

# INFINITE GENERATION OF NON-COCOMPACT LATTICES ON RIGHT-ANGLED BUILDINGS

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ABSTRACT. Let  $\Gamma$  be a non-cocompact lattice on a locally finite regular right-angled building  $X$ . We prove that if  $\Gamma$  has a strict fundamental domain then  $\Gamma$  is not finitely generated. We use the separation properties of subcomplexes of  $X$  called tree-walls.

Tree lattices have been well-studied (see [BL]). Less understood are lattices on higher-dimensional CAT(0) complexes. In this paper, we consider lattices on  $X$  a locally finite, regular right-angled building (see Davis [D] and Section 1 below). Examples of such  $X$  include products of locally finite regular or biregular trees, or Bourdon's building  $I_{p,q}$  [B], which has apartments hyperbolic planes tessellated by right-angled  $p$ -gons, and all vertex links the complete bipartite graph  $K_{q,q}$ .

Let  $G$  be a closed, cocompact group of type-preserving automorphisms of  $X$ , equipped with the compact-open topology, and let  $\Gamma$  be a lattice in  $G$ . That is,  $\Gamma$  is discrete, and the series  $\sum |\text{Stab}_\Gamma(\phi)|^{-1}$  converges, where the sum is over the set of chambers  $\phi$  of a fundamental domain for  $\Gamma$ . The lattice  $\Gamma$  is cocompact in  $G$  if and only if the quotient  $\Gamma \backslash X$  is compact.

If there is a subcomplex  $Y \subset X$  containing exactly one point from each  $\Gamma$ -orbit on  $X$ , then  $Y$  is called a *strict fundamental domain* for  $\Gamma$ . Equivalently,  $\Gamma$  has a strict fundamental domain if  $\Gamma \backslash X$  may be embedded in  $X$ .

Any cocompact lattice in  $G$  is finitely generated. We prove:

**Theorem 1.** *Let  $\Gamma$  be a non-cocompact lattice in  $G$ . If  $\Gamma$  has a strict fundamental domain, then  $\Gamma$  is not finitely generated.*

Our proof, in Section 3 below, uses the separation properties of subcomplexes of  $X$  which we call *tree-walls*. These generalize the tree-walls (in French, *arbre-murs*) of  $I_{p,q}$ , which were introduced by Bourdon in [B]. We define tree-walls and establish their properties in Section 2 below.

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The following examples of non-cocompact lattices on right-angled buildings are known to us.

- (1) For  $i = 1, 2$ , let  $G_i$  be a rank one Lie group over a nonarchimedean locally compact field whose Bruhat–Tits building is the locally finite regular or biregular tree  $T_i$ . Then any irreducible lattice in  $G = G_1 \times G_2$  is finitely generated (Raghunathan [Ra]). Hence by Theorem 1 above, such lattices on  $X = T_1 \times T_2$  cannot have strict fundamental domain.
- (2) Let  $\Lambda$  be a minimal Kac–Moody group over a finite field  $\mathbb{F}_q$  with right-angled Weyl group  $W$ . Then  $\Lambda$  has locally finite, regular right-angled twin buildings  $X_+ \cong X_-$ , and  $\Lambda$  acts diagonally on the product  $X_+ \times X_-$ . For  $q$  large enough:
  - (a) By Theorem 0.2 of Carbone–Garland [CG] or Theorem 1(i) of Rémy [Ré], the stabilizer in  $\Lambda$  of a point in  $X_-$  is a non-cocompact lattice in  $\text{Aut}(X_+)$ . Any such lattice is contained in a negative maximal spherical parabolic subgroup of  $\Lambda$ , which has strict fundamental domain a sector in  $X_+$ , and so any such lattice has strict fundamental domain.
  - (b) By Theorem 1(ii) of Rémy [Ré], the group  $\Lambda$  is itself a non-cocompact lattice in  $\text{Aut}(X_+) \times \text{Aut}(X_-)$ . Since  $\Lambda$  is finitely generated, Theorem 1 above implies that  $\Lambda$  does not have strict fundamental domain in  $X = X_+ \times X_-$ .
- (3) In [T], the first author constructed a functor from graphs of groups to complexes of groups, which extends the corresponding tree lattice to a lattice in  $\text{Aut}(X)$  where  $X$  is a regular right-angled building. The resulting lattice in  $\text{Aut}(X)$  has strict fundamental domain if and only if the original tree lattice has strict fundamental domain.

