones such that $c'(w) \neq c(w)$. Since $t > k$ It can be seen that w is contained in a $k+1$ dimensional face F such that every element in $F \setminus \{w\}$ has at most t ones and furthermore there is at least one point if F with 0 in its last coordinates. Such a face can be found by choosing the last $k+1$ elements from the support of w and then changing those coordinates in w.

We know that the restriction of c to F is in $C^{k+1}(N)$. Since there is only one way of completing $F \setminus \{w\}$ to a cube the proof is complete. $\square$

2.5 Linear and higher degree abelian groups

We will see that abelian groups appear in the structures of nilspaces in various ways as building blocks. Every abelian group A has a natural nilspace structure that we call "linear". Cubes in $C^n(A)$ are functions $f : \{0,1\}^n \to A$ satisfying

$$f(e_1, e_2, \ldots, e_n) = a_0 + \sum_{i=1} e_i a_i \tag{1}$$

for some elements $a_0, a_1, \ldots, a_n \in A$. There is however another way of describing these functions. If f satisfies (1) then every morphism $\phi : \{0,1\}^2 \to \{0,1\}^n$ in $\mathbf{Cubes}$ satisfies the property that

$$f(\phi(0,0)) - f(\phi(0,1)) - f(\phi(1,0)) + f(\phi(1,1)) = 0$$

and it is easy to see that it gives an alternative characterization for linear cubes. The advantage of the second description is that it can be naturally generalized. For an arbitrary map $f : \{0,1\}^n \to A$ to an abelian group let us introduce the weight of f by

$$w(f) = \sum_{v \in \{0,1\}^n} f(v)(-1)^{h(v)} \tag{2}$$

where $h(v) = \sum_{i=1}^n v_i$.

Definition 2.5. *For every $k \in \mathbb{N}$ and abelian group A let us define the nilspace $\mathcal{D}_k(A)$ on the point set A in the following way. A map $f : \{0,1\}^n \to A$ is in $C^n(\mathcal{D}_k(A))$ if and only if for every morphism $\phi : \{0,1\}^{k+1} \to \{0,1\}^n$ we have that $w(\phi \circ f) = 0$. We say that $\mathcal{D}_k(A)$ is the k*-**th degree structure** *on A.*

To check the gluing axiom in $\mathcal{D}_k(A)$ is a straightforward calculation.

Lemma 2.8. *One step nilspaces are affine abelian groups with the linear nilspace structure.*

Proof. Let N be a one step nilspace. Let us distinguish an arbitrary element $e \in N$ and call it identity. For every $x, y \in N$ we define $x + y$ as the unique extension of the morphism defined by $f(0,0) = e, f(1,0) = x, f(0,1) = y$ (of the corner of the two dimensional cube) to $(1,1)$. We need to check the abalian group axioms.

Commutativity of $+$ follows directly from the symmetry of $\{0,1\}^2$ interchanging $(1,0)$ and $(0,1)$.

If $x, y, z \in N$ the we can extend the map $g(0,0,0) = e, g(1,0,0) = x, g(0,1,0) = y, g(0,0,1) = z$ to the full cube $\{0,1\}^3$. Let g_2 denote the extension. The composition of g_2 by the maps $\phi_1, \phi_2 : \{0,1\}^2 \to \{0,1\}^3$, $\phi_1(a,b) = (a,a,b)$ and $\phi_2(a,b) = (a,b,b)$ shows associativity.

If $f(0,0) = x, f(1,0) = e, f(0,1) = e$ then the unique extension $y = f(1,1)$ satisfies $x + y = e$.

□

A generalization of the previous lemma will be important.

Definition 2.6. *Let N be a nilspace and $x \in N$. Then we define a cube-space $\partial_x(N)$ on the point set of N such that $f : \{0,1\}^n \to N$ is a cube in $\partial_x(N)$ if and only if the map $f' : \{0,1\}^{n+1} \to N$ defined by $f'(v,0) = f(v), f'(v,1) = x$ is a cube in N.*

It is easy to see from lemma 2.1 that $\partial_x(N)$ satisfies the gluing axiom however it is not necessarily ergodic. Nevertheless if N is a k-step nilspace then clearly all the ergodic components of $\partial_x(N)$ are $k-1$ step nilspaces.

Lemma 2.9. *If a k-step nilspace N satisfies that $x \sim_{k-1} y$ for every $x, y \in N$ then N is isomorphic to $\mathcal{D}_k(A)$ for some abelian group A.*

Proof. We use induction on k. Lemma 2.8 shows the statement for $k = 1$. Assume that $k \geq 2$ and the statement is already proved for $k-1$. Let e be a fixed element in N. After $k-1$ repeated applications of δ_e to N we obtain a 1-step nilspace $\partial_e^{k-1}N$. The condition that N is a single class of $\sim_{k-1}$ implies by lemma 2.2 that every function $f : \{0,1\}^k \to N$ is a cube. In particular $\partial_e^{k-1}N$ is ergodic. Lemma 2.8 implies that $\partial_e^{k-1}N$ is isomorphic to an abelian group A with the linear structure.

Let M be the arrow space over N. Since $k \geq 2$ we have that M is ergodic. Cubes of dimension $k+1$ in N are in a one to one correspondence with cubes of dimension k in M. We claim that two arrows $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in M are $\sim_{k-1}$ equivalent if and only if $x_1 - x_2 = y_1 - y_2$ in A. First notice that M is in a single $\sim_{k-2}$ class and so the factor $\mathcal{F}_{k-1}(M)$ satisfies the condition of the lemma with $k-1$. Let $f : \{0,1\}^k \to M$ be the map defined in a way that $f(0^k) = x$, $f(1,0,0,\dots,0) = y$ and $f(v) = e$ everywhere else. The induction hypothesis guarantees that $x = y$ in the factor $\mathcal{F}_{k-1}(M)$ if and only if f is a cube in M. This shows that $x \sim_{k-1} y$ if and only if x_1, x_2, y_1, y_2 form a two dimensional cube in $\partial_e^{k-1}(N) = A$. This proves the claim.

We obtain from the claim that if $c \in C^{k+1}(N)$ is an arbitrary cube then if we add the same element in $a \in A$ to the c values of two endpoints of an arbitrary edge in $\{0,1\}^{k+1}$ then the resulting new function is still a cube. By repeating this operation we can produce a new cube c' in which all but one of the vertices are mapped to e. Using that constant functions are all cubes and the unique closing property we obtain that c' has to be the constant function. In other words c can be obtained from the constant function with the inverses of the previous operations which shows that all the cubes are in $\mathcal{D}_k(A)$. The fact that every $2^{k+1} - 1$ points can be completed to a cube shows that the cubes in N are exactly the cubes in $\mathcal{D}_k(A)$.

□

Corollary 2.4. *If N is a k-step nilspace then every equivalence class of $\sim_{k-1}$ is an abelian group with the k-degree structure.*

2.6 Bundle decomposition of nilspaces

We give a structure theorem for k-step nilspaces which follows relatively easily from the axioms but which is useful as an intermediate step to prove deeper structure theorems.

Definition 2.7. *Let A be an abelian group. An (abstract) A-**bundle** over a set S is a set T with an action $\alpha : A \times T \to T$ and a bundle map $\pi : T \to S$ such that*

1. *the action α is free i.e. the stabilizer of every element is the trivial subgroup in A,*
2. *π gives a bijection between the orbits of A in T and the elements of S.*

*If the spaces A, S, T are topological then we will require that α is continuous. A k-**fold abelian bundle** with structure groups $A_1, A_2, \ldots, A_k$ is the last member of a sequence $T_0, T_1, \ldots, T_k$ of "factors" where T_0 is a one element set and T_i is an A_i bundle over T_{i-1}. k-fold abelian bundles come together with projections (bundle maps) $\pi_{i,j} : T_i \to T_j$ for $i \geq j$. By abusing the notation we use the short hand notation π_j for $\pi_{i,j}$.*

Note that if T is an A-bundle over S then fibres (preimages of points under π) can be regarded as affine versions of A. We will use the short hand notation $x + a$ for $\alpha(a, x)$. There is no distinguished bijection between the elements of a fibre F and A but there is a well defined difference map $F \times F \to A$ which, if $x, y \in F$, is given by the unique element in $a \in A$ satisfying $y + a = x$. We simply denote the difference of x and y by $x - y$.

Definition 2.8. *A **degree-k bundle** N is a k-fold abelian bundle with structure groups $A_1, A_2, \ldots, A_k$ and factors*
$T_0, T_1, \ldots, T_k = N$ such that N is a cubespace with the following property. For every $0 \leq i \leq k$, $n \in \mathbb{N}$ and $c \in C^n(T_{i+1})$ we have that

$$\{c_2 | c_2 \in C^n(T_{i+1})\ ,\ c \circ \pi_i = c_2 \circ \pi_i\} = \{c + c_3 | c_3 \in C^n(\mathcal{D}_{i+1}(A_{i+1}))\}$$

where $C^n(T_i) = \pi_i(C^n(N))$.

Theorem 1 (Bundle decomposition)**.** *A cubespace N is a degree-k bundle if and only if N is a k-step nilspace. Furthermore $\mathcal{F}_i(N)$ is equal to T_i for every $1 \leq i \leq k$.*

Proof. First we show that if N is a degree-k bundle then it is a k-step nilspace. It is clear that N satisfies the ergodicity axiom. It remains to show the gluing axiom. We use induction on i to prove it in T_i. If $i = 0$ then the statement is trivial.

Assume that we have gluing in T_i. Let $f : \{0,1\}^n \setminus \{1^n\} \to T_{i+1}$ be a morphism of the corner of the n-dimensional cube. The map $f \circ \pi_i$ has an extension $f_2 : \{0,1\}^n \to T_i$ to the full cube. Since $C^n(T_i) = \pi_i(C^n(N))$ we have that f_2 can be lifted (with respect to π_i) to a morphism $f_3 : \{0,1\}^n \to$

T_{i+1}. Let us consider $f_4 = f - f_3$ on $\{0,1\}^n \setminus \{1^n\}$. It follows by definition that f_4 is a morphism of the corner to $\mathcal{D}_{i+1}(A_{i+1})$ and so it can be extended to a morphism $f_5 : \{0,1\}^n \to \mathcal{D}_{i+1}(A_{i+1})$. Now it is clear that $f_3 + f_5$ is an extension of f to the full cube. The definition of degree-i bundles implies that $\mathcal{F}_i(N) = T_i$.

