HOM-LIE ALGEBRAS WITH SYMMETRIC INVARIANT NONDEGENERATE BILINEAR FORMS

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ABSTRACT. The aim of this paper is to introduce and study quadratic Hom-Lie algebras, which are Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms. We provide several constructions leading to examples and extend the double extension theory to Hom-Lie algebras. We reduce the case where the twist map is invertible to the study of involutive quadratic Lie algebras. We establish a correspondence between the class of involutive quadratic Hom-Lie algebras and quadratic simple Lie algebras with symmetric involution. Centerless involutive quadratic Hom-Lie algebras are characterized. Also elements of a representation theory for Hom-Lie algebras, including adjoint and coadjoint representations are supplied with application to quadratic Hom-Lie algebras.

INTRODUCTION

The Hom-algebra structures arose first in quasi-deformation of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie and quasi-Hom-Lie structures in which the Jacobi condition is twisted. The first examples of q-deformations, in which the derivations are replaced by σ-derivations, concerned the Witt and Virasoro algebras, see for example [1, 11, 12, 13, 14, 15, 16, 24, 29, 22]. A general study and construction of Hom-Lie algebras are considered in [20, 25, 26] and a more general framework bordering color and super Lie algebras was introduced in [20, 25, 26, 27]. In the subclass of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map.

The notion of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted by a linear map was introduced in [30], it turns out that the commutator bracket multiplication defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie algebras. This provided a different way of constructing Hom-Lie algebras. The Hom-Lie-admissible algebras and more general G-Hom-associative algebras with subclasses of Hom-Vinberg and pre-Hom-Lie algebras, generalizing to the twisted situation Lie-admissible algebras, G-associative algebras, Vinberg and pre-Lie algebras respectively, and shown that for these classes of algebras the operation of taking commutator leads to Hom-Lie algebras as well. The enveloping algebras of Hom-Lie algebras were discussed in [39]. The fundamentals of the formal deformation theory and associated cohomology structures for Hom-Lie algebras have been considered initially in [32] and completed in [2]. Simultaneously, in [40] elements of homology for Hom-Lie algebras have been developed. In [31] and [33], the theory of Hom-coalgebras and related structures are developed. Further development could be found in [3, 4, 10, 23, 41].

The quadratic Lie algebras, also called metrizable or orthogonal, are intensively studied, one of the fundamental results of constructing and characterizing quadratic Lie algebras is due to Medina and Revoy (see [34]) using double extension, while the concept of T^*-extension is due to Bordemann (see [9]). The T^*-extension concerns nonassociative algebras with nondegenerate associative symmetric bilinear form, such algebras are called metrizable algebas. In [9], the metrizable nilpotent associative algebras and metrizable solvable Lie algebras are described. The study of graded quadratic Lie algebras could be found in [6] .

The purpose of this paper is to study and construct quadratic Hom-Lie algebras. In the first Section we summarize the definitions and some key constructions of Hom-Lie algebras. Section 2 is dedicated to a theory of representations of Hom-Lie algebras including adjoint and coadjoint representations. In Section 3 we introduce the notion of quadratic Hom-Lie algebra and give some properties. Several procedures of construction leading to many examples are provided in Section 4. We show in Section 5 that there exists biunivoque correspondence between some classes of Lie algebras and classes of Hom-Lie algebras. In Section 6 we study simple and semisimple involutive Hom-Lie algebras. The last Section aims to extend double extension theory to Hom-Lie algebras. We give a characterization of a class of quadratic Hom-Lie algebras. Mainly we a give a structure theorem of centerless involutive quadratic Hom-Lie algebra.

1. Preliminaries

In the following we summarize the definitions of Hom-Lie and Hom-associative algebraic structures (see [30]) generalizing the well known Lie and associative algebras. Also we define the notion of modules over Hom-algebras.

Throughout the article we let $\mathbb{K}$ be an algebraically closed field of characteristic 0. We mean by a Hom-algebra a triple (A, μ, α) consisting of a vector space A, a bilinear map μ and a linear map α. In all the examples involving the unspecified products are either given by skewsymmetry or equal to zero.

1.1. **Definitions.** The notion of Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in [20, 25, 26] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields. In this article, we follow notations and a slightly more general definition of Hom-Lie algebras from [30].

Definition 1.1. A *Hom-Lie algebra* is a triple $(\mathfrak{g}, [\ ,\ ], \alpha)$ consisting of a linear space $\mathfrak{g}$ on which $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map and $\alpha : \mathfrak{g} \to \mathfrak{g}$ a linear map satisfying

(1.1) $$[x, y] = -[y, x], \quad \text{(skew-symmetry)}$$

(1.2) $$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0 \quad \text{(Hom-Jacobi condition)}$$

for all x, y, z from $\mathfrak{g}$, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z.

We recover classical Lie algebra when $\alpha = id_{\mathfrak{g}}$ and the identity (1.2) is the Jacobi identity in this case. Let $(\mathfrak{g}, \mu, \alpha)$ and $\mathfrak{g}' = (\mathfrak{g}', \mu', \alpha')$ be two Hom-Lie algebras. A linear map $f : \mathfrak{g} \to \mathfrak{g}'$ is a *morphism of Hom-Lie algebras* if

$$\mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

In particular, Hom-Lie algebras $(\mathfrak{g}, \mu, \alpha)$ and $(\mathfrak{g}, \mu', \alpha')$ are isomorphic if there exists a bijective linear map f such that

$$\mu = f^{-1} \circ \mu' \circ (f \otimes f) \quad \text{and} \quad \alpha = f^{-1} \circ \alpha' \circ f.$$

A subspace I of $\mathfrak{g}$ is said to be an ideal if for $x \in I$ and $y \in \mathfrak{g}$ we have $[x, y] \in I$ and $\alpha(x) \in I$. A Hom-Lie algebra in which the commutator is not identically zero and which has no proper ideals is called simple.