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## 1. RIGHT-ANGLED BUILDINGS

In this section we recall the basic definitions and some examples for right-angled buildings. We mostly follow Davis [D], in particular Section 12.2 and Example 18.1.10. See also Sections 1.2–1.4 of [KT].

Let  $(W, S)$  be a right-angled Coxeter system. That is,

$$W = \langle S \mid (st)^{m_{st}} = 1 \rangle$$

where  $m_{ss} = 1$  for all  $s \in S$ , and  $m_{st} \in \{2, \infty\}$  for all  $s, t \in S$  with  $s \neq t$ . We will discuss the following examples:

- $W_1 = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong D_\infty$ , the infinite dihedral group;

- $W_2 = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^2 = 1 \rangle \cong (C_2 \times C_2) * C_2$ , where  $C_2$  is the cyclic group of order 2;
- The Coxeter group  $W_3$  generated by the set of reflections  $S$  in the sides of a right-angled hyperbolic  $p$ -gon,  $p \geq 5$ . That is,  $W_3 = \langle s_1, \dots, s_p \mid s_i^2 = (s_i s_{i+1})^2 = 1 \rangle$  with cyclic indexing.

Fix  $(q_s)_{s \in S}$  a family of integers with  $q_s \geq 2$ . Given any family of groups  $(H_s)_{s \in S}$  with  $|H_s| = q_s$ , let  $H$  be the quotient of the free product of the  $(H_s)_{s \in S}$  by the normal subgroup generated by the commutators  $\{[h_s, h_t] : h_s \in H_s, h_t \in H_t, m_{st} = 2\}$ .

Now let  $X$  be the piecewise Euclidean CAT(0) geometric realization of the chamber system  $\Phi = \Phi(H, \{1\}, (H_s)_{s \in S})$ . Then  $X$  is a locally finite, regular right-angled building, with chamber set  $\text{Ch}(X)$  in bijection with the elements of the group  $H$ . Let  $\delta_W : \text{Ch}(X) \times \text{Ch}(X) \rightarrow W$  be the  $W$ -valued distance function and let  $l_S : W \rightarrow \mathbb{N}$  be word length with respect to the generating set  $S$ . Denote by  $d_W : \text{Ch}(X) \times \text{Ch}(X) \rightarrow \mathbb{N}$  the *gallery distance*  $l_S \circ \delta_W$ . That is, for two chambers  $\phi$  and  $\phi'$  of  $X$ ,  $d_W(\phi, \phi')$  is the length of a minimal gallery from  $\phi$  to  $\phi'$ .

Suppose that  $\phi$  and  $\phi'$  are  $s$ -adjacent chambers, for some  $s \in S$ . That is,  $\delta_W(\phi, \phi') = s$ . The intersection  $\phi \cap \phi'$  is called an  $s$ -panel. By definition, since  $X$  is regular, each  $s$ -panel is contained in  $q_s$  distinct chambers. For distinct  $s, t \in S$ , the  $s$ -panel and  $t$ -panel of any chamber  $\phi$  of  $X$  have nonempty intersection if and only if  $m_{st} = 2$ . Each  $s$ -panel of  $X$  is reduced to a vertex if and only if  $m_{st} = \infty$  for all  $t \in S - \{s\}$ .

For the examples  $W_1$ ,  $W_2$ , and  $W_3$  above, respectively:

- The building  $X_1$  is a tree with each chamber an edge, each  $s$ -panel a vertex of valence  $q_s$ , and each  $t$ -panel a vertex of valence  $q_t$ . That is,  $X_1$  is the  $(q_s, q_t)$ -biregular tree. The apartments of  $X_1$  are bi-infinite rays in this tree.
- The building  $X_2$  has chambers and apartments as shown in Figure 1 below. The  $r$ - and  $s$ -panels are 1-dimensional and the  $t$ -panels are vertices.
- The building  $X_3$  has chambers  $p$ -gons and  $s$ -panels the edges of these  $p$ -gons. If  $q_s = q \geq 2$  for all  $s \in S$ , then each  $s$ -panel is contained in  $q$  chambers, and  $X_3$ , equipped with the obvious piecewise hyperbolic metric, is Bourdon's building  $I_{p,q}$ .

