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A TOPOLOGICAL VERSION OF THE BOREL-TITS THEOREM ON ABSTRACT HOMOMORPHISMS OF SIMPLE ALGEBRAIC GROUPS

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Abstract. In this paper, we produce topologized versions of two theorems. One is due to Borel and Tits [2] and is concerned with abstract homomorphisms of absolutely almost simple algebraic groups. The other is a related result discussed by Tits in [8] concerned with isomorphisms between the buildings associated with such groups. In the latter case, in order to formulate the topologized version of the result, we define a notion of "localisability" for the topological building of an absolutely almost simple algebraic group over a Hausdorff topological field. The topologized result is that such a building is always localisable in this sense when the base field is non-discrete and the relative rank of the group is greater than one.

1. Introduction

Throughout the history of Lie theory there has been a notion of "local isomorphism". [1] Indeed, the original notion of Lie groups considered by Sophus Lie in [3] was an essentially local one. In the case of an algebraic group defined over a Hausdorff topological field k one may consider the notion of a local k-isogeny. This is a mapping defined on a nonempty open subset of the set of k-rational points of the group in the classical topology which "locally" acts as a k-isogeny, in the sense that it is a local group homomorphism and its range is not contained in any Zariski closed set of strictly smaller dimension than the codomain. As far as I know this notion has not been investigated systematically. If the field is not perfect then the range of a local k-isogeny need not be an open set in the classical topology; however if the field is perfect then it is not too hard to prove that this is the case, and locally compact Hausdorff topological fields are perfect.

In this paper we apply this notion in order to obtain a local version of the Borel-Tits theorem on abstract homomorphisms of isotropic simple algebraic groups in [2]. This theorem says that an abstract homomorphism from a certain subgroup of an isotropic absolutely almost simple algebraic group over a field k into an absolutely simple algebraic group G' over a field k', whose range is Zariski dense, arises from a field homomorphism together with a special k'-isogeny. In the same paper Borel and Tits also provide a generalisation of this to reductive groups. The result is similar except that, rather than speaking of a field homomorphism from k into k', one has a ring homomorphism from k into a finite-dimensional separable commutative k'-algebra L, and one then applies the "restriction of scalars" map. This generalisation is an easy corollary of the simple case. We briefly discuss

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a local version of this generalisation of the theorem in the main body of the paper. From our local version of the Borel-Tits theorem one can also obtain as a corollary a local version of Tits' theorem classifying isomorphisms of buildings of simple algebraic groups over a field k of k-rank greater than one.

The Borel-Tits result was used by Mostow [6] to prove his "strong rigidity theorem", and also underpins the work of Margulis on "superrigidity" (see [4]). More recently, using Tanaka's theory of prolongations of maps of filtered structures on manifolds [7], Yamaguchi [9] showed that smooth local maps preserving the fibrations in a Tits building in the real case must arise form the the action of the associated semisimple group. This suggests that there might be a "local rigidity theorem" for buildings which does not require any assumptions of smoothness, or even continuity, and our corollary establishes this.

2. Basic Definitions

In this section we review the definitions of the concepts introduced in [8].

Definition 2.1. *A complex is a partially ordered set $(P, \leq)$ such that*
(1) for all $v \in P$, the set $\{w \in P \mid w \leq v\}$ is order isomorphic to $(\mathfrak{P}(S), \subseteq)$ for some set S, $\mathfrak{P}(S)$ being the powerset of S
(2) if $A, B \in P$ then A and B have a greatest lower bound denoted by $A \cap B$.
If $A \leq B$ we say that A is contained in B or is a face of B.

A complex has just one minimal element called 0. The elements which are minimal nonzero elements are called vertices and the number of vertices contained in a given element of a complex is called its rank. Given an element A of a complex $(\Delta, \leq)$, the set $\{B \in \Delta \mid A \leq B\}$, with the order relation induced from Δ, is called St A or the star of A. It is also a complex. Given $B \in$ St A, the rank of B in St A is called the codimension of A in B. The greatest lower bound of two elements C, D in a complex (which always exists) is denoted $C \cap D$, and the least upper bound (when it exists) is denoted $C \cup D$.