Example 1.2. *Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional linear space $\mathfrak{g}$ over $\mathbb{K}$. The following bracket and linear map α on $\mathfrak{g} = \mathbb{K}^3$ define a Hom-Lie algebra over $\mathbb{K}$:*

$$
\begin{array}{rclcrcl}
[x_1, x_2] & = & ax_1 + bx_3 & \quad & \alpha(x_1) & = & x_1 \\
[x_1, x_3] & = & cx_2 & \quad & \alpha(x_2) & = & 2x_2 \\
[x_2, x_3] & = & dx_1 + 2ax_3, & \quad & \alpha(x_3) & = & 2x_3
\end{array}
$$

with $[x_2, x_1]$, $[x_3, x_1]$ and $[x_3, x_2]$ defined via skewsymmetry. It is not a Lie algebra if and only if $a \neq 0$ and $c \neq 0$, since

$$[x_1, [x_2, x_3]] + [x_3, [x_1, x_2]] + [x_2, [x_3, x_1]] = acx_2.$$

Example 1.3 (Jackson $\mathfrak{sl}_2$)**.** *The Jackson $\mathfrak{sl}_2$ is a q-deformation of the classical $\mathfrak{sl}_2$. It carries a Hom-Lie algebra structure but not a Lie algebra structure. It is defined with respect to a basis $\{x_1, x_2, x_3\}$ by the brackets and a linear map α such that*

$$
\begin{array}{rclcrcl}
[x_1, x_2] & = & -2qx_2 & \qquad & \alpha(x_1) & = & qx_1 \\
[x_1, x_3] & = & 2x_3 & \qquad & \alpha(x_2) & = & q^2 x_2 \\
[x_2, x_3] & = & -\frac{1}{2}(1+q)x_1, & \qquad & \alpha(x_3) & = & qx_3
\end{array}
$$

where q is a parameter in $\mathbb{K}$. if $q = 1$ we recover the classical $\mathfrak{sl}_2$.

For simplicity we will use in the sequel the following terminology and notations.

Definition 1.4. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. The Hom-algebra is called
- *multiplicative Hom-Lie algebra if $\forall x, y \in \mathfrak{g}$ we have $\alpha([x, y]) = [\alpha(x), \alpha(y)]$;*
- *regular Hom-Lie algebra if α is an automorphism;*
- *involutive Hom-Lie algebra if α is an involution, that is $\alpha^2 = id$.*

The *center* of the Hom-Lie algebra is denoted $\mathcal{Z}(\mathfrak{g})$ and defined by

$$\mathcal{Z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0 \; \forall y \in \mathfrak{g}\}.$$

We recall in the following the definition of Hom-associative algebra which provide a different way for constructing Hom-Lie algebras by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket multiplication. This structure was introduced by the second author and Silvestrov (see [30]).

Definition 1.5. A *Hom-associative algebra* is a triple (A, μ, α) consisting of a linear space A, $\mu : A \times A \to A$ is a bilinear map and $\alpha : A \to A$ is a linear map, satisfying

$$(1.3) \qquad\qquad \mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).$$

There is a functor from the category of Hom-associative algebras in the category of Hom-Lie algebras.

Proposition 1.6 ([30])**.** *Let (A, μ, α) be a Hom-associative algebra defined on the linear space A by the multiplication μ and a homomorphism α. Then the triple $(A, [\;,\;], \alpha)$, where the bracket is defined for $x, y \in A$ by $[x, y] = \mu(x, y) - \mu(y, x)$, is a Hom-Lie algebra.*

A structure of module over Hom-associative algebras is defined in [31] and [33] as follows.

Definition 1.7. Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra. A (left) $\mathcal{A}$-module is a triple (M, f, γ) where M is a $\mathbb{K}$-vector space and f, γ are $\mathbb{K}$-linear maps, $f : M \to M$ and $\gamma : \mathcal{A} \otimes M \to M$, such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{A} \otimes M & \xrightarrow{\mu \otimes f} & \mathcal{A} \otimes M \\
\downarrow{\scriptstyle \alpha \otimes \gamma} & & \downarrow{\scriptstyle \gamma} \\
\mathcal{A} \otimes M & \xrightarrow{\quad \gamma \quad} & M
\end{array}
$$

Remark 1.8. A Hom-associative algebra $(\mathcal{A}, \mu, \alpha)$ is a left $\mathcal{A}$-module with $M = \mathcal{A}$, $f = \alpha$ and $\gamma = \mu$.

The following result shows that Lie algebras deform into Hom-Lie algebras via endomorphisms.

Theorem 1.9 ([40])**.** *Let $(\mathfrak{g}, [\;,\;])$ be a Lie algebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra endomorphism. Then $\mathfrak{g}_\alpha = (\mathfrak{g}, [\;,\;]_\alpha, \alpha)$ is a Hom-Lie algebra, where $[\;,\;]_\alpha = \alpha \circ [\;,\;]$.*
Moreover, suppose that $(\mathfrak{g}', [\;,\;]')$ is another Lie algebra and $\alpha' : \mathfrak{g}' \to \mathfrak{g}'$ is a Lie algebra endomorphism. If $f : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$ then

$$f : (\mathfrak{g}, [\;,\;]_\alpha, \alpha) \longrightarrow (\mathfrak{g}', [\;,\;]'_{\alpha'}, \alpha')$$

is a morphism of Hom-Lie algebras.

Proof. Observe that $[\alpha(x), [y, z]_\alpha]_\alpha = \alpha[\alpha(x), \alpha[y, z]] = \alpha^2[x, [y, z]]$. Therefore the Hom-Jacobi identity for $\mathfrak{g}_\alpha = (\mathfrak{g}, [\;,\;]_\alpha, \alpha)$ follows obviously from the Jacobi identity of $(\mathfrak{g}, [\;,\;])$. The skew-symmetry and the second assertion are proved similarly. $\qquad\square$

In the sequel we denote by $\mathfrak{g}_\alpha$ the Hom-Lie algebra $(\mathfrak{g}, \alpha \circ [\ ,\ ], \alpha)$ corresponding to a given Lie algebra $(\mathfrak{g}, [\ ,\ ])$ and an endomorphism α. We say that the Hom-Lie algebra is obtained by composition.

Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a regular Hom-Lie algebra. It was observed in [19] that the composition method using α^{-1} leads to a Lie algebra.