We prove the other direction by induction on k. The step $k = 0$ is trivial. Assume that it holds for $k-1$ and N is a k-step nilspace. By induction we have the $k-1$ degree bundle structure on $\mathcal{F}_{k-1}(N)$.

Let $M = \{(x,y) | x, y \in N,\ x \sim_{k-1} y\} \subset N \times N$. Note that $F \times F \subset M$ holds for every class F of $\sim_{k-1}$. We introduce an equivalence relation $\sim$ on M. Let F_1, F_2 be two $\sim_{k-1}$ classes of a k-step nilspace N. If $x_1, x_2 \in F_1$ and $y_1, y_2 \in F_2$ then we say that $(x_1, x_2) \sim (y_1, y_2)$ if $(x_1, y_1) \sim_{k-1} (x_2, y_2)$ in the arrow space N' of N. Note that N' is not necessarily ergodic but it will not cause any problem.

By lemma 2.9 we get that if $F_1 = F_2$ then $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_2 - x_1 = y_2 - y_1$. In other words, inside one class of $\sim_{k-1}$ the $\sim$ classes of vectors are naturally parametrized by the elements of the abelian group constructed in lemma 2.9.

The unique closing property implies that for every $x_1, x_2 \in F_1$ and $y_1 \in N$ there is a unique y_2 such that $(x_1, x_2) \sim (y_1, y_2)$. This creates a bijection ϕ between $\sim$ classes inside $F_1 \times F_1$ and $\sim$ classes inside $F_2 \times F_2$. We show that this map gives an isomorphism between the corresponding abelian groups. The definition of $\sim$ shows that if $(x_1, x_2) \sim (y_1, y_2)$ and $(x_2, x_3) \sim (y_2, y_3)$ then $(x_1, x_3) \sim (y_1, y_3)$. Inside one fibre the class of (x_1, x_3) is the sum of the classes of (x_1, x_2) and (x_2, x_3). It follows that ϕ preserves addition in both directions and so it is a group isomorphism.

Let us denote by A the unique abelian group formed by the $\sim$ classes in $F \times F$ for each $\sim_{k-1}$ class F. The group A acts on each $\sim_{k-1}$ class and so on the whole space N. We denote this action by simple addition. This action satisfies that if $x \in F_1, y \in F_2$ the $(x, x+a) \sim (y, y+a)$ for every $a \in A$. It follows that if $c : \{0,1\}^{k+1} \to N$ is any cube and $a \in A$ then by applying the action of a to the two endpoint of an arbitrary edge in c we get a cube. Assume now that two cubes c_1 and c_2 in $C^{k+1}(N)$ satisfy that $c_1 \sim_{k-1} c_2$. Then by repeating the previous operations we can create a new cube c_2' from c_2 that differs from c_1 at most at one vertex. Using the unique closing property this implies that $c_2' = c_1$ and $c_2 - c_1 \in \mathcal{D}_k(A)$. □

An interesting consequence of theorem 1 is that in a k-step nilspace N the $\sim_{k-1}$ classes are all isomorphic abelian groups with k-degree structures and there is a distinguished set of affine isomorphisms between any two of them. Let F_1 and F_2 be $\sim_{k-1}$ classes and let us fix elements $x \in F_1$ and $y \in F_2$. Then the map $\phi(x+a) = y+a\ ,\ a \in A_k$ defines an affine morphism between F_1 and F_2. Such maps will be called **local translations**. The next characterization of local translations follows directly from theorem 1.

Lemma 2.10. *Let N be a k-step nilspace. Let us fix two $\sim_{k-1}$ classes F_1, F_2 and two elements $x \in F_1, y \in F_2$. For every $z \in F_1$ we denote by $\phi_{x,y}(z)$ the unique closure of the corner $c : \{0,1\}^{k+1} \setminus \{1^{k+1}\} \to N$ defined by $c(v,0) = x$ if $v \neq (1^k, 0)$, $c(1^k, 0) = z$ and $c(v,1) = y$ if $v \in \{0,1\}^k \setminus \{1^k\}$. Then the map $\phi_{x,y}$ is the local translation corresponding to x and y.*

2.7 Sub-bundles and bundle morphisms

Definition 2.9. *Let T_k be a k-fold abelian bundle with structure groups $A_1, A_2, \ldots, A_k$, factors $T_0, T_1, \ldots, T_k$ and projections $\pi_1, \pi_2, \ldots, \pi_k$. We define the notion of a* **sub-bundle** *of T_k with structure groups $A'_1 \leq A_1, A'_2 \leq A_2, \ldots, A'_k \leq A_k$ and factors $T'_0 = T_0, T'_1 \leq T_1, \ldots, T'_k \leq T_k$. If $k = 0$ then $T'_0 = T_0$ and both are equal to a one point space. For a general k we have the condition that $T'_{k-1} = \pi_{k-1}(T'_k)$ is already a sub-bundle and for every $x \in T'_k$ we have that*

$$\{a | a \in A_k, a + x \in T'_k\} = A'_k.$$

In particular if $k = 1$ then a sub-bundle is just a coset of A'_1.

An important example for sub-bundles is the following. Let $P = \{0,1\}^n$ be a cube and N be a k-step nilspace. Let us consider the natural embedding $\mathrm{Hom}(P, N)$ into the direct power N^P. This means that every homomorphism $\phi : P \to N$ is represented by the vector whose component at coordinate $p \in P$ is $\phi(p)$. According to theorem 1, $\mathrm{Hom}(P, N)$ is a sub-bundle in N^P with structure groups $\mathrm{Hom}(P, \mathcal{D}_i(A_i))$.

Definition 2.10. *Let $T = T_{k+1}$ and $T' = T'_{k+1}$ be two k-fold abelian bundles with structure groups $\{A_i\}_{i=1}^k$,$\{A'_i\}_{i=1}^k$ and factors $\{T_i\}_{i=0}^k$,$\{T'_i\}_{i=0}^k$. We define the notion of a* **bundle morphism** *$\phi : T \to T'$ with structure morphisms $\{\alpha_i : A_i \to A'_i\}$ by the next two axioms.*

1. *If $1 \leq i \leq k$ we have $\pi_i(x) = \pi_i(y)$ then $\pi_i(\psi(x)) = \pi_i(\psi(y))$. In other words ψ induces well defined maps $\psi_i : T_i \to T'_i$*
2. *$\psi_i(x + a) = \psi_i(x) + \alpha_k(a)$ where $x \in T_i$, $1 \leq i \leq k$ and $a \in A_k$.*

We say that ψ is **totally surjective** *if all the structure morphisms are surjective.*

Lemma 2.11. *Let $\psi : T \to T'$ be a totally surjective bundle morphism between two k-fold bundles. Then*

1. *For every $t \in T'$ and $i \leq k$ we have that $\psi_i^{-1}(\pi_i(t)) = \pi_i(\psi^{-1}(t))$*
2. *For every $t \in T'$ we have that $\psi^{-1}(t)$ is a sub-bundle in T with structure groups $\{\ker(\alpha_i)\}_{i=1}^k$.*

Proof. Let us start with the first statement. We do downwards induction on i. Case $i = k$ is trivial. Assume that we have the statement for $i + 1$. It is clear that $\pi_i(\psi^{-1}(t)) \subset \psi_i^{-1}(\pi_i(t))$ so we have to prove the other containment. If $x \in T_i$ is an element with $\psi_i(x) = \pi_i(t)$ then for an arbitrary lift $y \in T_{i+1}$ with $\pi_i(y) = x$ we have that $\psi_{i+1}(y) = \pi_{i+1}(t) + a'$ for some $a' \in A'_{i+1}$. Using total surjectivity, there is an element $a \in A_{i+1}$ with $\alpha_{i+1}(a) = a'$ and so $\psi_{i+1}(y-a) = \pi_{i+1}(t)$. By induction we have that $y - a \in \pi_{i+1}(\psi^{-1}(t))$ and so $x = \pi_i(y-a) \in \pi_i(\psi^{-1}(t))$.

We prove that second statement by induction on k. Assume that it is true for $k-1$. By the first statement we have that $\psi_{k-1}^{-1}(\pi_{k-1}(t)) = \pi_{k-1}(\psi^{-1}(t))$ and so $\pi_{k-1}(\psi^{-1}(t))$ is a sub-bundle in T_{k-1}. If $x \in \psi^{-1}(t)$ then $x + a \in \psi^{-1}(t)$ for $a \in A_k$ if and only if $\alpha_k(a) = 0$. This means that $\psi^{-1}(t)$ is a sub-bundle of T and the kernel of α_k is the k-th structure group. □

Lemma 2.12. *A morphism ψ between two k-step nilspaces N and N' is a bundle morphisms between the corresponding k-degree bundles T and T'.*

Proof. Lemma 2.3 shows that if $x \sim_i y$ then $\psi(x) \sim_i \psi(y)$. This verifies the first axiom.

First we prove the second axiom when the nil-spaces are of the form $\mathcal{D}_i(A_i)$ and $\mathcal{D}_i(A'_i)$. The abelian group structure of A_i and A'_i can be recovered by applying ∂_x^{i-1} to the cubic structure with some fixed element x in A_i or A'_i. It is clear that ψ_i preserves this structure and so ψ_i has to be an affine homomorphism between the two abelian groups which means that $\psi_i(x+a) = \psi_i(x) + \alpha(a)$ where α is a homomorphism.

Now let F be a $\sim_{i-1}$ class in T_i. Then $F = \mathcal{D}(A_i)$ and by the first part of the proof we have that ψ_i restricted to F satisfies $\psi_i(x+a) = \psi_i(x) + \alpha_F(a)$ where $x \in F, a \in A_k$ and $\alpha_F : A_i \to A'_i$ is a group homomorphism.