Definition 2.2. *We say that a complex Δ is a chamber complex if every element is contained in a maximal element and if, given two maximal elements C, C', there exists a finite sequence $C = C_0, C_1, \ldots C_m = C'$, called a gallery of length m, such that for all integers i such that $1 \leq i \leq m$, the codimension of $C_{i-1} \cap C_i$ in either C_{i-1} or C_i is at most one. The maximal elements are called chambers. We write* Cham Δ *for the set of chambers of Δ.*

An element of a chamber complex has the same codimension in any chamber which contains it; this quantity is called the codimension of the element of the complex. A chamber complex is called thick (respectively, thin) if every element of codimension one is contained in at least three (respectively, exactly two) chambers. The diameter of a chamber complex is the supremum of the lengths of all the galleries connecting two chambers of the chamber complex.

A flag complex is a complex in which any family of elements any two of which has an upper bound has an upper bound.

Definition 2.3. *A morphism of chamber complexes Δ, Δ' is a mapping $\phi : \Delta \to \Delta'$ such that the restriction of ϕ to the simplex of all faces of any given element $A \in \Delta$ is an isomorphism from the ordered set of all faces of A to the ordered set of all faces of $\phi(A)$, and ϕ maps* *chambers onto chambers. A subcomplex of a chamber complex is a chamber subcomplex if the inclusion mapping is a morphism of chamber complexes.*

Definition 2.4. *If Δ is a chamber complex and $\mathfrak{A}$ is a family of chamber subcomplexes of Δ called apartments then we say that $(\Delta, \mathfrak{A})$ is a building if*

(B1) Δ is thick;
(B2) The elements of $\mathfrak{A}$ are thin chamber complexes;
(B3) Any two elements of Δ belong to an apartment;
(B4) If two apartments Σ and Σ' contain two elements $A, A' \in \Delta$, there exists an isomorphism of Σ onto Σ' which leaves invariant A, A' and all their faces.

It is clear that all the apartments of a building are isomorphic. The isomorphism class of the apartments is called the Weyl complex of the building. The rank of the Weyl complex is also called the rank of the building. In this paper all buildings have finite rank. It can be shown that a building of finite rank is a flag complex.

If a complex Δ admits a set of apartments $\mathfrak{A}$ which makes it into a building then the union of all such sets also makes the complex Δ into a building; hence the complex Δ has at most one "maximal building structure". If the diameter of Δ is finite, then the set of apartments $\mathfrak{A}$ is unique when it exists. For these reasons we sometimes by abuse of notation speak of "the building Δ".

Definition 2.5. *Suppose that G is a reductive group defined over a field k. Then $\Delta(G, k)$ is defined to be the set of all k-parabolic subgroups of G with the reverse of inclusion as the order relation. For each maximal k-split torus of G we define the apartment corresponding to this torus to be the set of all k-parabolic subgroups containing this torus. The complex $\Delta(G, k)$ with this collection of apartments is a building.*

Given a building Δ and a chamber $C \in \Delta$ there exists a unique retraction λ_C of Δ onto the simplex of all faces of C. Two elements of the building are said to be of the same type if their image by λ_C is the same; this does not depend on the choice of C. If we take the quotient of the building by the equivalence relation "A and A' have the same type" then we obtain the typical simplex of the building typ Δ, and there is a canonical mapping $\text{typ} : \Delta \to \text{typ}\ \Delta$.