Proposition 1.10. *Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a regular Hom-Lie algebra. Then $(\mathfrak{g}, [\ ,\ ]_{\alpha^{-1}} = \alpha^{-1} \circ [\ ,\ ])$ is a Lie algebra.*

Proof. It follows from

$$\circlearrowleft_{x,y,z} [x, [y,z]_{\alpha^{-1}}]_{\alpha^{-1}} = \circlearrowleft_{x,y,z} \alpha^{-1}([x, \alpha^{-1}([y,z])]) = \circlearrowleft_{x,y,z} \alpha^{-2}[\alpha(x), [y,z]] = 0.$$

$\square$

Remark 1.11. In particular the proposition is valid when α is an involution.

We may also derive new Hom-Lie algebras from a given multiplicative Hom-Lie algebra using the following procedure.

Definition 1.12 ([42])**.** Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a multiplicative Hom-Lie algebra and $n \geq 0$. The nth derived Hom-algebra of $\mathfrak{g}$ is defined by

$$(1.4) \qquad \mathfrak{g}_{(n)} = \left(\mathfrak{g}, [\ ,\ ]^{(n)} = \alpha^n \circ [\ ,\ ], \alpha^{n+1}\right),$$

Note that $\mathfrak{g}_{(0)} = \mathfrak{g}$ and $\mathfrak{g}_{(1)} = \left(\mathfrak{g}, [\ ,\ ]^{(1)} = \alpha \circ [\ ,\ ], \alpha^2\right)$.

Observe that for $n \geq 1$ and $x, y, z \in \mathfrak{g}$ we have

$$\begin{aligned} [[x,y]^{(n)}, \alpha^{n+1}(z)]^{(n)} &= \alpha^n([\alpha^n([x,y]), \alpha^{n+1}(z)]) \\ &= \alpha^{2n}([[x,y], \alpha(z)]). \end{aligned}$$

Hence, one obtains the following result.

Theorem 1.13 ([42])**.** *Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a multiplicative Hom-Lie algebra. Then its nth derived Hom-algebra is a Hom-Lie algebra.*

In the following we construct Hom-Lie algebras involving elements of the centroid of Lie algebras. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra. An endomorphism $\theta \in End(\mathfrak{g})$ is said to be an element of the centroid if $\theta[x,y] = [\theta(x), y]$ for any $x, y \in \mathfrak{g}$. The centroid is defined by

$$Cent(\mathfrak{g}) = \{\theta \in End(\mathfrak{g}) : \theta[x,y] = [\theta(x), y], \ \forall x, y \in \mathfrak{g}\}.$$

The same definition is assumed for Hom-Lie algebra.

Proposition 1.14. *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra and $\theta \in Cent(\mathfrak{g})$. Set for $x, y \in \mathfrak{g}$*

$$\{x, y\} = [\theta(x), y],$$
$$[x, y]_\theta = [\theta(x), \theta(y)].$$

Then $(\mathfrak{g}, \{\cdot, \cdot\}, \theta)$ and $(\mathfrak{g}, [\cdot, \cdot]_\theta, \theta)$ are Hom-Lie algebras.

Proof. For $\theta \in Cent(\mathfrak{g})$ we have $[\theta(x), y] = \theta([x,y]) = -\theta([y,x]) = -[\theta(y), x] = [x, \theta(y)]$. Then

$$\{x, y\} = [\theta(x), y] = -[\theta(y), x] = -\theta[y, x] = -\{y, x\}.$$

Also we have

$$\begin{aligned} \{\theta(x), \{y, z\}\} &= [\theta^2(x), \{y,z\}] = [\theta^2(x), [\theta(y), z]] \\ &= \theta([\theta(x), [\theta(y), z]]) = [\theta(x), \theta([\theta(y), z])] \\ &= [\theta(x), [\theta(y), \theta(z)]]. \end{aligned}$$

It follows $\circlearrowleft_{x,y,z} \{\theta(x), \{y, z\}\} = \circlearrowleft_{x,y,z} [\theta(x), [\theta(y), \theta(z)]] = 0$ since $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra. Therefore the Hom-Jacobi is satisfied. Thus $(\mathfrak{g}, \{\cdot, \cdot\}, \theta)$ is a Hom-Lie algebra.

Similarly we have the skewsymmetry and the Hom-Jacobi identity satisfied for $(\mathfrak{g}, [\cdot, \cdot]_\theta, \theta)$. Indeed

$$[x, y]_\theta = [\theta(x), \theta(y)] = -[\theta(y), \theta(x)] = -[y, x]_\theta.$$

and

$$[\theta(x), [y, z]_\theta]_\theta = [\theta^2(x), \theta([y, z]_\theta)] = [\theta^2(x), \theta([\theta(y), \theta(z)])] = \theta^2([\theta(x), [\theta(y), \theta(z)]]).$$

which leads to $\circlearrowleft_{x,y,z} [\theta(x), [y, z]_\theta]_\theta = \theta^2(\circlearrowleft_{x,y,z} [\theta(x), [\theta(y), \theta(z)]]) = 0.$ $\square$

2. Representations of Hom-Lie algebras

In this section we introduce a representation theory of Hom-Lie algebras and discuss the cases of adjoint and coadjoint representations for Hom-Lie algebras. The representations of Hom-Lie algebras were considered independently in a general framework in [36]. Moreover the author give the corresponding coboundary operator, the cohomologies associated to the adjoint representations and the trivial representation are explicitly defined. Notice that the coadjoint representations are not discussed there.

Definition 2.1. Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Lie algebra. A *representation* of $\mathfrak{g}$ is a triple (V, ρ, β), where V is a $\mathbb{K}$-vector space, $\beta \in End(V)$ and $\rho : \mathfrak{g} \to End(V)$ is a linear map satisfying

(2.1) $$\rho([x, y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x) \quad \forall x, y \in \mathfrak{g}$$

One recovers the definition of a representation in the case of Lie algebras by setting $\alpha = Id_\mathfrak{g}$ and $\beta = Id_V$.