It remains to show that we have the same group homomorphism α_F corresponding to each $\sim_{i-1}$ class. This follows from the fact that the relation $\sim$ defined in the proof of 1 is preserved under ψ_i because it is defined through cubes. □

Definition 2.11. *A morphism $\psi : N_1 \to N_2$ between two nilspaces will be called* **fibre surjective** *if for every $n \in \mathbb{N}$ the image of a $\sim_n$ class in N_1 is a $\sim_n$ class in N_2.*

The next lemma follows immediately from lemma 2.12

Lemma 2.13. *A fiber surjective map between two k-step nilspaces is a totally surjective bundle morphism between the corresponding k-fold bundles.*

We will need the next lemma.

Lemma 2.14. *Let $\phi : N \to N'$ be a fibre surjective morphism between two k-step nilspaces. Then every cube $c \in C^n(N')$ can be lifted to a cube $c' \in C^n(N_1)$ such that $c' \circ \phi = c$.*

Proof. The proof is an induction on k. If $k = 0$ then there is nothing to prove. Assume that we have the statement for $k-1$. The map ϕ induces a map ϕ' from $\mathcal{F}_{k-1}(N)$ to $\mathcal{F}_{k-1}(N')$. This means (using the lifting property of cubes) that there is a cube $c_2 \in C^n(N)$ such that $c_2 \circ \phi \sim_{k-1} c$ and so $c_3 = c_2 \circ \phi - c$ is in $C^n(\mathcal{D}(A'_k))$. Now it is enough to find a lift c_4 of c_3 under the surjective homomorphism $\alpha_k : A_k \to A'_k$ because then $c_2 - c_4$ is a lift of c.

the existence of c_4 follows by first considering an arbitrary lift of a k-dimensional corner of c_3 and then by extending it (uniquely) to an n-dimensional cube. □

The previous lemma together with lemma 2.1 implies the next corollary.

Corollary 2.5. *Let $\phi : N \to N'$ be a fibre surjective morphism between two k-step nilspaces. Then every morphisms $m : P \to N'$ of a simplicial cube space can be lifted as a morphism $m' : P \to N$ with $m' \circ \phi = m$.*

An important example of a fiber surjective map is the following. Let N be a k-step nilspace with structure groups $A_1, A_2, \dots, A_k$ and let $B \subseteq A_k$ be a subgroup of A. We introduce a nilspace denoted by N/B in the following way. Let us say that two elements $x, y \in N$ satisfy $x \sim_B y$ if $x \sim_{k-1} y$ and $x - y \in B$. The elements of N/B are the equivalence classes of $\sim_B$. It follows from theorem 1 that N/B is a factor of N and the projection $N \to N/B$ is fibre surjective.

2.8 Restricted morphisms

Definition 2.12. *Let $P_2 \subset P$ be a subset of the cubespace P and let $f : P_2 \to N$ be an arbitrary function. We define the* **restricted homomorphism set** $\mathrm{Hom}_f(P, N)$ *as the collection of those homomorphisms whose restrictions to P_2 is equal to f.*

Note that the restricted homomorphism sets might be empty.

Lemma 2.15. *Let C_1, C_2 be two elements in $\mathbf{Cubes}$ and let $\phi : C_1 \to C_2$ be an injective morphism. Then there is an endomorphism $\psi : C_2 \to \phi(C_1)$ such that $\phi \circ \psi = \phi$. If $f : C_1 \to P$ is any morphism to a cubespace P then there is a morphism $m : C_2 \to P$ such that $f = \phi \circ m$.*

Proof. Assume that $C_1 = \{0,1\}^a$ and $C_2 = \{0,1\}^b$. The morphism ϕ is of the form $\phi(x_1, x_2, \dots, x_a) = (y_1, y_2, \dots, y_b)$ where each y_i is equal to one of $x_j, 1-x_j, 0, 1$ for some $1 \le j \le a$. Now let $V_1, V_2, \dots, V_a, W_0, W_1$ be the partition of $[b]$ defined in a way that $j \in V_i$ if $y_j = x_i$ or $y_j = 1-x_i$, $j \in W_0$ if $y_j = 0$ and $j \in W_1$ if $y_j = 1$. We define a further partition $V_i = V_i^0 \cup V_i^1$ such that $j \in V_i^0$ if and only if $y_j = x_i$. Let us choose a representative system $\{t_i \in V_i\}_{i=1}^a$ and assume that $t_i \in V_i^{e_i}$ for some $e \in \{0,1\}$. Now we define ψ in the following way. The value $q = \psi_j(z_1, z_2, \dots, z_b)$ satisfies

1. $q = 0$ if $z_j \in W_0$
2. $q = 1$ if $z_j \in W_1$

3. $q = z_{t_i}$ if $j \in V_i^{e_i}$
4. $q = 1 - z_{t_i}$ if $j \in V_i^{1-e_i}$.

It can be seen easily that ψ satisfies the requirement.

The second statement follows from the first one. Let $f' : \phi(C_1) \to P$ be defined as $\phi^{-1} \circ f$. Let $m = \psi \circ f'$. Then it is clear that m satisfies the requirement. □

Lemma 2.16. *Let C_1 and C_2 be as in lemma 2.15 and let us identify C_1 with $\phi(C_1)$. Let N be a k-step nilspace and $f : C_1 \to N$ be a morphism. Then $\mathrm{Hom}_f(C_2, N)$ is a sub-bundle of N^{C_2}.*

Proof. We proceed by induction on k. There is nothing to prove for $k = 0$. Assume that the statement if true for $k-1$. We have that $H = \mathrm{Hom}_{f\circ\pi_{k-1}}(C_2, N)$ is a sub-bundle of $\mathcal{F}_{k-1}(N)^{C_2}$. First we show that every element h in H can be lifted to an element m in $\mathrm{Hom}_f(C_2, N)$. The morphism h is a cube in $\mathcal{F}_{k-1}(N)$ so it can be lifted to a cube $h' : C_2 \to N$. We have that $f - h'$ on C_1 is a morphism of C_1 into $\mathcal{D}_k(A_k)$. Then by lemma 2.15 we get that $f - h'$ can be extended to a morphism $m' : C_2 \to \mathcal{D}_k(A_k)$. It is clear now that $m = m' + h'$ is in $\mathrm{Hom}_f(C_2, N)$.

A function $g : C_2 \to N$ is a lift of h to a morphism in $\mathrm{Hom}_f(C_2, N)$ if it differs from m by a morphism in $H_2 = \mathrm{Hom}_z(C_2, \mathcal{D}_k(A_k))$ where $z : C_1 \to \mathcal{D}_k(A_k)$ is the function mapping every element into to 0. It is clear that H_2 is an abelian group. □

Lemma 2.17. *Let $P = \{0,1\}^n$ be a cube and P_2 be a subcube. Let $\psi : N \to N'$ be a fibre surjective morphism between two k-step nilspaces. Then*

1. *$\mathrm{Hom}(P, N)$ is a sub-bundle in the direct power N^P with structure groups $\mathrm{Hom}(P, \mathcal{D}_i(A_i))$*
2. *$\psi^P : \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, N')$ is a totally surjective bundle morphism with structure morphisms*

$$\alpha_i^P : \mathrm{Hom}(P, \mathcal{D}_i(A_i)) \to \mathrm{Hom}(P, \mathcal{D}_i(A_i'))$$

3. *The preimage of $t \in \mathrm{Hom}(P, N')$ under $(\psi^P)^{-1}$ is a bundle with structure groups $\mathrm{Hom}(P, \mathcal{D}_i(\ker(\alpha_i)))$.*
4. *Let $t \in \mathrm{Hom}(P, N')$ and let $t_2 \in \mathrm{Hom}(P_2, N')$ be its restriction to P_2. Then the projection π_{P_2} from $(\psi^P)^{-1}(t)$ to $(\psi^{P_2})^{-1}(t_2)$ is a totally surjective bundle morphism.*

Proof. We prove the first statement by induction on k. For $k = 0$ it is trivial. If we know the statement for $k - 1$ then we have by the lifting property of cubes that $\mathrm{Hom}(P, \mathcal{F}_{k-1}(N)) = \pi_{k-1}(\mathrm{Hom}(P, N))$ and so we have that $\pi_{k-1}(\mathrm{Hom}(P, N))$ is a sub-bundle of $\mathcal{F}_{k-1}(N)^P$. Let $\psi \in \mathrm{Hom}(P, \mathcal{F}_{k-1}(N))$. If ψ' is any lift of ψ to N then by theorem 1 the other preimages of ψ are exactly those that differ from ψ' by an element in $\mathrm{Hom}(P, \mathcal{D}_k(A_k))$, which is clearly a subgroup in A_k^P.

For the second statement we check the two axioms of bundle morphisms. The first axiom follows from the fact (use lemma 2.3) that the map ψ^P preserves the relation $\sim_i$. Let $c \in \mathrm{Hom}(P, \mathcal{F}_i(N))$. It is clear that the structure morphisms are given by α_i^P on $\mathrm{Hom}(P, \mathcal{D}_i(A_i))$ but we have to show that they map surjectively to $\mathrm{Hom}(P, \mathcal{D}_i(A'_i))$. This follows by taking an arbitrary preimage of an i-dimensional full corner of P under α_i^{-1} and then by extending it in a unique way to a full morphism of P.

The third statement follows directly from lemma 2.11.