Definition 2.6. *Suppose that Δ is a building of finite rank and that an ordering of* typ Δ *is given. We define $\Delta_1 = \{A \in \Delta \mid A \text{ has rank one }\}$ and for an integer r such that $1 < r \leq$* rank Δ *we define $\Delta_r = \{(v_1, v_2, \ldots v_r) \in (\Delta_1)^r \mid$ there exists an A of rank r such that the vertex set of A is $\{v_1, v_2, \ldots v_r\}$ and $\text{typ}\ v_1 < \text{typ}\ v_2 < \ldots < \text{typ}\ v_r\}$. We say that Δ is a topological buliding if there is given a Hausdorff topology on Δ_1, such that, for each positive integer r such that $1 < r \leq \text{rank}\ \Delta$, the set Δ_r is closed in $(\Delta_1)^r$.*

If k is a Hausdorff topological field and G is an absolutely almost simple algebraic group defined over k then the building $\Delta(G, k)$ becomes a topological building in a natural way. Specifically we consider an affine space $(\overline{k})^n$ with the natural k-structure, where $\overline{k}$ is an algebraic closure of k, and a k-isomorphism of G onto a k-subvariety V of $(\overline{k})^n$. The topology on $G(k)$ induced by the subspace topology on $V(k)$ from the product topology on k^n is the classical topology on $G(k)$. This topology does not depend on the choice of the variety V or the k-isomorphism onto V, and if we assume that k is Hausdorff then it is finer than the k-Zariski topology. Given a positive integer r, the set of elements of rank r of $\Delta(G,k)$ may be identified with a disjoint union of quotients of $G(k)$ by k-parabolic subgroups, and each of the quotients may be given the quotient topology from the classical topology on $G(k)$. This gives rise to a topology on Δ_r for each r which makes $\Delta(G,k)$ into a topological building.

Definition 2.7. *Let Δ, Δ' be buildings. A mapping $\phi : \Delta \to \Delta'$ is a quasi-isomorphism if it is an isomorphism of chamber complexes onto its range and the range of the restriction of ϕ to the set of chambers of Δ is not contained in the star of any nonzero element of Δ'.*

Jacques Tits classifies the isomorphisms $\Delta(G,k) \to \Delta(H,k')$, where k and k' are fields and G and H are absolutely almost simple algebraic groups defined over k, in Chapter 5 of [8]. It is easy to modify his argument slightly so as to obtain a classification of the quasi-isomorphisms; the only change necessary is that one allows field homomorphisms which are not necessarily surjective rather than field isomorphisms.

Definition 2.8. *Let Δ, Δ' be topological buildings. A mapping $\phi : U \to \Delta'$ is a local quasi-isomorphism if U is the set of elements of Δ contained in some open subset of the set of chambers of Δ, ϕ is an isomorphism of chamber complexes onto its range, and the range of ϕ is not contained in the star of any nonzero element of Δ'.*

Definition 2.9. *Suppose that k is a topological field. A good base $\mathfrak{B}$ for the topology on k is a base for the topology such that, if $U \in \mathfrak{B}$ and σ is an affine transformation of k, then $\sigma(U) \in \mathfrak{B}$.*

Definition 2.10. *Let k be a topological field, let $\mathfrak{B}$ be a good base for the topology on k, and let V be a k-subvariety of $(\overline{k})^n$ where $\overline{k}$ is an algebraic closure of k. An open rectangle in k^n is a nonempty open subset of k^n (in the product topology) which is a product of n members of $\mathfrak{B}$. A nonempty subset $U \subseteq V(k)$, which is open (in the classical topology on $V(k)$), is said to be quasi-connected with respect to $\mathfrak{B}$ if, given any two points $p_1, p_2 \in U$, there exists a finite sequence of open rectangles $R_1, R_2, \ldots R_m \subseteq U$, such that for each i such that $1 \leq i < m$ the set $R_i \cap R_{i+1}$ is nonempty, and $p_1 \in R_1, p_2 \in R_m$. We extend this notion to arbitrary varieties over k in the obvious way. If G is an absolutely almost simple algebraic group over a Hausdorff topological field, we say that an open subset (in the classical topology) of the set of chambers of $\Delta(G,k)$ is quasi-connected with respect to a good base $\mathfrak{B}$ for the topology of k if it is quasi-connected with respect to $\mathfrak{B}$ when viewed as an open subset (in the classical topology) of the set of chambers of $\Delta(G,k)$, viewed as a projective k-variety.*

