

ALGEBRAIC DENSITY PROPERTY OF DANILOV-GIZATULLIN SURFACES

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ABSTRACT. A Danilov-Gizatullin surface is an affine surface V which is the complement of an ample section S of a Hirzebruch surface. The remarkable theorem of Danilov and Gizatullin states that the isomorphism class of V depends only on the self-intersection number S^2 . In this paper we apply their theorem to present V as the quotient of an affine threefold by a torus action, and to prove that the Lie algebra generated by the complete algebraic vector fields on V coincides with the set of all algebraic vector fields.

1. INTRODUCTION

Andersén-Lempert theory and density property.

Let X be a complex manifold. We say that a holomorphic vector field μ on X is complete if the solution of the associated first order ODE exists for all complex time, for any choice of initial point on X . In other words, μ induces a holomorphic action of the additive group \mathbb{C}_+ on X ; conversely, given an holomorphic \mathbb{C}_+ -action ψ_t on X there is a unique complete holomorphic vector field μ such that its flow coincides with ψ_t .

For $n \geq 2$, the abundance (in the sense of the next definition) of complete vector fields on \mathbb{C}^n was a crucial observation in the work of Andersén and Lempert on the group $\text{Aut}_{hol}(\mathbb{C}^n)$ of holomorphic automorphisms of \mathbb{C}^n [2], further developed by Forstneric and Rosay [14].

1.1. Definition. A complex manifold X is said to have the density property if the Lie algebra $\text{Lie}_{hol}(X)$ generated by complete holomorphic vector fields on X is dense, with respect to the compact-open topology, in the Lie algebra $\text{VF}_{hol}(X)$ of all holomorphic vector fields on X .

The density property for \mathbb{C}^n has many implications. An example is the existence of non-equivalent holomorphic embeddings of \mathbb{C}^k into \mathbb{C}^n [7], a result that was used by Kutzschebauch and Derksen to construct non-linearizable holomorphic actions of reductive Lie groups on affine spaces [9, 10].

The density property was introduced by Varolin [22], who was the first to extend the results of Andersén and Lempert to Stein manifolds different from \mathbb{C}^n . An interesting application from the prospective of the present paper concerns k -homogeneity.

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1.2. Definition. Let X be a complex manifold, k a positive integer. We say that a group G of holomorphic automorphisms of X acts k -transitively on X if given two collections $(x_1, \dots, x_k), (y_1, \dots, y_k)$ of pairwise distinct points, there exists an automorphism $f \in G$ such that $f(x_i) = y_i$.

1.3. Proposition. *If a Stein manifold X has the density property, then the group $\text{Aut}_{hol} X$ of holomorphic automorphisms of X acts k -transitively for all k .*

Varolin found various examples of Stein manifolds with the density property, such as some homogeneous spaces of complex semisimple Lie groups (see the paper with Toth, [23]).

If we are interested in affine algebraic varieties, we can refine the definition as follows.

1.4. Definition. A smooth affine algebraic variety X is said to have the algebraic density property if the Lie algebra $\text{Lie}_{alg}(X)$ generated by complete algebraic vector fields on X coincides with the Lie algebra $\text{VF}_{alg}(X)$ of all algebraic vector fields on X .

Since the ring of regular functions of an affine variety is dense in the ring of the holomorphic functions, the algebraic density property implies the density property. Kaliman and Kutzschebauch proved that all complex linear algebraic groups with the exception of the tori $(\mathbb{C}^*)^n$ and \mathbb{C}_+ have the algebraic density property [18]. The tori do not have the algebraic density property since Andersén [1], by using some results from Nevanlinna theory, proves that all complete polynomial vector fields on $(\mathbb{C}^*)^n$ must have divergence zero with respect to the canonical volume form, but we do not know if the tori have the (not necessarily algebraic) density property. Dvorsky, Kaliman and the author [8] prove the algebraic density property for homogeneous spaces not isomorphic to $(\mathbb{C}^*)^n$, of dimension at least three that are quotient of linear algebraic groups by a reductive subgroup. The proof relies on the methods developed in [18], but it also requires some non trivial facts from Lie theory, and an application of the Luna's slice theorem. Finally, we mention the affine surface X in \mathbb{C}^3 given by the equation $x + y + xyz = 1$. By using results of Brunella on holomorphic foliations on rational surfaces [5, 6], Kaliman and Kutzschebauch [20] prove that a complete vector field on X must have divergence zero with respect to the volume form $\frac{dx}{x} \wedge \frac{dy}{y}$, and hence that X does not have the density property. In the same preprint, the authors show that $\text{Aut}_{hol}(X)$ acts k -transitively for all k . Since we do not know if X has the density property, X could be counterexample to the converse of Proposition 1.3 (together with the tori). For a good survey of Andersén-Lempert Theory and its applications, we refer the reader to [20].

