A CONSISTENCY PROOF FOR SOME RESTRICTIONS OF TAIT'S REFLECTION PRINCIPLES

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ABSTRACT. In [4] Tait identifies a set of reflection principles which he calls $\Gamma_n^{(2)}$-reflection principles which Peter Koellner has shown to be consistent relative to an Erdös cardinal $\kappa(\omega)$ in [1]. Tait also goes on in the same work to define a set of reflection principles which he calls $\Gamma_n^{(m)}$-reflection principles; however Koellner has shown that these are inconsistent when $m > 2$ in [2], but identifies restricted versions of them which he proves consistent relative to $\kappa(\omega)$. In this paper we introduce a new large-cardinal property with an ordinal parameter α, calling those cardinals which satisfy it strongly α-reflecting cardinals. Its definition is motivated by the remarks Tait makes in [4] about why reflection principles must be restricted when parameters of third or higher order are introduced. We prove that if κ is $(\alpha+1)$-strong and $\alpha < \kappa$ then κ is strongly α-reflecting. Furthermore we show that strongly α-reflecting cardinals relativize to L, and that if $\kappa(\omega)$ exists and α is a countable ordinal such that all $\beta \leq \alpha$ are absolutely definable then there exists a $\lambda < \kappa(\omega)$ which is strongly α-reflecting in L. We also introduce a weaker version of the property, calling those cardinals which satisfy it weakly α-reflecting cardinals. We prove that if κ is a remarkable cardinal and α is a countable ordinal such that all $\beta \leq \alpha$ are absolutely definable then κ is weakly α-reflecting. From this it follows that if $\kappa(\omega)$ exists and α is a countable ordinal as before then there is a cardinal λ such that $\lambda < \kappa(\omega)$, and λ is weakly α-reflecting. We show that a weakly ω-reflecting cardinal satisfies some restricted versions of $\Gamma_n^{(m)}$-reflection, as well as all the reflection properties which Koellner proves consistent in [2].

1. Introduction

We are going to investigate reflection principles, which postulate the existence of a level of the universe V_κ, whose properties reflect down to some lower level V_β where $\beta < \kappa$. It is useful to begin by considering reflection principles involving second-order parameters only. In later sections we will consider the issues which arise when one introduces higher-order parameters.

The cardinals yielded by these reflection principles involving second-order parameters only are called "indescribable cardinals". These principles assert the existence of a cardinal κ such that certain statements true when relativized to V_κ hold when relativized to a level V_β where $\beta < \kappa$. The strength of the reflection principles increase as one increases the expressive power of the language in which the statements are formulated, and the complexity of the formulas which express them. For example, one may consider the case where the language L in which the statements are expressed is the first-order language of set theory extended by variables of all finite orders. We denote the order of a variable with a superscript, so that $X^{(m)}$ is a variable of mth order. If a formula φ in the language L is relativized to V_κ, then the variables of mth order range over $V_{\kappa+m-1}$.

Definition 1.1. *We say that a formula in the language* L *is a* Π^m_0*-formula if the only quantified variables it contains are at most* m*th order.*
We say that a formula in the language L *is a* Π^m_1*-formula if it consists of a block of universal* $(m+1)$*th order quantifiers tacked on to the beginning of a* Π^m_0*-formula.*
We say that a formula in the language L *is* Σ^m_{k+1} *if it consists of a block of existential* $(m+1)$*th-order quantifiers tacked on to the beginning of a* Π^m_k*-formula.*
We say that a formula in the language L *is* Π^m_{k+1} *if it consists of a block of universal* $(m+1)$*th-order quantifiers tacked on to the beginning of a* Σ^m_k*-formula.*

Definition 1.2. *If* φ *is formula in the language* L*, we denote by* φ^β *the result of relativizing every* m*th-order quantifier to* $V^{\beta+m-1}$*. If* $X^{(2)}$ *is a second-order variable we abbreviate* $X^{(2)} \cap V^\beta$ *to* $X^{(2),\beta}$*.*

