DEPENDENCE OF BETTI NUMBERS ON CHARACTERISTIC

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ABSTRACT. We study the dependence of graded Betti numbers of monomial ideals on the characteristic of the base field. The examples we describe include bipartite ideals, Stanley–Reisner ideals of vertex-decomposable complexes and ideals with componentwise linear resolutions. We give a description of bipartite graphs and, using discrete Morse theory, provide a way of looking at the homology of arbitrary simplicial complexes through bipartite ideals. We also prove that the Betti table of a monomial ideal over the field of rational numbers can be obtained from the Betti table over any field by a sequence of consecutive cancellations.

1. Introduction

Let $R = \Bbbk[V]$ be a polynomial ring with a finite set V of indeterminates over a field $\Bbbk$. We consider R to be *standard graded*, *i.e.*, $\deg x = 1$ for all $x \in V$. Write $\mathfrak{m}$ for the unique homogeneous maximal ideal $(V)R$. Let M be a finitely generated graded R-module. A *minimal graded free resolution* of M is a complex

$$F_\bullet : \qquad 0 \longrightarrow F_p \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0 \tag{1}$$

of finitely generated graded free R-modules and homomorphisms such that (a) for all $i \geq 1$, ϕ_i is of degree 0, (b) for all $i \geq 1$, $\phi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$, and (c) $\mathrm{H}_0(F_\bullet) \simeq M$ and $\mathrm{H}_i(F_\bullet) = 0$ for all $i \geq 1$. The numerical information of a free resolution, *i.e.*, the degrees of minimal generators of the F_i is captured in the list of Betti numbers of M; the (i, j)*th graded Betti number* of M, denoted $\beta_{i,j}(M)$, is the number of minimal homogeneous generators of F_i of degree j. The i*th total Betti number* of M is $\beta_i(M) = \sum_j \beta_{i,j}(M)$. We have $\beta_{i,j}(M) = \dim_\Bbbk \left[\operatorname{Tor}_i^R(\Bbbk, M)\right]_j$, so it is an invariant of M, independent of the choice of the free resolution $F_\bullet$. The set of graded Betti numbers is represented in terms of a *Betti table* $\beta(M)$, in which the entry at column i and row j is $\beta_{i,i+j}(M)$. Similarly, if $G_\bullet$ is a complex of finitely generated graded free R-modules and homomorphisms, we write $\beta(G_\bullet)$ for the Betti table of $G_\bullet$, in which the entry at column i and row j is $\dim_\Bbbk [G_i \otimes_R \Bbbk]_{i+j}$. Here we wish to understand the following question:

Question 1.1. Suppose that I is a monomial R-ideal. Under what conditions is $\beta(I)$ independent of the characteristic of $\Bbbk$?

We will see below (Proposition 2.1) that we can immediately reduce to the case that I is generated by squarefree monomials. Then using Stanley–Reisner theory (specifically, Hochster's formula relating Betti numbers to simplicial homology — see (2) below) we can translate the problem to one of determining whether certain simplicial complexes have torsion-free homology. Therefore, in principle, Question 1.1 has a straightforward answer; the purpose of this note is to describe some sufficient conditions that would guarantee the independence of $\beta(I)$ from $\operatorname{char} \Bbbk$. We will also give some examples of ideals with strong combinatorial properties, which, nonetheless, have Betti tables that depend on $\operatorname{char} \Bbbk$.

This work is motivated in part by questions raised by J. Herzog and by the paper of M. Katzman [Kat06]. Various authors have studied the dependence of Betti tables on the characteristic. In [TH96], N. Terai and T. Hibi showed that if I is generated by quadratic square-free monomials, then $\beta_2(I)$ and $\beta_3(I)$ do not depend on $\operatorname{char} \Bbbk$. It follows from a result of B. Xu [Xu01, Lemma 26] that if I is generated by quadratic square-free monomials and the 1-skeleton of the Stanley–Reisner complex of I is a planar graph, then all the Betti numbers of I are independent of the characteristic.

We begin with describing polarization and quoting some relevant results in combinatorial commutative algebra. In Section 3, we will give a construction of vertex-decomposable (Definition 3.3) simplicial complexes whose Stanley–Reisner ideals have Betti tables that depend on $\operatorname{char} \Bbbk$. Section 4 describes bipartite ideals. Given a simplicial complex, we construct a bipartite ideal whose Betti numbers give the homology of the simplicial complex, using which we exhibit an example of a bipartite ideal I such that $\beta(I)$ depends on $\operatorname{char} \Bbbk$. In Section 5, we look at consecutive cancellations in Betti tables (Definition 5.2) and show that ideals with componentwise linear resolution have Betti tables independent of the characteristic. We will use [Eis95] as a general reference in commutative algebra, and [BH93] and [MS05] for its relation to combinatorics.