Danilov-Gizatullin surfaces.

A Danilov-Gizatullin surface is an affine surface which is the complement of an ample section S in a Hirzebruch surface Σ_d . The ampleness of S implies that $n > d$ and $n \geq 2$. We have the following remarkable result of Danilov and Gizatullin [13] (see also [17] for a short proof):

Theorem 1. *Let S (S') be an ample section of Σ_d ($\Sigma_{d'}$).*

Then $\Sigma_d - S$ is isomorphic to $\Sigma_{d'} - S'$ if and only if $S^2 = S'^2$.

Consequently, for all $n \geq 2$ we will denote by V_n the surface complementary to a section S with $S^2 = n$. The Danilov-Gizatullin surfaces belong to the class of the affine rational surfaces with trivial Makar-Limanov invariant (the subring of regular functions that are invariant with respect to all \mathbb{C}_+ -actions consists of constants only) [16]. Moreover, V_n is a flexible affine variety, that is for any point $x \in V_n$ there are pairs of \mathbb{C}_+ -actions, such that the corresponding complete vector fields span the tangent space at x . On a forthcoming paper of Arzhantsev, Flenner, Kaliman, Kutzschebauch and Zaidenberg [3], it is proven that the flexibility implies that the subgroup G of $\text{Aut}(V_n)$ generated by \mathbb{C}_+ -actions acts k -transitively on V_n , for all k (see Definition 1.2). Moreover, the set of complete vector fields whose flow is an algebraic \mathbb{C}_+ -action generates an infinite-dimensional Lie algebra. In this paper (Theorem 3) we prove that for all $n \geq 2$, V_n has the algebraic density property.

Sketch of the proof of Theorem 3.

By presenting the Hirzebruch surface Σ_d as a quotient of an open toric variety in \mathbb{C}^4 by a two dimensional torus T^2 , we can describe any section S of the ruling as the zero locus of a T^2 -invariant polynomial in \mathbb{C}^4 . The Danilov-Gizatullin theorem now plays a crucial role: by choosing to embed V_n in Σ_{n-2} , and by making a specific choice for a section S with $S^2 = n$, we prove that V_n is isomorphic to the algebraic quotient by a one-dimensional torus T of a smooth affine threefold F_n (Theorem 2), and we find a set of generators for $\mathbb{C}[F_n]^T$, the ring of T -invariant regular functions on F_n (Proposition 2.4), which is isomorphic to $\mathbb{C}[V_n]$. Next, we construct some complete T -invariant vector fields on F_n , that descends to complete vector fields on V_n (Proposition 3.1). We then perform some computations, involving those fields and the generators of $\mathbb{C}[F_n]^T \cong \mathbb{C}[V_n]$, until we construct a non-zero $\mathbb{C}[V_n]$ -submodule \mathfrak{N} of all algebraic vector fields that is contained in the Lie algebra generated by the complete ones (Theorem 6). We then extend \mathfrak{N} to a $\mathbb{C}[V_n]$ -submodule \mathfrak{M} such that the fiber of \mathfrak{M} at a point x of V_n generates the tangent space at x . The transitivity of the action of the group $\text{Aut}_{alg} V_n$ of algebraic automorphisms of V_n allows then to apply to \mathfrak{M} a technical principle (Theorem 4) to prove the algebraic density property of V_n (Theorem 3).

