

ADDENDUM TO “ENERGIES OF ZEROS OF RANDOM SECTIONS ON RIEMANN SURFACES”. INDIANA UNIV. MATH. J. 57 (2008), NO. 4, 1753–1780

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The purpose of this note is to resolve an apparent discrepancy between the calculations in the article [ABS] of Armentano- Beltran-Shub (henceforth ABS) and that in Qi Zhong’s article [Zh] of the asymptotics of the expected energy of zeros of random polynomials on the Riemann sphere S^2 with respect to the log chordal distance $\log[z, w]$. For the sake of brevity, we do not repeat the statement of the problem but refer to [Zh, ABS] for the background and notation. We show that the calculation in [Zh] which uses the general method of Green’s function and correlation functions gives the same answer as in [ABS].

The discrepancy between [ABS] and [Zh] arose because [Zh] actually contains two calculations of the asymptotic energy with respect to $\log[z, w]$, one explicit and one implicit. The implicit one is indicated in the Remarks after [Zh] Theorem 1.2. There, it is pointed out that the general Green’s energy method of the paper applies to the log chordal energy on S^2 if one adds a constant to $\log[z, w]$ to convert it to the Green’s function. The Green’s energy asymptotics in [Zh] are correct and their application to the log chordal energy does give the correct answer, as we verify in this note. But the energy asymptotics were not computed by that method in [Zh]. Rather in Section 5.2 of [Zh], the energy asymptotics were calculated without converting $\log[z, w]$ to the Green’s function, and there are some errors in the calculation of the integrals that arise. As a result, the asymptotics stated in Theorem 1.3 (2) (see (1.13)) are incorrect. The correct asymptotics are given here.

Besides correcting the calculation of the $\log[z, w]$ energy in [Zh], the purpose of this note is to clarify what is correct and what is incorrect in [Zh]. Most importantly, the main result of [Zh] is correct, and proves the correct asymptotics for the Green’s energy of zeros of random holomorphic sections of powers of positive Hermitian line bundles for all Hermitian metrics of positive $(1, 1)$ curvature over all Riemann surfaces. By comparison, the later result of [ABS] only give the asymptotics in one special case, the round metric on S^2 . However, as pointed out to us by M. Shub, the formula in [ABS] is exact: the $o(N)$ error term in Zhong’s formula is zero. We do not prove that here, because we derive the asymptotics from a general formula for metrics on Riemann surfaces, where in general the $o(N)$ term is non-zero.

The discrepancy between the asymptotics stated in Theorem 1.3 (2) of [Zh] and those of [ABS] was spotted by D. Hardin and E. Saff while Q. Zhong was a postdoc at Vanderbilt. They also informed the authors of [ABS] about the apparent discrepancy. A reference is made in [ABS] to an erratum by Zhong. The present note supplants his erratum.

0.1. ABS versus Zhong. The calculation in question concerns the expected logarithmic energy of zero sets of polynomials of degree N on the Riemann sphere. The first important point of comparison is the normalizations of the energy in [ABS] and [Zh].

In both articles, the Riemann sphere is identified with the sphere $S(\frac{1}{2})$ of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$. The logarithm chordal energy is then given by

$$-\log[z, w], \quad [z, w] = \frac{|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}.$$

For this kernel on $S(\frac{1}{2})$, the expected log energy calculated in [ABS] is

$$\text{ABS} \quad \mathbf{E}\mathcal{E}_{ABS}^N \sim \frac{N^2}{4} - \frac{N \log N}{4} - \frac{N}{4}. \quad (1)$$

Qi Zhong calculated the energy not for $\log[z, w]$ but for minus the Green's function of $S(\frac{1}{2})$. Following [Zh], we denote minus the Green's function of a metric g by G_g and denote the usual Green's function by \hat{G}_g . Thus, $G_g = -\hat{G}_g$. We denote the Green's function of $S(\frac{1}{2})$ by $\hat{G}_{\frac{1}{2}}$. Then $-G_{\frac{1}{2}}(z, w) := -\frac{1}{2\pi} \log[z, w] - C_{\frac{1}{2}}$ for a certain constant $C_{\frac{1}{2}}$. For the G_g -Green's energy, Zhong prove the general (and correct) asymptotics,

$$\text{ZHONG} \quad \mathbf{E}\mathcal{E}_{ZH}^N = -\frac{1}{4\pi} N \log N - \frac{N}{4\pi} - N \int F_g(z, z) \omega_h / \pi + o(N), \quad (2)$$

where F_g is the Robin constant of g . We denote the Robin constant of $S(\frac{1}{2})$ by $F_{\frac{1}{2}}$; note that it is a constant for the round metric.

To compare the two formulae (1)-(2), we need to calculate the constants $C_{\frac{1}{2}}$ and $F_{\frac{1}{2}}$, add $C_{\frac{1}{2}}$ to $G_{\frac{1}{2}}$ to convert it to $-\frac{1}{2\pi} \log[z, w]$, and substitute the value of $F_{\frac{1}{2}}$. In addition, as detailed in §0.2, we need to multiply by $\frac{1}{\pi}$ to convert ABS to ZHONG. Thus, the following Lemma asserts that the calculations of [Zh] and [ABS] agree:

LEMMA 1. $C_{\frac{1}{2}} = \frac{1}{4\pi}$ and $F_g(z, z) = -\frac{1}{4\pi}$. Hence,

$$\pi \left(\text{ZHONG} + C_{\frac{1}{2}} N(N - 1) \right) = \text{ABS}.$$

0.2. Normalizations. To make the normalizations in [ABS] and [Zh] consistent we need to observe that:

- (i) Zhong defines the energy using $-\frac{1}{2\pi} \log[z, w] - C_{\frac{1}{2}}$ while ABS use $-\log[z, w]$.
- (ii) ABS sums over $i < j$ while Zhong sums over $i \neq j$.