In the fourth statement the structural maps are computed as

$$\mathrm{Hom}(P, \mathcal{D}_i(\ker(\alpha_i))) \to \mathrm{Hom}(P_2, \mathcal{D}(\ker(\alpha_i))).$$

It follows from lemma 2.15 that these are surjective maps. □

Lemma 2.18. *Let*

$$P = \{0,1\}^{2n},\ P_2 = \{(0,1),(1,0)\}^n,\ P_3 = \{(0,0),(1,0)\}^n,$$

and $u = (1,0)^n = P_3 \cap P_2$. *If* $f : P_2 \to N$ *is a morphism into a* k*-step nilspace* N *then the projection* $\mathrm{Hom}_f(P, N) \to \mathrm{Hom}_{f|_u}(P_3, N)$ *is a totally surjective bundle morphism.*

Proof. Lemma 2.16 shows that $\mathrm{Hom}_f(P, N)$ and $\mathrm{Hom}_{f|_u}(P_3, N)$ are subbundles in the spaces N^P and N^{P_3}. The structure groups are $\mathrm{Hom}_{z_1}(P, \mathcal{D}_i(A_i))$ and $\mathrm{Hom}_{z_2}(P_3, \mathcal{D}_i(A_i))$ where z_1 is the 0 map on P_2 and z_2 is the 0 map on u. We have to show that the natural projection between the structure groups is surjective. Similarly to the proof of lemma 2.3 we consider the map $\phi = f^n : T_n \to \{0,1\}^n$ on the 3-cube $T_n \subset P$ so that $f(-1) = 1, f(0) = 0, f(1) = 1$. By identifying $\{0,1\}^n \subset T_n$ with P_3 we get that any morphism $g : P_3 \to \mathcal{D}_i(A_i)$ with $g(u) = 0$ can be lifted to the three cube T_n as $g_2 = \phi \circ g$. It is clear that g_2 restricted to P_2 is the constant 0 function. Then lemma 2.1 says that we can further extend g_2 to P as a morphism $g_3 : P \to N$. We have that $g_3|_{P_2}$ is the 0 map. This proves the surjectivity in question. □

2.9 Extensions and cohomology

Definition 2.13. *Let* N *be an arbitrary nilspace. A degree* k*-extension of* N *is an abelian bundle* M *over* N *which is a cube space with the following properties.*

1. *For every* $n \in \mathbb{N}$ *and* $c \in C^n(N)$ *there is* $c' \in C^n(N)$ *such that* $\pi(c') = c$,
2. *If* $c_1 \in C^n(M)$ *and* $c_2 : \{0,1\}^n \to M$ *with* $\pi(c_1) = \pi(c_2)$ *then* $c_2 \in C^n(M)$ *if and only if* $c_1 - c_2 \in C^n(\mathcal{D}_k(A))$.

The map π is the projection from M to N. The extension M is called a split extension if there is a cube preserving morphism $m : N \to M$ such that $m \circ \pi$ is the identity map of N.

A motivation to study such extensions is that we can obtain every k-step nilspace from a trivial nilspace by k consecutive extensions of increasing degree. In the rest of the chapter we assume that M is a k-degree extension of N and that $\sim$ is the equivalence relation whose classes are the fibres of π.

The main idea of describing extensions in the following. Let us choose a representative system $S \subset M$ for the $\sim$ classes and let $r : M \to S$ be the function such that $r(x)$ is the representative of the class containing x. Then we define the function $f : M \to A$ by $f(x) = x - r(x)$. For an arbitrary cube $c \in C^{k+1}(M)$ we define its weight $\varrho(c)$ as the weight (see (2)) of the function $c \circ f$.

We have from the definition 2.13 that $\varrho(c)$ is determined by $c \circ \pi$. In other words ϱ can be defined as a function $\varrho : C^{k+1}(N) \to A$. Two natural questions arise.

Which functions $\varrho : C^{k+1}(N) \to A$ arise from some k-degree extension of N by A?

What happens to ϱ if we change the representative system S?

The answer to the second question is quite easy. Let $B(N, A)$ denote the set of functions $h : C^{k+1}(N) \to A$ such that there is some function $f : N \to A$ with $h(c) = w(c \circ f)$. The elements of $B(N, A)$ form an abelian group with respect to point wise addition. It is clear that if we modify S then the new function ϱ_2 differs from the original by an element in $B(N, A)$.

To answer the first question we need to understand the properties of weight functions arising from extensions. We define a subgroup $H_{p,q}$ of $A^{\mathrm{Hom}(C_1,C_2)}$ where $C_1 = \{0,1\}^p$ and $C_2 = \{0,1\}^q$. A map $m : \mathrm{Hom}(C_1, C_2) \to A$ is in $H_{p,q}$ if and only if there is a function $f : C_2 \to A$ such that $m(\phi) = w(\phi \circ f)$.

It is clear from the definition of ϱ and the lifting property of cubes that if $c \in C^n(N)$ is an arbitrary cube then the map $m_c : \mathrm{Hom}(\{0,1\}^{k+1}, \{0,1\}^n) \to A$ defined by $m_c(\phi) = \varrho(\phi \circ c)$ is an element in $H_{k+1,n}$. We define (degree k) cocycles as functions $\varrho : C^{k+1}(N) \to A$ satisfying this property. Let $Y(N, A)$ denote the set of degree-k cocycles. It is clear from the definition that they form an abelian group and that $B(N, A) \subset Y(N, A)$. The cohomology group $H(N, A)$ is defined as the factor group $Y(N, A)/B(N, A)$.

Our goal is to show that every element in $H(N, A)$ represents an extension. Let ϱ be a cocycle in $Y(N, A)$. We define a cubspace structure on the point set $M = N \times A$ in the following way. A map $c : \{0,1\}^n \to M$ is a cube if its projection to N is a cube and for every morphism $\phi : \{0,1\}^{k+1} \to \{0,1\}^n$ we have that $w(\phi \circ c \circ \pi_A) = \varrho(\phi \circ c \circ \pi_N)$. It is clear that it creates a cubespace structure on M. Simple calculation show that the gluing axiom is also true.

Now we give another description of the group $Y(N,A)$ in terms of finitely many equations.

Definition 2.14. *A function $g : C^n(N) \to A$ is said to be automorphism consistent if it satisfies the next condition. If $c : \{0,1\}^n \to N$ is in $C^n(N)$ and $\phi : \{0,1\}^n \to \{0,1\}^n$ is an automorphism then $g(\phi \circ c) = g(c)(-1)^q$ where q is the number of 1's in $\phi(0^n)$*

Definition 2.15. *A function $\varrho : C^{k+1}(N) \to A$ is a cocycle if it satisfies the next three axioms.*

1. *ϱ is automorphism consistent*
2. *For every pair of equivalent cubes $c_1, c_2 \in C^{k+1}(N)$ let us define $\varrho'(c_1c_2) = \varrho(c_1) - \varrho(c_2)$. Then $\varrho' : C^{k+2}(N) \to A$ is automorphism consistent.*
3. *If $c_1, c_2, c_3 \in C^k(N)$ are three equivalent cubes then $\varrho(c_1c_2)+\varrho(c_2c_3) = \varrho(c_1c_3)$.*

2.10 Translations

For an arbitrary subset F in $\{0,1\}^n$ and map $\alpha : N \to N$ we define the map α^F from $C^n(N)$ to $N^{\{0,1\}^n}$ such that $\alpha^F(c)(v) = \alpha(c(v))$ if $v \in F$ and $\alpha^F(c)(v) = c(v)$ if $v \notin F$.

Definition 2.16. *Let N be a nilspace. A map $\alpha : N \to N$ is called a translation of hight i if for every natural number $n \geq i$, $n-i$ dimensional face $F \subseteq \{0,1\}^n$ and $c \in C^n(N)$ the map $\alpha^F(c)$ is in $C^n(N)$. We denote the set of hight i translations by $\mathrm{Trans}_i(N)$. We will use the short hand notation $\mathrm{Trans}(N)$ for $\mathrm{Trans}_1(N)$.*

It is clear from this definition that

$$\mathrm{Trans}_1(N) \supseteq \mathrm{Trans}_2(N) \supseteq \mathrm{Trans}_3(N) \supseteq \dots.$$

Lemma 2.19. *A map $\alpha : N \to N$ is in $\mathrm{Trans}_i(N)$ if and only if the map $h : N \to N \times N$ defined by $h(n) = (n, \alpha(n))$ is a morphism into the i-th arrow space.*

Proof. It is clear that $\alpha \in \mathrm{Trans}_i(N)$ implies that h is a morphism. For the other direction assume that h is a morphism. Let $c \in C^n(N)$ be such that $n \geq i$. Let $F \subset \{0,1\}^n$ be the $n-i$ dimensional face with 0's in the last i coordinates. Using the symmetries of cubes it is enough to show that for this particular face $\alpha^F(c) \in C^n(N)$.

Let $Q = \{0,1\}^{n-i} \times \{-1,0,1\}^i = \{0,1\}^{n-i} \times T_i$, let f_1 be the identity on $\{0,1\}$ and f_2 be the function with $f_2(-1) = 1$, $f_2(0) = 0$, $f_2(1) = 0$. Let

$f = f_1^{n-i} \times f_2^i$. The function $h = f \circ c$ is a morphism from Q to N. Let h' be the function obtained from h by applying α to the values on $\{0,1\}^{n-i} \times 1^i$.

It is easy to see from our assumption that h' is also a morphism to N. On the other hand by lemma 2.1 the restriction of h' to $\{0,1\}^{n-i} \times \{-1,1\}^i$ is a morphism to N. This restriction is equal to $\alpha^F(c)$. □

Note that definition 2.16 implies that translations preserve cubes. Recall that two cubes in $C^n(N)$ are called equivalent if they are two opposite faces of a cube in $C^{n+1}(N)$. It is clear that a map α is a translation if and only if $\alpha(c)$ is equivalent with c for every cube $c \in C^n(N)$. The next lemma shows a strengthening of this fact for k-step nilspaces.

Lemma 2.20. *Let N be a k-step nilspace. An arbitrary map $\alpha : N \to N$ is a in $\mathrm{Trans}_i(N)$ if and only if for every $c \in C^k(N)$ we have that $(c, \alpha(c))_i \in C^{k+i}(N)$.*

Proof. Let $c \in C^n(N)$ be an arbitrary cube and let $c' = (c, \alpha(c))_i$. By lemma 2.19 it is enough to prove that $c' \in C^{n+i}(N)$. formed by c and $\alpha(c)$ as two faces. Using lemma 2.7 it is enough to show that c' restricted to $k+1$ dimensional faces in $\{0,1\}^n$ with at least one point with 0 in the last coordinate are cubes. This follows immediately from the condition of the lemma. □

Lemma 2.21. *Let N be a k-step nilspace. Then translations restricted to $\sim_{k-1}$ classes are local translations.*

Proof. It follows from lemma 2.3 that if $x \sim_{k-1} y$ then $\alpha(x) \sim_{k-1} \alpha(y)$. Lemma 2.10 shows that if the $\sim_{k-1}$ classes of x and $\alpha(x)$ are F_1 and F_2 then $\alpha(x+a) = \alpha(x) + a$ for an arbitrary element a in the structure group A_k. □

Lemma 2.22. *If N is a k-step nilspace then $\mathrm{Trans}(N)$ is a group.*

Proof. By induction on k and using lemma 2.21 we get that translations are invertible transformations. We need to show that the inverse of a translation α is again a translation. We go by induction on k. Assume that we have the statement for $k-1$. Then in particular we have that the image of a k dimensional cube c under α^{-1} is a cube modulo $\sim_{k-1}$. This means by lemma 2.2 that $\alpha^{-1}(c)$ is also in $C^k(N)$. Since $\alpha(\alpha^{-1}(c)) = c$ we obtain that $(\alpha^{-1}(c), c) \in C^{k+1}(N)$. By lemma 2.20 applied with $i = 1$ the proof is complete. □

2.11 Translation bundles

Let N be a k-step nilspace and let α be an element in $\mathrm{Trans}_i(\mathcal{F}_{k-1}(N))$. We say that α can be lifted to $\mathrm{Trans}_i(N)$ if there is an element $\alpha' \in \mathrm{Trans}_i(N)$ such that $\pi_{k-1}(\alpha'(n)) = \alpha(\pi_{k-1}(n))$ holds for every $n \in N$. Recall that π_{k-1} is the projection to $\mathcal{F}_{k-1}(N)$. Our goal is to understand when can α be lifted this way. We introduce a nilspace whose algebraic properties decide if there is such a lift or not.