2. CONSTRUCTION OF V_n AS A QUOTIENT BY A TORUS ACTION

A Hirzebruch surface can be realized as a quotient of a toric variety by a two-dimensional torus ([21], Section 1): consider the action of $T^2 \cong \mathbb{C}_{t_1}^* \times \mathbb{C}_{t_2}^*$ on $\mathbb{C}_{(a_1, a_2, a_3, a_4)}^4$ given by

$$(t_1, t_2) \cdot (a_1, a_2, a_3, a_4) = (t_1 t_2^d a_1, t_2 a_2, t_1 a_3, t_2 a_4)$$

and let

$$Z = \{a_1 = a_3 = 0\} \cup \{a_2 = a_4 = 0\}.$$

Then we have that

$$\Sigma_d \cong (\mathbb{C}^4 - Z)/T^2.$$

The isomorphism above can be understood by defining coordinates

$$\begin{aligned} t_0 &= \frac{a_1}{a_2^d a_3} \quad \text{for } a_2 \neq 0, \\ t_\infty &= \frac{a_1}{a_4^d a_3} \quad \text{for } a_4 \neq 0, \\ v &= \frac{a_2}{a_4} \in \mathbb{P}^1 \end{aligned}$$

(since they are T^2 -invariant, they are well defined).

The quotient space then is isomorphic to the Hirzerbruch surface Σ_d , with ruling

$$\begin{aligned} \pi : \Sigma_d &\rightarrow \mathbb{P}^1 \\ \pi(a_1, a_2, a_3, a_4) &= (a_2, a_4) \end{aligned}$$

and transition function

$$t_\infty = v^d t_0.$$

Recall now ([4], chapter IV) that the Picard group of Σ_d is generated by the linear equivalence class F of the fiber of π , and by the class of the unique irreducible curve C with self-intersection $C^2 = -d$. It follows that S is linear equivalent to $C + bF$, and that $n = 2b - d$.

Theorem 1 implies that $\Sigma_d - S \cong V_n$ for any section S with $S^2 = n$, and we claim that

$$t_\infty = v^b$$

is a local equation (for $v \neq \infty$) of a section S of the ruling π with self intersection $n = 2b - d$. In fact, the base locus of the family of section $t_\infty = cv^b$ consists of two multiple points, namely the point $A(v = 0, t_\infty = 0)$ with multiplicity b , and $B(v = \infty, t_0 = 0)$ with multiplicity $b - d$: if we sum the intersection multiplicities at A and B , we obtain that $S^2 = 2b - d = n$.

In the coordinates (a_1, a_2, a_3, a_4) the defining equation of S takes the form

$$(1) \quad a_1 a_4^{b-d} - a_2^b a_3 = 0$$

(the ampleness of S implies that $b > d$).

Let S' be the hypersurface in \mathbb{C}^4 given by equation (1). Then the inverse image $\rho^{-1}(S) = S' - Z$ under the quotient map is the closed T -invariant subset of \mathbb{C}^4 given by the equation (1), and

$$V_n \cong \rho(\mathbb{C}^4 - (S' \cup Z)) = (\mathbb{C}^4 - (S' \cup Z))/T^2.$$

Since $Z \subset S'$, the variety $\mathbb{C}^4 - (S' \cup Z)$ is isomorphic to the affine manifold $V' = \mathbb{C} - S'$.

2.1. Lemma. *The geometric quotient V'/T^2 is isomorphic to the algebraic quotient $V'//T^2 = \text{Spec } \mathbb{C}[V']^{T^2}$, where $\mathbb{C}[V']^{T^2}$ denotes the subring of the T^2 -invariant regular functions of V' .*

Proof. The orbits of the T^2 -action restricted to V' are all closed. Therefore the lemma follows from the Luna's slice theorem ([11], Theorem 5.4). \square