Definition 2.11. *Let G be an absolutely almost simple algebraic group over a Hausdorff topological field k. We say that the building $\Delta(G,k)$ is localisable if, given any good base $\mathfrak{B}$ for the topology on k, and any absolutely almost simple algebraic group H over a field k', and any local quasi-isomorphism $\phi : U \to \Delta(H,k')$, defined on the set U of elements of $\Delta(G,k)$ contained in a nonempty open subset of the set of chambers of $\Delta(G,k)$ which is quasi-connected with respect to $\mathfrak{B}$, ϕ extends to a quasi-isomorphism $\Delta(G,k) \to \Delta(H,k')$.*

It is worth noting that in the case where $k = \mathbb{R}$ or $\mathbb{C}$ and we let $\mathfrak{B}$ be the usual base for the topology arising from the standard metric on $\mathbb{R}$ or $\mathbb{C}$, every nonempty connected open set is quasi-connected with respect to $\mathfrak{B}$. So in that case "quasi-connected" may be replaced by "connected" in the definition of localisability for the purposes of Theorem 2.12 below.

Our goal in this paper is the following theorem, which generalises results given in [5].

Theorem 2.12. *Suppose that k is a non-discrete Hausdorff topological field and G is an absolutely almost simple algebraic group defined over k, such that the building $\Delta(G,k)$ has rank greater than one. Then the building $\Delta(G,k)$ is localisable.*

3. Proof of the Main Theorem

We will follow reasoning presented in [8] and [2], adapting the reasoning in order to obtain our topologised version of the result.

In Chapter 5 of [8] Jacques Tits proves a theorem which we state below, with the final statement Tits gives being ommitted:

Theorem 3.1. *Let G (respectively G') be an adjoint absolutely simple algebraic group defined over a field k (respectively k') and of relative rank ≥ 2 and let $\psi : \Delta(G,k) \to \Delta(G',k')$ be an isomorphism. Then, there exists a unique isomorphism $\alpha : k \to k'$ and a unique special k'-isogeny $\phi : {}^{\alpha}G \to G'$ such that ϕ induces ψ.*

Tits begins by identifying $G(k)$ and $G'(k')$ with their images in Aut Δ and Aut Δ', and observes that the mapping ψ induces an isomorphism $\psi_* : \text{Aut } \Delta \to \text{Aut } \Delta'$. If ψ is merely a quasi-isomorphism, then this will be an isomorphism $\psi_* : \text{Aut } \Delta \to \text{Aut } \psi(\Delta)$. He then uses G^+ (respectively G'^+) to denote the group generated by the groups of rational points of the unipotent radicals of the k (respectively k') -parabolic subgroups of G (respectively G'). Suppose that S is a maximal k-split torus of G and that $\Phi = \Phi(S,G)$ is the set of relative roots of G with respect to S, $\Phi^{red} = \{a \in \Phi \mid \frac{1}{2}a \notin \Phi\}$. For $a \in \Phi$, denote by $U_{(a)}$ the unipotent "root group" corresponding to a. The group G^+ is generated by the groups $U_{(a)}(k)$, for all possible choices of the maximal k-split torus S of G and of the root $a \in \Phi^{red}(G,S)$, and similar remarks may be made about G'^+. He then appeals to Proposition 5.6 of [8] which is as follows:

Theorem 3.2. *Suppose that Σ is the apartment of Δ corresponding to the torus S. Let $R_a \subset \Sigma$ consist of all k-parabolic subgroups of G which contain $S \cdot U_{(a)}$.*
(i) Let $a \in \Phi^{red}$ and let $C \in \text{Cham } R_a$ be such that no face of codimension 1 of G is contained in ∂R_a. Then $U_{(a)}(k)$ is the group of all automorphisms of Δ fixing R_a and all chambers adjacent to C ; it operates effectively on the set of all apartments containing R_a, and induces

a simply transitive permutation group of this set.
(ii) For $a \in \Phi^{red}$, a necessary and sufficient condition for the existence of a chamber C satisfying the condition of (i), is that a does not belong to a direct factor of rank 1 of the root system Φ.

