Fano 3-folds in codimension 4, Tom and Jerry. Part I

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Abstract

This work is part of the Graded Ring Database project [GRDB], and is a sequel to [A0], [A] and [ABR]. We introduce a strategy based on Kustin–Miller unprojection [KM], [PR], [Ki] that allows us to construct many hundreds of Gorenstein codimension 4 ideals with 9×16 resolutions (that is, 9 equations and 16 first syzygies). Our two basic games are called Tom and Jerry; the main application is the biregular construction of most of the anticanonically polarised Mori Fano 3-folds of Altınok's thesis [A0]. There are 115 cases whose numerical data (in effect, the Hilbert series) allow a Type I projection. In every case, at least one Tom and one Jerry construction works, providing at least two deformation families of quasismooth Fano 3-folds having the same numerics but different topology.

Contents

1 Ancestral examples

1.1 Linear subspaces of $\mathrm{Grass}(2,5)$

A del Pezzo variety of degree 5 is an n-fold $Y_5^n \subset \mathbb{P}^{n+3}$ of codimension 3, defined by 5 quadrics that are Pfaffians of a 5×5 skew matrix of linear forms. Thus Y is a linear section of Plücker $\mathrm{Grass}(2,5) \subset \mathbb{P}(\bigwedge^2 V)$ (here $V = \mathbb{C}^5$). We want to unproject a projective linear subspace $\mathbb{P}^{n-1}$ contained as a divisor in Y to construct a degree 6 del Pezzo variety $X_6^n \subset \mathbb{P}^{n+4}$. The crucial point is the following.

Lemma 1.1 *The Plücker embedding* $\mathrm{Grass}(2,5)$ *contains two families of maximal linear subspaces. These arise from*

(I) The 4-dimensional vector subspace $v \wedge V \subset \bigwedge^2 V$ *for a fixed* $v \in V$.

(II) The 3-dimensional subspace $\bigwedge^2 U \subset \bigwedge^2 V$ *for a fixed 3-dimensional vector subspace* $U \subset V$.

Thus there are two different formats to set up $\mathbb{P}^{n-1} \subset Y$. Case I gives $\mathbb{P}^3_v \subset \mathrm{Grass}(2,5)$. A section of $\mathrm{Grass}(2,5)$ by a general $\mathbb{P}^7$ containing $\mathbb{P}^3_v$ is a 4-fold Y^4 whose unprojection is $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. Case II gives $\mathrm{Grass}(2,U) = \mathbb{P}^2_U \subset \mathrm{Grass}(2,5)$. A section of $\mathrm{Grass}(2,5)$ by a general $\mathbb{P}^6$ containing $\mathbb{P}^2_U$ is a 3-fold Y^3 whose unprojection is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$.

The proof is a lovely exercise. Hint: use local and Plücker coordinates

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 & a_3 \\ 0 & 1 & b_1 & b_2 & b_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3 \\ & & m_{12} & m_{13} \\ & & & m_{23} \end{pmatrix} \tag{1.1}$$

with Plücker equations $m_{12} = a_1 b_2 - a_2 b_1$, etc.; permute the indices and choose signs pragmatically to make this true. Prove that in Plücker $\mathbb{P}^9$, the tangent plane $m_{12} = m_{13} = m_{23} = 0$ intersects $\mathrm{Grass}(2,5)$ in the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

1.2 Tom_1 and Jer_{12} in equations

Tom_1 is

$$\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix} \tag{1.2}$$

with $y_{1\dots4}$ arbitrary elements, and the six entries m_{ij} linear combinations of a regular sequence $x_{1\dots4}$ of length four. Expressed vaguely, there are "two constraints on these six entries"; these two coincidences take the simplest form when $m_{23} = m_{45} = 0$. In this case, the Pfaffian equations all reduce to binomials, and can be seen as the 2×2 minors of an array:

$$4 \times 4 \text{ Pfaffians of } \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ & 0 & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & 0 \end{pmatrix} = 2 \times 2 \text{ minors of } \begin{pmatrix} * & y_3 & y_4 \\ y_1 & m_{24} & m_{25} \\ y_2 & m_{34} & m_{35} \end{pmatrix}. \tag{1.3}$$

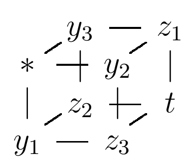
To see the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and its linear projection from a single point, replace the star by the unprojection variable s.

In a similar style, Jer_{12} is

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & y_{34} & y_{35} \\ & & & y_{45} \end{pmatrix} \tag{1.4}$$

with y_{34}, y_{35}, y_{45} arbitrary, and the seven entries m_{ij} linear combinations of $x_{1\dots4}$. Vaguely, "three constraints on these seven entries"; most simply, these take the form $m_{15} = m_{23} = 0$, $m_{24} = m_{14}$. We leave you to see this as the linear projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, starting from the hint:

$$4 \times 4 \text{ Pfaffians of } \begin{pmatrix} t & z_1 & z_2 & 0 \\ & 0 & z_2 & z_3 \\ & & y_3 & y_2 \\ & & & y_1 \end{pmatrix} = 2 \times 2 \text{ minors of}$$



(1.5)

then replacing the star by an unprojection variable.

1.3 General conclusions

Definition 1.2 Tom_i and Jer_{ij} are matrix formats that specify unprojection data, namely a codimension 3 scheme Y defined by a 5×5 Pfaffian ideal, containing a codimension 4 complete intersection D. Given a regular sequence $x_{1\dots4}$ in a regular ambient ring R generating the ideal I_D, the ideal of Y is generated by the Pfaffians of a 5×5 skew matrix M with entries in R, subject to the conditions

Tom_i: the 6 entries $m_{jk} \in I_D$ for all $j, k \neq i$; in other words, the 4 entries m_{ij} of the ith row and column are free choices, but the other entries of M are required to be in I_D. See (3.6) for an example.

Jer_{ij}: the 7 entries $m_{kl} \in I_D$ if either k or l equals i or j. See (3.7) for an example. The bound entries are the *pivot* m_{ij} and the two rows and columns through it. The 3 free entries are the Pfaffian partners m_{kl}, m_{km}, m_{lm} of the pivot, where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. In Y, the pivot vanishes twice on D.

Case I in 1.1 is the ancestor of our Tom constructions and II that of Jerry. Our main aim in what follows is to work out several hundred applications of the same formalism to biregular models of Fano 3-folds, when our "constraints"

$$m_{ij} = \text{linear combination of } x_{1\ldots4} \tag{1.6}$$

are not linear, do not necessarily reduce to a simple normal form, and display a rich variety of colourful and occasionally complicated behaviour.

Nevertheless, the same general tendencies recur again and again. Tom tends to be fatter than Jerry. Jerry tends to have a singular locus of bigger degree than Tom, and the unprojected varieties X have different topologies, in fact different Euler numbers. For example, Y^4 in Case I has two lines of transversal nodes; the Y^3 in Case II has three nodes. If we only look at 3-folds in 1.1 (cutting Y^4 by a hyperplane), the unprojected varieties X are then the familiar del Pezzo 3-folds of index 2, namely the flag manifold of $\mathbb{P}^2$ versus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; see Remark 6.2 for the number of nodes (2 and 3 in the two cases) via enumerative geometry. Tom equations often relate to extensions of $\mathbb{P}^2 \times \mathbb{P}^2$ such as the "extrasymmetric 6×6 format"; Jerry equations often relate to extensions of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such as the "rolling factors format" (an anticanonical divisor in a scroll) or the "double Jerry format"; Section 8 gives a brief discussion.

2 The main result

2.1 Fano 3-fold

A *Fano 3-fold* X is a normal projective 3-fold whose anticanonical divisor $-K_X$ is $\mathbb{Q}$-Cartier and ample; we usually write $-K_X = A$ with A ample (or $-K_X = fA$ if it is divisible in $\operatorname{Cl} X$, with $f \geq 1$ the *Fano index* – our main interest is $f = 1$). We eventually impose additional conditions on the singularities and class group of X, such as terminal, $\mathbb{Q}$-factorial, quasismooth, prime (that is, class group $\operatorname{Cl} X$ of rank 1) or $\operatorname{Cl} X = \mathbb{Z} \cdot A$, but more general cases occur in the course of our arguments.

We study X via its graded ring $R(X, A)$. Choosing generators of $R(X, A)$ embeds X as a projectively normal subvariety $X \subset \mathbb{P}(a_1, \dots, a_n)$ in weighted projective space. The anticanonical ring $R(X, A)$ is known to be Gorenstein, and we say that $X \subset \mathbb{P}(a_1, \dots, a_n)$ is projectively Gorenstein.

As explained in [ABR], the numerical data of X consists of an integer genus $g \geq -2$ plus a basket $\mathcal{B} = \{\frac{1}{r}(1, a, r - a)\}$ of terminal cyclic orbifold points; this data determines the Hilbert series $P_X(t) = \sum_{a \geq 0} h^0(X, nA)$ of $R(X, A)$, and is equivalent to it. At present we only treat cases when the ring is generated as simply as possible; that is, not degenerate cases of hypersurface or codimension 2 numerical cases that fall (say) in a monogonal or hyperelliptic special case. The database [GRDB] lists cases of small codimension, including 145 candidate cases in codimension 4 from Altınok's thesis [A0]. We sometimes say Fano 3-fold to mean numerical candidate; the abuse of terminology is fairly harmless, because practically all the candidates in codimension ≤ 5 (possibly all of them) give rise to quasismooth Fano 3-folds; in fact usually more than one family, as we now relate.

2.2 Type I centre and Type I projection

An orbifold point $P \in X$ of type $\frac{1}{r}(1, a, r-a)$ is a *Type I centre* if its orbinates are restrictions of global forms $x \in H^0(A)$, $y \in H^0(aA)$, $z \in H^0((r-a)A)$ of the same weight. The condition means that after projecting, the exceptional locus of the projection is a weighted projective plane $\mathbb{P}(1, a, r - a)$ that is embedded projectively normally.

One may view a projection $P \in X \dashrightarrow Y \supset D$ in simple terms: as the map $(x_1, \dots, x_n) \mapsto (x_1, \dots, \widehat{x_i}, \dots, x_n)$ analogous to linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ from centre $P_i = (0, \dots, 1, \dots, 0)$; or in algebra, as eliminating a variable, corresponding to passing to a graded subring $k[x_1, \dots, \widehat{x_i}, \dots, x_n]$; to be clear, the distinguishing characteristic is not the eliminated variable x_i, rather the point P_i and the complementary system of variables x_j that vanish there.