We can invoke Theorem 1 again and fix from now on that V_n is embedded in Σ_{n-2} , that is $n = d + 2$, and $b = d + 1$. Under this assumption, the equation of S' becomes

$$(2) \quad a_1 a_4 - a_2^b a_3 = 0.$$

2.2. Proposition. *Consider the smooth affine threefold*

$$(3) \quad F_n = \{a_1 a_4 - a_2^b a_3 = 1\} \subset \mathbb{C}^4.$$

Then V' is equivariantly isomorphic to $\mathbb{C}^ \times F_n$, where the action of T^2 on $\mathbb{C}^* \times F_n$ is defined by:*

$$(4) \quad (t_1, t_2) \cdot (w, a_1, a_2, a_3, a_4) = (t_1^{-1} t_2^{-b} w, t_2^{-1} a_1, t_2 a_2, t_2^{-b} a_3, t_2 a_4).$$

Proof. The isomorphism (as affine varieties) is given by

$$(5) \quad (a_1, a_2, a_3, a_4) \mapsto \left(a_1 a_4 - a_3^b a_2, \frac{a_1}{a_1 a_4 - a_3^b a_2}, a_2, \frac{a_3}{a_1 a_4 - a_3^b a_2}, a_4 \right)$$

Then it is easy to check that the induced T^2 action on $\mathbb{C}^* \times F_n$ by the isomorphism is given by (4). \square

When we pass to the quotient, we get rid of the factor \mathbb{C}^* , as follows.

Theorem 2. *For any $n \geq 2$, the Danilov-Gizatullin surface V_n is isomorphic to the algebraic quotient $F_n//T$, where F_n is the affine threefold given by the equation*

$$(6) \quad a_1 a_4 - a_2^b a_3 = 1$$

and the torus $T \cong \mathbb{C}^$ acts on F_n via*

$$(7) \quad t.(a_1, a_2, a_3, a_4) = (t^{-1}a_1, ta_2, t^{-b}a_3, ta_4)$$

Proof. Lemma 2.1 implies that $V'/T^2 \cong V''/T^2$. Observe that the T^2 -action of V' (equation 4) is such that the t_1 -variable acts non-trivially only on the w -coordinate. Therefore there are no non-constant T^2 -invariant polynomials depending on w , and the result follows. \square

Perhaps, the above theorem has been already established, but we do not know a reference for it.

2.3. Remark. Theorem 2 presents V_n as a quotient of a hypersurface in \mathbb{C}^4 given by an equation of the type $wv = p(y, z)$, for p a polynomial with smooth zero locus. The affine manifolds of this type have the algebraic density property, as shown by Kaliman and Kutzschebauch [19]. Furthermore, it is clear from its defining equation that F is a ramified b -sheeted covering of SL_2 , another manifold with the algebraic density property [23]. The behavior of the density property with respect to quotients or coverings is not known in general.

2.4. Proposition. $\mathbb{C}[V_n] \cong \mathbb{C}[F_n]^T$ is generated by the monomials

$$(8) \quad y = a_1a_2, \quad z = a_1a_4, \quad x_k = a_2^{b-k}a_3a_4^k \quad (0 \leq k \leq b).$$

Proof. Since the action is monomial, it is sufficient to look for monomial generators. The action of t on $a_1^X a_2^Y a_3^Z a_4^W$ is

$$t.(a_1^X a_2^Y a_3^Z a_4^W) = t^{-X+Y-bZ+W} a_1^X a_2^Y a_3^Z a_4^W,$$

therefore a monomial is invariant if and only if

$$(9) \quad -X + Y - bZ + W = 0.$$

We show that every solution (with non-negative integers) of equation (9) is a linear combination with non-negative integer coefficients of the vectors

$$\begin{aligned} &(1, 1, 0, 0) \quad (1, 0, 0, 1), \\ &(0, b - k, 1, k) \quad (\text{for } 0 \leq k \leq b). \end{aligned}$$

Write $W = NbZ + r$ and $Y = MbZ + r'$, for $0 \leq r, r' \leq bZ$. Suppose that $W < bZ$: choose Z integers k_i , with $0 \leq k_i \leq b$, such that $\sum_i^Z k_i = W$. Then

$$(X, Y, Z, W) = X(1, 1, 0, 0) + \sum_i^Z (0, b - k_i, 1, k_i)$$

Suppose instead that $W \geq bZ$. Then $X - MbZ = W - bZ + r' \geq 0$; choose Z integers k_i , with $0 \leq k_i \leq b$, such that $\sum_i^Z (b - k_i) = r'$. Then we can write

$$(X, Y, Z, W) = (X - MbZ)(1, 0, 0, 1) + \sum_i^Z (0, b - k_i, 1, k_i) + MbZ(1, 1, 0, 0).$$

□

3. ALGEBRAIC DENSITY PROPERTY OF V_n

In this section we prove the main result of the paper.