Let $\mathcal{T} = \mathcal{T}(\alpha, N, i)$ be the set of pairs $(x, y) \in N^2$ where $\alpha(\pi_{k-1}(x)) = \pi_{k-1}(y)$. We interpret $\mathcal{T}$ as a subset of the i-th arrow space over N. It is easy to see that if $k \geq i+1$ then $\mathcal{T}$ is an ergodic nilspace with the inherited cubic structure.

We define $\mathcal{T}^*$ as $\mathcal{F}_{k-1}(\mathcal{T})$. We will use the next two algebraic properties of $\mathcal{T}^*$.

1. The group $A_k \times A_k$ acts on the space $\mathcal{T}$ by

$$(x, y) \mapsto (x + a_1, y + a_2).$$

   This action induces an action of A_k on $\mathcal{T}^*$. For $a_1, a_2 \in A_k$ we have that $(x + a_1, y + a_2) \sim_{k-1} (x, y)$ if and only if $a_1 = a_2$. It follows that the elements of $\mathcal{T}^*$ represent local translations $\phi : F_1 \to F_2$ where F_1, F_2 are $\sim_{k-1}$ classes in N with $\alpha(F_1) = F_2$.

2. The map $(x, y) \mapsto x$ creates a map $\mathcal{T} \to N$. It induces a map $\gamma : \mathcal{T}^* \to \mathcal{F}_{k-1}(N)$.

Combining these two facts one can see easily that $\mathcal{T}^*$ is a degree $k - i$ extension of $\mathcal{F}_{k-1}(N)$ by A_k.

Proposition 2.1. *Let N be a k-step nilspace and $\alpha \in \mathrm{Trans}_i(\mathcal{F}_{k-1}(N))$. If $\mathcal{T}^* = \mathcal{T}^*(\alpha, N, i)$ is a split extension then α lifts to an element $\beta \in \mathrm{Trans}_i(N)$.*

Proof. Let $\gamma' : \mathcal{F}_{k-1}(N) \to \mathcal{T}^*$ be a morphism such that $\gamma' \circ \gamma$ is the identity map. The element $\gamma'(\pi_{k-1}(x))$ in $\mathcal{T}^*$ represents a local translation from the $\sim_{k-1}$ class F_1 of x to the class $\alpha(F_1)$. Let $\beta(x)$ denote the image of x under this local translation. We claim that the map β is in $\mathrm{Trans}_i(N)$. Let $h : N \to N \times N$ be the map defined by $h(n) = (n, \beta(n))$. According to lemma 2.20 it is enough to show that for every $c \in C^k(N)$ we have that $c \circ h$ is a cube in the i-th arrow space on $N \times N$. Since γ' is a morphism we have that $\gamma'(\pi_{k-1}(c))$ is in $C^k(\mathcal{T}^*)$. By lemma 2.2 we obtain that any lift of $\gamma(\pi_{k-1}(c))$ to $\mathcal{T}$ is in $C^k(\mathcal{T})$. The pairs $\{(c(v), \beta(c(v)))|v \in \{0,1\}^k\}$ form such a lift. This shows that $h \circ c$ in a cube in $\mathcal{T}$. $\square$

The condition of lemma 2.1 holds for α if and only if $\mathcal{T}_0(\alpha, N)$ is a split extension. A way of checking the condition is to show that the cocycle describing $\mathcal{T}_0(\alpha, N)$ as an extension of $\mathcal{F}_{k-1}$ by A_k is a coboundary.

2.12 Nilpotency

Let N be a k-step nilspace. In this part we investigate the properties of the groups $\mathrm{Trans}_i(N)$.

Lemma 2.23. *We have that* $[\mathrm{Trans}_i(N), \mathrm{Trans}_j(N)] \subseteq \mathrm{Trans}_{i+j}(N)$.

Proof. Let F be a face in $\{0,1\}^n$ of codimension $i+j$. Then $F = F_1 \cap F_2$ where F_1 is a face of codimension i and F_2 is a face of codimension j. Assume that $\alpha_1 \in \mathrm{Trans}_i(N)$ and $\alpha_2 \in \mathrm{Trans}_j(N)$. Then $[\alpha_1^{F_1}, \alpha_2^{F_2}] = [\alpha_1, \alpha_2]^F$. This implies that if $c \in C^n(N)$ then $[\alpha_1, \alpha_2]^F(c) \in C^n(N)$. □

Corollary 2.6. *The group* $\mathrm{Trans}(N)$ *is* k*-nilpotent and* $\{\mathrm{Trans}_i(N)\}_{i=1}^{k+1}$ *is a central series in it.*

Lemma 2.24. *if* $k \geq i$ *then the action of* A_k *is in* $\mathrm{Trans}_i(N)$.

Proof. It follows directly from theorem 1. □

Definition 2.17. *We say that two cubes* $c_1, c_2 \in C^n(N)$ *are translation equivalent if* c_2 *can be obtained from* c_1 *be a sequence applications of operations* α^F *where* $\alpha \in \mathrm{Trans}_i(N)$ *and* F *is a face in* $\{0,1\}^n$ *of codimension* i. *Note that the number* i *can be different in the above operations. A cube is called translation cube if it is translation equivalent with a constant cube.*

3 Compact nilspaces

In this part of the paper we study compact topological versions of nilspaces.

Definition 3.1. *A nilspace is called compact if all the sets* $C^n(N)$ *are second countable Hausdorff topological spaces and the maps* $\hat{\phi} : C^m(N) \to C^n(N)$ *(defined in the introduction) are continuous for every* $n, m \in \mathbb{N}$ *and morphism* $\phi : \{0,1\}^n \to \{0,1\}^m$.

An important consequence of compactness is that $\mathcal{F}_k(N)$ is compact for every $k \in \mathcal{N}$. Furthermore all the abelian groups occurring in theorem 1 are compact abelian groups.

3.1 Haar measure on abelian bundles and nilspaces

Compact k-step nilspaces are generalizations of compact abelian groups. It will be important to generalize the normalized Haar measure to them. Recall that the normalized Haar measure is a shift invariant Borel probability measure. Such measures always exist on compact groups and they are unique.

First we define the Haar measure for compact abelian bundles. Let T be an A bundle over a set S and action $\alpha : A \times T \to T$. Assume that T, S and A are compact Hausdorff spaces, A is a topological group and α is continuous. Assume that S has a Borel probability measure μ_S. Then we introduce the extension μ of μ_S as the unique Borel probability measure on T which is A invariant. The measure μ can be defined through the property that

$$\mu(H) = \int_{s\in S} \mu_A(\pi_S^{-1}(s) \cap H)\, d\mu \tag{3}$$

where H is a Borel set of T and π is the projection to S.

We define the Haar measure on a compact k-fold abelian bundle iteratively. If it is already defined for $k-1$ fold bundles then we use (3) to extend it from the factor T_{k-1} to T_k. We use theorem 1 to define (normalized) Haar measures for k-step nilspaces.

By abusing the notation we will always denote the Haar measure by μ. Since we never define two different measure on one structure it will not cause any problem.

The following fact is well known for compact abelian groups.

Lemma 3.1. *Surjective continuous (affine) homomorphisms between compact abelian groups are measure preserving.*

We will need a generalization of this fact for k-fold compact abelian bundles.

Lemma 3.2. *Let $\phi : T \to T'$ be a totally surjective continuous map between two compact k-fold abelian bundles. Then ϕ preserves the Haar measure. This means that for an arbitrary Borel set $H \subset T'$ we have $\mu(H) = \mu(\phi^{-1}(H))$.*

Proof. The proof is an induction using lemma 3.1. The map ϕ induces a map ϕ' from T_{k-1} to T'_{k-1}. If we know the statement for $k-1$ then ϕ' is measure preserving. On the other hand it is measure preserving on the fibres so the integral in (3) is preserved. □

The next lemma follow directly form lemma 3.2 and lemma 2.12

Lemma 3.3. *Continuous fibre surjective morphisms between k-step nilspaces are measure preserving.*

3.2 Fibre bundles

In this part we study A bundles where A is a compact abelian group of finite dimension. We say that A is of finite rank if the dual group $\hat{A}$ is finitely generated. It is well known that A is of finite rank if and only if it is finite dimensional. Finite rank compact abelian groups are direct products circles ($\mathbb{R}/\mathbb{Z}$) and finite cyclic groups. Their dual groups are direct products of cyclic groups. The main result in this chapter is the following lemma.

Lemma 3.4. *Let C be a compact second-countable Hausdorff topological space which is an A bundle for some finite rank compact abelian group. Then the bundle is locally trivial.*

Proof. Since A is of finite rank the dual group $\hat{A}$ is the direct product of finitely many, say n, cyclic group. Let us pick generators $\chi_1, \chi_2, \dots, \chi_n$ one for each cyclic component. Note that the map $\tau : A \to \mathbb{C}^n$ defined by $\tau_i(a) = \chi_i(a)$ defines an isomorphism between A and a subgroup of Q^n where Q is the unit circle (with multiplication) in the complex plane. If the dual group is torsion free then $\tau(A) = Q^n$.