We take the more sophisticated view of [CPR], 2.6.3 of a projection as an intrinsic biregular construction of Mori theory; namely a diagram

$$\begin{array}{lll} & & P \in X \subset \mathbb{P}(a_0, \dots, a_n) \\ & \nearrow & \\ E \subset X_1 & & \\ & \searrow & \\ & & D \subset Y \subset \mathbb{P}(a_0, \dots, \widehat{a_k}, \dots, a_n) \end{array} \tag{2.1}$$

consisting of an extremal extraction $\sigma\colon X_1 \to X$ followed by the anticanonical morphism $\varphi\colon X_1 \to Y$.

In more detail, we assume that $P \in X$ is a $\frac{1}{r}(1, a, r-a)$ orbifold point; by a theorem of Kawamata [Ka] (discussed also in [CPR], Proposition 3.4.1), the $(1, a, r-a)$ weighted blowup $\sigma\colon X_1 \to X$ is the unique Mori extremal extraction whose centre meets P. It has exceptional divisor the weighted plane $E = \mathbb{P}(1, a, r-a)$ with discrepancy $\frac{1}{r}$. Thus $-K_{X_1} = -K_X - \frac{1}{r}E$, and the anticanonical ring of X_1 consists of forms of weight d in $R(X, K_X)$ vanishing to order $\geq \frac{d}{r}$ on E. The homogenising variable x_k of degree r with $x_k(P) = 1$ does not vanish at all, so is eliminated. The Type I assumption gives orbinates at P as global forms x, y, z of weights $1, a, r-a$ vanishing to order exactly $\frac{1}{r}, \frac{a}{r}, \frac{r-a}{r}$, so these extend to regular elements of $R(X_1, -K_{X_1})$; appropriate monomials in x, y, z base the sheaves $\mathcal{O}_X(d)$ modulo any power of the maximal ideal m_P, so we can adjust the remaining generators x_l of $R(X, -K_X)$ to vanish to order $\geq \frac{\operatorname{wt} x_l}{r}$, so they lift to $R(K_1, -K_{X_1})$. It follows that $-K_{X_1}$ is semiample and the anticanonical morphism $\varphi\colon X_1 \to Y$ takes E isomorphically to $D \subset Y$.

The image Y is again a Fano 3-fold in its anticanonical model; it is not $\mathbb{Q}$-factorial because the divisor $D \subset Y$ is not $\mathbb{Q}$-Cartier. As in [CPR], 4.1 (3), Y is the *midpoint of a Sarkisov link*; we return to this in Part II. The morphism $\varphi\colon X_1 \to Y$ contracts finitely many flopping curves Γ_i to points of Y. In the ideal case, the Γ_i are copies of $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1,-1)$ in X_1 meeting E transversally, or equivalently, Y has only ordinary nodes on D. We prove that this happens generically in all our families.

Example Consider the general codimension 2 complete intersection

$$X_{12,14} \subset \mathbb{P}(1, 1, 4, 6, 7, 8)_{\langle x,a,b,c,d,e \rangle}. \tag{2.2}$$

The coordinate point $P_e = (0, \dots, 0, 1)$ is necessarily contained in X: near it, the two equations $f_{12} : be = F_{12}$ and $g_{14} : ce = G_{14}$ express b and c as implicit functions of the other variables, so that X is locally the orbifold point $\frac{1}{8}(1,1,7)$ with orbinates x, a, d.

Eliminating e from f_{12}, g_{14} projects $X_{12,14}$ birationally to the hypersurface $Y_{18} : (bG - cF = 0) \subset \mathbb{P}(1,1,4,6,7)_{\langle x,a,b,c,d\rangle}$. Note that Y contains the plane $D = \mathbb{P}(1,1,7)_{\langle x,a,d\rangle} = V(b,c)$, and has in general $24 = \frac{1}{7} \times 12 \times 14$ ordinary nodes at the points $F = G = 0$ of D.

In this case, the Kustin–Miller unprojection of the "opposite" divisor $(b = F = 0) \subset Y$ completes the 2-ray game on X_1 to a Sarkisov link, in the style of Corti and Mella [CM]: the flop $X_1 \to Y \leftarrow Y^+$ blows this up to a $\mathbb{Q}$-Cartier divisor, and the unprojection variable $z_2 = c/b = G/F$ then contracts it to a nonorbifold terminal point $P_z \in Z_{14} \subset \mathbb{P}(1,1,4,7,2)_{\langle x,a,b,d,z\rangle}$.

2.3 Main theorem

Write $P \in X$ for the numerical type of a codimension 4 Fano 3-fold of index 1 marked with a Type I centre. There are 115 or 116 candidates for X (depending on how you count an easy initial case); some have two or three centres, and treating them separately makes 162 cases for $P \in X$.

Theorem 2.1 *Let $P \in X$ be as above; then the projected variety is realised as a codimension 3 Fano $Y \subset w\mathbb{P}^6$, and Y can be made to contain a coordinate stratum $D = \mathbb{P}(1, a, r-a)$ of $w\mathbb{P}^6$ in several ways.*

For every numerical case $P \in X$, there are several formats, at least one Tom and one Jerry, for which the general $D \subset Y$ only has nodes on D, and unprojects to a quasismooth Fano 3-fold $X \subset w\mathbb{P}^7$. In different formats, the resulting Y have different numbers of nodes on D, so that the unprojected quasismooth varieties X have different Betti numbers. Therefore in each of the 115 numerical cases for X, the Hilbert scheme has at least two components containing quasismooth Fano 3-folds.

2.4 Discussion of the result

The theorem constructs around 320 different families of quasismooth Fano 3-folds. We do not burden the journal pages with the detailed lists; the case worked out in Section 3 may be adequate for most readers. Our data and the software tools for manipulating them are available from the Graded Ring Database `http://grdb.lboro.ac.uk/`. Section 7 gives "quick start-up" instructions; do not under any circumstances read the `README` file.

Our 162 cases for $P \in X$ project to $D \subset Y \subset w\mathbb{P}^6$; of the 69 codimension 3 families of Fanos Y that are 5×5 Pfaffians, 67 are the images of projections, each having up to four candidate planes $D \subset Y$. For each of the 162 candidate pairs $D \subset Y$, we study 5 Tom and 10 Jerry formats, of which at least one Tom and one Jerry is successful (often one more, occasionally two), so that Theorem 2.1 describes around 450 successful constructions of pairs $P \in X$ of quasismooth Fano 3-folds with marked centre of projection, giving around 320 different families of X; not all the X are prime.

Theorem 2.1 covers codimension 4 Fano 3-folds of index 1 for which there exists a Type I centre. If one believes the possible conjecture raised in [ABR], 4.8.3 that every Fano 3-fold in the Mori category (that is, with terminal singularities) admits a $\mathbb{Q}$-smoothing, this also establishes the components of the Hilbert scheme of codimension 4 Fano 3-folds in these numerical cases. The main novelty of this paper (and this was a big surprise to us) is that in every case, the moduli space has 2, 3 or 4 different components.

Flowchart Our proof in Sections 4–6 applies computer algebra calculations and verifications to a couple of thousand cases; any of these could in principle be done by hand. We go to the database for candidates for $P \in X$, figure out the weights of the coordinates of $D \subset Y \subset w\mathbb{P}^6$ and the matrix of weight, and list all inequivalent Tom and Jerry formats. Section 4 gives criteria for a format to fail. In the cases that pass these tests, Section 5 contains an algorithm to produce $D \subset Y$ in the given format, and to prove that it has only allowed singularities (that is, only nodes on D). Section 6 contains the Chern class calculation for the number of nodes.

Further outlook In small codimension we can write down hypersurfaces, codimension 2 complete intersections and codimension 3 Pfaffian varieties fluently – in codimension 3, with one or two simple exceptions, only the 5×5 Pfaffians appears among varieties of low coindex. Gorenstein in codimension 4 remains one of the frontiers of science: there is no automatic structure theory, and deformations are almost always obstructed. Type I projection and Kustin–Miller unprojection is a substitute that is sometimes adequate.

This paper concentrates on 115 numerical cases of codimension 4 Fano 3-folds of index 1. Most of the remaining numerical cases from Altınok's list of 145 [A0] can be studied in terms of more complicated Type II or Type IV unprojections, when the unprojection divisor is not projectively normal; see

[Ki] for an introduction. We believe that codimension 5 is basically similar: most cases have two or more Type I centres that one can project to smaller codimension, leading to parallel unprojection constructions.

The methods of this paper apply also to other categories of varieties, most obviously K3 surfaces and Calabi–Yau 3-folds. K3 surfaces are included as general elephants $S \in |-K_X|$ in our Fano 3-folds, although the K3 is unobstructed, so that passing to the elephant conceals the distinction between Tom and Jerry. We can also treat some of the Fano 3-folds of index > 1 of Suzuki's thesis [S]; we have partial results on the existence of some of these families, and hope eventually to cover the cases not excluded by Prokhorov's birational methods [Pr].

This paper uses Type I projections $X \dashrightarrow Y$ to study the biregular question of the existence and moduli of X; however, in each case, the Kawamata blowup $X_1 \to X$ initiates a 2-ray game on X_1, with the anticanonical model $X_1 \to Y$ and its flop $Y \leftarrow Y^+$ as first step. In many cases, we know how to complete this to a Sarkisov link using Cox rings, in the spirit of [CPR], [CM], [BCZ] and [BZ]; we return to this in Part II.