Definition 2.2. Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Lie algebra. Two representations (V, ρ, β) and (V', ρ', β') of $\mathfrak{g}$ are said to be isomorphic if there exists a linear map $\phi : V \to V'$ such that

$$\forall x \in \mathfrak{g} \ \ \rho'(x) \circ \phi = \phi \circ \rho(x) \quad \text{and} \quad \phi \circ \beta = \beta' \circ \phi.$$

in the following, we discuss some properties of Hom-Lie algebras representations.

Proposition 2.3. *Let* $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g}, \alpha)$ *be a Hom-Lie algebra and* (V, ρ, β) *be a representation of* $\mathfrak{g}$. *The direct summand* $\mathfrak{g} \oplus V$ *with a bracket defined by*

(2.2) $$[x + u, y + w] := [x, y]_\mathfrak{g} + \rho(x)(w) - \rho(y)(u) \quad \forall x, y \in \mathfrak{g} \ \forall u, w \in V$$

and the twisted map $\gamma : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ *defined by*

(2.3) $$\gamma(x + w) = \alpha(x) + \beta(u) \quad \forall x \in \mathfrak{g} \ \forall u \in V.$$

is a Hom-Lie algebra.

Proof. The skew-symmetry of the bracket is obvious. We show that the Hom-Jacobi identity is satisfied: Let $x, y, z \in \mathfrak{g}$ and $\forall u, v, w \in V$.

$$\begin{aligned}
\circlearrowleft_{(x,u),(y,v),(z,w)} [\gamma(x + u), [y + v, z + w]] =& \circlearrowleft_{(x,u),(y,v),(z,w)} [\alpha(x) + \beta(u), [y, z]_\mathfrak{g} + \rho(y)(w) - \rho(z)(v)] \\
=& \circlearrowleft_{(x,u),(y,v),(z,w)} [\alpha(x), [y, z]_\mathfrak{g}]_\mathfrak{g} + \rho(\alpha(x)(\rho(y)(w) - \rho(z)(v)) - \rho([y, z]_\mathfrak{g})(\beta(u)) \\
=& \circlearrowleft_{(x,u),(y,v),(z,w)} \rho(\alpha(x)(\rho(y)(w)) - \rho(\alpha(x)(\rho(z)(v)) - \rho(\alpha(y)(\rho(z)(u)) + \rho(\alpha(z)(\rho(y)(u)) \\
=& \ \rho(\alpha(x)(\rho(y)(w)) - \rho(\alpha(x)(\rho(z)(v)) - \rho(\alpha(y)(\rho(z)(u)) + \rho(\alpha(z)(\rho(y)(u)) \\
& + \rho(\alpha(y)(\rho(z)(u)) - \rho(\alpha(y)(\rho(x)(w)) - \rho(\alpha(z)(\rho(x)(v)) + \rho(\alpha(x)(\rho(z)(v)) \\
& + \rho(\alpha(z)(\rho(x)(v)) - \rho(\alpha(z)(\rho(y)(u)) - \rho(\alpha(x)(\rho(y)(w)) + \rho(\alpha(y)(\rho(x)(w)) \\
& = 0
\end{aligned}$$

where $\circlearrowleft_{(x,u),(y,v),(z,w)}$ denotes summation over the cyclic permutation on $(x, u), (y, v), (z, w)$. $\square$

Now, we discuss the adjoint representations of a Hom-Lie algebra.

Proposition 2.4. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha)$ *be a Hom-Lie algebra and* $\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$ *be an operator defined for* $x \in \mathfrak{g}$ *by* $\mathrm{ad}(x)(y) = [x, y]$. *Then* $(\mathfrak{g}, \mathrm{ad}, \alpha)$ *is a representation of* $\mathfrak{g}$.

Proof. Since $\mathfrak{g}$ is Hom-Lie algebra, the Hom-Jacobi condition on $x, y, z \in \mathfrak{g}$ is

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$$

and may be written

$$ad[x, y](\alpha(z)) = ad(\alpha(x))(ad(y)(z)) - ad(\alpha(y))(ad(x)(z))$$

Then the operator ad satisfies

$$ad[x, y] \circ \alpha = ad(\alpha(x)) \circ ad(y) - ad(\alpha(y)) \circ (ad(x).$$

Therefore, it determines a representation of the Hom-Lie algebra $\mathfrak{g}$.   □

We call the representation defined in the previous proposition *adjoint representation* of the Hom-Lie algebra.

In the following, we explore the dual representations and coadjoint representations of Hom-Lie algebras. Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Lie algebra and (V, ρ, β) be a representation of $\mathfrak{g}$. Let V^* be the dual vector space of V. We define a linear map $\widetilde{\rho} : \mathfrak{g} \to End(V^*)$ by $\widetilde{\rho}(x) = -{}^t\rho(x)$.

Let $f \in V^*$, $x, y \in \mathfrak{g}$ and $u \in V$. We compute the right hand side of the identity (2.1)

$$
\begin{aligned}
(\widetilde{\rho}(\alpha(x)) \circ \widetilde{\rho}(y) - \widetilde{\rho}(\alpha(y)) \circ \widetilde{\rho}(x))(f)(u) &= (\widetilde{\rho}(\alpha(x))(\widetilde{\rho}(y)(f)) - \widetilde{\rho}(\alpha(y))(\widetilde{\rho}(x)(f)))(u) \\
&= -\widetilde{\rho}(y)(f)(\rho(\alpha(x))(u)) + \widetilde{\rho}(x)(f)(\rho(\alpha(y))(u)) \\
&= f(\rho(y)\rho(\alpha(x))(u)) - f(\rho(x)\rho(\alpha(y))(u)) \\
&= f(\rho(y)\rho(\alpha(x)) - \rho(x)\rho(\alpha(y))(u)).
\end{aligned}
$$

On the other hand, we set that the twisted map for $\widetilde{\rho}$ is $\widetilde{\beta} = {}^t\beta$, then the left hand side of (2.1) writes

$$
\begin{aligned}
((\widetilde{\rho}([x, y])\widetilde{\beta})(f))(u) &= (\widetilde{\rho}([x, y])(f \circ \beta)(u), \\
&= -f \circ \beta(\rho([x, y])(u)).
\end{aligned}
$$

Therefore, we have the following proposition:

Proposition 2.5. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and (V, ρ, β) be a representation of $\mathfrak{g}$. The triple $(V^*, \widetilde{\rho}, \widetilde{\beta})$, where $\widetilde{\rho} : \mathfrak{g} \to End(V^*)$ is given by $\widetilde{\rho}(x) = -{}^t\rho(x)$, defines a representation of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ if and only if*

$$(2.4) \qquad \beta \circ \rho([x, y]) = \rho(x)\rho(\alpha(y)) - \rho(y)\rho(\alpha(x)).$$

We obtain the following characterization in the case of adjoint representation.