Zhong assumes that the Riemannian area form is $dV = \omega_h$ with $\int_{\mathbb{C}P^1} \omega_h = \pi$; see (2.2) of [Zh]. Since this is the area of the sphere of radius $\frac{1}{2}$, [Zh] and ABS are working on $S(\frac{1}{2})$.

To emphasize that the metric quantities pertain to the sphere of radius $\frac{1}{2}$, we subscript all metric quantities by the respective radius, except for the geodesic distance, which we denote by r .

0.3. Proof of Lemma 1. Both constants depend on the radius we pick for S^2 , namely radius $\frac{1}{2}$. To keep track of constants, we denote by A the area of S^2 in the given metric. For the round metric of area π , $G_{\frac{1}{2}}(z, w)$ is a function of the geodesic distance $r(z, w)$ and hence of the chordal distance $[z, w]$. Subscripting with the radius, we have

$$\hat{G}_{\frac{1}{2}}(z, w) = \frac{1}{2\pi} \log[z, w]_{\frac{1}{2}} + C_{\frac{1}{2}}.$$

0.4. **The constant $C_{\frac{1}{2}}$.** The constant is determined by the fact that $\int_{\mathbb{C}\mathbb{P}^1} G_{\frac{1}{2}}(z, w)\omega_w = 0$ for all z , which becomes

$$\begin{aligned}
 A(S(\tfrac{1}{2}))C_{\frac{1}{2}} &= -\frac{1}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \log[0, r]_{\frac{1}{2}} (\tfrac{1}{2} \sin 2r dr) d\theta = -\int_0^{\frac{\pi}{2}} \log[0, r]_{\frac{1}{2}} (\tfrac{1}{2} \sin 2r dr) \\
 &= -\int_0^{\pi} \log(\tfrac{1}{2} \sin \varphi) (\tfrac{1}{4} (\sin \varphi) d\varphi) \\
 &= -(\tfrac{1}{4} \int_0^{\pi} \log(\sin \varphi) (\sin \varphi) d\varphi) - (\tfrac{1}{4} (\log \tfrac{1}{2}) \int_0^{\pi} (\sin \varphi) d\varphi). \\
 &= -(\tfrac{1}{4} \int_0^{\pi} (\log \sqrt{2(1 - \cos \varphi)}) (\sin \varphi) d\varphi) - (\tfrac{1}{4} (\log \tfrac{1}{2}) \int_0^{\pi} (\sin \varphi) d\varphi) \\
 &= -\tfrac{1}{4} \log \tfrac{4}{e} - (\tfrac{1}{4} (\log \tfrac{1}{2}) \int_0^{\pi} (\sin \varphi) d\varphi) = -\tfrac{1}{4} \log \tfrac{4}{e} - (\tfrac{1}{4} (\log \tfrac{1}{2}) 2).
 \end{aligned}$$

We conclude that

$$C_{\frac{1}{2}} = -\frac{1}{4\pi} \log \frac{4}{e} - (\frac{1}{4\pi} (\log \frac{1}{2}) 2) = \frac{1}{4\pi}. \quad (3)$$

Then Zhong’s minus Green’s function is given by,

$$G_{\frac{1}{2}}(z, w) = -\frac{1}{2\pi} \log[z, w] + \frac{1}{4\pi} \log \frac{4}{e} + (\frac{1}{4\pi} (\log \frac{1}{2}) 2) = -\frac{1}{2\pi} \log[z, w] - \frac{1}{4\pi}. \quad (4)$$

It follows that,

$$-\frac{1}{2\pi} \log[z, w] = G_{\frac{1}{2}}(z, w) + \frac{1}{4\pi}. \quad (5)$$

0.5. **Robin constant.** We further show that

$$F_{\frac{1}{2}}(z, z) = -\frac{1}{4\pi}. \quad (6)$$

Here, $F_{\frac{1}{2}}(z, z)$ is the constant in the expansion

$$G_{\frac{1}{2}}(z, w) = -\frac{1}{2\pi} \log r + F_{\frac{1}{2}}(z, z) + O(r), \quad r \rightarrow 0.$$

In fact, it is the same constant that we just calculated.

Indeed, $\hat{G}_{\frac{1}{2}} = \frac{1}{2\pi} \log(\frac{1}{2} \sin 2r) + \frac{1}{4\pi}$, and $\frac{1}{2} \sin 2r = r f(r)$, $f(0) = 1$. So $\log f(0) = 0$ and

$$F_{\frac{1}{2}} = -\frac{1}{4\pi},$$

as desired.

0.6. **Conclusion.** Adding $\frac{1}{4\pi}$ to $G_{\frac{1}{2}}$ in (5) results in adding $\frac{1}{4\pi} N(N-1)$ to Zhong’s asymptotics (2). We then substitute (6) to find that (2) equals

$$\frac{1}{4\pi} N(N-1) - \frac{1}{4\pi} N \log N - \frac{N}{4\pi} - N(-\frac{1}{4\pi}) + o(N).$$

Finally, we multiply by π , to get

$$\frac{N^2}{4} - \frac{N \log N}{4} - \frac{N}{4} + o(N),$$

which is the same as (1). This completes the proof of Lemma 1, and proves that the calculations of [ABS] and [Zh] agree.

Finally, we thank B. Shiffman for checking over the calculations.

REFERENCES

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- [Zh] Q. Zhong, Energies of zeros of random sections on Riemann surfaces. Indiana Univ. Math. J. 57 (2008), no. 4, 1753–1780.

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