2. Preliminaries

We will use V to denote an arbitrary set of vertices, as well as the variables in the polynomial ring $R = \Bbbk[V]$. Write $V = \{x_1, \dots, x_n\}$. For a monomial R-ideal I, a *polarization* of I in a larger polynomial ring R' is the squarefree monomial ideal I' generated by monomials $\prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{i,j}$ for every minimal monomial generator $x_1^{a_1} \cdots x_n^{a_n}$ of I. For example, a polarization of (x_1^2, x_1x_2, x_2^3) is $(x_{1,1}x_{1,2}, x_{1,1}x_{2,1}, x_{2,1}x_{2,2}x_{2,3})$. See [MS05, Section 3.2 and Exercise 3.15] for details. We can get a minimal free resolution of R/I from a minimal free resolution of R'/I'. Thus:

Proposition 2.1. *Suppose that I is a monomial R-ideal. Let I' be a polarization of I in a larger polynomial ring R'. Then, $\beta(I)$ depends on $\operatorname{char} \Bbbk$ if and only if $\beta(I')$ depends on $\operatorname{char} \Bbbk$.* ☐

Hochster's Formula. (See [MS05, Corollary 5.12 and Corollary 1.40].) For $\sigma \subseteq V$, we denote by $\Delta|_\sigma$ the simplicial complex obtained by taking all the faces of Δ whose vertices belong to σ. Note that $\Delta|_\sigma$ is the Stanley-Reisner complex of the ideal $I \cap \Bbbk[\sigma]$. First, the multidegrees σ with $\beta_{i,\sigma}(R/I) \neq 0$ are squarefree. Secondly, for all squarefree multidegrees σ,

$$\beta_{i,\sigma}(R/I) = \dim_{\Bbbk} \widetilde{\mathrm{H}}_{|\sigma|-i-1}(\Delta|_\sigma; \Bbbk) \tag{2}$$

Let $I \subseteq R = \Bbbk[V]$ be a squarefree monomial ideal. Let $W \subseteq V$ and $J = (I \cap \Bbbk[W])R$. Then,

$$\beta_{i,\sigma}(R/J) = \begin{cases} 0, & \sigma \not\subseteq W, \\ \beta_{i,\sigma}(R/I), & \sigma \subseteq W. \end{cases} \tag{3}$$

Remark 2.2. Let Δ be a simplicial complex with Stanley–Reisner ideal I. Then, by (2) and the universal coefficient theorem for homology [Hat02, Theorem 3A.3], we see that $\beta(I)$ depends on $\operatorname{char} \Bbbk$ if and only if the groups $\mathrm{H}_*(\Delta; \mathbb{Z})$ have torsion. ☐

Example 2.3 (G. Reisner [BH93, Section 5.3])**.** Let Δ be the minimal triangulation of $\mathbb{RP}^2$ on the vertex set $V = \{x_1, \dots, x_6\}$ with facets $x_4x_5x_6$, $x_3x_5x_6$, $x_2x_4x_6$, $x_1x_3x_6$, $x_1x_2x_6$, $x_1x_4x_5$, $x_2x_3x_5$, $x_1x_2x_5$, $x_2x_3x_4$ and $x_1x_3x_4$. Then $I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6)$. The Betti table of I depends on $\operatorname{char} \Bbbk$, owing to the fact that $\widetilde{\mathrm{H}}_1(\mathbb{RP}^2; \mathbb{Z}) \simeq \mathbb{Z}/2$. When $\operatorname{char} \Bbbk = 2$ and when $\operatorname{char} \Bbbk \neq 2$, $\beta(I)$ is, respectively:

	0	1	2	3	4
total	1	10	15	7	1
0	1	.	.	.	.
1	.	.	.	.	.
2	.	10	15	6	1
3	.	.	.	1	.

or

	0	1	2	3
total	1	10	15	6
0	1	.	.	.
1	.	.	.	.
2	.	10	15	6

☐

Remark 2.4. Let Δ be any simplicial complex on V and $x \in V$. Then there exists a decomposition $\Delta = \operatorname{star}_\Delta(x) \cup \operatorname{del}_\Delta(x)$, where $\operatorname{star}_\Delta(x) = \{F \in \Delta : F \cup \{x\} \in \Delta\}$ and $\operatorname{del}_\Delta(x) = \Delta|_{V \smallsetminus \{x\}}$. Note that $\operatorname{star}_\Delta(x) \cap \operatorname{del}_\Delta(x) = \operatorname{lk}_\Delta(x)$, called the *link* of x in Δ. Its Stanley-Reisner ideal in $\Bbbk[V \smallsetminus \{x\}]$ is $(I : x) \cap \Bbbk[V \smallsetminus \{x\}]$. ☐

Discussion 2.5. Let Δ be any simplicial complex on V and $x \in V$. Since $\operatorname{star}_\Delta(x)$ is a cone over x, we obtain, from the Mayer–Vietoris sequence on homology [Hat02, Section 2.2], the following exact sequence:

$$\cdots \longrightarrow \widetilde{\mathrm{H}}_i(\operatorname{lk}_\Delta(x); \mathbb{Z}) \longrightarrow \widetilde{\mathrm{H}}_i(\operatorname{del}_\Delta(x); \mathbb{Z}) \longrightarrow \widetilde{\mathrm{H}}_i(\Delta; \mathbb{Z}) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{H}}_0(\Delta; \mathbb{Z}) \longrightarrow 0. \tag{4}$$