Corollary 2.6. *Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Lie algebra and $(\mathfrak{g}, \mathrm{ad}, \alpha)$ be the adjoint representation of $\mathfrak{g}$, where $\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$. We set $\widetilde{\mathrm{ad}} : \mathfrak{g} \to End(\mathfrak{g}^*)$ and $\widetilde{\mathrm{ad}}(x)(f) = -f \circ \mathrm{ad}(x)$. Then $(\mathfrak{g}^*, \widetilde{\mathrm{ad}}, \widetilde{\alpha})$ is a representation of $\mathfrak{g}$ if and only if*

$$(2.5) \qquad \alpha([[x, y], z]) = [x, [\alpha(y), z]] - [y, [\alpha(x), z]] \quad \forall x, y, z \in \mathfrak{g}.$$

3. Definition and properties of Quadratic Hom-Lie algebras

In this section we extend the notion of quadratic Lie algebra to Hom-Lie algebras and provide some properties.

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a symmetric nondegenerate bilinear form satisfying

$$(3.1) \qquad B([x, y], z) = B(x, [y, z]) \quad \forall x, y, z \in \mathfrak{g}.$$

The identity (3.1) may be written $B([x, y], z) = -B(y, [x, z])$ and is called invariance of B. The bilinear form B is called, with misuse of language, invariant scalar product. The triple $(\mathfrak{g}, [\ ,\ ], B)$ is called quadratic Lie algebra or sometimes orthogonal Lie algebra.

More generally, for nonassociative algebras $(A, \cdot)$, a triple $(A, \cdot, B)$ where B is a symmetric nondegenerate bilinear form satisfying

$$(3.2) \qquad B(x \cdot y, z) = B(x, y \cdot z) \quad \forall x, y, z \in A$$

defines a quadratic algebra, called also metrizable algebra. A bilinear form B satisfying (3.2) is said either associative form or invariant form.

Definition 3.1. Let $(\mathfrak{g}, [\ , \ ], \alpha)$ be a Hom-Lie algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be an invariant symmetric nondegenerate bilinear form satisfying

$$(3.3) \qquad\qquad B(\alpha(x), y) = B(x, \alpha(y)) \quad \forall x, y \in \mathfrak{g}.$$

The quadruple $(\mathfrak{g}, [\ , \ ], \alpha, B)$ is called *quadratic* Hom-Lie algebra.

If α is an involution (resp. invertible), the quadratic Hom-Lie algebra is said to be involutive (resp. regular) quadratic Hom-Lie algebra and we write for shortness IQH-Lie algebra (resp. RQH-Lie algebra).

We recover the classical notion of quadratic Lie algebra when α is the identity map. One may consider a larger class with a definition without condition (3.3). We may also introduce in the following a generalized quadratic Hom-Lie algebra notion where the invariance is twisted by a linear map.

Definition 3.2. A Hom-Lie algebra $(\mathfrak{g}, [\ , \ ], \alpha)$ is called *Hom-quadratic* if there exist a pair (B, γ) where $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is a symmetric nondegenerate bilinear form and $\gamma : \mathfrak{g} \to \mathfrak{g}$ is a linear map satisfying

$$(3.4) \qquad\qquad B([x, y], \gamma(z)) = -B(\gamma(y), [x, z]) \quad \forall x, y, z \in \mathfrak{g}$$

We call the identity (3.4) the γ-invariance of B. We recover the quadratic Hom-Lie algebras when $\gamma = id$.

3.1. **Quadratic Hom-Lie algebras and Representation theory.** We establish in the following a connection between quadratic Hom-Lie algebras and representation theory.

Proposition 3.3. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. If there exists $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a bilinear form such that the quadruple $(\mathfrak{g}, [\cdot, \cdot], \alpha, B)$ is a quadratic Hom-Lie algebra then*

    (1) $(\mathfrak{g}^*, \widetilde{\mathrm{ad}}, \widetilde{\alpha})$ *is a representation of $\mathfrak{g}$*
    (2) *The representations $(\mathfrak{g}, \mathrm{ad}, \alpha)$ and $(\mathfrak{g}^*, \widetilde{\mathrm{ad}}, \widetilde{\alpha})$ are isomorphic.*

Proof. To prove the first assertion, we should show that for any z we have

$$(3.5) \qquad\qquad \alpha \circ ad([x, y])(z) - \rho(x) ad(\alpha(y))(z) + ad(y) ad(\alpha(x))(z) = 0,$$

that is

$$\alpha[[x, y], z] - [x, [\alpha(y), z]] + [y, [\alpha(x), z]] = 0.$$

Let $u \in \mathfrak{g}$

$$
\begin{aligned}
B(\alpha[[x, y], z] - [x, [\alpha(y), z]] + [y, [\alpha(x), z]], u) =& B(\alpha[[x, y], z], u) - B([x, [\alpha(y), z]], u) + B([y, [\alpha(x), z]], u) \\
=& B([[x, y], z], \alpha(u)) + B([\alpha(y), z], [x, u]) - B([\alpha(x), z], [y, u]) \\
=& -\left(B(z, [[x, y], \alpha(u)]) + B(z, [\alpha(y), [x, u]]) - B(z, [\alpha(x), [y, u]])\right) \\
=& -\left(B(z, [[x, y], \alpha(u)] + [\alpha(y), [x, u]]) - [\alpha(x), [y, u]]\right) \\
=& -\left(B(z, [\alpha(u), [y, x]]) + [\alpha(y), [x, u]]) + [\alpha(x), [u, y]]\right) \\
=& 0.
\end{aligned}
$$

The identity 3.5 since B is nondegenerate.