For every $1 \leq i \leq n$ we introduce the averaging operator $\mathcal{A}_i$ on the space of continuous functions on C by

$$\mathcal{A}_i(f)(x) = \int_{a \in A} \overline{\chi_i(a)} f(x+a)\, d\mu.$$

It is easy to see that the continuity of f implies that $\mathcal{A}_i(f)$ is continuous. It is also clear that

$$\mathcal{A}_i(f)(x+a) = \mathcal{A}_i(f)(x)\chi_i(a) \tag{4}$$

holds for every $x \in X, a \in A$.

Claim: *If $\{f_i\}_{i=1}^n$ is a system of continuous functions on C such that $\mathcal{A}_i(f_i)(y) \neq 0$ for every $1 \leq i \leq n$ for some $y \in C$ then y has an open neighborhood U such that U is the union of A orbits and the bundle restricted to U is trivial.*

Let $U_1 = \cap_{i=1}^n \{x | \mathcal{A}_i(f_i)(x) \neq 0\}$. It is clear that U_1 is an open set containing y which is, by (4), the union of A orbits. Let us introduce the map $\phi : C \to Q^n$ defined by $\phi_i(x) = \mathcal{A}_i(f_i)(x)/|\mathcal{A}_i(f_i)(x)|$ on U_1.

First note that if all the characters χ_i are of infinite order then $\phi \circ \tau^{-1}$ proves the triviality of the fibration restricted to U_2.

If χ_i is of finite order for some i then the image of ϕ does not coincide with $\tau(A)$ and so for every A orbit $x+A$ in C we will need a third map which creates an affine isomorphism between $\phi(x+A)$ and $\tau(A)$. Furthermore we need to choose these maps in a continuous way. Note that $\phi(x+A)$ is always of the form $w\tau(A)$ where $w \in Q^n$. Our goal is to choose a $w \in Q^n$ with $w\phi(x+A) = \tau(A)$ continuously for every A orbit in a small neighborhood of y.

Let $I \subseteq [n]$ be the set of indices i for which χ_i is of finite order and let ϕ_I be the composition of ϕ with the projection $Q^{[n]} \to Q^I$. For every $x \in U_1$ the image $\phi_I(x+A)$ is (affine) isomorphic to the torsion part of $\hat{A}$. For every ϵ we can choose a neighborhood U_ϵ of y inside U_1 such that for every $x \in U_\epsilon$ the Hausdorff distance between $\phi_I(y+A)$ and $\phi_I(x+A)$ is at most ϵ. If ϵ is smaller than half of the minimal distance inside $\phi_I(y+A)$ then for every $x \in U_\epsilon$ there is a unique element $v_x \in Q^I$ (depending only on the orbit of x such that $\phi_I(x)v$ is the nearest element in $\phi_I(y+A)$ from $\phi_I(x)$. It is clear that v_x depends continuously on the orbit of x. Let v'_x be the element in Q^n whose coordinates in I are given by v_x and coordinates outside I are all 1's. Now the map $x \mapsto \tau^{-1}(v'_x\phi(x))$ proves the triviality of the fibration on U_ϵ.

Now we can finish the proof of the lemma. It is enough to find continuous functions $\{f_i\}_{i=1}^n$ from C to $\mathbb{C}$ such that property in the claim holds. Let us take a separating family of functions $\{f_i\}_{i\in I}$ on C. Then the functions $f_i' : A \to \mathbb{C}$ defined by $f_i'(a) = f_i(y+a)$ also form a separating family on A. According to the Stone-Weierstrass theorem, for every $\chi \in \hat{A}$ there is an element g in the function algebra generated by $\{f_i'\}_{i\in I}$ such that $\|\chi - g\|_\infty < 1$. This means that $(g,\chi) \neq 0$. We can use the same polynomial which produces g for the functions $\{f_i\}_{i\in I}$ and obtain a continuous function h on C with the property that $h(y+a) = g(a)$ for every $a \in A$. It is clear now that the integral $\int_A \overline{\chi(a)} h(y+a)\, d\mu$ is not zero. This completes the proof.

□

3.3 Finite rank nilspaces and averaging

Let N be a compact k-step nilspace. We have from theorem 1 that N is a degree k-bundle with structure groups $A_1, A_2, \dots, A_k$. The compactness of N implies that the structure groups are compact abelian groups. We define the rank $\mathrm{rk}(N)$ by

$$\mathrm{rk}(N) = \sum_{i=1}^{k} \mathrm{rk}(\hat{A}_i)$$

where $\hat{A}_i$ is the Pontrjagin dual of A_i and $\mathrm{rk}(\hat{A}_i)$ is the minimal number of generators of $\hat{A}_i$. According to lemma 3.4 we have that finite rank nilspaces are iterated locally trivial fibrations of finite dimensional compact abelian groups. Topologically, they are finite dimensional manifolds.

Finite rank abelian groups are direct products of finite dimensional tori's and finite abelian groups. There is a natural way of metrizing them. For two elements $x, y \in \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}_n$ we define their distance $d_2(x,y)$ as the minimal possible Euclidean distance between a preimage of x and a preimage of y under the map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. If the abelian group is not connected then points in different connected components have infinite distance.

Let X_1 and X_2 be two Borel random variables taking values in a finite rank compact abelian group A. In general there is no natural way of defining their expected values. However if they take values in small diameter sets in A then there is a canonical way of defining their expected value and it will satisfy $\mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2)$.

Let $a \in \mathbb{T}^n$ be an element and $B_r(a)$ be the open ball of radius r around a. Let $a' \in \mathbb{R}^n$ be an arbitrary preimage of a under the homomorphism $\mathbb{R}^n \to \mathbb{T}_n$. If a Borel random variable X takes all its values in $B_{1/4}(a)$ then there is a unique way of lifting X to a random variable X' on $\mathbb{R}^n$ in a way that the values are closer than $1/4$ to a'. We define $\mathbb{E}(X)$ as the image of $\mathbb{E}(X')$ under the map $\mathbb{R}^n \to \mathbb{T}_n$. It is easy to see that $\mathbb{E}(X)$ does not depend on the choice of a. If m random variables take their values in sets of diameter at most $1/5n$ then the additivity of the expected value is guaranteed.

The next lemma is an important application of averaging.

Lemma 3.5. *Let N be l-step nilspace and A be a finite rank abelian group. Then there is an ϵ such that every Borel measurable cocycle $\sigma : C^{k+1}(N) \to A$ of degree k with $d_2(\sigma(c), 0) \leq \epsilon$ for every $c \in C^{k+1}(M)$ is a coboundary.*

Proof. We define $g : N \to A$ by

$$g(m) = \mathbb{E}_{c \in \mathrm{Hom}_f(\{0,1\}^{k+1}, N)} \sigma(c)$$

where f maps the point 1^{k+1} to m. The expected value makes sense because σ is always close to 0 and by lemma 2.16 the set $\mathrm{Hom}_f(\{0,1\}^{k+1}, N)$ is a sub-bundle in $N^{\{0,1\}^{k+1}}$ so the Haar measure gives a probability space.

We claim that σ is a coboundary corresponding to the function g. Let $c \in C^{k+1}(N)$ be an arbitrary element. Let us use the notation of lemma 2.18. for an arbitrary morphism $\gamma : P \to N$ and $v \in \{-1, 1\}^{k+1}$ let us denote by γ_v the restriction of γ to the cube $\Psi_{k+1}(v)$. Observe that if $\gamma|_{P_2} = c$ then

$$\sigma(c) = \sum_{v \in \{0,1\}^{k+1}} \sigma(\gamma_v)(-1)^{h(v)} \tag{5}$$

where $h(v) = \sum v_i$. By averaging the equation over the set $\mathrm{Hom}_c(P, N)$ and using lemma 2.18 we obtain the claim. □

3.4 The Inverse limit theorem

Theorem 2 (Inverse limit theorem)**.** *Every k-step compact nilspace is an inverse limit of finite rank nilspaces. The maps used in the inverse system are all fiber surjective morphisms.*

This whole chapter deals with the proof of this statement.

We prove the theorem by induction on k. If $k = 0$ then there is nothing to prove. Assume that it is true for $k - 1$. Let N be a k-step nilspace with structure groups $A_1, A_2, \ldots, A_k$. By induction $M = \mathcal{F}_{k-1}(N)$ is the inverse limit of a system $M_1 \leftarrow M_2 \leftarrow \ldots$ where the maps are all fiber surjective morphisms. Let us denote by τ_i the projection to M_i and let π be the projection $N \to M$. Let $\mathcal{Q}_i$ denote the collection of open sets of the form $\tau_i^{-1}(U)$ where U is open in M_i. Since M is a compact Hausdorff space, its topology is generated by the system $\{\mathcal{Q}_i\}_{i=1}^{\infty}$.

Since A_k is a compact abelian group we have that A_k is the inverse limit of finite rank compact abelian groups. This implies that there is a descending chain $A_k = B_0 > B_1 > \ldots$ of subgroups with trivial intersection such that each factor A_k/B_i is of finite rank. The nilspace N is the inverse limit of the nilspaces N/B_i and all the maps $N \to N/B_i$ are fibre surjective. It follows that it is enough to prove the theorem for the special case when $A = A_k$ is already of finite rank.

From theorem 3.4 we have that N as an A-bundle is locally trivial. Let d be a metrization of N. For an arbitrary epsilon and every point $p \in M$ we can choose an open neighborhood U_p of p with the following three properties.

1. there is a continuous cross section $S_p : U_p \to N$ above U_p
2. $S_p(U_p)$ has diameter at most ϵ
3. $U_p \in \mathcal{Q}_{t(p)}$ for some $t(p) \in \mathbb{N}$.

It is clear that we can guarantee the first two properties. The last property follows from the fact that the topology on M is generated by the topologies on M_i.

The compactness of M implies that there are finitely many points $p_1, p_2, \ldots, p_n$ such that $\{U_{p_i}\}_{i=1}^n$ is a covering system of M. Let $t = \max\{t(p_i)\}_{i=1}^n$. We have that every set in $\{U_{p_i}\}_{i=1}^n$ is in $\mathcal{Q}_t$.