In particular, if $\widetilde{\mathrm{H}}_*(\operatorname{del}_\Delta(x); \mathbb{Z}) = 0$, then $\widetilde{\mathrm{H}}_{i+1}(\Delta; \mathbb{Z}) \simeq \widetilde{\mathrm{H}}_i(\operatorname{lk}_\Delta(x); \mathbb{Z})$, for all $i \geq 0$. ☐

3. Ideals containing powers

Let I be a monomial R-ideal containing x_1^i for some $i \geq 1$. In Theorem 3.1 we describe when $\beta(I)$ would be independent of the characteristic, from which we derive a result of M. Katzman and construct examples of vertex-decomposable simplicial complexes whose free resolution depends on the characteristic.

Theorem 3.1. *Let I be a monomial R-ideal containing x_1^i for some $i \geq 1$. Write $I = (J, x_1^t)$ minimally, i.e., t is the least integer such that $x_1^t \in I$ and J is generated by the elements of I not divisible by x_1^t. Then the following are equivalent:*

(a) $\beta(I)$ is independent of $\operatorname{char} \Bbbk$.

(b) Both $\beta(J)$ and $\beta((I :_R x_1))$ are independent of $\operatorname{char} \Bbbk$.

Proof. If $t = 1$, then $I = (J, x_1)$ and J is an ideal extended from $\Bbbk[x_2, \ldots, x_n]$. Since x_1 is a nonzerodivisor on R/J, we see that $\beta(I)$ depends on the characteristic if and only if $\beta(J)$ depends on the characteristic. Therefore we may assume that $t \geq 2$.

We will use polarization (by Proposition 2.1) to reduce to the case of squarefree monomial ideals. Let I' be a polarization of I in a polynomial ring R'; we will denote the variables that correspond to x_1 by $y_1, \ldots, y_t$ and those that correspond to $x_2, \ldots, x_n$ by $z_1, \ldots, z_m$. Write I' minimally as $(J', y_1 \cdots y_t)$. Note that J' and $(I' :_{R'} y_1)$ are, respectively, the polarization of J and $(I :_R x_1)$ in R'. By Proposition 2.1, it suffices to show that $\beta(I')$ is independent of $\operatorname{char} \Bbbk$ if and only if both $\beta(J')$ and $\beta((I' :_{R'} y_1))$ are independent of $\operatorname{char} \Bbbk$. Since $J' = (I' \cap \Bbbk[y_1, \ldots, y_{t-1}, z_1, \ldots, z_m])R'$, we see, by (3), that if $\beta(I')$ is independent of $\operatorname{char} \Bbbk$ then $\beta(J')$ is independent of $\operatorname{char} \Bbbk$. Therefore, we will assume that $\beta(J')$ is independent of $\operatorname{char} \Bbbk$ and show that $\beta(I')$ is independent of $\operatorname{char} \Bbbk$ if and only if $\beta((I' :_{R'} y_1))$ is independent of $\operatorname{char} \Bbbk$.

Suppose that $\beta_{i,\tau}(I')$ depends on $\operatorname{char} \Bbbk$, for some $\tau \subseteq \{y_1, \ldots, y_t, z_1, \ldots, z_m\}$ and i. Then $\{y_1, \ldots, y_t\} \subseteq \tau$, for, otherwise, $\beta_{i,\tau}(I') = \beta_{i,\tau}(J')$. Let Δ be the Stanley–Reisner complex of $I' \cap \Bbbk[\tau]$ on the vertex set τ. Since every generator of I' that is divisible by y_2 is also divisible by y_1, we see that $\operatorname{del}_\Delta(y_1)$ is a cone over y_2; in fact, it is a cone over the simplex on $y_2, \ldots, y_t$. On the other hand, $\operatorname{lk}_\Delta(y_1)$ is the Stanley–Reisner complex (on $\tau \smallsetminus \{y_1\}$) of $(I' :_{R'} y_1) \cap \Bbbk[\tau \smallsetminus \{y_1\}]$. By Discussion 2.5, Remark 2.2 and (3), we see that $\beta((I' :_R y_1))$ depends on $\operatorname{char} \Bbbk$.

Conversely, assume that $\beta_{i,\sigma}((I' :_R y_1))$ depends on $\operatorname{char} \Bbbk$ for some $\sigma \subseteq \{y_1, \ldots, y_t, z_1, \ldots, z_m\}$ and i. Then $y_1 \notin \sigma$. Write $\tau = \sigma \cup \{y_1\}$. Now, reversing the above argument, we see that $\beta(I')$ depends on $\operatorname{char} \Bbbk$. ☐

Corollary 3.2 (Katzman [Kat06, Corollary 1.6])**.** *Let I be quadratic squarefree monomial R-ideal, and let y be algebraically independent over R. Then $\beta((IR[y], x_1 y))$ is independent of* $\operatorname{char} \Bbbk$ *if and only if $\beta(I)$ is independent of* $\operatorname{char} \Bbbk$.