For the second assertion we consider the map $\phi : \mathfrak{g} \to \mathfrak{g}^\star$ defined by $x \to B(x, \cdot)$ which is bijective since B is nondegenerate and prove that it is also a module morphism. $\qquad\square$

3.2. **Ideals and Decompositions of Quadratic Hom-Lie algebras.** We introduce the following definitions and give some properties related to ideals of Hom-Lie algebras.

Definition 3.4. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha, B)$ be a quadratic Hom-Lie algebra.

    (1) An ideal I of $\mathfrak{g}$ is said to be nondegenerate if $B_{|I \times I}$ is nondegenerate.
    (2) The quadratic Hom-Lie algebra is said to be irreducible (or B-irreducible) if $\mathfrak{g}$ doesn't contain any nondegenerate ideal I such that $I \neq \{0\}$ and $I \neq \mathfrak{g}$.
    (3) Let I be an ideal of $\mathfrak{g}$. The orthogonal $I^\perp$ of I with respect to B is defined by $\{x \in \mathfrak{g} : B(x, y) = 0 \ \forall y \in I\}$.

Remark 3.5. Let I be a nondegenerate ideal of a quadratic Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha, B)$. Then $(I, [\ ,\ ]_{|I \times I}, \alpha_{|}I, B_{|I \times I})$ is a quadratic Hom-Lie algebra.

Lemma 3.6. *Let* $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ *be a multiplicative Hom-Lie algebra. Then the center* $\mathcal{Z}(\mathfrak{g})$ *is an ideal of* $\mathfrak{g}$.

Proof. We have $[\mathfrak{g}, \mathcal{Z}(\mathfrak{g})] = \{0\} \subseteq \mathcal{Z}(\mathfrak{g})$. Let $x \in \mathcal{Z}(\mathfrak{g})$ and $y \in \mathfrak{g}$. For any $z \in \mathfrak{g}$ the invariance and the symmetry of B leads to $B([\alpha(x), y], z) = B(\alpha(x), [y, z]) = B(x, \alpha([y, z])) = B(x, [\alpha(y), \alpha(z)]) = B([x, \alpha(y)], \alpha(z)]) = 0$ (since $x \in \mathcal{Z}(\mathfrak{g})$).
Then for any $y \in \mathfrak{g}$ we have $[\alpha(x), y] = 0$ since B is nondegenerate. Thus $\alpha(x) \in \mathcal{Z}(\mathfrak{g})$. $\qquad\square$

Lemma 3.7. *Let* $(\mathfrak{g}, [\cdot, \cdot], \alpha, B)$ *be a quadratic Hom-Lie algebra and I be an ideal of $\mathfrak{g}$. Then the orthogonal* $I^\perp$ *of I with respect to B is an ideal of $\mathfrak{g}$.*

Proof. It is clear that $[\mathfrak{g}, I^\perp] \subseteq I^\perp$. Let $y \in I$ and $z \in I^\perp$, then $B(\alpha(y), z) = B(y, \alpha(z)) = 0$ since $\alpha(I) \subseteq I$. We conclude that $I^\perp$ is an ideal of $\mathfrak{g}$. $\qquad\square$

Proposition 3.8. *Let* $(\mathfrak{g}, [\cdot, \cdot], \alpha, B)$ *be a quadratic Hom-Lie algebra. Then* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ *such that*
  (1) $\mathfrak{g}_i$ *is an irreducible ideal of* $\mathfrak{g}$, *for any* $i \in \{1, \cdots, n\}$,
  (2) $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$, *for any* $i, j \in \{1, \cdots, n\}$ *such that* $i \neq j$,
  (3) $(\mathfrak{g}_i, [\cdot, \cdot]_{|\mathfrak{g}_i \times \mathfrak{g}_i}, \alpha_{|\mathfrak{g}_i}, B_{|\mathfrak{g}_i \times \mathfrak{g}_i})$ *is an irreducible quadratic Hom-Lie algebra.*

Proof. By induction on the dimension of $\mathfrak{g}$. $\qquad\square$

Now, let $\mathfrak{g} = (\mathfrak{g}, [\ ,\ ], \alpha, B)$ be a quadratic multiplicative Hom-Lie algebra. We provide in the following some observations.

Proposition 3.9. *If the linear map α is an automorphism and the center* $\mathcal{Z}(\mathfrak{g}) = \{0\}$ *then α is an involution i.e.* $\alpha^2 = id$.

Proof. Let $x, y, z \in \mathfrak{g}$, we have

$$
\begin{aligned}
B([\alpha(x), y], z) \quad &= B(\alpha(x), [y, z]) = B(x, \alpha([y, z]) \\
&= B(x, [\alpha(y), \alpha(z)]) = B([x, \alpha(y)], \alpha(z)) \\
&= B(\alpha([x, \alpha(y)]), z) = B([\alpha(x), \alpha^2(y)], z).
\end{aligned}
$$

Then $B([\alpha(x), y] - [\alpha(x), \alpha^2(y)], z) = 0$ which may be written $B([\alpha(x), y - \alpha^2(y)], z) = 0$. Hence, for any $x, y \in \mathfrak{g}$ we have $[\alpha(x), (id - \alpha^2)(y)] = 0$. Since α is bijective and $\mathcal{Z}(\mathfrak{g}) = \{0\}$ then $\alpha^2 = id$. $\qquad\square$

Proposition 3.10. *There exist two nondegenerate ideals I, J of $\mathfrak{g} = (\mathfrak{g}, [\ ,\ ], \alpha, B)$ such that*
  (1) $B(I, J) = \{0\}$,
  (2) $\mathfrak{g} = I \oplus J$,
  (3) $\alpha_{|I}$ *is nilpotent and $\alpha_{|J}$ is invertible.*