3 Extended example

The case $g = 0$ plus basket $\left\{\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{5}(1,1,4)\right\}$ gives the codimension 4 candidate $X \subset \mathbb{P}^7(1,1,2,3,3,4,4,5)$ with Hilbert numerator

$$1 - 2t^6 - 3t^7 - 3t^8 - t^9 + t^9 + 4t^{10} + 6t^{11} + \cdots + t^{22}. \tag{3.1}$$

It has three different possible Type I centres, namely the $\frac{1}{3}$, $\frac{1}{4}$ or $\frac{1}{5}$ points. We project away from each of these, obtaining consistent results; each case leads to four unprojection constructions for X, two Toms and two Jerries:

from $\frac{1}{3}$: gives $\mathbb{P}(1,1,2) \subset Y \subset \mathbb{P}(1,1,2,3,4,4,5)$ with matrix of weights

$$\begin{pmatrix} 2 & 2 & 3 & 4 \\ & 3 & 4 & 5 \\ & & 4 & 5 \\ & & & 6 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \text{Tom}_2 \text{ has 13 nodes} \\ \text{Tom}_1 \text{ has 14 nodes} \\ \text{Jer}_{45} \text{ has 16 nodes} \\ \text{Jer}_{25} \text{ has 17 nodes} \end{array} \tag{3.2}$$

from $\frac{1}{4}$: gives $\mathbb{P}(1,1,3) \subset Y \subset \mathbb{P}(1,1,2,3,3,4,5)$ with matrix of weights

$$\begin{pmatrix} 2 & 3 & 3 & 4 \\ & 3 & 3 & 4 \\ & & 4 & 5 \\ & & & 5 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \text{Tom}_3 \text{ has 9 nodes} \\ \text{Tom}_1 \text{ has 10 nodes} \\ \text{Jer}_{35} \text{ has 12 nodes} \\ \text{Jer}_{15} \text{ has 13 nodes} \end{array} \tag{3.3}$$

from $\frac{1}{5}$: gives $\mathbb{P}(1,1,4) \subset Y \subset \mathbb{P}(1,1,2,3,3,4,4)$ with matrix of weights

$$\begin{pmatrix} 2 & 2 & 3 & 3 \\ & 3 & 4 & 4 \\ & & 4 & 4 \\ & & & 5 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \text{Tom}_4 \text{ has 8 nodes} \\ \text{Tom}_2 \text{ has 9 nodes} \\ \text{Jer}_{24} \text{ has 11 nodes} \\ \text{Jer}_{14} \text{ has 12 nodes} \end{array} \tag{3.4}$$

Specifically, we assert that *in each of these 12 cases, if we pour general elements of the ideal I_D and general elements of the ambient ring into the Tom or Jerry matrix M as specified in Definition 1.2, the Pfaffians of M define a Fano 3-fold Y having only the stated number of nodes on D, and the resulting X is quasismooth.* Section 5 verifies this claim by cheap computer algebra, although we work out particular cases here without such assistance. Section 6 computes the number of nodes in each case from the numerical data. Imposing the unprojection plane D on the general quasismooth Y_t introduces singularities on $Y = Y_0$, nodes in general, which are then resolved on the quasismooth X_1. Each node thus gives a conifold transition, replacing a vanishing cycle S^3 by a flopping line $\mathbb{P}^1$, and therefore adds 2 to the Euler number of X; so the four different X have different topology.

The unprojection formats and nonsingularity algorithms establish the existence of four different families of quasismooth Fano 3-folds X. The rest of this section analyses these in reasonably natural formats; an ideal would be to free ourselves from unprojection and computer algebra, although we do not succeed completely.

For illustration, work from $\frac{1}{3}$; take $X \subset \mathbb{P}^7(1,1,2,3,3,4,4,5)_{\langle x,a,b,c,d,e,f,g\rangle}$, and assume that $P_d = (0,0,0,0,1,0,0,0)$ is a Type 1 centre on X of type $\frac{1}{3}(1,1,2)$. The assumption means that $P \in X$ is quasismooth with orbinates x, a, b. The cone over X is thus a manifold along the d-axis, and therefore, by the implicit function theorem, four of the generators of I_X form a regular sequence locally at P_d, with independent derivatives, say $cd = \cdots$, $de = \cdots$,

$df = \cdots$, $dg = \cdots$ of degrees $6, 7, 7, 8$. Eliminating d gives the Type I projection $X \dashrightarrow Y$ where $Y \subset \mathbb{P}^6(1, 1, 2, 3, 4, 4, 5)$ has Hilbert numerator

$$1 - t^6 - t^7 - 2t^8 - t^9 + t^{10} + 2t^{11} + t^{12} + t^{13} - t^{19}. \tag{3.5}$$

Let Y be a 5×5 Pfaffian matrix with weights as in (3.2). Since rows 2 and 3 have the same weights, we can interchange the indices 2 and 3 throughout; thus Tom_2 is equivalent to Tom_3, Jer_{25} to Jer_{35}, and so on.

3.1 Failure

Some Tom and Jerry cases fail, either for coarse or for more subtle reasons; for example, it sometimes happens that for reasons of weight, one of the variables x_i cannot appear in the matrix, so the variety is a cone, which we reject. Section 4 discusses failure systematically.

In the present case $D = \mathbb{P}(1, 1, 2)_{\langle x,a,b\rangle}$, the generators of $I_D = (c, e, f, g)$ all have weight ≥ 3, but $\operatorname{wt} m_{12}, m_{13} = 2$. Thus requiring $m_{12}, m_{13} \in I_D$ forces them to be zero, making the Pfaffians $\mathrm{Pf}_{12.34}$ and $\mathrm{Pf}_{12.35}$ reducible. This kills Tom_4, Tom_5, Jer_{1i} for any i and Jer_{23}. The same argument says that Tom_2 has $m_{13} = 0$ and Jer_{25} has $m_{12} = 0$, a key simplification in treating them: a zero in M makes three of the Pfaffians binomial.

We see below that Jer_{24} fails for an interesting new reason. The other cases all work, as we could see from the nonsingularity algorithm of Section 5. Tom_2 and Jer_{25} are simpler, and we start with them, whereas Tom_1 and Jer_{45} involve heavier calculations; they are more representative of constructions that possibly lead to prime X.

3.2 Tom_2

The analysis of the matrix proceeds as:

$$\begin{pmatrix} K_2 & 0 & c & e \\ & L_3 & M_4 & N_5 \\ & & f & g \\ & & & \langle c,e,f,g\rangle_6 \end{pmatrix} \mapsto \begin{pmatrix} b & 0 & c & e \\ & L_3 & M_4 & N_5 \\ & & f & g \\ & & & 0 \end{pmatrix} \mapsto \begin{pmatrix} b & c & e \\ d & M & N \\ L & f & g \end{pmatrix} \tag{3.6}$$

here $m_{13} = 0$ is forced by low degree, K_2, L_3, M_4, N_5 are general forms of the given degrees, that we can treat as tokens (independent indeterminates), and the four entries $m_{14}, m_{15}, m_{34}, m_{35}$ are general elements of I_D that we write c, e, f, g by choice of coordinates. Next, m_{45} can be whittled away to 0 by successive row-column operations that do not harm the remaining format; seeing this is a "crossword puzzle" exercise that uses the fact that $m_{13} = 0$ and all the entries in Row 2 are general forms. For example, subtracting a suitable multiple of Row 1 from Row 5 (and then the same for the columns) kills the c in m_{45}, while leaving m_{15} and m_{35} unchanged (because $m_{11} = m_{13} = 0$) and modifying N_5 by a multiple of K_2, which is harmless because N_5 is just a general ring element of weight 5.

The two zeros imply that all the Pfaffians are binomial, and, as in 1.2, putting in the unprojection variable d of weight 4 gives the 2×2 minors of the matrix on the right. The equations describe X inside the projective cone over $w(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}(2, 3^3, 4^3, 5^2)$ with vertex $\mathbb{P}^1_{\langle x,a\rangle}$ as the complete intersection of three general forms of degree $3, 4, 5$ expressing L, M, N in terms of the other variables. (It is still considerably easier to do the nonsingularity computation after projecting to smaller codimension.)

3.3 $\mathbf{Jer}_{25}$

We start from

$$\begin{pmatrix} 0 & b & L_3 & f \\ & c & e & g \\ & & M_4 & \lambda_1 e \\ & & & \mu_3 c + \nu_2 e \end{pmatrix} \tag{3.7}$$

where $m_{12} = 0$ is forced by low degree, and we put tokens b, L, M for the free entries m_{13}, m_{14}, m_{34}. We have cleaned out m_{35} and m_{45} as much as we can; the quantities $b, L, M, \lambda, \mu, \nu$ are general ring elements of the given weights.

We have to adjoin d together with unprojection equations for dc, de, df, dg. There are various ways of doing this, including the systematic method of writing out the Kustin–Miller homomorphism between resolution complexes, that we use only as a last resort. An ad hoc parallel unprojection method is to note that g appears only as the entry m_{25}, so we can project it out to a codimension 2 c.i. containing the plane $c = e = f = 0$:

$$\begin{pmatrix} \mu b & \nu b - \lambda L & M \\ L & -b & 0 \end{pmatrix} \begin{pmatrix} c \\ e \\ f \end{pmatrix} = 0. \tag{3.8}$$

The equations for dc, de, df come from Cramer's rule, and we can write the unprojection in rolling factors format:

$$\bigwedge^2 \begin{pmatrix} b & L & f & d \\ c & e & g & M \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \mu b^2 + \nu bL - \lambda L^2 + df, \\ \mu bc + \nu cL - \lambda eL + Mf, \\ \mu c^2 + \nu ce - \lambda e^2 + Mg. \end{array} \tag{3.9}$$

The first set of equations of (3.9), with the entries viewed as indeterminates, defines $w(\mathbb{P}^1 \times \mathbb{P}^3) \subset \mathbb{P}(2,3,3,3,4,4,4,5)_{\langle b,c,d,L,e,f,M,g\rangle}$; the second set is a single quadratic form evaluated on the rows, so defines a divisor in the cone over this with vertex $\mathbb{P}^1_{\langle x,a\rangle}$. Finally, setting L, M general forms gives X as a complete intersection in this.

3.4 $\mathbf{Jer}_{24}$ fails

The matrix has the form

$$\begin{pmatrix} 0 & b & c & L_4 \\ & c & f & g \\ & & e & M_5 \\ & & & \langle c,e,f,g\rangle_6 \end{pmatrix} \mapsto \begin{pmatrix} 0 & b & c & L_4 \\ & c & f & g \\ & & e & M_5 \\ & & & 0 \end{pmatrix} \tag{3.10}$$

The entries in the rows and columns through the pivot $m_{24} = f$ are general elements of the ideal $I_D = (c,e,f,g)$. As before, $m_{12} = 0$ is forced by degrees. Although 4.2, (5) fails this for a mechanical reason, we discuss it in more detail as an instructive case, giving a perfectly nice construction of the unprojected variety X, that happens to be slightly too singular. First, please check that the entry m_{45} can be completely taken out by row and column operations. For example, to get rid of the e term in m_{45}, add α_3 times Row 3 to Row 5; in m_{25} this changes g to $g + \alpha c$, that we rename g.