Proof. It suffices to show that if $\beta(I)$ is independent of $\operatorname{char} \Bbbk$, then $\beta((I :_R x_1))$ is independent of $\operatorname{char} \Bbbk$. Let $\sigma = \{x_i : x_1 x_i \notin I\}$. Then $(I :_R x_1) = (I \cap \Bbbk[\sigma])R + (\sigma)R$. If $\beta(I)$ is independent of $\operatorname{char} \Bbbk$, then $\beta((I \cap \Bbbk[\sigma])R)$, and, hence, $\beta((I :_R x_1))$ are independent of $\operatorname{char} \Bbbk$. ☐

Definition 3.3 ([PB80, Definition 2.1])**.** Let Δ be a d-dimensional simplicial complex on a vertex set V. We say that Δ is *vertex-decomposable* if it is pure-dimensional and either Δ is the d-simplex, or there exists $x \in V$ such that (a) $\operatorname{lk}_\Delta(x)$ is $(d-1)$-dimensional and vertex-decomposable, and (b) $\operatorname{del}_\Delta(x)$ is d-dimensional and vertex-decomposable.

Note that $\operatorname{lk}_\Delta(x)$ is $(d-1)$-dimensional and vertex-decomposable if and only if $\operatorname{star}_\Delta(x)$ is d-dimensional and vertex-decomposable. If Δ is vertex-decomposable, then it is shellable and, hence, Cohen-Macaulay in all characteristics.

We say that a R-ideal I is *primary* if R/I has a unique associated prime. A monomial R-ideal I is primary (with associated prime ideal $\mathfrak{p}$) if and only if $\mathfrak{p}$ is the radical of I and no minimal monomial generator of I is divisible by a variable not in $\mathfrak{p}$. (Note that $\mathfrak{p}$ is generated by a subset of the variables.)

Proposition 3.4. *Stanley–Reisner complexes of the polarization of primary monomial ideals are vertex-decomposable.*

Proof. Let $\mathcal{S}$ be the set of simplicial complexes on a vertex set V. This is a poset, under inclusion: $\Delta' \subseteq \Delta$ if $F \in \Delta$ for every $F \in \Delta'$. By induction on $\mathcal{S}$, it suffices to show that if Δ is the Stanley–Reisner complex of the polarization of a primary ideal, then there exists $x \in V$ such that the Stanley–Reisner ideals of $\operatorname{star}_\Delta(x)$ and $\operatorname{del}_\Delta(x)$ are also obtained through polarization.

Let $1 \leq c \leq |V|$, and I a squarefree monomial ideal with $\operatorname{ht} I = c$. Then I is the polarization of a primary monomial ideal if and only if there exists a partition $V = \bigsqcup_{i=1}^{c} \{x_{i,1}, \ldots, x_{i,n_i}\}$ of the vertex set such that for every $1 \leq i \leq c$ and for every generator f of I, if $x_{i,j} \mid f$ for some $1 \leq j \leq n_i$, then $x_{i,k} \mid f$ for every $1 \leq k \leq j$. Moreover, if this holds, we may assume that I is the polarization of an monomial ideal primary to $(x_{1,1}, \ldots, x_{c,1})R$.

Let $\mathfrak{a}$ be an $(x_{1,1}, \ldots, x_{c,1})$-primary monomial ideal and I its polarization. Let Δ be the Stanley–Reisner complex of I. The Stanley–Reisner ideal of $\operatorname{star}_\Delta(x_{1,1})$ is $(I : x_{1,1})$, which is a polarization of $(\mathfrak{a} : x_{1,1})$. The Stanley–Reisner ideal of $\operatorname{del}_\Delta(x_{1,1})$ is $(I, x_{1,1})$, which is a polarization of $(\mathfrak{a}, x_{1,1})$. Both $(\mathfrak{a} : x_{1,1})$ and $(\mathfrak{a}, x_{1,1})$ are primary. □

Remark 3.5. We now see that vertex-decomposability does not ensure that Betti tables are independent of $\operatorname{char} \Bbbk$. For, let I be as in Example 2.3. Let $S = R[y_1, \ldots, y_n]$. Let $J = IS + (x_1y_1, \ldots, x_ny_n)$; it is the polarization of $I + (x_1^2, \ldots, x_n^2)$ which is $(x_1, \ldots, x_n)$-primary. Therefore Δ_J is vertex-decomposable, while $\beta(J)$ depends on $\operatorname{char} \Bbbk$, by Theorem 3.1 and Proposition 2.1. This behaviour is already known for shellable complexes [TH96, Examples 3.3, 3.4]. □

4. Bipartite Ideals

We say that a quadratic monomial ideal I is *bipartite* if there exists a partition $V = V_1 \sqcup V_2$ such that every minimal generator of I is of the form xy for some $x \in V_1$ and $y \in V_2$. Construction 4.4 describes all bipartite ideals. In Theorem 4.7, we give a method to calculate the homology of arbitrary simplicial complexes, similar to the method of nerve complexes.