Proof. The fitting decomposition with respect to the linear map α leads to the existence of an integer n such that $\mathfrak{g} = I \oplus J$, where $I = Ker(\alpha^n)$ and $J = Im(\alpha^n)$, satisfying
  • $\alpha(I) \subseteq I$,
  • $\alpha(J) \subseteq I$,
  • $\alpha_{|I}$ is nilpotent,
  • $\alpha_{|J}$ is invertible.
Let $x \in \mathfrak{g}$, $y \in I$. We have $\alpha^n([x, y]) = [\alpha^n(x), \alpha^n(y)] = 0$ since $\alpha^n(y) = 0$, and $[x, y] \in I$. Then $[\mathfrak{g}, I] \subseteq I$. In addition $\alpha^n(\alpha(y)) = \alpha^{n+1}(y) = 0$ which implies that $\alpha(y) \in Ker(\alpha^n)$. Therefore I is an ideal of $\mathfrak{g}$.
Let $x, y \in J$ then there exist $x', y' \in \mathfrak{g}$ such that $x = \alpha^n(x')$ and $y = \alpha^n(y')$. We have $[x, y] = [\alpha^n(x'), \alpha^n(y')] = \alpha^n([x', y']) \in J$. In addition $\alpha(J) \subseteq J$. Therefore J is a subalgebra.
Let $x \in I$ and $y \in J$. There exists $y' \in \mathfrak{g}$ such that $y = \alpha^n(y')$. For any $z \in \mathfrak{g}$, we have $B([x, y], z) = -B([y, x], z) = -B(y, [x, z]) = -B(\alpha^n(y'), [x, z]) = -B(y', \alpha^n([x, z])) = -B(y', [\alpha^n(x), \alpha^n(z)]) = 0$. Then $[x, y] = 0$, since B is a nondegenerate bilinear form. We conclude that $I = Im(\alpha^n)$ is an ideal of $\mathfrak{g}$ and $[I, J] = 0$.
Now let $x \in I$ and $y = \alpha^n(y') \in J$, where $y' \in \mathfrak{g}$. We have $B(x, y) = B(x, \alpha^n(y')) = B(\alpha^n(x), y') = 0$ since $\alpha^n(x) = 0$. Therefore $B(I, J) = 0$. $\qquad\square$

Corollary 3.11. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha, B)$ be a quadratic Hom-Lie algebra which is B-irreducible. Then either α is nilpotent or α is an automorphism of $\mathfrak{g}$.*

4. Construction of Quadratic Hom-Lie algebras and Examples

We show in the following some constructions leading to new examples of quadratic Hom-Lie algebras. We use Theorem 1.9 and Theorem 1.13 to provide some classes of quadratic Hom-Lie algebras starting from an ordinary quadratic Lie algebras, respectively from any multiplicative quadratic Hom-Lie algebra. Also we provide constructions using elements in the centroid of a Lie algebras and constructions of T^*-extension type.

Let $(\mathfrak{g}, [\,,\,], B)$ be a quadratic Lie algebra. We denote by $Aut_S(\mathfrak{g}, B)$ the set of symmetric automorphisms of $\mathfrak{g}$ with respect of B, that is automorphisms $f : \mathfrak{g} \to \mathfrak{g}$ such that $B(f(x), y) = B(x, f(y))$, $\forall x, y \in \mathfrak{g}$.

Proposition 4.1. *Let $(\mathfrak{g}, [\,,\,], B)$ be a quadratic Lie algebra and $\alpha \in Aut_S(\mathfrak{g}, B)$. Then $\mathfrak{g}_\alpha = (\mathfrak{g}, [\,,\,]_\alpha, \alpha, B_\alpha)$, where for any $x, y \in \mathfrak{g}$*

$$(4.1) \qquad\qquad [x, y]_\alpha = [\alpha(x), \alpha(y)]$$

$$(4.2) \qquad\qquad B_\alpha(x, y) = B(\alpha(x), y),$$

is a quadratic Hom-Lie algebra.

Proof. The triple $(\mathfrak{g}, [\,,\,]_\alpha, \alpha)$ is a Hom-Lie algebra by Theorem 1.9.
The linear form B_α is nondegenerate since B is nondegenerate and α bijective.
We show that the identity (3.1) is satisfied by $\mathfrak{g}_\alpha = (\mathfrak{g}, [\,,\,]_\alpha, \alpha, B_\alpha)$. Let $x, y, z \in \mathfrak{g}$, then

$$
\begin{aligned}
B_\alpha([x, y]_\alpha, z) &= B(\alpha([\alpha(x), \alpha(y)]), z) \\
&= B([\alpha(x), \alpha(y)], \alpha(z)) \\
&= B(\alpha(x), [\alpha(y), \alpha(z)]) \quad \text{(Invariance of } B) \\
&= B(\alpha(x), [y, z]_\alpha) \\
&= B_\alpha(x, [y, z]_\alpha).
\end{aligned}
$$

Therefore B_α is invariant.
We have $\alpha \in Aut_S(\mathfrak{g}_\alpha, B_\alpha)$. Indeed

$$\alpha([x, y]_\alpha) = \alpha([\alpha(x), \alpha(y)]) = [\alpha^2(x), \alpha^2(y)] = [\alpha(x), \alpha(y)]_\alpha,$$

and

$$B_\alpha(\alpha(x), y) = B(\alpha(\alpha(x)), y) = B(\alpha(x), \alpha(y)) = B_\alpha(x, \alpha(y)).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem permits to obtain new quadratic Hom-Lie algebras starting from a multiplicative quadratic Hom-Lie algebra.

Proposition 4.2. *Let $(\mathfrak{g}, [\,,\,], \alpha, B)$ be a multiplicative quadratic Hom-Lie algebra. For any $n \geq 0$, the quadruple*

$$(4.3) \qquad\qquad \mathfrak{g}_{(n)} = \left(\mathfrak{g}, [\,,\,]^{(n)} = \alpha^n \circ [\,,\,], \alpha^{n+1}, B_{\alpha^n} \right),$$

where B_{α^n} is defined for $x, y \in \mathfrak{g}$ by $B_{\alpha^n}(x, y) = B(\alpha^n(x), y)$, determine a multiplicative quadratic Hom-Lie algebra.