One sees that the equations of the unprojected variety X take the form

$$\bigwedge^2 \begin{pmatrix} b & c & e & f \\ d & L & M & g \end{pmatrix} = 0 \quad \text{and} \quad \begin{cases} bf = c^2, \\ bg = cL, \\ dg = L^2. \end{cases} \tag{3.11}$$

(exercise, hint: project out f or g). In straight projective space, these equations define $\mathbb{P}^1 \times Q \subset \mathbb{P}^1 \times \mathbb{P}^3$ where $Q \subset \mathbb{P}^3$ is the quadric cone. This is singular in codimension 2, so the 3-fold X cannot have isolated singularities.

3.5 $\mathbf{Tom}_1$

The matrix and its clean form are

$$\begin{pmatrix} b & K_2 & L_3 & M_4 \\ & c & e & g \\ & & f & \langle c,e,f,g\rangle_5 \\ & & & \langle c,e,f,g\rangle_6 \end{pmatrix} \mapsto \begin{pmatrix} b & K & L & M \\ & c & e & g \\ & & f & \lambda_1 e \\ & & & \mu_3 c + \nu_2 e \end{pmatrix} \tag{3.12}$$

where K, L, M and λ, μ, ν are general forms, that we treat as tokens. We add a multiple of Column 2 to Column 5 to clear c from m_{35}, so we cannot use the same operation to clear e from m_{45}. The nonsingularity algorithm of Section 5 ensures that for general choices this has only nodes on D.

We show how to exhibit X as a triple parallel unprojection from a hypersurface in the product of three codimension 2 c.i. ideals (compare 8.1). Since g only appears as m_{25}, it is eliminated by writing the two Pfaffians $\mathrm{Pf}_{12.34}$ and $\mathrm{Pf}_{13.45}$ as:

$$\begin{pmatrix} L & -K & b \\ \mu K & \nu K - \lambda L & M \end{pmatrix} \begin{pmatrix} c \\ e \\ f \end{pmatrix} = 0; \tag{3.13}$$

in the same way, $\mathrm{Pf}_{12.45}$ and $\mathrm{Pf}_{12.35}$ eliminate f:

$$\begin{pmatrix} M & \lambda b & -K \\ \mu b & M + \nu b & -L \end{pmatrix} \begin{pmatrix} c \\ e \\ g \end{pmatrix} = 0. \tag{3.14}$$

Cramer's rule applied to these gives the unprojection equations for d:

$$\begin{aligned} dc &= KM + \nu bK - \lambda bL, & df &= -\mu K^2 + \nu KL - \lambda L^2, \\ de &= LM - \mu bK, & dg &= M^2 + \nu bM - \lambda\mu b^2. \end{aligned} \tag{3.15}$$

The combination eliminating d, f and g is

$$eKM - cLM - \lambda beL + \mu bcK + \nu beK = 0. \tag{3.16}$$

This is a hypersurface $Z_{10} \subset \mathbb{P}^4(1,1,2,3,4)_{\langle x,a,b,c,e\rangle}$ contained in the product ideal of $I_d = (c,e)$, $I_f = (b, M_4)$, $I_g = (K_2, L_3)$. The unprojection planes Π_d, Π_f, Π_g are projectively equivalent to $\mathbb{P}(1,1,2)$, $\mathbb{P}(1,1,3)$, $\mathbb{P}(1,1,4)$, but we cannot normalise all three of them to coordinate planes at the same time. Their pairwise intersection is:

$$\begin{aligned} \Pi_d \cap \Pi_f &= \text{the 4 zeros of } M_4 \text{ on the line } b = c = e = 0, \\ \Pi_d \cap \Pi_g &= \text{the 3 zeros of } L_3 \text{ on the line } c = e = K = 0, \\ \Pi_f \cap \Pi_g &= \text{the 2 zeros of } K_2 \text{ on the line } b = L = M = 0. \end{aligned}$$

Nonsingularity based on (3.16) All the assertions we need for Y and X are most simply derived from (3.16). The linear system $|I_d \cdot I_f \cdot I_g \cdot \mathcal{O}_{\mathbb{P}}(10)|$ of hypersurfaces through the three unprojection planes has base locus the planes themselves, together with the curve $(b = c = K_2 = 0)$, which is in the base locus because the term $eLM \in I_d \cdot I_f \cdot I_g$ has degree 11 and so does not appear in the equation of Z. This curve is a pair of generating lines $(K = 0) \subset \mathbb{P}(1,1,4)_{\langle x,a,e\rangle}$. One sees that for general choices, one of the terms cLM or λbeL in Z provides a nonzero derivative LM or λeL at every point along this curve away from the three planes.

The singular locus of Z on $\Pi_d = \mathbb{P}(1,1,2)$ is given by

$$\frac{\partial Z}{\partial c} = -LM + \mu bK = 0, \quad \frac{\partial Z}{\partial e} = KM - \lambda bL + \nu bK = 0. \tag{3.17}$$

For general choices, these are $21 = \frac{7\times 6}{2}$ reduced points of $\mathbb{P}(1,1,2)$, including the 4 points of $\Pi_d \cap \Pi_f$ and the 3 points of $\Pi_d \cap \Pi_g$; after unprojecting Π_f and Π_g, this leaves 14 nodes of Tom_1, as we asserted in (3.2). The calculations for the other planes are similar.

We believe that $Z_{10} \subset \mathbb{P}^4(1,1,2,3,4)$ has class group $\mathbb{Z}^4$ generated by the hyperplane section $A = -K_Z$ and the three planes Π_d, Π_f, Π_g, so that X is prime.

3.6 Jer_{45}

The tidied up matrix is

$$\begin{pmatrix} b & -L_2 & c & e \\ & M_3 & e & g \\ & & f & \lambda_2 c \\ & & & m_{45} \end{pmatrix}, \tag{3.18}$$

with pivot $m_{45} = \delta_3 c + \gamma_2 e + \beta_2 f + \alpha_1 g$; we use row and column operations and changes of coordinates in $I_D = (c,e,f,g)$ to clean c and f out of m_{24}, but we cannot modify the pivot m_{45} without introducing multiples of b, L, M into Row 4 or Row 5, spoiling the Jer_{45} format.

We get parallel unprojection constructions for X by eliminating f or g or both. First, subtract α times Row 2 from Row 4, and ditto with the columns, to take g out of m_{45}. This spoils the format by $c \mapsto c - \alpha b \notin I_D$ in m_{14}, but does not change the Pfaffian ideal. The new matrix only contains g in m_{25}; the two Pfaffians not involving it are $\mathrm{Pf}_{12.34}$ and the modified $\mathrm{Pf}_{13.45}$, giving

$$\begin{pmatrix} M & L & b \\ \delta L + \lambda c - \alpha\lambda b & \gamma L - \alpha M & \beta L - e \end{pmatrix} \begin{pmatrix} c \\ e \\ f \end{pmatrix} = 0. \tag{3.19}$$

Eliminating $f = m_{34}$ is similar, with $\mathrm{Pf}_{12.35}$ and modified $\mathrm{Pf}_{12.45}$ giving

$$\begin{pmatrix} \lambda b & M & L \\ \delta b - \beta M & \gamma b + e - \beta L & \alpha b - c \end{pmatrix} \begin{pmatrix} c \\ e \\ g \end{pmatrix} = 0. \tag{3.20}$$

We derive the unprojection equations for d using Cramer's rule:

$$\begin{aligned} dc &= -L(e - \beta L) - \gamma Lb + \alpha Mb, \\ de &= M(e - \beta L) + \lambda b(c - \alpha b) + \delta Lb, \\ df &= -\lambda L(c - \alpha b) - \delta L^2 + \gamma LM - \alpha M^2, \\ dg &= \lambda b(e - \beta L) + M(g - \delta b) + \gamma\lambda b^2 + \beta M^2. \end{aligned} \tag{3.21}$$

This is also a triple parallel unprojection, but with a difference: the hypersurface $Z_{10} \subset \mathbb{P}(1, 1, 2, 3, 4)$ obtained by eliminating f from (3.19) or g from (3.20) or d from the first two rows of (3.21) is now

$$e(e - \beta L)L + \delta cbL + \gamma ebL + \lambda bc(c - \alpha b) + M(ce - \beta cL - \alpha be) = 0. \tag{3.22}$$

It is in the intersection of the three codimension 2 c.i. unprojection ideals $I_d = (c, e)$, $I_f = (b, e - \beta L)$, $I_g = (c - \alpha b, L)$, but not in their product: the first 4 terms are clearly in the product ideal. The interesting part is the bracket in the last term, which cannot be in the product since it has terms of degree 2, but is in $I_d \cap I_f \cap I_g$, because

$$c(e - \beta L) - \alpha be = e(c - \alpha b) - \beta Lc. \tag{3.23}$$

The slogan is *like lines on a quadric*; the three ideals have linear combinations of b, c as first generator, and of e, L as second generator, like three disjoint lines $x = z = 0$, $y = t = 0$ and $x = t, y = z$ on $Q : (xy = zt)$. One analyses the singularities of Z_{10} from this much as before; we believe that $\mathrm{Cl}\, Z = \langle A, D_1, D_2, D_3 \rangle$, so that the triple unprojection X is prime.

4 Failure

We give reasons for failure following the introductory discussion in Section 3; we don't need to treat all the possible tests in rigorous detail, or the logical relations between them. For the structure of our proof, the point of this section is merely to give cheap preliminary tests to exclude all the candidates $D \subset Y$ that will not pass the nonsingularity algorithm in Section 5.

4.1 Easy fail at a coordinate point

Consider a coordinate point $P_i = P_{x_i} \in Y$. In either of the following cases, P_i cannot be a hyperquotient point, let alone terminal, and we can safely fail the candidate $D \subset Y$:

(1) x_i does not appear in the matrix M.

(2) x_i does not appear as a pure power in any entry of M, which thus has rank zero at P_i.

4.2 Fishy zero in M and excess singularity

Suppose we can arrange that $m_{12} = 0$, if necessary after row and column operations; then the subscheme $Z = V(\{m_{1i}, m_{2i} \mid i = 3, 4, 5\})$ is in the singular locus of Y. Indeed, the three Pfaffians Pf $12, ij$ are in I_Z^2, so do not contribute to the Jacobian at points of Z. The case that $\dim Z = 0$ and $Z \subset D$ is perfectly acceptable and happens in a fraction of our successful constructions (see Tom_2 and Jer_{25} in Section 3). Notice that $\dim Z = 0$ if and only if the 6 forms m_{1i}, m_{2i} make up a regular sequence for $\mathbb{P}^6$; in the contrary case, the zero is *fishy*. Thus any little coincidence between the six m_{1i}, m_{2i} fails $D \subset Y$. The tests we implement are:

(3) Two collinear zeros in M; see 3.1 for an example.