Construction 4.1. Let Γ be a simplicial complex on $V_1 := \{x_1, \ldots, x_n\}$. Let $\Gamma_j, 1 \leq j \leq m$ be a collection of simplicial subcomplexes of Γ such that $\Gamma = \cup_{j=1}^{m} \Gamma_j$. Let $V_2 = \{y_1, \ldots, y_m\}$ be a set of m new vertices. Define

$$\widetilde{\Gamma} = \{\sigma \cup \tau : \sigma \in \Gamma, \tau \subseteq \{y_j : \sigma \in \Gamma_j\}\}. \tag{5}$$

Lemma 4.2. *With notation as above, $\widetilde{\Gamma}$ is contractible.*

Proof. We prove this using discrete Morse theory developed by R. Forman [For98]. Refer to the exposition in [For02] for unexplained terminology. Specifically, we will exhibit a complete acyclic matching on the Hasse diagram of $\widetilde{\Gamma}$; see [Cha00, Section 3] and [For02, Section 6] for the interpretation of acyclic matchings of the Hasse diagram in terms of discrete Morse theory.

Let $\sigma \in \Gamma$. Let $Y_\sigma = \{y_j : \sigma \in \Gamma_j\}$ and $\mathcal{F}_\sigma = \{\sigma \cup \tau : \tau \subseteq Y_\sigma\}$. Then $\widetilde{\Gamma} = \bigsqcup_{\sigma \in \Gamma} \mathcal{F}_\sigma$ is a partition. Let j be the smallest integer such that $y_j \in Y_\sigma$. We define a complete matching on $\mathcal{F}_\sigma$ by connecting $\sigma \cup \tau$ with $\sigma \cup \tau \cup \{y_j\}$ for all $\tau \subseteq Y_\sigma$ with $y_j \notin \tau$. Repeating this for all $\sigma \in \Gamma$, we obtain a complete matching of the Hasse diagram of $\widetilde{\Gamma}$. We now claim that this is an acyclic matching. Assume the claim; then $\widetilde{\Gamma}$ is contractible, by [For02, Theorem 6.4].

To prove the claim, we let, for a face F of $\widetilde{\Gamma}$,

$$j_F = \begin{cases} \min\{j : y_j \in F\}, & \text{if there exists } j \text{ such that } y_j \in F \\ \infty, & \text{otherwise.} \end{cases}$$

Let $F \to F' \to F''$ be edges in the Hasse diagram (modified, as in [For02, Section 6], to include the matchings), such that one of them is an up arrow and the other is a down arrow. Then $j_F > j_{F''}$. Since every edge in the Hasse diagram connects two faces whose sizes differ exactly by one, we see that every cycle has an even number of edges. Since no two up arrows share a vertex (the up arrows form the matching), the up and the down arrows alternate in every directed cycle. Hence the Hasse diagram does not have directed cycles. □

Remark 4.3. Note that there may exists j such that $\Gamma_j = \{\varnothing\}$. □

Construction 4.4. Let Γ be a simplicial complex on $V_1 := \{x_1, \dots, x_n\}$. Denote the number of facets of Γ by m. Let $G_j, 1 \leq j \leq m$ be such that for all $1 \leq j \leq m$, $V_1 \smallsetminus G_j$ is a face of Γ and such that every facet of Γ is of the form $V_1 \smallsetminus G_j$ for some j. Let $y_1, \dots, y_m$ be new vertices. Let Δ_{V_1} be the $(n-1)$-simplex on $x_1, \dots, x_n$. Define

$$\Delta' = \{\sigma \cup \tau : \sigma \in \Gamma, \tau \subseteq \{y_j : \sigma \subseteq (V_1 \smallsetminus G_j)\}\} \qquad \text{and} \qquad \Delta = \Delta' \bigcup \Delta_{V_1}. \tag{6}$$

Let I be the Stanley–Reisner ideal of Δ, in the ring $R = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_m]$. Let I_Γ denote the extension of the Stanley–Reisner ideal of Γ from the ring $\Bbbk[x_1, \dots, x_n]$ to R.

Proposition 4.5. *With notation as above, $I = (x_i y_j : 1 \leq j \leq m, x_i \in G_j)$. Moreover, $I = (I + I_\Gamma) \cap (y_1, \dots, y_m)$. Hence the Stanley–Reisner ideal of Δ' is $(I + I_\Gamma)$.*

Proof. We will first show that the minimal nonfaces of Δ are precisely $\{x_i, y_j\}, 1 \leq j \leq m, x_i \in G_j$. It follows from the definition of Δ that for every $1 \leq j \leq m$ and $x_i \in G_j$, $\{x_i, y_j\}$ is a nonface. Observe that $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ are faces of Δ. Let $\sigma \cup \tau$ with $\sigma \subseteq \{x_1, \dots, x_n\}$ and $\tau \subseteq \{y_1, \dots, y_m\}$ be a minimal nonface of Δ. Hence $\sigma \neq \varnothing \neq \tau$. Therefore there exists $y_j \in \tau$ such that $\sigma \notin V_1 \smallsetminus G_j$. Let $x_i \in \sigma \cap G_j$. Now, $\{x_i, y_j\} \subseteq \sigma \cup \tau$; by minimality of $\sigma \cup \tau$ we conclude that $\sigma \cup \tau = \{x_i, y_j\}$.