## 2. TREE-WALLS

We now generalize the notion of tree-wall due to Bourdon [B]. We will use basic facts about buildings, found in, for example, Davis [D]. Our main results concerning tree-walls are Corollary 3 below, which

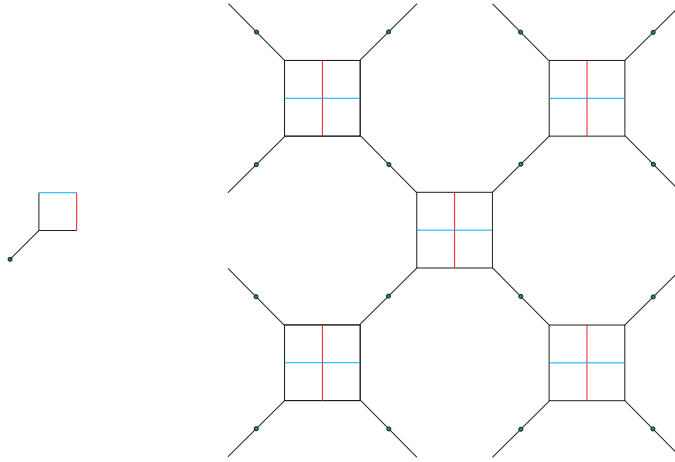


FIGURE 1. A chamber (on the left) and part of an apartment (on the right) for the building  $X_2$ .

describes three possibilities for tree-walls, and Proposition 6 below, which generalizes the separation property 2.4.A(ii) of [B].

Let  $X$  be as in Section 1 above and let  $s \in S$ . As in Section 2.4.A of [B], we define two  $s$ -panels of  $X$  to be *equivalent* if they are contained in a common wall of type  $s$  in some apartment of  $X$ . A *tree-wall of type  $s$*  is then an equivalence class under this relation. We note that in order for walls and thus tree-walls to have a well-defined type, it is necessary only that all finite  $m_{st}$ , for  $s \neq t$ , be even. Tree-walls could thus be defined for buildings of type any even Coxeter system, and they would have similar properties to those below. We will however only explicitly consider the right-angled case.

Let  $\mathcal{T}$  be a tree-wall of  $X$ , of type  $s$ . We define a chamber  $\phi$  of  $X$  to be *epicormic at  $\mathcal{T}$*  if the  $s$ -panel of  $\phi$  is contained in  $\mathcal{T}$ , and we say that a gallery  $\alpha = (\phi_0, \dots, \phi_n)$  *crosses  $\mathcal{T}$*  if, for some  $0 \leq i < n$ , the chambers  $\phi_i$  and  $\phi_{i+1}$  are epicormic at  $\mathcal{T}$ .

By the definition of tree-wall, if  $\phi \in \text{Ch}(X)$  is epicormic at  $\mathcal{T}$  and  $\phi' \in \text{Ch}(X)$  is  $t$ -adjacent to  $\phi$  with  $t \neq s$ , then  $\phi'$  is epicormic at  $\mathcal{T}$  if and only if  $m_{st} = 2$ . Let  $s^\perp := \{t \in S \mid m_{st} = 2\}$  and denote by  $\langle s^\perp \rangle$  the subgroup of  $W$  generated by the elements of  $s^\perp$ . If  $s^\perp$  is empty then by convention,  $\langle s^\perp \rangle$  is trivial. For the examples in Section 1 above:

- in  $W_1$ , both  $\langle s^\perp \rangle$  and  $\langle t^\perp \rangle$  are trivial;
- in  $W_2$ ,  $\langle r^\perp \rangle = \langle s \rangle \cong C_2$  and  $\langle s^\perp \rangle = \langle r \rangle \cong C_2$ , while  $\langle t^\perp \rangle$  is trivial; and
- in  $W_3$ ,  $\langle s_i^\perp \rangle = \langle s_{i-1}, s_{i+1} \rangle \cong D_\infty$  for each  $1 \leq i \leq p$ .