(4) Two of the m_{1i}, m_{2i} coincide; see Section 3, Jer_{24}.

(5) An entry m_{1i} or m_{2i} is in the ideal generated by the other five.

In fact, the tricky point here is how to read our opening 'Suppose we can arrange that $m_{12} = 0$'. The row and column operations clearly need a modicum of care to preserve the format (i.e., the entries we require to be in I_D). The harder point is that we may need a particular change of basis in I_D for the zero to appear. For example, in the Tom_5 format for $\mathbb{P}^2 \subset Y \subset \mathbb{P}(1^6, 2)$, with matrix of weights $\begin{smallmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 2 \\ & & & 2 \end{smallmatrix}$, the lowest degree Pfaffian is quadratic in three variables of weight 1, so we can write it $xy - z^2$. Mounting this as a Pfaffian in these coordinates, we can force a fishy zero, with two equal entries z arising from the term z^2. (The same applies to several candidates, but this is the only one that fails solely for this reason.)

4.3 More sophisticated and ad hoc reasons for failure

For the unprojected X to have terminal singularities, Y itself must also: it is the anticanonical model of the weak Fano 3-fold X_1. We can test for this at a coordinate point P of index $r > 1$: by Mori's classification, Y is either quasismooth at P, or a hyperquotient singularity with local weights $\frac{1}{r}(1, a, r-a, 0)$ or $\frac{1}{4}(1, 1, 3, 2)$. Thus we can fail the candidate $D \subset Y$ if:

(6) A coordinate point off D is a nonterminal hyperquotient singularity.

(7) A coordinate point on D is a nonterminal hyperquotient singularity.

These tests dispatch most of the remaining failing candidates.

(8) Ad hoc fail. Just two cases have nonisolated singularities not revealed by the elementary tests so far:

   (a) Tom_4 for $\mathbb{P}(1,2,3) \subset Y \subset \mathbb{P}(1^2, 2, 3^2, 4^2)$ with weights $\begin{smallmatrix} 2 & 2 & 3 & 3 \\ & 3 & 4 & 4 \\ & & 4 & 4 \\ & & & 5 \end{smallmatrix}$;

   (b) Jer_{12} for $\mathbb{P}(1,2,3) \subset Y \subset \mathbb{P}(1^2, 2^2, 3^2, 4)$ with weights $\begin{smallmatrix} 2 & 2 & 2 & 3 \\ & 3 & 3 & 4 \\ & & 3 & 4 \\ & & & 4 \end{smallmatrix}$.

   Each of these has a $\frac{1}{2}(1,1,1,0;0)$ hyperquotient singularity at the $\frac{1}{2}$ point of D. Such a point may be terminal if it is an isolated double point, but the format of the matrix prevents this. The second case also fails at the index 4 point P_7 lying off D: it is a hyperquotient singularity of the exceptional type $\frac{1}{4}(1,1,3,2;2)$ with the right quadratic part to be terminal. However, it lies on a curve of double points along the line $\mathbb{P}(2,4)$ joining P_7 to the $\frac{1}{2}$ point on D: in local coordinates x, a, e, b at P_7, the equation is $xa = e^2 + b \times \text{terms in } (x, a, e)^2$.

5 Nonsingularity and proof of Theorem 2.1

To prove Theorem 2.1, we need to run through a long list of candidate 3-folds $D \subset Y \subset w\mathbb{P}^6$ with choice of format Tom_i or Jer_{ij}. We exclude many of these by the automatic methods of Section 4. In every remaining case, we run a nonsingularity algorithm to confirm that the candidate can be unprojected to a codimension 4 Fano 3-fold X with terminal singularities (in fact, we conclude also quasismooth). For the proof of Theorem 2.1, we check that at least one Tom and one Jerry works for each case $D \subset Y$.

We outline the proof as a pseudocode algorithm; our implementation is discussed in Section 7. The justification of the algorithm is that it works in practice. A priori, it could fail, e.g., the singular locus of Y on D could be more complicated than a finite set of nodes, or all three coordinate lines of D could contain a node, but by good luck such accidents never happen.

5.1 Nonsingularity analysis

We work with any $D \subset Y$ not failed in Section 4. The homogeneous ideal I_Y is generated by the 4×4 Pfaffians of M. Differentiating the 5 equations Pf with respect to the seven variables gives the 5×7 Jacobian matrix $J(\mathrm{Pf})$. Its ideal $I_{\mathrm{Sing}\,Y} = \bigwedge^3 J(\mathrm{Pf})$ of 3×3 minors defines the singular locus of Y; more precisely, it generates the ideal sheaf $\mathcal{I}_{\mathrm{Sing}\,Y} \subset \mathcal{O}_{\mathbb{P}^6}$. Our claim is that the only singularities of Y lie on D, and are nodes. For this, we check that

(a) $\mathrm{Sing}\,Y \subset D$, or equivalently $I_D \subset \mathrm{Rad}(I_{\mathrm{Sing}\,Y})$.

(b) The restriction $\mathcal{I}_{\mathrm{Sing}\,Y} \cdot \mathcal{O}_D$ defines a reduced subscheme of D.

In fact (b) together with Lemma 6.1 imply that Y has only nodes. In practice, we may work on a standard affine piece of D containing all the singular points: it turns out in every case that some 1-strata of D is disjoint from the singular locus.

5.2 Proof of Theorem 2.1

We start with the data for a candidate $P \in X \subset w\mathbb{P}^7$: a genus $g \geq -2$ and a basket $\mathcal{B}$ of terminal quotient singularities, or equivalently, the resulting Hilbert series (see [ABR]). We give a choice of 8 ambient weights W_X of $w\mathbb{P}^7$ and a choice of Type I centre $P = \frac{1}{r}(1, a, r-a)$ from the basket. The Type I definition predicts that the ambient weights of $Y \subset w\mathbb{P}^6$ are $W_X \setminus \{r\}$ and that $D = \mathbb{P}(1, a, r-a)$ can be chosen to be a coordinate stratum of $w\mathbb{P}^6$. We analyse all possible Tom and Jerry formats for $D \subset Y \subset w\mathbb{P}^6$.

Step 1 Set up coordinates $x_1, x_2, x_3, x_4,\ y_1, y_2, y_3$ on $w\mathbb{P}^6$; here $x_{1\dots4}$ is a regular sequence generating I_D, and y_1, y_2, y_3 are coordinates on D.

Step 2 The numerics of [CR] determine the weights d_{ij} of the 5×5 skew matrix M from the Hilbert numerator of $Y \subset w\mathbb{P}^6$.

Step 3 Set each entry m_{ij} of M equal to a general form, respectively a general element of the ideal I_D of the given degree d_{ij}, according to the chosen Tom or Jerry format (see Definition 1.2).

Tidy up the matrix M as much as possible while preserving its Tom or Jerry format. Some entries of M may already be zero. Use coordinate changes on $w\mathbb{P}^6$ to set some entries of M equal to single variables. If possible, use row and column operations to simplify M further. Check every zero of M for failure for the mechanical reasons discussed in 4.2, followed by the other failing conditions of 4.1. Now any candidate that passes these tests actually works.

Step 4 Carry out the singularity analysis of 5.1.

Step 5 Calculate the number of nodes as in Section 6; check that no two sets of unprojection data give the same number of nodes.

Step 6 (optional) Apply the Kustin–Miller algorithm [KM] to construct the equations of X. This is not essential to prove that X exists, but knowing the full set of equations is useful if we want to put the equations in a codimension 4 format, for example by projecting from another Type I centre.

6 Number of nodes

The unprojection divisor $D = V(x_{1\dots4}) \subset \mathbb{P}^6$ is a codimension 4 c.i., with conormal bundle $\mathcal{I}_D/\mathcal{I}_D^2$ the direct sum of four orbifold line bundles $\mathcal{O}_D(-x_i)$ on D. The ideal sheaf $\mathcal{I}_Y$ is generated by 5 Pfaffians that vanish on D, so each is $\mathrm{Pf}_i = \sum a_{ij}x_j$. Thus the Jacobian matrix Jac restricted to D is the 5×4 matrix $(\overline{a}_{ij})$, where bar is restriction mod $I_D = (x_{1\dots4})$; the induced homomorphism to the conormal bundle

$$\mathcal{J}\colon \bigoplus_5 \mathcal{O}_{\mathbb{P}}(-\mathrm{Pf}_i) \twoheadrightarrow \mathcal{I}_Y/(\mathcal{I}_D\cdot\mathcal{I}_Y) \to \mathcal{I}_D/\mathcal{I}_D^2 \tag{6.1}$$

has generic rank 3. Its cokernel $\mathcal{N}$ is the conormal sheaf to D in Y. It is a rank 1 torsion free sheaf on D whose second Chern class $c_2(\mathcal{N})$ counts the nodes of Y on D. The more precise result is as follows:

Lemma 6.1 *(I) The cokernel $\mathcal{N}$ is an orbifold line bundle at points of D where* $\operatorname{rank}\mathcal{J} = 3$*, that is, at quasismooth points of Y.*

(II) Assume that $P \in D$ is a nonsingular point (not orbifold), and that $\operatorname{rank}\mathcal{J} = 2$ *at P and $= 3$ in a punctured neighbourhood of P in D; then $\mathcal{N}$ is isomorphic to a codimension 2 c.i. ideal (f, g) locally at P. This coincides locally with the ideal $\bigwedge^3 \mathrm{Jac}\cdot\mathcal{O}_D$ generated by the 3×3 minors of the Jacobian matrix.*

(III) Assume that $\bigwedge^3 \mathrm{Jac}\cdot\mathcal{O}_D$ is reduced (locally the maximal ideal m_P at each point). Then Y has an ordinary node at P.

(IV) If this holds everywhere then $c_2(\mathcal{N})$ is the number of nodes of Y on D.

Proof The statement is the hard part; the proof is just commutative algebra over a regular local ring. The rank 1 sheaf $\mathcal{N}$ is the quotient of a rank 4 locally free sheaf by the image of the 5×4 matrix $\mathrm{Jac} = (\overline{a}_{ij})$, of generic rank 3. It is a line bundle where the rank is 3, and where it drops to 2, we can use a 2×2 nonsingular block to take out a rank 2 locally free summand. The cokernel is therefore locally generated by 2 elements, so is locally isomorphic to an ideal sheaf (f, g), a c.i. because the rank drops only at P.