Definition 1.3. *If* Ω *is a class of formulas, we say that* κ *is* Ω*-indescribable if for all formulas* $\varphi \in \Omega$ *whose only free variable is second-order, for all sets* $U \subset V^\kappa$*,* $\varphi^\kappa(U) \implies \exists\beta < \kappa$ $\varphi^\beta(U^\beta)$*. We say that* κ *is totally indescribable if it is* Π^m_n*-indescribable for all* $m, n > 0$*.*

Definition 1.4. *Suppose that* α *is an ordinal. We say that* κ *is* α*-indescribable if for all* Π^1_0 *formulas* φ *in the language* L *whose only free variable is second-order, for all sets* $U \subset V_\kappa$*,* $V_{\kappa+\alpha} \models \varphi(U) \implies \exists\beta < \kappa$ $V_{\beta+\alpha} \models \varphi(U^\beta)$ *for some* $\beta < \kappa$*.*

Definition 1.5. *We say that* κ *is absolutely indescribable if* κ *is* α*-indescribable for all* $\alpha < \kappa$*.*

Definition 1.6. *We say that* κ *is extremely indescribable if for all formulas* Π^1_0 *formulas* φ *in the language* L *whose only free variable is second-order, for all sets* $U \subset V_\kappa$*,* $V_{\kappa+\kappa} \models \varphi(U) \implies \exists\beta < \kappa$ $V_{\beta+\beta} \models \varphi(U^\beta)$*.*

Here we are giving examples of cardinals κ such that V_κ satisfies reflection of formulas with second-order parameters. Let us next consider what happens when we move to parameters of third or higher order.

2. Reflection involving parameters of third or higher order

We have already defined $A^{(2),\beta}$ when $A^{(2)}$ is a second-order parameter. We define $A^{(m+1),\beta} = \{B^{(m),\beta} \mid B^{(m)} \in A^{(m+1)}\}$ for all integers $m \geq 2$. We say that κ satisfies reflection with mth-order parameters for all formulas in a class Ω if, whenever $\varphi^\kappa(U^{(m)})$ for some $U^{(m)} \subset V^{\kappa+m-1}$, there exists a $\beta < \kappa$ such that $\varphi^\beta(U^{(m),\beta})$. It is inconsistent to postulate the existence of cardinal κ which satisfies reflection for all first-order formulas with third-order parameters. To see this, let $A^{(3)}$ be a third-order parameter and let φ be the assertion that every element of $A^{(3)}$ is a bounded subset of On. This assertion can be written as a sentence in L with a third-order parameter, and all quantifiers first-order. Now, suppose that κ satisfies reflection for such sentences with third-order parameters. Let $U^{(3)} = \{\{\xi \mid \xi < \alpha\} \mid \alpha \in \mathrm{On} \cap \kappa\}$. We have $\varphi^\kappa(U^{(3)})$. So by the hypothesis about κ we must have $\varphi^\beta(U^{(3),\beta})$ for some $\beta < \kappa$. But this is impossible because $U^{(3),\beta}$ contains the set $\{\xi \mid \xi < \beta\}$, which is not bounded in $\mathrm{On} \cap V_\beta$. Thus no ordinal κ satisfies reflection for first-order formulas with third-order parameters.

This means that in order to formulate consistent reflection principles for formulas with third-order parameters or higher one must constrain the formulas relativized in some way. Let us consider what Tait writes in [4] about this issue.

"One plausible way to think about the difference between reflecting $\varphi(A)$ when A is second-order and when it is of higher-order is that, in the former case, reflection is asserting that, if $\varphi(A)$ holds in the structure $\langle R(\kappa), \in, A\rangle$, then it holds in the substructure $\langle R(\beta), \in, A^\beta\rangle$ for some $\beta < \kappa \ldots$ But, when A is higher-order, say of third-order this is no longer so. Now we are considering the structure $\langle R(\kappa), R(\kappa+1), \in, A\rangle$ and $\langle R(\beta), R(\beta+1), \in, A^\beta\rangle$. But, the latter is not a substructure of the former, that is the 'inclusion map' of the latter structure into the former is no longer single-valued: for subclasses X and Y of $R(\kappa), X \neq Y$ does not imply $X^\beta \neq Y^\beta$. Likewise for $X \in R(\beta+1), X \notin A$ does not imply $X^\beta \notin A^\beta$. For this reason, the formulas that we can expect to be preserved in passing from the former structure to the latter must be suitably restricted and, in particular, should not contain the relation $\notin$ between second- and third-order objects or the relation $\neq$ between second-order objects."