In order to prove that $I = (I + I_\Gamma) \cap (y_1, \dots, y_m)$, it suffices to show that $f \in I$ for all monomials $f \in I_\Gamma \cap (y_1, \dots, y_m)$. Since the generators of I_Γ are monomials in V_1, write $f = f' y_j$ for some $f' \in I_\Gamma$. Let f' correspond to a nonface σ of Γ. Therefore $\sigma \cup \{y_j\}$ is a nonface of Δ, so $f \in I$.

Note that $(I + I_\Gamma) \nsubseteq (y_1, \dots, y_m)$. Hence the intersection $(I + I_\Gamma) \cap (y_1, \dots, y_m)$ corresponds to the union $\Delta' \cup \Delta_{V_1}$; see [MS05, Theorem 1.7]. Therefore the Stanley–Reisner ideal of Δ' is $(I + I_\Gamma)$. ☐

Remark 4.6. Every bipartite R-ideal I, with the partition $\{x_1, \dots, x_n\} \sqcup \{y_1, \dots y_m\}$, arises through Construction 4.4. Write $I = (x_i y_j : 1 \leq j \leq m, x_i \in G_j)$, where the G_j are subsets of V_1. Let

$$J = \left(\prod_{x_i \in F} x_i : F \cap G_j \neq \varnothing \text{ for all } j \right) = \left(\prod_{x_i \in F} x_i : F \cap G_j \neq \varnothing \text{ for all minimal } G_j \right).$$

Then $I = (I + J) \cap (y_1, \dots y_m)$. Let Γ be the Stanley–Reisner complex of J on the vertex set V_1. The facets of Γ are $V_1 \smallsetminus G_j$ for G_j minimal. To see this, it suffices to show that

$$\operatorname{Ass}(R/J) = \{(G_j)R : G_j \text{ minimal}\}$$

or, equivalently, that

$$J = \left(\prod_{\substack{x \in G_j \\ G_j \text{ minimal}}} x \right)^{\vee} \qquad \text{(here } (-)^{\vee} \text{ denotes taking the Alexander dual)}$$

which follows from the definition of J and [Far02, Proposition 1]. Now apply Construction 4.4 with the G_j as above. ☐

Theorem 4.7. *Let Γ and Δ be as in Construction 4.4. Then for all $i \geq 0$, $\widetilde{\mathrm{H}}_{i+1}(\Delta; \mathbb{Z}) \simeq \widetilde{\mathrm{H}}_i(\Gamma; \mathbb{Z})$.*

Proof. Notice that $\Delta' \cap \Delta_{V_1} = \Gamma$, or, equivalently, that $(I + I_\Gamma) + (y_1, \dots, y_m) = I_\Gamma + (y_1, \dots, y_m)$. From the Mayer–Vietoris sequence on homology [Hat02, Section 2.2], it suffices to prove that $\widetilde{\mathrm{H}}_i(\Delta'; \mathbb{Z}) = 0$ for all $i \geq 0$. This follows from Lemma 4.2. ☐

J. Herzog raised the question whether the Betti tables of bipartite ideals are independent of the characteristic.

Example 4.8. Let $R = \mathbb{Z}[x_1, \dots, x_6, y_1, \dots, y_{10}]$. Let Γ be the minimal triangulation of $\mathbb{RP}^2$ on the vertices $x_1, \dots, x_6$, given in Example 2.3 (and called Δ there). By Theorem 4.7, $\widetilde{\mathrm{H}}_*(\Delta; \mathbb{Z})$ is not torsion-free so, $\beta(I)$ depends on $\operatorname{char} \Bbbk$. ☐

5. Componentwise linear resolutions

We look at consecutive cancellation in Betti tables, and use it to show that the Betti tables of ideals with componentwise linear resolution are independent of the characteristic. For $t \in \mathbb{N}$, we write $(I_t)R$ for the ideal generated by the vector space I_t of polynomials of degree t in I. We say that the resolution of I is *t-linear* if $I = (I_t)R$ and $\beta_{i,j}(I) = 0$ for all $j \neq i+t$ and for all i. We say that an R-ideal I has a *componentwise linear resolution* (see [HH99]) if, for all $t \in \mathbb{N}$, the resolution of $(I_t)R$ is t-linear.

Theorem 5.1. *Suppose that I is a monomial R-ideal that has a componentwise linear resolution, in all characteristics. Then $\beta(I)$ does not depend on* $\operatorname{char} \Bbbk$.