The minimal free resolution of $\mathcal{N}$ is the Koszul complex of f, g; now (6.1) is also part of a free resolution of $\mathcal{N}$, so covers the Koszul complex. This means that the matrix $\mathrm{Jac} = (\overline{a}_{ij})$ can be written as its 2×2 nonsingular block and a complementary 2×3 block of rank 1, whose two rows are $g\cdot v$ and $-f\cdot v$ for v a 3-vector with entries generating the unit ideal. Therefore $\bigwedge^3 \mathrm{Jac}$ generates the same ideal (f, g).

If $(f, g) = (y_1, y_2)$ is the maximal ideal at $P \in D$ then the shape of $\bigwedge^3 \mathrm{Jac}$ says that two of the Pfaffians $\mathrm{Pf}_1, \mathrm{Pf}_2$ express two of the variable x_1, x_2 as implicit functions; then a linear combination p of the remaining three has $\partial p/\partial x_3 = y_1$ and $\partial p/\partial x_4 = y_2$, so that Y is a hypersurface with an ordinary node at P. QED

We now show how to resolve $\mathcal{N}$ by an exact sequence involving direct sums of orbifold line bundles on D, and deduce a formula for $c_2(\mathcal{N})$.

Tom$_1$ The matrix is

$$M = \begin{pmatrix} K & L & M & N \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix} \tag{6.2}$$

where m_{ij} are linear forms in $x_{1\dots4} \in \mathcal{I}_D$ with coefficients in the ambient ring. When we write out $\mathrm{Jac} = (\overline{a}_{ij})$, the only terms that contribute are the derivatives $\partial/\partial x_{1\dots4}$, with the x_i set to zero; thus only the terms that are exactly linear in the x_i contribute. Since Pf_1 is of order ≥ 2 in the x_i, the corresponding row of the matrix J is zero and we omit it in (6.3). Moreover, the first row K, L, M, N of M provides a syzygy $\Sigma_1 = K\,\mathrm{Pf}_2 + L\,\mathrm{Pf}_3 + M\,\mathrm{Pf}_4 + N\,\mathrm{Pf}_5 \equiv 0$ between the 4 remaining Pfaffians. Hence we can replace J by the resolution

$$\mathcal{N} \leftarrow \sum_{1\dots4} \mathcal{O}(-d_i) \leftarrow \sum_{j\neq1} \mathcal{O}(-a_j) \leftarrow \mathcal{O}(-\sigma_1) \leftarrow 0 \tag{6.3}$$

where $d_i = \mathrm{wt}\, x_i$, $a_j = \mathrm{wt}\,\mathrm{Pf}_j$ and $\sigma_1 = \mathrm{wt}\,\Sigma_1$, and leave the reader to think of names for the maps. Therefore $\mathcal{N}$ has total Chern class

$$\prod_{i=1}^{4}(1 - d_i h) \times (1 - \sigma_1 h) \Big/ \prod_{j\neq1}(1 - a_j h) \tag{6.4}$$

The number of nodes $c_2(\mathcal{N})$ is then the h^2 term in the expansion of (6.4); recall that we view $h = c_1(\mathcal{O}_D(1))$ as an orbifold class, so that $h^2 = 1/ab$ for $D = \mathbb{P}(1, a, b)$.

Jer$_{12}$ The pivot m_{12} appears in three Pfaffians $\mathrm{Pf}_i = \mathrm{Pf}_{12,jk}$ for $\{i, j, k\} = \{3, 4, 5\}$ as the term $m_{12}m_{jk}$, together with two other terms $m_{1j}m_{2k}$ of order ≥ 2 in $x_{1\dots4}$. The Jacobian matrix restricted to D thus has three corresponding rows that are m_{jk} times the same vector $\partial m_{12}/\partial x_{1\dots4}$. This proportionality gives three syzygies Σ_l between these three rows, yoked by a second syzygy T in degree $t =$ adjunction number $-$ wt m_{12}. In other words, the conormal bundle has the resolution

$$\mathcal{N} \leftarrow \bigoplus_4 \mathcal{O}(-d_i) \leftarrow \bigoplus_5 \mathcal{O}(-a_j) \leftarrow \bigoplus_3 \mathcal{O}(-\sigma_l) \leftarrow \mathcal{O}(-t) \leftarrow 0, \tag{6.5}$$

so that the total Chern class of $\mathcal{N}$ is the alternate product

$$\frac{\prod_4 (1-d_i h) \prod_3 (1-\sigma_l h)}{\prod_5 (1-a_j h)(1-th)}, \tag{6.6}$$

with $c_2(\mathcal{N})$ equal to the h^2 term in this expansion.

Example 6.2 We read the number of nodes mechanically from the Hilbert numerator, the matrix of weights and the choice of format. As a baby example, the "interior" projections of the two del Pezzo 3-folds of degree 6 discussed in 1.2 have 2 and 3 respective nodes. These numbers are the coefficient of h^2 in the formal power series

$$\frac{(1-h)^4(1-3h)}{(1-2h)^4} = 1+h+2h^2 \text{ and } \frac{(1-h^4)(1-3h)^3}{(1-2h)^5(1-4h)} = 1+h+3h^2. \tag{6.7}$$

As a somewhat more strenuous example, in (3.2),

Tom_1 has wt $x_{1\dots4} = 3,4,4,5$, wt $\mathrm{Pf}_{2\dots5} = 8,8,7,6$, $\Sigma_1 = 10$, so that $c(\mathcal{N}) = \frac{\prod_{a\in[3,4,4,5,10]}(1-ah)}{\prod_{b\in[6,7,8,8]}(1-bh)} = 1+3h+28h^2$, giving $\frac{28}{1\cdot1\cdot2} = 14$ nodes.

Jer_{25} has the same x_i, $\mathrm{Pf}_{1\dots5} = 9,8,8,7,6$, $\Sigma_l = 10,11,12$, adjunction number $= 19$, wt $m_{25} = 5$, so $c(\mathcal{N}) = \frac{\prod_{a\in[3,4,4,5,10,11,12]}(1-ah)}{\prod_{b\in[6,7,8,8,9,14]}(1-bh)} = 1+3h+34h^2$, giving $\frac{34}{1\cdot1\cdot2} = 17$ nodes.

Try the other cases in (3.2)–(3.4) as homework.

7 Computer code and the GRDB database

A Big Table with the detailed results of the calculations proving Theorem 2.1 is online at the Graded Ring Database webpage

`http://grdb.lboro.ac.uk` + Downloads.

This website makes available computer code implementing our calculations systematically, together with the Big Table they generate. The code is for the Magma system [Ma], and installation instructions are provided; at heart, it only uses primary elements of any computer algebra system, such as polynomial ideal calculations and matrix manipulations. The code runs online in the Magma Calculator

`http://magma.maths.usyd.edu.au/calc`

All the data on the codimension 4 Fano 3-folds we construct is available on webloc. cit.: follow the link to Fano 3-folds, select Fano index $f = 1$ (the default value), codimension = 4 and Yes for Projections of Type I, then submit. The result is data on the 116 Fano 3-folds with a Type I projection (the 116th is an initial case with 7×12 resolution, that projects to the complete intersection $Y_{2,2,2} \subset \mathbb{P}^6$ containing a plane, so is not part of our story here). The + link reveals additional data on each Fano.

The computer code follows closely the algorithm outlined as the proof of Theorem 2.1. For each Tom and Jerry format, we build a matrix with random entries; some of these can be chosen to be single variables, since we assume Y is general for its format. We use row and column operations to simplify the matrix further without changing the format. The first failure tests (fishy zeroes, cone points and points of embedding dimension 6) are now easy, and inspection of the equations on affine patches at coordinate points on Y is enough to determine whether their local quotient weights are those of terminal singularities. An ideal inclusion test checks that the singularities lie on D. By good fortune, in every case that passes the tests so far, the singular locus lies on one standard affine patch of D. We pass to this affine patch and check that $\mathcal{I}_{\operatorname{Sing} Y} \cdot \mathcal{O}_D$ defines a reduced scheme there. We calculate the length of the quotient $\mathcal{O}_D/(\mathcal{I}_{\operatorname{Sing} Y} \cdot \mathcal{O}_D)$ on this patch, providing an alternative to the computation of Section 6 (and a comforting sanity check).

The random entries in the matrix are not an issue: our nonsingularity requirements are open, so if one choice leads to a successful $D \subset Y$, any general choice also works. The only concern is false negative reports, for example, an alleged nonreduced singular locus on D. To tackle such hiccups, if a candidate fails at this stage (in practice, a rare occurrence), we simply rerun the code with a new random matrix; the fact that the code happens to terminate justifies the proof.

The conclusion is that every possible Tom and Jerry format for every numerical Type I projection either fails one of the human-readable tests of

Section 4 (and we have made any number of such hand calculations), or is shown to work by constructing a specific example.

To complete the proof of Theorem 2.1, we check that the final output satisfies the following two properties:

(a) Every numerical candidate admits at least one Tom and one Jerry unprojection.

(b) Whenever a candidate has more than one Type I centre, the successful Tom and Jerry unprojections of any two correspond one-to-one, with compatible numbers of nodes: the difference in Euler number computed by the nodes is the same whichever centre we calculate from; compare (3.2)–(3.4).

The polynomial ideal calculations of Nonsingularity analysis 5.1 (that is, the inclusion $I_D \subset \mathrm{Rad}(I_{\mathrm{Sing}\,Y})$ and the statement that $\mathcal{I}_{\mathrm{Sing}\,Y} \cdot \mathcal{O}_D$ is reduced) are the only points where we use computer power seriously (other than to handle hundreds of repetitive calculations accurately). In cases with 2 or 3 centres, even this could be eliminated by projecting to a complete intersection and applying Bertini's theorem, as in Section 3.

8 Appendix: Some favourite formats

The Segre embeddings $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ are well known codimension 4 projectively Gorenstein varieties with 9×16 resolution. Singularity theorists consider the affine cones over them to be rigid, because they have no nontrivial infinitesimal deformations or small analytic deformation. Nevertheless, both are sections of higher dimensional graded varieties in many different nontrivial ways. Each of these constructions appears at many points in the study of algebraic surfaces by graded rings methods.