**Lemma 2.** *Let  $\mathcal{T}$  be a tree-wall of  $X$  of type  $s$ . Let  $\phi$  be a chamber which is epicormic at  $\mathcal{T}$  and let  $A$  be any apartment containing  $\phi$ .*

- (1) *The intersection  $\mathcal{T} \cap A$  is a wall of  $A$ , hence separates  $A$ .*
- (2) *There is a bijection between the elements of the group  $\langle s^\perp \rangle$  and the set of chambers of  $A$  which are epicormic at  $\mathcal{T}$  and in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ .*

*Proof.* Part (1) is immediate from the definition of tree-wall. For Part (2), let  $w \in \langle s^\perp \rangle$  and let  $\psi = \psi_w$  be the unique chamber of  $A$  such that  $\delta_W(\phi, \psi) = w$ . We claim that  $\psi$  is epicormic at  $\mathcal{T}$  and in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ .

For this, let  $s_1 \cdots s_n$  be a reduced expression for  $w$  and let  $\alpha = (\phi_0, \dots, \phi_n)$  be the minimal gallery from  $\phi = \phi_0$  to  $\psi = \phi_n$  of type  $(s_1, \dots, s_n)$ . Since  $w$  is in  $\langle s^\perp \rangle$ , we have  $m_{s_i s} = 2$  for  $1 \leq i \leq n$ . Hence by induction each  $\phi_i$  is epicormic at  $\mathcal{T}$ , and so  $\psi = \phi_n$  is epicormic at  $\mathcal{T}$ . Moreover, since none of the  $s_i$  are equal to  $s$ , the gallery  $\alpha$  does not cross  $\mathcal{T}$ . Thus  $\psi = \psi_w$  is in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ .

It follows that  $w \mapsto \psi_w$  is a well-defined, injective map from  $\langle s^\perp \rangle$  to the set of chambers of  $A$  which are epicormic at  $\mathcal{T}$  and in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ . To complete the proof, we will show that this map is surjective. So let  $\psi$  be a chamber of  $A$  which is epicormic at  $\mathcal{T}$  and in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ , and let  $w = \delta_W(\phi, \psi)$ .

If  $\langle s^\perp \rangle$  is trivial then  $\psi = \phi$  and  $w = 1$ , and we are done. Next suppose that the chambers  $\phi$  and  $\psi$  are  $t$ -adjacent, for some  $t \in S$ . Since both  $\phi$  and  $\psi$  are epicormic at  $\mathcal{T}$ , either  $t = s$  or  $m_{st} = 2$ . But  $\psi$  is in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ , so  $t \neq s$ , hence  $w = t$  is in  $\langle s^\perp \rangle$  as required. If  $\langle s^\perp \rangle$  is finite, then finitely many applications of this argument will finish the proof. If  $\langle s^\perp \rangle$  is infinite, we have established the base case of an induction on  $n = l_S(w)$ .

For the inductive step, let  $s_1 \cdots s_n$  be a reduced expression for  $w$  and let  $\alpha = (\phi_0, \dots, \phi_n)$  be the minimal gallery from  $\phi = \phi_0$  to  $\psi = \phi_n$  of type  $(s_1, \dots, s_n)$ . Since  $\phi$  and  $\psi$  are in the same component of  $A - \mathcal{T} \cap A$  and  $\alpha$  is minimal, the gallery  $\alpha$  does not cross  $\mathcal{T}$ . We claim that  $s_n$  is in  $s^\perp$ . First note that  $s_n \neq s$  since  $\alpha$  does not cross  $\mathcal{T}$  and  $\psi = \phi_n$  is epicormic at  $\mathcal{T}$ . Now denote by  $\mathcal{T}_n$  the tree-wall of  $X$  containing the  $s_n$ -panel  $\phi_{n-1} \cap \phi_n$ . Since  $\alpha$  is minimal and crosses  $\mathcal{T}_n$ , the chambers  $\phi = \phi_0$  and  $\psi = \phi_n$  are separated by the wall  $\mathcal{T}_n \cap A$ . Thus the  $s$ -panel of  $\phi$  and the  $s$ -panel of  $\psi$  are separated by  $\mathcal{T}_n \cap A$ . As the  $s$ -panels of both  $\phi$  and  $\psi$  are in the wall  $\mathcal{T} \cap A$ , it follows that the walls  $\mathcal{T}_n \cap A$  and  $\mathcal{T} \cap A$  intersect. Hence  $m_{s_n s} = 2$ , as claimed.