Now we can create a Borel measurable cross section $S : M \to N$ with the following properties.

1. S is continuous on every preimage $\tau_t^{-1}(v)$ where $v \in M$
2. The diameter of $S(\tau_t^{-1}(v))$ is at most ϵ for every $v \in M$.

This can be constructed by dividing M into the atoms of the Boolean algebra generated by $\{U_{p_i}\}_{i=1}^n$ and then using one type of cross section for each atom. The cross section S generates a cocycle $\varrho : C^{k+1}(M) \to A$ on M.

If ϵ is small enough than we can guarantee that for any two cubes $c_1, c_2 \in C^{k+1}(M)$ with $c_1 \circ \tau_t = c_2 \circ \tau_t$ we have

$$d_2(\varrho(c_1) - \varrho(c_2)) \leq \epsilon_2. \tag{6}$$

Let $P = \{0,1\}^{k+1}$. We have by lemma 2.17 that the map $\beta : \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M_t)$ given by the restriction of τ_t^P to $\mathrm{Hom}(P, M)$ is totally surjective and preimages of elements in $\mathrm{Hom}(P, M_t)$ are $k-1$-fold sub-bundles of $\mathrm{Hom}(P, M)$. We define the function $\varrho' : C^{k+1}(M) \to A$ by

$$\varrho'(c) = \mathbb{E}_{c' \in \beta^{-1}(\beta(c))}(\varrho(c')).$$

It makes sense to use the expected value because (6) implies that $\{\varrho(c')|c' \in \beta^{-1}(\beta(c))\}$ has small diameter if ϵ_2 is small enough. Note that we compute the expected value according to the Haar measure on $\beta^{-1}(\beta(c))$.

We claim that ϱ' is a cocycle on M. This follows basically from the fact that the cocycle axioms are all linear equations on cubes of dimension $k+1$ and $k+2$ and expected value is additive. However we need to use the fourth point of lemma 2.17 to connect the probability spaces of $k+2$ dimensional cubes and $k+1$ dimensional cubes.

Now we have that ϱ' is a cocycle and so $\varrho'' = \varrho' - \varrho$ is also a cocycle. We have by (6) that $d_2(\varrho''(c), 0) \leq \epsilon_2$ holds for every $c \in C^{k+1}(M)$. By lemma 3.5 we get that ϱ'' is a coboundary.

Since the difference of ϱ and ϱ' is a coboundary corresponding to a function g we have that by adding g to our cross section S we get a new cross section S' such that the cocycle corresponding to S' is equal to ϱ'. The way we produced S' (see the proof of lemma 3.5) guarantees that it is continuous on the preimages of points in M_t under τ_t. Let us define the map $q : N \to A$ by $q(x) = x - S'(\pi(x))$. We say that $x \sim_q y$ for two elements $x, y \in N$ if $\tau_t(x) = \tau_t(y)$ and $q(x) = q(y)$. It is now easy to see that $\sim_q$ creates a factor which is isomorphic to the extension of M_n with the cocycle ϱ'. (Note that ϱ' can be interpreted as a cocycle on M_t.)

It is also clear that factoring by $\sim_q$ provides a fibre surjective morphism of N to a finite rank nilspace. By repeating the argument for some infinite increasing sequence of t's the proof is complete.

3.5 Rigidity of morphisms

Let N and M be compact k-step nilspaces and let d be a metric on M (metrizing its topology). We say that a map $\phi : N \to M$ is an ϵ-almost morphism if for an arbitrary $c \in C^{k+1}(N)$ there is $c' \in C^{k+1}(N)$ such that $d(c \circ \phi, c') \leq \epsilon$ point wise.

An ϵ modification of a map $\phi : N \to M$ is another map ϕ' satisfying $d(\phi(x), \phi'(x)) \leq \epsilon$ for every $x \in N$.

Theorem 3. *For every finite rank k-step nilspace M with metric d there is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{x \to 0} f(x) = 0$ and $\epsilon_0 > 0$ such that if $\phi : N \to M$ is a Borel ϵ-almost morphism with $\epsilon < \epsilon_0$ from a compact k-step nilspace N to M then it can be $f(\epsilon)$-modified to a morphism ϕ'.*

In the rest of this chapter we prove this theorem.

We go by induction on k. For $k = 0$ there is nothing to prove. Assume that we have the statement for $k - 1$. The metric d induces another metric d' on $\mathcal{F}_{k-1}(M)$ such that

$$d'(x', y') = \min\{d(x, y) | x, y \in M,\ \pi_{k-1}(x) = x', \pi_{k-1}(y) = y'\}.$$

The assumption that ϕ is an ϵ-morphism trivially implies that $\phi \circ \pi_{k-1}$ is an ϵ-morphism into $\mathcal{F}_{k-1}(M)$. By induction we can $f'(\epsilon)$-modify ϕ' to get a morphism $\phi_2 : N \to \mathcal{F}_{k-1}(M)$. It is easy to see that there is a Borel measurable lift of ϕ_2 to $\phi_3 : N \to M$ which is an at most $\epsilon_2 = f'(\epsilon) + 2\epsilon$ almost-morphism.

Now we introduce an averaging process to get a function ϕ_4 in the following way. Let $P_2 = \{0, 1\}^{k+1} \setminus \{1^{k+1}\}$ be the corner of the $k+1$ dimensional cube P. Using corollary 2.2 and the fact that ϕ_2 is a morphism we get that ϕ_3 takes k-dimensional cubes in N into k-dimensional cubes in M. This means that for every morphism $\gamma : P \to N$ the composition $\gamma|_{P_2} \circ \phi_3$ is a morphism of the corner P_2. For a morphism $\gamma : P \to N$ We denote by $Q(\gamma) \in M$ the unique completion of $\gamma|_{P_2} \circ \phi_3$ in M.

Now we define

$$\phi_4(x) = \mathbb{E}_{\gamma \in \mathrm{Hom}_f(P, N)}(Q(\gamma))$$

where f maps the point 1^{k+1} to x. The averaging makes sense because all the elements $Q(\gamma)$ are in the same fibre. If ϵ_2 is small enough then we can average. It is easy to see that ϕ_4 is continuous. It remains to show that ϕ_4 is a morphism.

According to lemma 2.7 we need to show that $k+1$ dimensional cubes map to cubes under ϕ_4. Let $c \in C^{k+1}(N)$. Let T_{k+1} be the 3-cube embedded into $B = \{0,1\}^{2k+2}$. Let $B_2 = \omega(\{0,1\}^{k+1})$ and $B_3 = T_{k+1} \setminus B_2$. By abusing the notation the cube c can be interpreted as a function $c : B_2 \to N$. For every element $\kappa \in \mathrm{Hom}_c(B,N)$ we denote by $Q(\kappa) \in C^{k+1}(M)$ the cube obtained by first taking the unique extension of $\kappa|_{B_3} \circ \phi_3$ to a morphism $B \to M$ and then restricting it to B_2. Now

$$c_2 = \mathbb{E}_{\kappa \in \mathrm{Hom}_c(B,N)} Q(\kappa)$$

makes sense if ϵ_4 is small enough and by theorem 1 it will be a cube. On the other hand By lemma 2.18 we obtain that $c_2 = c \circ \phi_4$.

3.6 Nilspaces as nilmanifolds

Let N be a compact k-step nilspace. By abusing the notation we denote by $\mathrm{Trans}(N)$ the set of translations of which are continuous functions from N to N. A simple induction on k together with lemma 2.21 shows that every element of $\mathrm{Trans}(N)$ is measure preserving.

Let d be a metrization of the topology on N. This induces a metric t on $\mathrm{Trans}(N)$ defined by

$$t(g,h) = \max_{x \in N} d(g(x), h(x)).$$

It is easy to see that $\mathrm{Trans}(N)$ is a Polish group with this metrization. Similarly we will denote by $\mathrm{Trans}_i(N)$ the set of continuous translations of hight i.

From now on we assume that N is a finite rank k-step nilspace. Our goal is to show that $\mathrm{Trans}(N)$ is a k-nilpotent Lie group which acts transitively on the connected components of N.

From lemma 2.21 we obtain that $\sim_{k-1}$ classes are imprimitivity domains of $\mathrm{Trans}(N)$. This means that the action on $\sim_{k-1}$ classes induces a homomorphism $h : \mathrm{Trans}(N) \to \mathrm{Trans}(\mathcal{F}_{k-1}(N))$. It is clear that $h(\mathrm{Trans}_i(N)) \subseteq \mathrm{Trans}_i(\mathcal{F}_{k-1}(N))$. Let $M = \mathcal{F}_{k-1}(N)$ and let us denote the version of the t metric on M by t'.

Lemma 3.6. *Let i be a natural number. There is a positive number $\epsilon > 0$ such that if $\alpha \in \mathrm{Trans}_i(M)$ satisfies $t'(\alpha, 1) \leq \epsilon$ then there is $\beta \in \mathrm{Trans}_i(N)$ with $h(\beta) = \alpha$.*

Proof. The translation bundle $\mathcal{T}^* = \mathcal{T}^*(\alpha, N, i)$ is a $k-i$ degree extension of M by A_k. Our goal is to show if ϵ is small enough then the cocycle describing the extension is a coboundary. If ϵ is small enough then we can choose a Borel representative system S for the fibres of the map $\mathcal{T}^* \to M$ such that $(x,y) \in \mathcal{T}$ represents an element in S then $d(x,y) \leq \epsilon_2$. A standard compactness argument shows that if ϵ_2 is small enough then the cocycle corresponding to S is also small. Then lemma 3.5 and lemma 2.1 finish the proof.