From this theorem and the fact that G and G' both have relative rank ≥ 2 it follows that $\psi_*(G^+) = G'^+$ (and G'^+ is Zariski dense in G'). If ψ is only a quasi-isomorphism, then $\psi_*(G^+)$ is a subset of G'^+ which is still Zariski dense in G'. We should now consider what happens if $\psi : V \subset \Delta(G,k) \to \Delta(G',k')$ is a local quasi-isomorphism. In that case ψ will still induce a mapping ψ_* from an open subset $W \subset G^+$ (in the classical topology) onto an open subset $W' \subset H$ (again in the classical topology) where H is a subset of G'^+ which is Zariski dense in G'. So we now need to prove the following topologised version of part of Theorem 8.1 of [2].

Theorem 3.3. *Suppose that G is an isotropic absolutely almost simple k-group, and that G' is an adjoint absolutely simple k'-group and that $\psi : W \subset G^+ \to G'(k')$ is a local homomorphism, W being an open subset of G^+ in the classical topology, whose range is Zariski dense in G'. Then there exists a unique field homomorphism $\alpha : k \to k'$ and a unique special k'-isogeny $\phi : {}^{\alpha}G \to G'$ such that $\psi = \phi \circ \alpha^{\circ} |_{G^+}$.*

As in Section 8.1 of [2], we let S_m be a maximal k-split torus of G, a_m the dominant weight of S_m and $a_m^{\vee} : K^* \to S_m$ the dual weight, K being an algebraic closure of k. We let $S = a_m^{\vee}(K^*)$ and $a = a_m |_S$. Then, as in Section 8.3 of [2], we let $\Phi^+ = \Phi(S,G) \cap (\mathbb{R}_+ \cdot a)$, $U = G^{*(S)}_{\Phi^+}$, $U^- = G^{*(S)}_{-\Phi^+}$, $U_2 = G^{*(S)}_{\{a\}}$. If we let $\epsilon \in \{1,2\}$, and $b \in \Psi_\epsilon$, where Ψ_ϵ denotes the set of weights of $S' = \overline{\psi(S \cap G^+)}^{\circ}$ in U'_ϵ, where $U'_2 = \overline{\psi(U_2 \cap G^+)}$, $U' = \overline{\psi(U \cap G^+)}$, $U'_1 = U'/U'_2$, we may establish that there exists a local field homomorphism $\alpha_{\epsilon,b} : W \subset k \to K'$, K' being an algebraic closure of k' and W being an open subset of k, such that $\{\alpha_{\epsilon,b}(t)\} = b(\psi((\frac{1}{2}\epsilon a)^{-1}(t)))$ for all $t \in k^{*\epsilon}$. When we say that $\alpha_{\epsilon,b}$ is a local field homomorphism we mean that $\alpha_{\epsilon,b}(x+y) = \alpha_{\epsilon,b}(x) + \alpha_{\epsilon,b}(y)$ if $x, y, x+y \in W$, $\alpha_{\epsilon,b}(xy) = \alpha_{\epsilon,b}(x)\alpha_{\epsilon,b}(y)$ if $x, y, xy \in W$, and $\alpha_{\epsilon,b}(xy^{-1}) = \alpha_{\epsilon,b}(x)\alpha_{\epsilon,b}(y^{-1})$ if $x, y, xy^{-1} \in W$. This is all easily proved by the same reasoning as in Section 8.3 of [2].

We now observe that local field homomorphisms extend to global field homomorphisms. Consider all the mappings $\alpha_{\epsilon,b,y} : Wy^{-1} \to K'$ such that $\alpha_{\epsilon,b,y}(xy^{-1}) = \alpha_{\epsilon,b,y}(x)\alpha_{\epsilon,b,y}(y^{-1})$, as y ranges over $W \backslash \{0\}$. Since $\alpha_{\epsilon,b}$ is a local field homomorphism it is easy to see that these mappings are all compatible and so they can be glued together into a field homomorphism $k \to K'$.