Now, suppose that we are reflecting a formula φ of the form

$$\forall X_1^{(m_1)} \exists Y_1^{(n_1)} \forall X_2^{(m_2)} \exists Y_2^{(n_2)} \cdots \forall X_k^{(m_k)} \exists Y_k^{(n_k)}$$
$$\psi(X_1^{(m_1)}, Y_1^{(n_1)}, X_2^{(m_2)}, Y_2^{(n_2)}, \ldots X_k^{(m_k)}, Y_k^{(n_k)}, A_1^{(l_1)}, A_2^{(l_2)}, \ldots A_j^{(l_j)})$$

This can be re-written as

$$\exists f_1 \exists f_2 \cdots \exists f_k \forall X_1^{(m_1)} \forall X_2^{(m_2)} \cdots \forall X_k^{(m_k)}$$
$$\psi(X_1^{(m_1)}, f_1(X_1^{(m_1)}), X_2^{(m_2)}, f_2(X_1^{(m_1)}, X_2^{(m_2)}), \ldots X_k^{(m_k)}, f_k(X_1^{(m_1)}, X_2^{(m_2)}, \ldots X_k^{(m_k)}),$$
$$A_1^{(l_1)}, A_2^{(l_2)}, \ldots A_j^{(l_j)})$$

The point is that if this formula, without the existential function quantifiers, is conceived of as holding in the structure $\langle V_\kappa, V_{\kappa+}, \ldots V_{\kappa+l}, \in, f_1, \ldots f_k, A_1^{(l_1)}, A_2^{(l_2)}, \ldots A_j^{(l_j)}\rangle$, where $l = \max(m_1, n_1, \ldots m_k, n_k, l_1 - 1, \ldots l_j - 1) - 1$, and we try to reflect down to the structure $\langle V_\beta, V_{\beta+1}, \ldots V_{\beta+l}, \in, f_1^\beta, \ldots f_k^\beta, A_1^{(l_1),\beta}, A_2^{(l_2),\beta}, \ldots A_j^{(l_j),\beta}\rangle$ for some $\beta < \kappa$, then the functions f_i^β are no longer necessarily single-valued. This consideration suggests the following reflection principle.

Definition 2.1. *We define $l(\gamma) = \gamma - 1$ if $\gamma < \omega$ and $l(\gamma) = \gamma$ otherwise. We extend the definition $A^{(m+1),\beta} = \{B^{(m),\beta} \mid B^{(m)} \in A^{(m+1)}\}$ to $A^{(\alpha),\beta} = \{B^\beta \mid B \in A^{(\alpha)}\}$ for all ordinals $\alpha > 1$, it being understood that if V_κ is the domain of discourse then $A^{(\alpha)}$ ranges over $V_{\kappa+l(\alpha)}$.*

Definition 2.2. *Suppose that α, κ are ordinals such that $\alpha < \kappa$ and that*