□

Lemma 3.7. *Assume that $i > k$. Then*

$$\ker(h) \cap \mathrm{Trans}_i(N) = \hom(M, \mathcal{D}_{k-i}(A_k)).$$

Proof. The elements of $\ker(h)$ are those translations which stabilize every $\sim_{k-1}$ class in N. It follows that if $\alpha \in \ker(h)$ then the map $\alpha' : x \mapsto \alpha(x) - x$ can be viewed as a map from M to A_k. Lemma 2.20 implies that α' arises this way if it is a homomorphism of M to $\mathcal{D}_{k-1}(A_k)$. It is easy to see that if in addition $\alpha' \in \mathrm{Trans}_i(N)$ then it is a morphism to $\mathcal{D}_{k-i}(A_k)$. □

Lemma 3.8. *Let $k, r \geq 1$ be two natural numbers and A, B two compact abelian groups. Assume that B is finite dimensional. Then there is a constant $\epsilon = \epsilon(r, B) > 0$ such that if $\phi \in \mathrm{Hom}(\mathcal{D}_k(A), \mathcal{D}_r(B))$ satisfies $d(\phi(x), \phi(y)) \leq \epsilon$ for every $x, y \in A$ then ϕ is a constant function.*

Proof. Using that $\mathrm{Hom}(\mathcal{D}_k(A), \mathcal{D}_r(B)) \subseteq \mathrm{Hom}(\mathcal{D}_1(A), \mathcal{D}_r(B))$ we can assume that $k = 1$. Let ϕ be an arbitrary non-constant morphism from $\mathcal{D}_1(A)$ to $\mathcal{D}_r(B)$.

For any $t \in A$ and function $f : A \to B$ we denote by $\Delta_t f$ the function $x \to f(x) - f(x+t)$. With this notation we have that if $f \in \mathrm{Hom}(\mathcal{D}_1(A), \mathcal{D}_i(B))$ then $\Delta_t f \in \mathrm{Hom}(\mathcal{D}_1(A), \mathcal{D}_{i-1}(B))$. for every $t \in A$. It follows that $\Delta_{t_1, t_2, \ldots, t_r} \phi$ is constant for every r-tuple of elements $t_1, t_2, \ldots, t_r$ in A. We obtain that there is a number $i < r$ and elements $t_1, t_2, \ldots, t_i \in A$ such that $\phi' = \Delta_{t_1, t_2, \ldots, t_i} \phi$ is non-constant but $\Delta_t \phi'$ is constant for every $t \in A$. It follows that ϕ' is a non-constant affine group homomorphism from A to B. In particular there is a constant c depending only on B such that there are $x, y \in A$ with $d(\phi'(x), \phi'(y)) \geq c$. We get that if the variation $\max_{x,y} d(\phi(x), \phi(y))$ is too small this is impossible. In other words there is a non-zero lower bound (depending only on B and r) for the variation of ϕ. □

Corollary 3.1. *Let $r \geq 1$ be a natural numbers and B a compact finite dimensional abelian groups. Let N be a k-step compact nilspace. Then there is a constant $\epsilon = \epsilon(r, B) > 0$ such that if $\phi \in \mathrm{Hom}(N, \mathcal{D}_r(B))$ satisfies $d(\phi(x), \phi(y)) \leq \epsilon$ for every $x, y \in N$ then ϕ is a constant function.*

Proof. Assume that $d(\phi(x), \phi(y)) < \epsilon$ for every $x, y \in N$ where $\epsilon = \epsilon(r, B)$ is the constant from lemma 3.8. We prove by induction on k that ϕ is constant.

If $k = 1$ then N is abelian and lemma 3.8 finishes the proof. Assume that the statement holds for $k-1$. We get from lemma 3.8 that ϕ is constant on the $\sim_{k-1}$ classes of N. This means that ϕ can be regarded as a function on $\mathcal{F}_{k-1}(N)$. Then our assumption finishes the proof. ☐

Lemma 3.9. *The group* $\ker(h)$ *is a Lie group.*

Proof. Let $x \in N$ be an arbitrary element and let F be the stabilizer of x in $\ker(h)$. Then by lemma 3.7 we obtain that $\ker(h) = F \times A_k$. It follows from corollary 3.1 that F is discrete and since A_k is a Lie-group the proof is complete. ☐

Theorem 4. *Let i be a natural number. Then the following statements hold.*

1. $\mathrm{Trans}_i(N)$ *and* $\mathrm{Trans}_i(N)^0$ *are Lie groups,*
2. $h(\mathrm{Trans}_i(N)^0) = \mathrm{Trans}_i(M)^0$.

Proof. We prove the statements by induction on k. If $k = 1$ then N is an abelian Lie-group and all statements are clear. Assume that the statements hold for $k-1$. In particular we have that $\mathrm{Trans}_i(M)$ is a Lie-group.

First we show that

$$\mathrm{Trans}_i(M)^0 \subseteq h(\mathrm{Trans}_i(N)) \tag{7}$$

To see this we use that $\mathrm{Trans}_i(M)$ is a Lie group and so every element $\alpha \in \mathrm{Trans}_i(M)^0$ is connected with the unit element with a continuous path $p : [0,1] \to \mathrm{Trans}_i(M)$ with $p(0) = 1$ and $p(1) = \alpha$. Let $n \in \mathbb{N}$ be sufficiently big and let $\alpha_i = p((i-1)/n)^{-1}p(i/n)$. Then $\alpha = \prod_{i=1}^{n} \alpha_i$. Lemma 3.6 implies that if n is big enough then for every α_i there is $\beta_i \in \mathrm{Trans}_i(N)$ with $h(\beta_i) = \alpha_i$. Let $\beta = \prod_{i=1}^{n} \beta_i$. We have that $h(\beta) = \alpha$.

The see the first statement we observe that (7) implies that $h(\mathrm{Trans}_i(N))$ is a Lie-group. It follows from lemma 3.9 that $\mathrm{Trans}_i(N)$ is an extension of a Lie-group with a Lie-group. Since $\mathrm{Trans}_i(N)$ is a Polish group we get that it is a Lie-group.

Now we show the second statement. Since h is continuous we have that $h(\mathrm{Trans}_i(N)^0) = h(\mathrm{Trans}_i(N))^0$. Equation (7) implies that $\mathrm{Trans}_i(M)^0 \subseteq h(\mathrm{Trans}_i(N))^0$ and so $\mathrm{Trans}_i(M)^0 \subseteq h(\mathrm{Trans}_i(N)^0)$. The other containment is trivial. ☐

Corollary 3.2. *The action* $\mathrm{Trans}(N)^0$ *is transitive on the connected components of N.*

Proof. By induction $\mathrm{Trans}(M)^0$ acts transitively on the connected components of M and furthermore $A_k \subseteq \mathrm{Trans}(N)$. By theorem 4 $\mathrm{Trans}(M)^0 = h(\mathrm{Trans}(N)^0)$. It follows that the group T generated by A_k and $\mathrm{Trans}(N)^0$ is transitive on the connected components of N. Since A_k^0 is a finite index subgroup in A_k we have that $\mathrm{Trans}(N)^0$ is of finite index in T. This is only possible if $\mathrm{Trans}(N)^0$ is already transitive on the connected components. □

Definition 3.2. *A k-step nilspace is called* **torsion free** *if all the structure groups A_i have torsion free dual groups.*

Note that a compact finite dimensional abelian group A has torsion free dual group if and only if A is isomorphic to $(\mathbb{R}/\mathbb{Z})^n$ for some natural number n.

Theorem 5. *If N is finite rank torsion free k-step nilspace then N is a nilmanifold with structure corresponding to the central series $\{\mathrm{Trans}_i(N)^0\}_{i=1}^k$ in $\mathrm{Trans}(N)^0$.*

Proof. We prove the statement by induction on k. If $k = 1$ then N is an abelian group and the statement is trivial. Assume that it is true for $k-1$. Let $x \in N$ be any fixed point. From theorem 4 and our induction hypothesis it follows that for every cube $c \in C^n(N)$ there is a cube $c' \in C^n(N)$ such that c' is translation equivalent with the constant x cube and $\pi_{k-1}(c) = \pi_{k-1}(c')$. It follows from theorem 1 that $c - c' \in C^n(\mathcal{D}_k(A_k))$. Since $A_k \subset \mathrm{Trans}_k(N)$ it is easy to that c is translation equivalent with c' with translations from A_k. □

References

[1] R. Brown, P. J. Higgins, R. Sivera, *Nonabelian Algebraic Topology: Filtered spaces, crossed complexes, cubical homotopy groupoids*, EMS Tracts in Mathematics, volume 15, to appear. Draft available from `http://www.bangor.ac.uk/~mas010/nonab-a-t.html`

[2] G. Elek, B. Szegedy: *A measure-theoretic approach to the theory of dense hypergraphs*, arXiv:0810.4062

[3] T. Gowers, *A new proof of Szemerédi's theorem*, Geom. Funct. Anal. 11 (2001), no 3, 465-588

[4] T. Gowers, *Fourier analysis and Szemerédi's theorem*, Proceedings of the International Congress of Mathematics, Vol. I (Berlin 1998).

[5] B. Green, T. Tao, *An inverse theorem for the gowers $U_3(G)$ norm* Proc. Edinb. Math. Soc. (2) 51 (2008), no. 1, 73–153.

[6] B. Green, T. Tao, T. Ziegler, *An inverse theorem for the Gowers $U^{s+1}[N]$-norm (announcement)*, arXiv:1006.0205

[7] M. Gromov, *Almost flat manifolds*, J. Differential Geom. 13 (1978), no. 2, 231–241.

[8] B. Host, B. Kra, *Nonconventional ergodic averages and nilmanifolds*, Ann. of Math. (2) 161 (2005), no. 1, 397–488

[9] B. Host, B. Kra, *Parallelepipeds, nilpotent groups and Gowers norms*, Bulletin de la Socit Mathmatique de France 136, fascicule 3 (2008), 405-437

[10] A. I. Mal'cev, *On a class of homogeneous spaces*, AMS Translation No. 39 (1951).

[11] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. 21 (1976), no. 3, 293–329.

[12] M.S. Raghunathan, *Discrete Subgroups in Lie Groups*, Springer Verlag 1972

[13] B. Szegedy, *Higher order Fourier analysis as an algebraic theory I.*, arXiv:0903.0897

[14] B. Szegedy, *Higher order Fourier analysis as an algebraic theory II.*, arXiv:0911.1157

[15] B. Szegedy, *Higher order Fourier analysis as an algebraic theory III.*, arXiv:1001.4282

[16] B. Szegedy, *Limits and regularization of functions on abelian groups*, in preparation

[17] B. Szegedy. *Limits of functions on groups*, in preparation

[18] T. Ziegler, *Universal characteristic factors and Fürstenberg averages*, J. Amer. Math. Soc. 20 (2007), no. 1, 53-97