Having extended the local field homomorphisms $\alpha_{\epsilon,b}$ in this way, we can then see that the following equation in Section 8.3 of [2] remains true on all of the field k.

$$\alpha_{\eta,d}(t)^{2\eta} = \alpha_{\epsilon,b}(t)^{2\epsilon} \cdot \Pi_{i=1}^{m} \alpha_{\epsilon_i,b_i}(t)^{2\epsilon_i} \tag{1}$$

We can then deduce as in Section 8.3 of [2] the existence of a field homomorphism $\alpha : k \to k'$ and non-negative integers $r(\epsilon, b)$ such that

(2) $$\alpha_{\epsilon,b} = Fr^{r(\epsilon,b)} \circ \alpha$$

where Fr is the Frobenius automorphism of K' or the identity automorphism if K' has characteristic zero.

The rest of the reasoning in [2] up to section 8.9 may now be modified to obtain a proof of our Theorem 3.3 without difficulty. In Section 8.5 a morphism ϕ_u is constructed for each $u \in U(k) - \{e\}$ which agrees with α on ${}^S u \cup \{e\}$. One performs a "localised" version of this construction. In Section 8.6 and 8.7 these various morphisms are used to construct morphisms ϕ_U, ϕ_{U^-}, and ϕ_Z, Z being the connected centre of G, which are finally used in Section 8.8 to obtain a special isogeny ϕ with the properties required in the theorem, which can be proved to be unique. One "localises" all of these constructions.

At this stage we briefly discuss a generalisation of this result to reductive groups.

Definition 3.4. *Suppose that k' is a finite field extension of k. If G is an algebraic group over k', then we write $R_{k'/k}G$ for the group resulting from G under restriction of scalars from k' to k, matrix entries in k' being replaced by square matrices with entries in k. We write $R^\circ_{k'/k}$ for the natural isomorphism $G(k') \to (R_{k'/k}G)(k)$.*

Theorem 3.5. *Suppose that G is an isotropic absolutely almost simple k-group and that G' is a reductive k'-group. Suppose further that either G is simply connected or G' is adjoint. Let $G'_i (1 \leq i \leq m)$ be the normal k'-subgroups of G' that are k'-simple (perhaps not absolutely simple). Suppose that ψ is as in the statement of Theorem 3.3. Then there exist finite separable extensions $k_i (1 \leq i \leq m)$, field homomorphisms $\phi_i : k \to k_i$, and a special k'-isogeny $\beta : \Pi_{i=1}^m R_{k_i/k'}({}^{\phi_i}G) \to G'$ and a homomorphism $\mu : H \to Z(G')(k')$ such that $\beta(R_{k_i/k'}({}^{\phi_i}G)) = G'_i$ and $\psi(h) = \mu(h) \cdot \beta(R^\circ_{k_i/k'}(\phi^\circ_i(h)))$ for all $h \in H$.*

This theorem is a local version of Theorem 8.16 in [2]. We briefly discuss the proof in the case G' is adjoint. In that case G' is the direct product of the G'_i's, and $G'_i = R_{k_i/k'}G''_i$ with k_i/k' finite separable and G''_i absolutely simple. Let $\pi_i : G' \to G'_i$ be the natural projection. One can then obtain the result by applying Theorem 3.3 to each of the maps $(R^\circ_{k_i/k'})^{-1} \circ \pi_i \circ \alpha : G \to G''_i$.

Returning to the proof of Theorem 2.12, we have established that ψ_* extends to a homomorphism $G^+ \to G'^+$ such that there is a unique field homomorphism $\alpha : k \to k'$ and a unique special k'-isogeny $\phi : {}^\alpha G \to G'$ such that $\phi\,|_{G^+} = \psi_*\,|_{G^+}$. If ψ' denotes the restriction to U of the building morphism induced by ϕ and α, then $\psi^{-1} \circ \psi'$ centralises V. But it follows from 4.20 and 5.19 of [1] that two distinct elements of Δ have different stabilisers in G^+, and it can also be shown that they have different stabilisers in V, given that V is an open subset of G^+ in the classical topology. It then follows that ψ' agrees with ψ on U and the theorem is proved.

This completes our proof of the main theorem. It would be of interest to know whether this result has a generalization to other buildings. Investigations along these lines would be worth pursuing.

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