Theorem 3. *The Danilov-Gizatullin surfaces V_n have the algebraic density property.*

We make use of the following principle [18].

Theorem 4. *Let X be a smooth affine algebraic variety, such that the group $\text{Aut}_{\text{alg}}(X)$ of algebraic automorphisms acts transitively on it. Let \mathfrak{M} be a submodule of the $\mathbb{C}[X]$ -module of all algebraic vector fields such that $\mathfrak{M} \subset \text{Lie}_{\text{alg}}(X)$. Suppose that the fiber of \mathfrak{M} at a point $x_0 \in X$ generates the tangent space $T_{x_0}M$. Then X has the algebraic density property.*

In our case the transitivity of the action of $\text{Aut}_{\text{alg}}(V_n)$ follows from the following result of Gizatullin [12].

Theorem 5. *Let X be an irreducible smooth affine variety over an algebraically closed field of characteristic zero. If X can be completed by a smooth rational curve, then $\text{Aut}_{\text{alg}}(X)$ acts transitively on X .*

We start by listing the complete vector fields that will be used to construct the module \mathfrak{M} of the theorem above.

3.1. Proposition. *The following vector fields on F_n are complete and T -invariant, and they descend to non-identically zero complete vector fields on V_n .*

$$(10) \quad \delta = ba_2^{b-1}a_3 \frac{\partial}{\partial a_1} + a_4 \frac{\partial}{\partial a_2}$$

$$(11) \quad \delta' = a_1^{b-1}a_2^b \frac{\partial}{\partial a_4} + a_1^b \frac{\partial}{\partial a_3}$$

$$(12) \quad \varepsilon = a_1 \frac{\partial}{\partial a_1} - a_4 \frac{\partial}{\partial a_4}$$

Moreover, δ and δ' are locally nilpotent derivations (their flow is an algebraic action of \mathbb{C}_+).

Proof. The vector fields are tangent to F_n , since they annihilate the defining equation of the threefold. The invariance and their completeness is a straightforward check left to the reader. Finally, the vector fields descends non-trivially on V_n because they are

generically transversal to the vector field $-a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} - ba_3 \frac{\partial}{\partial a_3} + a_4 \frac{\partial}{\partial a_4}$ that defines the T -action on F_n . □

We collect in the next two lemmas the formulas upon which the proofs of lemmas 4.5-4.9 are based.

3.2. Lemma. *For the surface V_n :*

$$(13) \quad \varepsilon(x_k) = -kx_k$$

$$(14) \quad \varepsilon(y) = y$$

$$(15) \quad \delta(y) = 1 + nx_0$$

$$(16) \quad \delta(x_k) = (b - k)x_{k+1}$$

$$(17) \quad \delta'(x_0) = y^b$$

$$(18) \quad \delta'(y) = 0$$

3.3. Lemma. *For the surface V_n :*

$$(19) \quad [\varepsilon, \delta] = -\delta$$

$$(20) \quad [\varepsilon, \delta'] = b\delta' \quad (n = b + 1)$$

We will also need a well-known fact ([25], Proposition 3.1) about differential equations that will be used implicitly in some of the remaining statements.

3.4. Proposition. *Let μ be a complete vector field on a complex manifold X , and f be a holomorphic function. Then the vector field $f\mu$ is complete if and only if $\mu^2(f) = 0$.*

3.5. Lemma. *The following algebraic vector fields belong to $\text{Lie}_{\text{alg}}(V_n)$:*

$$(1) x_0\varepsilon, (2) x_b\varepsilon, (3) x_0x_b\varepsilon.$$

Proof. Statement (1) follows from $\varepsilon(x_0) = 0$ (in particular, $x_0\varepsilon$ is complete).