(1) $S = \langle\{V_{\kappa+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \ldots f_k, A_1, A_2, \ldots A_n\rangle$ is a structure where each f_i is a function $V_{\kappa+l(\gamma_1)} \times V_{\kappa+l(\gamma_2)} \times \ldots V_{\kappa+l(\gamma_i)} \to V_{\kappa+\zeta_i}$ for some ordinals $\gamma_1, \gamma_2, \ldots \gamma_i, \zeta_i$ such that $l(\gamma_1), l(\gamma_2), \ldots l(\gamma_i), \zeta_i < \alpha$, and each A_i is a subset of $V_{\kappa+l(\delta_i)}$ for some $\delta_i < \alpha$
(2) φ is a formula true in the structure S, of the form
$$\forall X_1^{(\gamma_1)} \forall X_2^{(\gamma_2)} \cdots \forall X_k^{(\gamma_k)}$$
$$\psi(X_1^{(\gamma_1)}, f_1(X_1^{(\gamma_1)}), X_2^{(\gamma_2)}, f_2(X_1^{(\gamma_1)}, X_2^{(\gamma_2)}), \ldots X_k^{(\gamma_k)}, f_k(X_1^{(\gamma_1)}, X_2^{(\gamma_2)}, \ldots X_k^{(\gamma_k)}),$$
$A_1, A_2, \ldots A_j)$ with ψ a formula with first-order quantifiers only
(3) there exists a $\beta < \kappa$ and a mapping $j : V_{\beta+\alpha} \to V_{\kappa+\alpha}$, such that $j(X) \in V_{\kappa+\gamma}$ whenever $X \in V_{\beta+\gamma}$, $j(X) = X$ for all $X \in V_\beta$, $X \subset j(X)$ for all $X \in V_{\beta+1}$ and $j(X) \in j(Y)$ whenever $X \in Y$, and such that, in the structure

$S^\beta = \langle V_\beta, \{V_{\beta+\gamma} \mid 0 < \gamma < \alpha\}, \{V_{\kappa+\gamma} \mid 0 < \gamma < \alpha\}, \in, j, f_1, f_2, \ldots f_k, A_1, A_2 \ldots A_n\rangle$, with variables of order γ ranging over $V_{\beta+l(\gamma)}$, we have

$$\forall X_1^{(\gamma_1)} \forall X_2^{(\gamma_2)} \cdots \forall X_k^{(\gamma_k)}$$
$$\psi(j(X_1^{(\gamma_1)}), f_1(j(X_1^{(\gamma_1)})), j(X_2^{(\gamma_2)}), f_2(j(X_1^{(\gamma_1)}), j(X_2^{(\gamma_2)})), \ldots j(X_k^{(\gamma_k)}), f_k(j(X_1^{(\gamma_1)}), j(X_2^{(\gamma_2)}), \ldots j(X_k^{(\gamma_k)})), A_1, A_2, \ldots A_n)$$

Then we say that the formula φ reflects down from S to β. If for all formulas φ of the above form true in the structure S, this occurs for some $\beta < \kappa$, then κ is said to be strongly α-reflecting.

We now give a consistency proof for this large cardinal property.

Theorem 2.3. *Suppose that $\omega \le \alpha < \kappa$ and κ is $\alpha+1$-strong. Then κ is strongly α-reflecting.*

Proof. Suppose that $\omega \le \alpha < \kappa$ and κ is $\alpha + 1$-strong. Then there exists an elementary embedding $k : V \to M$ with critical point κ such that $V_{\kappa+\alpha+1} \subset M$. Let $S = \langle\{V_{\kappa+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \ldots f_k, A_1, A_2, \ldots A_n\rangle$ and φ be as in the definition of an α-reflecting cardinal. Working in M, consider the structure $k(S)$. Since $V_{\kappa+\alpha+1} \subset M$, the elementary embedding k induces a mapping $j \in M$ as in the definition of an α-reflecting cardinal such that the structure $k(S)$ reflects down to κ in M. Since k is an elementary embedding we may infer that there exists a $\delta < \kappa$ such that S reflects down to δ in V. This completes the proof. □

Theorem 2.4. *Suppose that $\omega \le \alpha < \kappa$ and κ is strongly α-reflecting. Then κ is strongly α-reflecting in the constructible universe L.*