Now let  $w' = ws_n = s_1 \cdots s_{n-1}$  and let  $\psi'$  be the unique chamber of  $A$  such that  $\delta_W(\phi, \psi') = w'$ . Since  $s_n$  is in  $s^\perp$  and  $\psi'$  is  $s_n$ -adjacent

to  $\psi$ , the chamber  $\psi'$  is epicormic at  $\mathcal{T}$  and in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ . Moreover  $s_1 \cdots s_{n-1}$  is a reduced expression for  $w'$ , so  $l_S(w') = n - 1$ . Hence by the inductive assumption,  $w'$  is in  $\langle s^\perp \rangle$ . Therefore  $w = w's_n$  is in  $\langle s^\perp \rangle$ , which completes the proof.  $\square$

**Corollary 3.** *The following possibilities for tree-walls in  $X$  may occur.*

- (1) *Every tree-wall of type  $s$  is reduced to a vertex if and only if  $\langle s^\perp \rangle$  is trivial.*
- (2) *Every tree-wall of type  $s$  is finite but not reduced to a vertex if and only if  $\langle s^\perp \rangle$  is finite but nontrivial.*
- (3) *Every tree-wall of type  $s$  is infinite if and only if  $\langle s^\perp \rangle$  is infinite.*

*Proof.* Let  $\mathcal{T}$ ,  $\phi$ , and  $A$  be as in Lemma 2 above. The set of  $s$ -panels in the wall  $\mathcal{T} \cap A$  is in bijection with the set of chambers of  $A$  which are epicormic at  $\mathcal{T}$  and in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ .  $\square$

For the examples in Section 1 above:

- in  $X_1$ , every tree-wall of type  $s$  and of type  $t$  is a vertex;
- in  $X_2$ , the tree-walls of types both  $r$  and  $s$  are finite and 1-dimensional, while every tree-wall of type  $t$  is a vertex; and
- in  $X_3$ , all tree-walls are infinite, and are 1-dimensional.

**Corollary 4.** *Let  $\mathcal{T}$ ,  $\phi$ , and  $A$  be as in Lemma 2 above and let*

$$\rho = \rho_{\phi, A} : X \rightarrow A$$

*be the retraction onto  $A$  centered at  $\phi$ . Then  $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$ .*

*Proof.* Let  $\psi$  be any chamber of  $A$  which is epicormic at  $\mathcal{T}$  and is in the same component of  $A - \mathcal{T} \cap A$  as  $\phi$ . Then by the proof of Lemma 2 above,  $w := \delta_W(\phi, \psi)$  is in  $\langle s^\perp \rangle$ . Let  $\psi'$  be a chamber in the preimage  $\rho^{-1}(\psi)$  and let  $A'$  be an apartment containing both  $\phi$  and  $\psi'$ . Since the retraction  $\rho$  preserves  $W$ -distances from  $\phi$ , we have that  $\delta_W(\phi, \psi') = w$  is in  $\langle s^\perp \rangle$ . Again by the proof of Lemma 2, it follows that the chamber  $\psi'$  is epicormic at  $\mathcal{T}$ . But the image under  $\rho$  of the  $s$ -panel of  $\psi'$  is the  $s$ -panel of  $\psi$ . Thus  $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$ , as required.  $\square$

**Lemma 5.** *Let  $\mathcal{T}$  be a tree-wall and let  $\phi$  and  $\phi'$  be two chambers of  $X$ . Let  $\alpha$  be a minimal gallery from  $\phi$  to  $\phi'$  and let  $\beta$  be any gallery from  $\phi$  to  $\phi'$ . If  $\alpha$  crosses  $\mathcal{T}$  then  $\beta$  crosses  $\mathcal{T}$ .*