Definition 5.2 ([Pee04])**.** Let β and β' be Betti tables. We say that β' is obtained from β by a *consecutive cancellation* if there exists i, j such that $\beta'_{i,j} = \beta_{i,j} - 1$, $\beta'_{i+1,j} = \beta_{i+1,j} - 1$ and $\beta'_{k,l} = \beta_{k,l}$ if $(k,l) \neq (i,j)$ and $(k,l) \neq (i+1,j)$.

For instance, in Example 2.3, the Betti table of R/I in characteristic 0 is obtained from its Betti table in characteristic 2 by a consecutive cancellation; we have $i = 3$ and $j = 6$.

Proposition 5.3. *Let $A = \mathbb{Z}[x_1, \ldots, x_n]$. Let $\mathfrak{a}$ be a homogeneous A-ideal such that every integer is a nonzerodivisor on $A/\mathfrak{a}$. Then, for all primes p, $\beta^{A \otimes_{\mathbb{Z}} \mathbb{Q}}((A/\mathfrak{a}) \otimes_{\mathbb{Z}} \mathbb{Q})$ can be obtained from $\beta^{A/pA}((A/\mathfrak{a}) \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z}))$ by a sequence of consecutive cancellations.*

Proof. Note that $A/\mathfrak{a}$ is a flat $\mathbb{Z}$-algebra. Let $\mathbb{F}_\bullet$ be a minimal graded free $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-resolution of $(A/\mathbf{a}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Then $\beta^{A/pA}((A/\mathfrak{a}) \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})) = \beta(\mathbb{F}_\bullet)$. Now, $\mathbb{F}_\bullet \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is a graded free $(A \otimes_{\mathbb{Z}} \mathbb{Q})$-resolution of $(A/\mathfrak{a}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore we can write $\mathbb{F}_\bullet \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} = \mathbb{G}_\bullet \oplus \mathbb{G}'_\bullet$ where $\mathbb{G}_\bullet$ is a minimal graded free $(A \otimes_{\mathbb{Z}} \mathbb{Q})$-resolution of $(A/\mathfrak{a}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{G}'_\bullet$ is graded trivial complex of free $(A \otimes_{\mathbb{Z}} \mathbb{Q})$-modules [Eis95, Theorem 20.2]. Therefore $\beta(\mathbb{G}_\bullet)$ can be obtained from $\beta(\mathbb{F}_\bullet)$ by a sequence of consecutive cancellations; now, note that $\beta^{A \otimes_{\mathbb{Z}} \mathbb{Q}}((A/\mathfrak{a}) \otimes_{\mathbb{Z}} \mathbb{Q}) = \beta(\mathbb{G}_\bullet)$. ☐

The following is an elaboration of the 'truncation principle' of D. Eisenbud, C. Huneke and B. Ulrich [EHU06, Proposition 1.6].

Lemma 5.4. *Let $t \in \mathbb{N}$. Then for all $i \geq 0$ and for all $j > i + t$, $\beta_{i,j}(I \cap \mathfrak{m}^t) = \beta_{i,j}(I)$*

Proof. The lemma follows by repeatedly applying (finitely many times) the following. Claim: Suppose that I is minimally generated by $f_1, \ldots, f_r$. Write $\tilde{I} = (f_2, \ldots, f_r) + f_1\mathfrak{m}$. Then $\beta_{i,j}(\tilde{I}) = \beta_{i,j}(I)$ for all $i \geq 0$ and for all $j > i + \deg f_1 + 1$. To prove the claim, consider the exact sequence

$$0 \longrightarrow \frac{R}{(\tilde{I} :_R f_1)}(-\deg f_1) \longrightarrow R/\tilde{I} \longrightarrow R/I \longrightarrow 0,$$

and the associated exact sequence of Tor,

$$\longrightarrow \operatorname{Tor}_i(\Bbbk, \Bbbk(-\deg f_1))_j \longrightarrow \operatorname{Tor}_i(\Bbbk, R/\tilde{I})_j \longrightarrow \operatorname{Tor}_i(\Bbbk, R/I)_j \longrightarrow \operatorname{Tor}_{i-1}(\Bbbk, \Bbbk(-\deg f_1))_j \longrightarrow \cdot$$

(Here, we use the fact that $(\tilde{I} :_R f_1) = \mathfrak{m}$.) Now, $\beta_{i,j}(\Bbbk(-\deg f_1)) = 0 = \beta_{i,j}(\Bbbk(-\deg f_1))$ for all $i \geq 0$ and for all $j > i + \deg f_1$, which proves the claim. ☐

For a homogeneous R-ideal I, set $d(I)$ to be the least degree of a minimal generator of I, *i.e.*, $d(I) = \min\{j : \beta_{0,j}(I) \neq 0\}$.

Proposition 5.5. *Let $\mathcal{C}$ be a class of monomial R-ideals such that for all $I \in \mathcal{C}$, (a) $\beta_{i,i+d(I)}(I)$ is independent of* $\operatorname{char} \Bbbk$*, and (b) $I \cap \mathfrak{m}^{d(I)+1} \in \mathcal{C}$. Then for all $I \in \mathcal{C}$, $\beta(I)$ is independent of* $\operatorname{char} \Bbbk$.