8.1 Parallel unprojection and extrasymmetric format

The extrasymmetric 6×6 format occurs frequently, possibly first in Dicks' thesis [D]. It is a particular case of triple unprojection from a hypersurface in the product of three codimension 2 c.i. ideals. Start from the "undeformed"

6×6 skew matrix

$$M_0 = \begin{pmatrix} b_3 & -b_2 & x_1 & a_3 & a_2 \\ & b_1 & a_3 & x_2 & a_1 \\ & & a_2 & a_1 & x_3 \\ & & & -b_3 & b_2 \\ & & & & -b_1 \end{pmatrix} \tag{8.1}$$

with the "extrasymmetric" property that the top right 3×3 block is symmetric, and the bottom right 3×3 block equals minus the top left block. So instead of 15 independent entries, it has only 9 independent entries and 6 repeats.

Direct computation reveals that the 4×4 Pfaffians of M_0 fall under the same numerics: of its 15 Pfaffians, 9 are independent and 6 repeats. One sees they generate the same ideal as the 2×2 minors of the 3×3 matrix

$$N_0 = \begin{pmatrix} x_1 & a_3 + b_3 & a_2 - b_2 \\ a_3 - b_3 & x_2 & a_1 + b_1 \\ a_2 + b_2 & a_1 - b_1 & x_3 \end{pmatrix} \tag{8.2}$$

N_0 is the generic 3×3 matrix (written as symmetric plus skew), with minors defining Segre $\mathbb{P}^2 \times \mathbb{P}^2$, and thus far we have not gained anything, beyond representing $\mathbb{P}^2 \times \mathbb{P}^2$ as a nongeneric section of $\mathrm{Grass}(2,6)$.

However M_0 can be modified to preserve the codimension 4 Gorenstein property while destroying the sporadic coincidence with $\mathbb{P}^2 \times \mathbb{P}^2$. The primitive one-parameter way of doing this is to choose the triangle $(1,2,6)$ and multiply the entries m_{12}, m_{16}, m_{26} by a constant r_3. This gives

$$M_1 = \begin{pmatrix} r_3 b_3 & -b_2 & x_1 & a_3 & r_3 a_2 \\ & b_1 & a_3 & x_2 & r_3 a_1 \\ & & a_2 & a_1 & x_3 \\ & & & -b_3 & b_2 \\ & & & & -b_1 \end{pmatrix} \tag{8.3}$$

One checks that the three Pfaffians $\mathrm{Pf}_{12.i6}$ for $i = 3,4,5$ are r_3 times others, whereas three other repetitions remain unchanged. So the 4×4 Pfaffians of M_1 still defines a Gorenstein codimension 4 subvariety with 9×16 resolution. We can view it as the unprojection of the codimension 3 Pfaffian ideal obtained by deleting the final column, with x_3 as unprojection variable.

If $r_3 = \rho^2$ is a perfect square then floating the factor ρ to the complementary entries m_{34}, m_{35}, m_{45} restores the original extrasymmetry. In general this is a "nonsplit form" of $\mathbb{P}^2 \times \mathbb{P}^2$.

A more elaborate version of this depends on 6 parameters:

$$M_2 = \begin{pmatrix} r_3 b_3 & -r_2 b_2 & x_1 & r_2 s_1 a_3 & r_3 s_1 a_2 \\ & r_1 b_1 & r_1 s_2 a_3 & x_2 & r_3 s_2 a_1 \\ & & r_1 s_3 a_2 & r_2 s_3 a_1 & x_3 \\ & & & -s_3 b_3 & s_2 b_2 \\ & & & & -s_1 b_1 \end{pmatrix} \tag{8.4}$$

Now the same three Pfaffians $\mathrm{Pf}_{12.i6}$ are divisible by r_3, and the complementary three are divisible by s_3 with the same quotient, so one has to do a little cancellation to see the irreducible component. The necessity of cancelling these terms (although cheap in computer algebra as the colon ideal) has been a headache in the theory for decades, since it introduces apparent uncertainty as to generators of the ideal.

An easy way of viewing this ideal is as the triple parallel unprojection of the hypersurface

$$V(a_1 a_2 b_3 r_3 s_3 + a_1 a_3 b_2 r_2 s_2 + a_2 a_3 b_1 r_1 s_1 + b_1 b_2 b_3) \tag{8.5}$$

in the product ideal $\prod_{i=1}^3 (a_i, b_i)$. Then $x_1 = \frac{a_2 a_3 r_1 s_1 + b_2 b_3}{a_1} = -\frac{a_2 b_3 r_2 s_2 + a_2 b_3 r_3 s_3}{b_1}$, etc., and all nine generators of the ideal are contained in the Pfaffians of the three matrixes

$$\begin{pmatrix} x_2 & b_1 & a_1 r_3 s_3 & a_3 \\ & -a_1 r_2 s_2 & -b_1 r_1 s_1 & b_3 \\ & & x_3 & a_2 \\ & & & b_2 \end{pmatrix} \begin{pmatrix} x_1 & b_3 & a_3 r_2 s_2 & a_2 \\ & -a_3 r_1 s_1 & -b_3 r_3 s_3 & b_2 \\ & & x_2 & a_1 \\ & & & b_1 \end{pmatrix} \begin{pmatrix} x_3 & b_2 & a_2 r_1 s_1 & a_1 \\ & -a_2 r_3 s_3 & -b_2 r_2 s_2 & b_1 \\ & & x_1 & a_3 \\ & & & b_3 \end{pmatrix}. \tag{8.6}$$

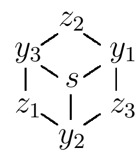
8.2 Double Jerry

The equations of Segre $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ are the minors of a $2 \times 2 \times 2$ array; they admit several extensions, and it seems most likely that there is no irreducible family containing them all. One family consists of various "rolling factors" formats discussed below; here we treat "double Jerry".

Start from the equations written as

$$\begin{aligned} s y_i &= x_j x_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \\ t x_i &= y_j y_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \\ st &= x_i y_i \quad \text{for } i = 1, 2, 3. \end{aligned} \tag{8.7}$$

corresponding to a hexagonal view of the cube centred at vertex s (with three square faces $\Box s y_i z_k y_j$, and t behind the page, c.f. (1.5)):



(8.8)

Eliminating both s and t gives the codimension 2 c.i.

$$(x_1y_1 = x_2y_2 = x_3y_3) \subset \mathbb{P}^5, \tag{8.9}$$

containing the two codimension 3 c.i.s $\mathbf{x} = 0$ and $\mathbf{y} = 0$ as divisors. We can view $\mathbf{x}$ as a row vector and $\mathbf{y}$ a column vector, and the two equations (8.9) as the matrix products

$$\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{8.10}$$

The unprojection equations for s and t separately take the form

$$t\mathbf{x} = (A\mathbf{y}) \times (B\mathbf{y}) \quad \text{and} \quad s\mathbf{y} = (\mathbf{x}A) \times (\mathbf{x}B), \tag{8.11}$$

where $\times$ is cross product of vectors in $\mathbb{C}^3$, with the convention that the cross product of two row vectors is a column vector and vice versa. For example, $\mathbf{x}A = (x_1, -x_2, 0)$, $\mathbf{x}B = (0, x_2, -x_3)$ and the equations $s\mathbf{y} = (\mathbf{x}A) \times (\mathbf{x}B)$ giving the first line of (8.7) are deduced via Cramer's rule from (8.9).

We can generalise this at a stroke to A, B general 3×3 matrixes. That is, for $\mathbf{x}$ a row vector and $\mathbf{y}$ a column vector, $\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0$ is a codimension 2 c.i.; since these are general bilinear forms in $\mathbf{x}$ and $\mathbf{y}$, it represents a universal solution to two elements of the product ideal $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3)$. It has two single unprojections:

$$\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0 \quad \text{and} \quad s\mathbf{y} = (\mathbf{x}A) \times (\mathbf{x}B), \tag{8.12}$$

$$\mathbf{x}A\mathbf{y} = \mathbf{x}B\mathbf{y} = 0 \quad \text{and} \quad t\mathbf{x} = (A\mathbf{y}) \times (B\mathbf{y}), \tag{8.13}$$

either of which is a conventional 5×5 Pfaffian, and a parallel unprojection putting those equations together with a 9th *long equation*

$$st = \text{something complicated}. \tag{8.14}$$

The equation certainly exists by the Kustin–Miller theorem. It can be obtained easily in computer algebra by coloning out any of $x_1, x_2, x_3, y_1, y_2, y_3$ from the ideal generated by the eight equations (8.12) and (8.13). Its somewhat amazing right hand side has 144 terms, each bilinear in x, y and biquadratic in A, B. Taking a hint from $144 = 12 \times 12$, we suspect that it may have a product structure of the form

$$\mathbf{x} * (A \wedge B) \times (A \wedge B) * \mathbf{y}, \tag{8.15}$$

but if so, "$*$", "$\times$" and "$\wedge$" still need to be invented.

The significance of the double Jerry parallel unprojection format is that it covers any Jerry case where the pivot is one of the generators of I_D. Indeed, if the regular sequence generating I_D is s, x_1, x_2, x_3, a Jerry matrix for D is

$$\begin{pmatrix} s & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & y_3 & -y_2 \\ & & & y_1 \end{pmatrix} \quad \text{where} \quad \begin{array}{l} (m_{13}, m_{14}, m_{15}) = \mathbf{x}A, \\ (m_{23}, m_{14}, m_{15}) = \mathbf{x}B. \end{array} \tag{8.16}$$

for some 3×3 matrixes A, B. Unprojecting D gives a double Jerry.

8.3 Rolling factors format

Rolling factors view a divisor $X \subset V$ on a normal projective variety $V \subset \mathbb{P}^n$ as residual to a nice linear system. This phenomenon occurs throughout the literature, with typical cases a divisor on the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$, or on a rational normal scroll $\mathbb{F}$, or on a cone over a Veronese embedding. A divisor $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ in the linear system $|ah_1 + (a+2)h_2| = |-K_V + bH|$ is of course defined by a single bihomogeneous equation in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^3$, but to get equations in the homogeneous coordinate ring of Segre $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ we have to add $|2h_1|$. This is a type of hyperquotient, given by one equation in a nontrivial eigenspace.

Dicks' thesis [D] discussed the generic pseudoformat

$$\bigwedge^2 \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} = 0, \quad \text{and}$$

$$\begin{array}{rcl} m_1a_1 + m_2a_2 + m_3a_3 + m_4a_4 & = & 0 \\ m_1b_1 + m_2b_2 + m_3b_3 + m_4b_4 & \equiv & n_1a_1 + n_2a_2 + n_3a_3 + n_4a_4 \;=\; 0 \\ & & n_1b_1 + n_2b_2 + n_3b_3 + n_4b_4 \;=\; 0. \end{array} \tag{8.17}$$

One sees that under fairly general assumptions the "scroll" V defined by the first set of equations of (8.17) is codimension 3 and Cohen–Macaulay, with resolution $\mathcal{O}_V \leftarrow R \leftarrow 6R \leftarrow 8R \leftarrow 3R \leftarrow 0$.