As for (2), let $X_1 = [\delta, x_0\varepsilon]$, $X_s = [\delta, X_{s-1}]$. By definition $X_s \in \text{Lie}_{\text{alg}}(V_n)$, for all s . Then by induction on $s \geq 2$ it follows that

$$X_s = sb(b-1)\dots(b-s+2)x_{s-1}\delta + b(b-1)\dots(b-s+1)x_s\varepsilon.$$

In particular, for $s = b$, one obtains $X_b = b!x_b\varepsilon + b^2(b-1)\dots(2)x_{b-1}\delta$. Since $\delta^2(x_{b-1}) = 0$, $x_{b-1}\delta$ is complete, and the statement (2) is proven.

From the first two facts, one has that (3) $[x_0\varepsilon, x_b\varepsilon] = -bx_0x_b\varepsilon \in \text{Lie}_{\text{alg}}(V_n)$. □

3.6. Lemma. *For $0 \leq k \leq b$, for all $N > 0$, $x_kx_b^N\delta \in \text{Lie}_{\text{alg}}(V_n)$.*

Proof. Now we calculate that

$$(21) \quad [x_0\varepsilon, x_b\delta] = -(1+b)x_0x_b\delta - bx_1x_b\varepsilon$$

$$(22) \quad [\delta, x_0x_b\varepsilon] = x_0x_b\delta + bx_1x_b\varepsilon$$

The sum of (21) and (22) shows that $x_0x_b\delta \in \text{Lie}_{\text{alg}}(V_n)$ (use Lemma 3.5). Then, for all $M \geq 0$, we have $[x_b^M\delta, x_0x_b\delta] = bx_1x_b^{M+1}\delta \in \text{Lie}_{\text{alg}}(V_n)$. By induction on k we then prove the statement for all positive k .

As for $k = 0$, observe that since $yx_b = (1+x_0)x_{b-1}$, we obtain $yx_b\delta = x_0x_{b-1}\delta + x_{b-1}\delta$. This equation shows that $yx_b\delta \in \text{Lie}_{\text{alg}}(V_n)$: in fact, $x_{b-1}\delta$ is complete, and $x_0x_{b-1}\delta \in \text{Lie}_{\text{alg}}(V_n)$ as follows from Lemma 3.5 and Lemma 3.6 applied to

$$[x_0\varepsilon, x_{b-1}\delta] = -bx_0x_{b-1}\delta - bx_1x_{b-1}\varepsilon = -bx_0x_{b-1}\delta - bx_0x_b\varepsilon,$$

the second equality being true since $x_1x_{b-1} = x_0x_b$. Hence, for $M \geq 0$ we obtain

$$[x_b^M\delta, yx_b\delta] = x_b^{M+1}(1+nx_0)\delta = x_b^{M+1}\delta + nx_0x_b^{M+1}\delta$$

which shows the case $k = 0$. □

3.7. Lemma. For $1 \leq k \leq b$, $N > 0$, $M \geq 0$, $x_0^M x_k x_b^N \varepsilon \in \text{Lie}_{\text{alg}}(V_n)$.

Proof. We know that $x_0x_b\varepsilon \in \text{Lie}_{\text{alg}}(V_n)$ (Lemma 3.5).

To show that $x_k x_b \varepsilon \in \text{Lie}_{\text{alg}}(V_n)$ for all k , apply induction on k and Lemma 3.6 to the equation

$$[\delta, x_{k-1}x_b\varepsilon] = x_{k-1}x_b\delta + (b-k+1)x_kx_b\varepsilon.$$

Next, the result for $N' \geq 0$, $M = 0$ follows from the equation

$$[x_b^{N'}\delta, x_kx_b\varepsilon] = (1+bN')x_kx_b^{N'+1}\delta + x_b^{N'+1}(b-k)x_{k+1}\varepsilon.$$

Finally, the case $M \geq 0$, $N > 0$ follows from

$$[x_0^M\varepsilon, x_kx_b^N\varepsilon] = (-Nb-k)x_0^Mx_kx_b^N\varepsilon. □$$

The next computations deal with the y -variable.