Proof. Let $I \in \mathcal{C}$. We prove the theorem by induction on $\operatorname{reg} I - d(I)$. If $\operatorname{reg} I = d(I)$, then the resolution of I is $d(I)$-linear. The only non-zero entries in $\beta(I)$ are $\beta_{i,i+d(I)}(I), i \geq 0$. Hence, by hypothesis (a), $\beta(I)$ is independent of $\operatorname{char} \Bbbk$.

If $\operatorname{reg} I > d(I)$, then, by (b) and the induction hypothesis, $\beta((I \cap \mathfrak{m}^{d(I)+1})$ is independent of $\operatorname{char} \Bbbk$. By Lemma 5.4, $\beta_{i,j}(I)$ is independent of $\operatorname{char} \Bbbk$ for all $i \geq 0$ and for all $j \geq i + d(I) + 2$. Proposition 5.3, along with (a), now finishes the proof. ☐

Lemma 5.6. *For all $i \geq 0$, $\beta_{i,i+d(I)}((I_{d(I)})R) = \beta_{i,i+d(I)}(I)$.*

Proof. Let $J \subseteq I$ be the subideal generated by the minimal generators of I of degree $d(I)+1$ or greater. Then $I = I_{d(I)} + J$. Consider the exact sequence

$$0 \to R/(I_{d(I)} \cap J) \to R/I_{d(I)} \oplus R/J \to R/I \to 0$$

and the associated exact sequence of Tor,

$$\longrightarrow \operatorname{Tor}_i(\Bbbk, \tfrac{R}{I_{d(I)} \cap J})_j \longrightarrow \begin{matrix} \operatorname{Tor}_i(\Bbbk, R/I_{d(I)})_j \\ \oplus \\ \operatorname{Tor}_i(\Bbbk, R/J)_j \end{matrix} \longrightarrow \operatorname{Tor}_i(\Bbbk, R/I)_j \longrightarrow \operatorname{Tor}_{i-1}(\Bbbk, \tfrac{R}{I_{d(I)} \cap J})_j \longrightarrow \cdot$$

Now, for all $i \geq 1$, $\beta_{i,i+d(I)}(\frac{R}{I_{d(I)} \cap J}) = \beta_{i-1,i+d(I)}(\frac{R}{I_{d(I)} \cap J}) = \beta_{i,i+d(I)}(R/J) = \beta_{i-1,i+d(I)}(R/J) = 0$. This proves the lemma. $\square$

Proof of Theorem 5.1. We will verify that ideals with componentwise linear resolution satisfy the hypotheses of Proposition 5.5. By definition, $(I_{d(I)})R$ has a $d(I)$-linear resolution in all characteristics. By Proposition 5.3, $\beta((I_{d(I)})R)$ does not depend on $\operatorname{char} \Bbbk$, so, by Lemma 5.6, we see that hypothesis (a) is satisfied. Hypothesis (b) is obtained from noting that for all $t \geq d(I)+1$, $I_t = (I \cap \mathfrak{m}^{d(I)+1})_t$. $\square$

Remark 5.7. We note that the proofs of Proposition 5.5 and Theorem 5.1 will hold, *mutatis mutandis*, if we replace the phrase "I is a monomial R-ideal" with the phrase "I is the image in R of a $\mathbb{Z}[x_1, \dots x_n]$-ideal $\mathfrak{a}$ such that $\mathbb{Z}[x_1, \dots x_n]/\mathfrak{a}$ is a flat $\mathbb{Z}$-algebra". $\square$

Examples. Theorem 5.1 shows that we cannot detect dependence on the characteristic using Alexander duality. For, let I be an ideal (such as the one in Remark 3.5) such that R/I is Cohen–Macaulay in all characteristics, but $\beta(I)$ depends on the characteristic. By a result of J. Eagon and V. Reiner [MS05, Theorem 5.56], its Alexander dual $I^\vee$ has a linear resolution in all characteristics. Hence $\beta(I^\vee)$ is independent of $\operatorname{char} \Bbbk$.

On the other hand, stable ideals have componentwise linear resolutions, given by S. Eliahou and M. Kervaire; see [MS05, Section 2.3] and [HH99, Example 1.1]. Therefore for any stable ideal I, $\beta(I)$ is independent of $\operatorname{char} \Bbbk$.

Now, as an application of Proposition 5.3, we obtain that if I is the edge ideal of a chordal graph G, then $\beta(I)$ does not depend on characteristic. T. Hibi, K. Kimura and S. Murai [HKM10, Theorem 2.1] show that the sequence $(\beta_i(R/I))$ of total Betti numbers depend only on I. By Proposition 5.3, $\beta(I)$ is independent of $\operatorname{char} \Bbbk$. As another corollary, we see that if R/I has a pure resolution in all characteristics, then $\beta(I)$ does not depend on the characteristic.

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