*Proof.* Suppose that  $\alpha$  crosses  $\mathcal{T}$ . Since  $\alpha$  is minimal, there is an apartment  $A$  of  $X$  which contains  $\alpha$ , and hence the wall  $\mathcal{T} \cap A$  separates  $\phi$  from  $\phi'$ . Choose a chamber  $\phi_0$  of  $A$  which is epicormic at  $\mathcal{T}$  and consider the retraction  $\rho = \rho_{\phi_0, A}$  onto  $A$  centered at  $\phi_0$ . Since  $\phi$  and  $\phi'$  are in  $A$ ,  $\rho$  fixes  $\phi$  and  $\phi'$ . Hence  $\rho(\beta)$  is a gallery in  $A$  from  $\phi$  to

$\phi'$ , and so  $\rho(\beta)$  crosses  $\mathcal{T} \cap A$ . By Corollary 4 above,  $\rho^{-1}(\mathcal{T} \cap A) = \mathcal{T}$ . Therefore  $\beta$  crosses  $\mathcal{T}$ .  $\square$

**Proposition 6.** *Let  $\mathcal{T}$  be a tree-wall of type  $s$ . Then  $\mathcal{T}$  separates  $X$  into  $q_s$  gallery-connected components.*

*Proof.* Fix an  $s$ -panel in  $\mathcal{T}$  and let  $\phi_1, \dots, \phi_{q_s}$  be the  $q_s$  chambers containing this panel. Then for all  $1 \leq i < j \leq q_s$ , the minimal gallery from  $\phi_i$  to  $\phi_j$  is just  $(\phi_i, \phi_j)$ , and hence crosses  $\mathcal{T}$ . Thus by Lemma 5 above, any gallery from  $\phi_i$  to  $\phi_j$  crosses  $\mathcal{T}$ . So the  $q_s$  chambers  $\phi_1, \dots, \phi_{q_s}$  lie in  $q_s$  distinct components of  $X - \mathcal{T}$ .

To complete the proof, we show that  $\mathcal{T}$  separates  $X$  into at most  $q_s$  components. Let  $\phi$  be any chamber of  $X$ . Then among the chambers  $\phi_1, \dots, \phi_{q_s}$ , there is a unique chamber, say  $\phi_1$ , at minimal gallery distance from  $\phi$ . It suffices to show that  $\phi$  and  $\phi_1$  are in the same component of  $X - \mathcal{T}$ .

Let  $\alpha$  be a minimal gallery from  $\phi$  to  $\phi_1$  and let  $A$  be an apartment containing  $\alpha$ . Then there is a unique chamber of  $A$  which is  $s$ -adjacent to  $\phi_1$ . Hence  $A$  contains  $\phi_i$  for some  $i > 1$ , and the wall  $\mathcal{T} \cap A$  separates  $\phi_1$  from  $\phi_i$ . Since  $\alpha$  is minimal and  $d_W(\phi, \phi_1) < d_W(\phi, \phi_i)$ , the Exchange Condition (see p. 35 [D]) implies that a minimal gallery from  $\phi$  to  $\phi_i$  may be obtained by concatenating  $\alpha$  with the gallery  $(\phi_1, \phi_i)$ . Since a minimal gallery can cross  $\mathcal{T} \cap A$  at most once,  $\alpha$  does not cross  $\mathcal{T} \cap A$ . Thus  $\phi$  and  $\phi_1$  are in the same component of  $X - \mathcal{T}$ , as required.  $\square$

### 3. PROOF OF THEOREM

Let  $G$  be as in the introduction and let  $\Gamma$  be a non-cocompact lattice in  $G$  with strict fundamental domain. Fix a chamber  $\phi_0$  of  $X$ . For each integer  $n \geq 0$  define

$$D(n) := \{ \phi \in \text{Ch}(X) \mid d_W(\phi, \Gamma\phi_0) \leq n \}.$$

Then  $D(0) = \Gamma\phi_0$ , and for every  $n > 0$  every connected component of  $D(n)$  contains a chamber in  $\Gamma\phi_0$ . To prove Theorem 1, we will show that there is no  $n > 0$  such that  $D(n)$  is connected.