3.8. Lemma. For all $R \geq 0$, $y^{b+R}\varepsilon \in \text{Lie}_{\text{alg}}(V_n)$.

Proof. Use the formula

$$[x_0\varepsilon, y^R\delta'] = (b+j)x_0y^R\delta' - y^{b+R}\varepsilon$$

and notice that $\delta'^2(y^R x_0) = 0$. □

3.9. Proposition. For $N > 0$, $M > 0$, $0 < k < b$, $R \geq 0$, $y^{b+R}x_0^M x_k x_b^N \varepsilon \in \text{Lie}_{\text{alg}}(V_n)$

Proof. Apply Lemma 3.7 and Lemma 3.8 to the equation

$$\begin{aligned} [y^{b+R}\varepsilon, x_0^M x_k x_b^N \varepsilon] = \\ (1 - k - bN - b - R)y^{b+R}x_0^M x_k x_b^N \varepsilon. \end{aligned}$$

□

Next, we make use of some relations between the invariant monomials.

3.10. Proposition. For any collection k_1, \dots, k_r , there are integers $N > 0$ and $M > 0$, $0 \leq h \leq b$, such that

$$(23) \quad x_0 x_{k_1} x_{k_2} \dots x_{k_r} x_b = x_0^M x_h x_b^N,$$

Proof. We observe that:

$$(24) \quad \text{if } h + k \leq b \quad x_k x_h = x_{h+k} x_0$$

$$(25) \quad \text{if } h + k \geq b \quad x_k x_h = x_b x_{h+k-b}$$

Write the monomial in the form

$$x_0^{N'} x_{k_1} \dots x_{k_s} x_b^{M'}$$

for $x_{k_1}, \dots, x_{k_s} \neq x_0, x_b$.

Then, by applying (24) and (25) we can eliminate the factors x_k not equal to x_0 and x_b until there will be left at most one.

□

Theorem 6. Let \mathfrak{N} be the $\mathbb{C}[V_n]$ -submodule of $\text{VF}_{\text{alg}}(V_n)$ generated by the vector field $x_0 x_1 \dots x_b y^b \varepsilon$. Then $\mathfrak{N} \subset \text{Lie}_{\text{alg}}(V_n)$.

Proof. We need to show that, for all monomials $z^T y^R x_{k_1} \dots x_{k_r}$,

$z^T y^R x_{k_1} \dots x_{k_r} x_0 x_1 \dots x_b y^b \varepsilon \in \text{Lie}_{\text{alg}}(V_n)$. Observe first that we can assume $T = 0$, since we can eliminate z via the relation $z = 1 + x_0$.

Then, according to Proposition 3.10, we can write

$$y^R x_{k_1} \dots x_{k_r} x_0 x_1 \dots x_b y^b = y^{b+R} x_0 x_{h_1} \dots x_{h_r} x_b = x_0^N x_h x_b^M,$$

for some $N, M > 0$, $0 \leq h \leq b$

The theorem follows from Proposition 3.9.

□

Conclusion of the proof of Theorem 3. We follow the idea of [18] to produce a module \mathfrak{M} satisfying the hypothesis of Theorem 4. Consider the regular function $f = x_b$. The flow of the vector field $f\delta$ is an algebraic action of \mathbb{C}_+ : this can be checked directly by solving the associated ODE, or by using the fact that δ is a locally nilpotent derivation and $\delta(x_b) = 0$ ([15], Principle 7 page 24). Let $p \in V_n$ be a point with $f(p) = 0$, $\delta_p \neq 0$, $\varepsilon_p \neq 0$, $\varepsilon_p(f) \neq 0$: then ([18], Claim page 4) the flow φ of $f\delta$ at time one induces an isomorphism φ_* on the tangent space at p , that map ε_p to $\varepsilon_p + \varepsilon_p(f)\delta_p$. Let $\mathfrak{M} = \mathfrak{N} + \varphi_*\mathfrak{N}$: then $\mathfrak{M} \subset \text{Lie}_{\text{alg}}(V_n)$, and since ε_p and δ_p are linearly independent vectors, its fiber at p spans the tangent space at p . Since V_n is homogeneous with respect to $\text{Aut}_{\text{alg}}(V_n)$, we can then apply Theorem 4 to conclude the proof of Theorem 3.

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