Proof. Suppose that $\omega \le \alpha < \kappa$. Let $S = \langle\{V_{\kappa+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \ldots f_k, A_1, A_2, \ldots A_n\rangle$, let $S^L = \langle\{V^L_{\kappa+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \ldots f_k, A_1, A_2, \ldots A_n\rangle \in L$ and φ be a formula as in the definition of an α-reflecting cardinal which is true in S^L. We may consider the formula φ^L with all α-order quantifiers relativized to $V^L_{\kappa+l(\alpha)}$. By introducing new Skolem functions for φ^L into the structure S to produce an expanded structure S', we may replace φ^L with a formula ψ which is true in the expanded structure S'. Then since κ is α-reflecting in V then there must be a mapping j which witnesses that ψ reflects down to some $\beta < \kappa$. One can ensure that $j \in L$ by defining j by means of the canonical well-ordering of L. This shows that φ reflects down from S^L to β in L. This completes the proof. □

Theorem 2.5. *Suppose that $\kappa(\omega)$ exists, and that α is a countable ordinal such that all ordinals $\beta \le \alpha$ are absolutely definable. Then there exists a $\lambda < \kappa(\omega)$ which is strongly α-reflecting in the constructible universe L.*

Proof. Suppose that $\kappa = \kappa(\omega)$. Then κ remains an ω-Erdös cardinal in L. Let $<$ be a well-ordering of L_κ and let $S = \{\iota_1, \iota_2, \ldots\}$ be a set of Silver indiscernibles for the structure $\langle L_\kappa, \epsilon, <\rangle$. Let M be the Skolem hull of S in this structure and let $\lambda = \iota_2$. Then the mapping $\iota_k \mapsto \iota_{k+1}$ induces an elementary embedding $j : M \to M$. If φ is a formula as in the definition of an α-reflecting cardinal where $\alpha < \lambda$ then φ will reflect down in M from ι_2 to ι_1 by means of a truncation of the mapping j. Then if one defines a mapping j' by means of the canonical well-ordering of M with the same property one will have $j' \in M$. (Given the hypothesis on α, one will only have to work with the Skolem hull of a finite fragment of S.) This shows that ι_2 is α-reflecting in M and hence in L. □

Definition 2.6. *Suppose that κ is a cardinal and that α is a countable ordinal such that all ordinals $\beta \leq \alpha$ are absolutely definable. Suppose that there exists an elementary embedding $\pi : M \to V_{\kappa+\omega+1}$ with $\pi(\lambda) = \kappa$ for some transitive set M and some $\lambda \in M$, and an elementary embedding $\sigma : M \to N$, such that if we let $S = \langle \{V^M_{\lambda+\gamma} \mid \gamma < \omega\}, \in, f_1, f_2, \ldots f_k, A_1, A_2, \ldots A_n\rangle \in M$, then in N all formulas φ as in the definition of a strongly α-reflecting cardinal true in the structure $\sigma(S)$ reflect down to some $\delta < \sigma(\lambda)$, by means of some mapping j which is not necessarily in N. Then κ is said to be weakly α-reflecting.*

Definition 2.7. *We say that a cardinal κ is remarkable* [3] *if for all regular cardinals $\theta > \kappa$, there exist $\pi, M, \lambda, \sigma, N$ and ρ such that*

(1) $\pi : M \to H_\theta$ is an elementary embedding.
(2) M is countable and transitive
(3) $\pi(\lambda) = \kappa$
(4) $\sigma : M \to N$ is an elementary embedding with critical point λ
(5) N is countable and transitive
(6) $\rho = M \cap \mathrm{On}$ is a regular cardinal in N
(7) $\sigma(\lambda) > \rho$
(8) $M = H^N_\rho$

We now prove the following.

Theorem 2.8. *Suppose that κ is a remarkable cardinal, and that α is a countable ordinal such that all $\beta \leq \alpha$ are absolutely definable. Then κ is weakly α-reflecting.*

Proof. Let κ be a remarkable cardinal. Let θ be a regular cardinal such that $\theta > \kappa$ and θ is strong limit. Let $\pi, M, \lambda, \sigma, N$ and ρ be as stipulated in the definition of a remarkable cardinal.

Working in M, let $S = \langle \{V^M_{\lambda+\gamma} \mid \gamma < \alpha\}, \in, f_1, f_2, \ldots f_k, A_1, A_2, \ldots A_n\rangle$ and φ be as in the definition of an α-reflecting cardinal.