On the right, the identity is a preliminary condition on quantities in the ambient ring. If we assume (say) that R is a regular local ring and $a_i, b_i, m_i, n_i \in R$ satisfy it (and are "fairly general"), the second set defines an elephant $X \in |-K_V|$ (anticanonical divisor) which is a codimension 4 Gorenstein variety with 9×16 resolution.

The identity in (8.17) is a quadric of rank 16. It is a little close-up view of the "variety of complexes" discussed in [Ki], Section 10. To use this method to build genuine examples, we have to decide how to map a regular ambient scheme into this quadric; there are several different solutions. If we take the a_i, b_i to be independent indeterminates, the first set of equations gives the cone on Segre $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, and the second set consists of a single quadratic form q in 4 variables evaluated on the two rows, so that $X \subset V$ is given by $q(\mathbf{a}) = \varphi(\mathbf{a}, \mathbf{b}) = q(\mathbf{b}) = 0$, with φ the associated symmetric bilinear form (cf. (3.9)). This format seems to be the only commonly occurring codimension 4 Gorenstein format that tends not to have any Type I projection.

On the other hand, if there are coincidences between the a_i, b_i, there may be other ways of choosing the m_i, n_i to satisfy the identity in (8.17) without the need to take m_i, n_i quadratic in the a_i, b_i: for example, if $a_2 = b_1$, we can roll $a_1 \to a_2$ and $b_1 \to b_2$.

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9 Appendix: Further outlook, loose ends

Our calculations leave many stubs to be continued, some of which we take up in Part II. We list a few as an Appendix, not intended as part of the submitted paper. As with [CPR] and its descendants, it seems possible to get an idea of general issues in each case, without yet knowing in detail the awkward customers and new phenomena hiding among the hundreds of calculations, so it is perhaps too early to make general or precise overall claims.

9.1 The prime question

Our current statement does not address the main remaining issue: which X are prime? The following narrative is a speculative attempt.

(a) If the matrix M has a zero, then at least 3 of the Pfaffians are binomial equations, so $\operatorname{Cl} Y$ is bigger than $\langle A, D\rangle$, and X is not prime.

(b) A recent reading of the data suggests that every numerical case of $P \in X$ or $D \subset Y$ (162 cases) has exactly one T and one J model without forced zeros (this needs checking, but it seems mostly correct).

(c) The converse, that no zero implies prime is a little way off. We assert this for Tom_1 of the Extended Example, but without a really convincing argument. A hypersurface having imposed divisors D_1, D_2, D_3 meeting pointwise, and only nodes on D_i and fairly simple singularities (also nodes?) at the intersection points. Some kind of Lefschetz argument should imply that $\operatorname{Cl} Y = \langle A, D_1, D_2, D_3\rangle$.

(d) The question might be settled by having bigger prime T and J formats directly in codimension 4, not via their projections. Or maybe the whole thing is too optimistic.

9.2 Higher index Fano 3-folds

Our unprojection arguments and computer algebra routines apply almost unchanged to many of the Fanos of higher index of Suzuki's thesis (although the projected codimension 3 variety Y is not Mori category). Brown and Suzuki [BS] compute 35 cases with Fano index 2 in codimension 4, and [BS2] finds a further 8 cases with index ≥ 3, and it seems that some of these also have topologically distinct models.

For example, the index 2 variety $\mathbb{P}(1,2,4) \subset Y \subset \mathbb{P}^6(1,2,2,3,3,3,4)$ with matrix of weights $\begin{smallmatrix} 2 & 3 & 3 & 3 \\ & 3 & 3 & 3 \\ & & 4 & 4 \\ & & & 5 \end{smallmatrix}$ leads to 2 families of index 2 codimension 4 Fanos: Tom_3 works with 4 nodes, Jer_{12} with 6 nodes.

On the other hand, $\mathbb{P}(1,2,2) \subset Y \subset \mathbb{P}^6(1,2,2,3,3,4,7)$ with matrix of weights $\begin{smallmatrix} 1 & 2 & 3 & 4 \\ & 3 & 4 & 5 \\ & & 5 & 6 \\ & & & 7 \end{smallmatrix}$ only leads to a single family of index 2 codimension 4 Fanos: only Tom_1 works.

In index 4, consider $\mathbb{P}(2,3,4) \subset Y \subset \mathbb{P}^6(2,3,3,4,4,5,5)$ with matrix of weights $\begin{smallmatrix} 3 & 3 & 4 & 4 \\ & 4 & 5 & 6 \\ & & 5 & 5 \\ & & & 6 \end{smallmatrix}$. This leads to 2 families of index 4 codimension 4 Fanos (listed as number 41314 in the graded ring database): Tom_1 works with 2 nodes, Jer_{35} with 4 nodes.

9.3 Completing the Sarkisov link

The Type I projection from a point $P \in X$ initiates a 2-ray game from X; see [CPR]. Playing this game out constructs a Sarkisov link from X to another Mori fibre space. We sketch the idea, which we return to in Part II. The main new point seems to be that in the various Tom and Jerry formats, forms acquire different order of vanishing along $D \subset Y$, pushing the Sarkisov link in different directions.

The following observation is suggestive, by analogy with the computation of [CPR], 4.4 and 4.10 or [BZ], Lemma 3.6: consider Type I unprojection data $D \subset Y \subset w\mathbb{P}^6$, with $I_D = (x_{1\ldots4})$. Then for Tom, each variable x_i usually vanishes exactly once along D. In many cases of Jerry format, one of the variables x_i vanishes twice along D.

When this holds, it follows from the shape of the formats. Since Y is nonsingular at the generic point of D, at least one of the x_i must vanish to order 1. The typical Tom format is (1.2), where $m_{ij} \in I_D$ and the free entries y_i do not vanish on D. In typical cases, the x_i appear linearly in some of the m_{ij}, so no x_i vanishes to higher order than the others.

In the Jerry case the typical format is (1.2) where each $m_{ij} \in I_D$ and the free entries y_{34}, y_{35}, y_{45} do not vanish on D. Again the x_i usually appear linearly in the entries m_{ij}. In particular, up to a factor that does not vanish on D, the pivot m_{12} can be taken as the variable x_4, so the Pfaffian equations show that x_4 vanishes to order ≥ 2.

Example Consider the $\frac{1}{7}(1,2,5)$ point $P \in X$ on

$$X \subset \mathbb{P}^8(1,2,3,3,4,5,5,7) \tag{9.1}$$

with variables $x, y, z, z_1, t, u, u_1, v$ in that order. Projecting from P, we expect a Tom or Jerry setup

$$\mathbb{P}(1,2,5) = D \subset Y \subset \mathbb{P}^7(1,2,3,3,4,5,5). \tag{9.2}$$

The weights of the syzygy matrix of Y are

$$\begin{pmatrix} 6 & 5 & 5 & 4 \\ & 5 & 5 & 4 \\ & & 4 & 3 \\ & & & 3 \end{pmatrix} \tag{9.3}$$

The calculations of this paper show that Tom_1, Jer_{24} and Jer_{45} all work.

As in [BCZ], [BZ], we expect to be able to follow the Sarkisov link by tracing a simple 2-ray game in an ambient rank 2 toric variety $\mathbb{F}$. (We identify such toric varieties by the weights defining their Cox ring.) In the Tom_1 case, $\mathbb{F}$ is determined by the table of $\mathbb{C}^\times \times \mathbb{C}^\times$ weights

$$\begin{matrix} \xi & v & x & y & u & u_1 & t & z_1 & z \\ 0 & 7 & 1 & 2 & 5 & 5 & 4 & 3 & 3 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{matrix} \tag{9.4}$$

Here $\xi\colon \mathcal{O}_{X_1} \hookrightarrow \mathcal{O}_{X_1}(E)$ is the equation of the exceptional divisor $E \subset X_1$ of the Kawamata blowup $X_1 \to X$. This produces a link from X to a del Pezzo fibration which is the composite of:

(1) a Kawamata blowup $X \leftarrow Y$ of $P \in X$.

(2) a flop $Y \dashrightarrow Y_1$ of finitely many curves (corresponding to the nodes on $D \subset Y$).

(3) a hypersurface flip $(5,2,1,-2,-1;4)$ $Y_1 \dashrightarrow Y_2$ which flips the union of two $\mathbb{P}^1$ meeting at a $\frac{1}{5}(1,2,3)$ point to a single $\mathbb{P}^1$ passing through a $\frac{1}{2}$ singularity.

(4) a map $Y_2 \to \mathbb{P}^1$ (given by the ratio $z : z_1$) whose general fibre is a del Pezzo surface of degree 1.

In contrast to this, the Jer_{24} produces a completely different link: $\mathbb{F}$ is determined by the weights

$$\begin{array}{ccccccccc} \xi & v & x & y & u & t & z_1 & z & u_1 \\ 0 & 7 & 1 & 2 & 5 & 4 & 3 & 3 & 5 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \end{array} \tag{9.5}$$

Here the pivot $u_1 = m_{24}$ of the Jer_{24} matrix vanishes twice on D. This time the link is the composite of:

(1) a Kawamata blowup $X \leftarrow Y$ of $P \in X$.

(2) a flop $Y \dashrightarrow Y_1$ of finitely many curves.

(3) The t-axis does not lie on Y_1 because t^2 appears in the equations of X, so although there is a birational modication of $\mathbb{F}$ centred there, Y_1 is disjoint from the modified locus.

(4) an extremal contraction $Y_1 \to X'$ of a surface to a line (the z, z_1 axis).

The end of the link X' is a Fano 3-fold: after undoing a triple Veronese embedding, it is of the form

$$X' = X'_{6,9} \subset \mathbb{P}(1,1,2,3,4,5), \tag{9.6}$$

containing the $\mathbb{P}(1,1)$ line (with coordinates z, z_1).

The Jer_{45} link is not quite so simple – it doesn't follow the obvious toric 2-ray game, and we haven't yet computed the complete link.

References

Only referred to in the appendix:

[Br3] Gavin Brown, Flips arising as quotients of hypersurfaces, Math. Proc. Camb. Phil. Soc. **127** (1999) 13–31

[BS2] Gavin Brown and Kaori Suzuki, Computing certain Fano 3-folds, Japan J. Indust. Appl. Math., **24** 3 (2007) 241–250