Let  $Y$  be a strict fundamental domain for  $\Gamma$  which contains  $\phi_0$ . For each chamber  $\phi$  of  $X$ , denote by  $\phi_Y$  the representative of  $\phi$  in  $Y$ .

**Lemma 7.** *Let  $\phi$  and  $\phi'$  be  $t$ -adjacent chambers in  $X$ , for  $t \in S$ . Then either  $\phi_Y = \phi'_Y$ , or  $\phi_Y$  and  $\phi'_Y$  are  $t$ -adjacent.*

*Proof.* It suffices to show that the  $t$ -panel of  $\phi_Y$  is the  $t$ -panel of  $\phi'_Y$ . Since  $Y$  is a subcomplex of  $X$ , the  $t$ -panel of  $\phi_Y$  is contained in  $Y$ .

By definition of a strict fundamental domain, there is exactly one representative in  $Y$  of the  $t$ -panel of  $\phi$ . Hence the unique representative in  $Y$  of the  $t$ -panel of  $\phi$  is the  $t$ -panel of  $\phi_Y$ . Similarly, the unique representative in  $Y$  of the  $t$ -panel of  $\phi'$  is the  $t$ -panel of  $\phi'_Y$ . But  $\phi$  and  $\phi'$  are  $t$ -adjacent, hence have the same  $t$ -panel, and so it follows that  $\phi_Y$  and  $\phi'_Y$  have the same  $t$ -panel.  $\square$

**Corollary 8.** *The fundamental domain  $Y$  is gallery-connected.*

**Lemma 9.** *For all  $n > 0$ , the fundamental domain  $Y$  contains a pair of adjacent chambers  $\phi_n$  and  $\phi'_n$  such that, if  $\mathcal{T}_n$  denotes the tree-wall separating  $\phi_n$  from  $\phi'_n$ :*

- (1) *the chambers  $\phi_0$  and  $\phi_n$  are in the same gallery-connected component of  $Y - \mathcal{T}_n \cap Y$ ;*
- (2)  *$\min\{d_W(\phi_0, \phi) \mid \phi \in \text{Ch}(X) \text{ is epicormic at } \mathcal{T}_n\} > n$ ; and*
- (3) *there is a  $\gamma \in \text{Stab}_\Gamma(\phi'_n)$  which does not fix  $\phi_n$ .*

*Proof.* Fix  $n > 0$ . Since  $\Gamma$  is not cocompact,  $Y$  is not compact. Thus there exists a tree-wall  $\mathcal{T}_n$  with  $\mathcal{T}_n \cap Y$  nonempty such that for every  $\phi \in \text{Ch}(X)$  which is epicormic at  $\mathcal{T}_n$ ,  $d_W(\phi_0, \phi) > n$ . Let  $s_n$  be the type of the tree-wall  $\mathcal{T}_n$ . Then by Corollary 8 above, there is a chamber  $\phi_n$  of  $Y$  which is epicormic at  $\mathcal{T}_n$  and in the same gallery-connected component of  $Y - \mathcal{T}_n \cap Y$  as  $\phi_0$ , such that for some chamber  $\phi'_n$  which is  $s_n$ -adjacent to  $\phi_n$ ,  $\phi'_n$  is also in  $Y$ . Now, as  $\Gamma$  is a non-cocompact lattice, the orders of the  $\Gamma$ -stabilizers of the chambers in  $Y$  are unbounded. Hence the tree-wall  $\mathcal{T}_n$  and chambers  $\phi_n$  and  $\phi'_n$  may be chosen so that  $|\text{Stab}_\Gamma(\phi_n)| < |\text{Stab}_\Gamma(\phi'_n)|$ .  $\square$