Now, working in N, consider the structure $\sigma(S)$. We have $\sigma(\alpha) = \alpha$. The formula φ is true in the structure $\sigma(S)$, and the elementary embedding σ induces a mapping j as in the definition of a α-reflecting cardinal from $V_{\lambda+\alpha} \to V_{\sigma(\lambda)+\alpha}$ (in N), and the formula φ reflects down (in N) from $\sigma(S)$ to λ by means of j, which is not necessarily in N. This shows that κ is weakly α-reflecting and completes the proof. $\square$

Corollary 2.9. *Suppose that $\kappa(\omega)$ exists, and that α is a countable ordinal as before. Then there exists a cardinal λ such that $\lambda < \kappa(\omega)$, and λ is weakly α-reflecting.*

Next we establish some properties of weakly ω-reflecting cardinals.

3. Restricted versions of Tait's reflection principles

In [4] Tait defines the following set of reflection principles.

Definition 3.1. *A formula in the language of finite orders is positive iff it is built up by means of the operations $\vee$, $\wedge$, $\forall$, $\exists$ from atoms of the form $x = y$, $x \neq y$, $x \in y$, $x \notin y$, $x \in Y^{(2)}$, $x \notin Y(2)$ and $X^{(m)} = X'^{(m)}$ and $X^{(m)} \in Y^{(m+1)}$, where $m \geq 2$.*

Definition 3.2. *For $0 < n < \omega$, $\Gamma_n^{(2)}$ is the class of formulas*

$$\forall X_1^{(2)} \exists Y_1^{(k_1)} \cdots \forall X_n^{(2)} \exists Y_n^{(k_n)} \varphi(X_1^{(2)}, Y_1^{(k_1)}, \ldots, X_n^{(2)}, Y_n^{(k_n)}, A^{(l_1)}, \ldots A^{(l_{n'})}) \tag{1}$$

where φ does not have quantifiers or second or higher-order and $k_1, \ldots k_n, l_1, \ldots l_{n'}$ are natural numbers.

Definition 3.3. *We say that V_α satisfies $\Gamma_n^{(2)}$-reflection if for each formula $\varphi \in \Gamma_n^{(2)}$, if $V_\alpha \models \varphi$ then there is a $\delta < \alpha$ such that $V_\alpha \models \varphi^\delta$.*

Theorem 3.4 (Koellner). *Suppose that $\kappa = \kappa(\omega)$ is the first ω-Erdös cardinal. Then there exists a $\delta < \kappa$ such that V_δ satisfies $\Gamma_n^{(2)}$-reflection for all n.*

Theorem 3.5 (Tait). *Suppose that $n < \omega$ and V_κ satisfies $\Gamma_n^{(2)}$-reflection. Then κ is n-ineffable.*

Theorem 3.6 (Tait). *Suppose that κ is measurable. Then V_κ satisfies $\Gamma_n^{(2)}$-reflection for all $n < \omega$.*

In [4] Tait proposes to define $\Gamma_n^{(m)}$ in the same way as the class of formulas $\Gamma_n^{(2)}$, except that universal quantifiers of order $\leq m$ are permitted. Koellner shows in [2] that this form of reflection is inconsistent when $m > 2$. We formulate a new form of reflection which we will be able to prove holds for an ω-reflecting cardinal.

Definition 3.7. *For $2 \leq m < \omega$, $0 < n < \omega$, $\Gamma_n^{*(m)}$ is the class of formulas*

$$\forall X_1^{(k_1)} \exists Y_1^{(l_1)} \cdots \forall X_n^{(k_n)} \exists Y_n^{(l_n)} \psi(X_1^{(k_1)}, Y_1^{(l_1)}, \ldots, X_n^{(k_n)}, Y_n^{(l_n)}, A^{(m_1)}, \ldots A^{(m_p)}) \tag{2}$$

where ψ does not have quantifiers or second or higher-order and $k_1, \ldots k_n, l_1, \ldots l_n, m_1, \ldots m_p$ are natural numbers such that $l_i \geq k_j$ whenever $0 < i \leq j \leq n$.

Definition 3.8. *We say that V_κ satisfies $\Gamma_n^{*(m)}$-reflection if, for all $\varphi \in \Gamma_n^{*(m)}$, if $V_\kappa \models \varphi(A^{(m_1)}, A^{(m_2)}, \ldots A^{(m_p)})$ then $V_\kappa \models \varphi^\delta(A^{(m_1),\delta}, A^{(m_2),\delta}, \ldots A^{(m_p),\delta}$ for some $\delta < \kappa$.*

We shall now prove that if κ is weakly ω-reflecting then V_κ satisfies $\Gamma_n^{*(m)}$-reflection for all $m \geq 2, n > 0$. Note that $\Gamma_n^{*(2)}$-reflection is the same as $\Gamma_n^{(2)}$-reflection.

Theorem 3.9. *Suppose that κ is weakly ω-reflecting. Then V_κ satisfies $\Gamma_n^{*(m)}$-reflection for all $m \geq 2, n > 0$.*

Proof. We will prove it when κ is strongly ω-reflecting, and the argument will be easily modifiable to prove the case where κ is weakly ω-reflecting. Suppose that $\varphi \in \Gamma_n^{*(m)}$ is true in V_κ and that φ is as in Formula 2. There must exist functions $f_1, f_2, \ldots f_n$ such that

$$\forall X_1^{(k_1)} \ldots \forall X_n^{(k_n)} \psi(X_1^{(k_1)}, f_1(X_1^{(k_1)}), \ldots X_n^{(k_n)}, f_n(X_1^{(k_1)}, X_2^{(k_2)}, \ldots X_n^{(k_n)}), A^{(m_1)}, \ldots A^{(m_p)}) \tag{3}$$

is true in V_κ. Since κ is ω-reflecting there will be some $\beta < \kappa$ and a function $j : V_{\beta+\omega} \to V_{\kappa+\omega}$ as in the definition of an ω-reflecting cardinal such that

(4) $\forall X_1^{(k_1)} \ldots \forall X_n^{(k_n)} \psi(j(X_1^{(k_1)}), f_1(j(X_1^{(k_1)})), \ldots j(X_n^{(k_n)}), f_n(j(X_1^{(k_1)}), j(X_2^{(k_2)}), \ldots j(X_n^{(k_n)})), A^{(m_1)}, \ldots A^{(m_p)})$

is true in V_β. As Koellner observes in [1], when $k_i = 2$ for each i this is enough to prove $\Gamma_n^{(2)}$-reflection because the map $X^{(2)} \mapsto j(X^{(2)}) \cap V_\beta$ is surjective on $V_{\beta+1}$. To establish $\Gamma_n^{*(m)}$-reflection for $m > 2$, we replace j in the above formula with the function j' which agrees with j on $V_{\beta+1}$, and on $V_{\beta+m} \setminus V_{\beta+m-1}$, satisfies $j'(X) = \{j'(Y) \mid Y \in X\}$. The part of the formula inside the quantifiers will certainly remain true in V_κ. Since there exists a function k such that $j = k \circ j'$, the formula will remain true in V_β as it is equivalent to a formula asserting the existence of certain Skolem functions pickinng out appropriate values for the first-order variables, and these Skolem functions can be composed on the right with k. This completes the proof. □

It is also easy to see by examining Koellner's proofs in [2] that weakly ω-reflecting cardinals satisfy the reflection principles which he proves consistent there.

It is plausible to regard strongly α-reflecting cardinals as the natural generalization of Tait's proposed reflection principles. It would be of interest to know whether the existence of $\kappa(\omega)$ implies the existence of strongly α-reflecting cardinals outright, as opposed to merely in the constructible universe L. In any event the results here shows that these cardinals do not break the $V = L$ barrier. This provides further evidence for the view that Koellner has expressed in [2] that reflection principles are not sufficient to effect a significant reduction in incompleteness of ZFC.

References

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