Let  $\phi_n, \phi'_n, \mathcal{T}_n$ , and  $\gamma$  be as in Lemma 9 above and let  $s = s_n$  be the type of the tree-wall  $\mathcal{T}_n$ . Let  $\alpha$  be a gallery in  $Y - \mathcal{T}_n \cap Y$  from  $\phi_0$  to  $\phi_n$ . The chambers  $\phi_n$  and  $\gamma \cdot \phi_n$  are in two distinct components of  $X - \mathcal{T}_n$ , since they both contain the  $s$ -panel  $\phi_n \cap \phi'_n \subseteq \mathcal{T}_n$ , which is fixed by  $\gamma$ . Hence the galleries  $\alpha$  and  $\gamma \cdot \alpha$  are in two distinct components of  $X - \mathcal{T}_n$ , and so the chambers  $\phi_0$  and  $\gamma \cdot \phi_0$  are in two distinct components of  $X - \mathcal{T}_n$ . Denote by  $X_0$  the component of  $X - \mathcal{T}_n$  which contains  $\phi_0$ , and put  $Y_0 = Y \cap X_0$ .

**Lemma 10.** *Let  $\phi$  be a chamber in  $X_0$  that is epicormic at  $\mathcal{T}_n$ . Then  $\phi_Y$  is in  $Y_0$  and is epicormic at  $\mathcal{T}_n \cap Y$ .*

*Proof.* We consider three cases, corresponding to the possibilities for tree-walls in Corollary 3 above.

- (1) If  $\mathcal{T}_n$  is reduced to a vertex, there is only one chamber in  $X_0$  which is epicormic at  $\mathcal{T}_n$ , namely  $\phi_n$ . Thus  $\phi = \phi_n = \phi_Y$  and we are done.



- (2) If  $\mathcal{T}_n$  is finite but not reduced to a vertex, the result follows by finitely many applications of Lemma 7 above.
- (3) If  $\mathcal{T}_n$  is infinite, the result follows by induction, using Lemma 7 above, on

$$k := \min\{d_W(\phi, \psi) \mid \psi \text{ is a chamber of } Y_0 \text{ epicormic at } \mathcal{T}_n \cap Y\}.$$

□

**Lemma 11.** *For all  $n > 0$ , the complex  $D(n)$  is not connected.*

*Proof.* Fix  $n > 0$ , and let  $\alpha$  be a gallery in  $X$  between a chamber in  $X_0 \cap \Gamma\phi_0$  and some chamber  $\phi$  in  $X_0$  that is epicormic at  $\mathcal{T}_n$ . Let  $m$  be the length of  $\alpha$ .

By Lemmas 7 and 10 above, the gallery  $\alpha$  projects to a gallery  $\beta$  in  $Y$  between  $\phi_0$  and a chamber  $\phi_Y$  that is epicormic at  $\mathcal{T}_n \cap Y$ . The gallery  $\beta$  in  $Y$  has length at most  $m$ .

It follows from (2) of Lemma 9 above that the gallery  $\beta$  in  $Y$  has length greater than  $n$ . Therefore  $m > n$ . Hence the gallery-connected component of  $D(n)$  that contains  $\phi_0$  is contained in  $X_0$ . As the chamber  $\gamma \cdot \phi_0$  is not in  $X_0$ , it follows that the complex  $D(n)$  is not connected. □

This completes the proof, as  $\Gamma$  is finitely generated if and only if  $D(n)$  is connected for some  $n$ .

## REFERENCES

- [BL] H. Bass and A. Lubotzky, *Tree Lattices*. Prog. in Math., 176, Birkhäuser.
- [B] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. Funct. Anal. **7** (1997), 245–268.
- [CG] L. Carbone and H. Garland, *Existence of lattices in Kac–Moody groups over finite fields*, Commun. Contemp. Math. **5** (2003), 813–867.
- [D] M. Davis, *The Geometry and Topology of Coxeter Groups*, LMS Monographs, Princeton University Press, 2008.
- [KT] A. Kubena and A. Thomas, *Density of commensurators for uniform lattices of right-angled buildings*. Preprint.
- [Ra] M.S. Raghunathan, *Discrete subgroups of algebraic groups over local fields of positive characteristics*, Proc. Indian Acad. Sci. Math. Sci. **99** (1989), 127–146.
- [Ré] B. Rémy, *Constructions de réseaux en théorie de Kac–Moody*, C. R. Acad. Sci. Paris **329** (1999), 475–478.
- [T] A. Thomas, *Lattices acting on right-angled buildings*, Algebr. Geom. Topol. **6** (2006), 1215–1238.

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