Proof. The triple $\mathfrak{g}_{(n)} = \left(\mathfrak{g}, [\,,\,]^{(n)} = \alpha^n \circ [\,,\,], \alpha^{n+1} \right)$ is a Hom-Lie algebra by Theorem 1.13.
Since $\alpha \in Aut(\mathfrak{g})$ by induction we have $\alpha^n \in Aut(\mathfrak{g})$. The bilinear form B_{α^n} is nondegenerate because B is nondegenerate and α^n is bijective. It is is symmetric. Indeed

$$B_{\alpha^n}(x, y) = B(\alpha^n(x), y) = B(x, \alpha^n(y)) = B(\alpha^n(y), x) = B_{\alpha^n}(y, x).$$

The invariance of B_{α^n} is given by

$$B_{\alpha^n}([x,y]^n, z) = B(\alpha^n \circ \alpha^n([x,y]), z) = B(\alpha^n([x,y]), \alpha^n(z)) = B([\alpha^n(x), \alpha^n(y)], \alpha^n(z))$$
$$= B(\alpha^n(x), [\alpha^n(y), \alpha^n(z)]) = B(\alpha^n(x), \alpha^n([y,z])) = B_{\alpha^n}(x, [y,z]^n).$$

We have also $B_{\alpha^n}(\alpha^n(x), y) = B_{\alpha^n}(x, \alpha^n(y))$, indeed

$$B_{\alpha^n}(\alpha^n(x), y) = B(\alpha^{2n}(x), y) = B(\alpha^n(x), \alpha^n(y)) = B_{\alpha^n}(x, \alpha^n(y)).$$

$\square$

We provide here a construction a Hom-Lie algebra $\mathcal{L}$ which is a generalization of the trivial T^*-extension introduced by M. Bordemann [9] and also the double extension of $\{0\}$ by $\mathcal{L}$ see [34].

Proposition 4.3. *Let $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g})$ be a Lie algebra and $\mathfrak{g}^*$ be the underlying dual vector space. The vector space $\mathcal{L} = \mathfrak{g} \oplus \mathfrak{g}^*$ equipped with the following product*

$$(4.4) \qquad [\ ,\ ] : \begin{array}{c} \mathcal{L} \times \mathcal{L} \to \mathcal{L} \\ (x+f, y+h) \mapsto [x,y]_\mathfrak{g} + f \circ ady - h \circ adx \end{array}$$

and a bilinear form

$$(4.5) \qquad B : \begin{array}{c} \mathcal{L} \times \mathcal{L} \to \mathbb{K} \\ (x+f, y+h) \mapsto f(y) + h(x) \end{array}$$

is a quadratic Lie algebra, which we denote by $\mathcal{L}$.

In the sequel we denote $\mathcal{L}$ by $T^*(\mathfrak{g})$ and B by B_0.

Theorem 4.4. *Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra and $\alpha \in Aut(\mathfrak{g})$. Then the endomorphism $\Omega := \alpha + {}^t\alpha$ of $T^*(\mathfrak{g})$ is a symmetric automorphism of $T^*(\mathfrak{g})$ with respect to B_0 if and only if $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$, where $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.*
Hence, if $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$ then $(T_0^*(\mathfrak{g})_\Omega, [\ ,\ ]_\Omega, \Omega, B_\Omega)$ is a RQH-Lie algebra where $\Omega = \alpha + {}^t\alpha$.

Proof. Let $x, y \in \mathfrak{g}$ and $f, h \in \mathfrak{g}^*$.

$$\begin{aligned} \Omega([x+f, y+h]) &= \Omega([x,y]_\mathfrak{g} + f \circ ady - h \circ adx) \\ &= \alpha([x,y]_\mathfrak{g}) + f \circ ady \circ \alpha - h \circ adx \circ \alpha, \end{aligned}$$

and

$$\begin{aligned} [\Omega(x+f), \Omega(y+h)] &= [\alpha(x) + f \circ \alpha, \alpha(y) + h \circ \alpha] \\ &= [\alpha(x), \alpha(y)]_\mathfrak{g} + f \circ \alpha \circ ad\alpha(y) - h \circ \alpha \circ ad\alpha(x), \end{aligned}$$

Then $\Omega([x+f, y+h]) = [\Omega(x+f), \Omega(y+h)]$ if and only if

$$\forall x, y \in \mathfrak{g}, \quad f \circ ady \circ \alpha - h \circ adx \circ \alpha = f \circ \alpha \circ ad\alpha(y) - h \circ \alpha \circ ad\alpha(x).$$

That is for all $z \in \mathfrak{g}$

$$f([y, \alpha(z)]) - h([x, \alpha(z)]) = f(\alpha[\alpha(y), z]) - h(\alpha[\alpha(x), z]).$$

Hence, Ω is an automorphism of $T^*(\mathfrak{g})$ if and only if $f([x, \alpha(y)]) = f(\alpha[\alpha(x), y])$, $\forall f \in \mathfrak{g}^*$ $\forall x, y \in \mathfrak{g}$, which is equivalent to $[x, \alpha(y)] = \alpha[\alpha(x), y]$ $\forall x, y \in \mathfrak{g}$.
As a consequence, $\Omega \in Aut(T_0^*(\mathfrak{g}))$ if and only if $[\alpha^2(x) - x, \alpha(y)]_\mathfrak{g} = 0$ $\forall x, y \in \mathfrak{g}$, ie. $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$, since $\alpha \in Aut(\mathfrak{g})$.
In the following we show that Ω is symmetric with respect to B_0. Indeed, let $x, y \in \mathfrak{g}$ and $f, h \in \mathfrak{g}^*$

$$
\begin{aligned}
B_0(\Omega(x+f), y+h) &= B_0(\alpha(x) + f \circ \alpha, y + h) \\
&= f \circ \alpha(y) + h(\alpha(x)) \\
&= f \circ \alpha(y) + h \circ \alpha(x) \\
&= B_0(x+f, \alpha(y) + h \circ \alpha) \\
&= B_0(x+f, \Omega(y+h)).
\end{aligned}
$$

The last assertion is a consequence of the previous calculations and Proposition 2.3. $\qquad\square$

In the following we provide examples which show that the class of Lie algebras with automorphisms satisfying the condition $Im(\alpha^2(x) - x) \in \mathcal{Z}(\mathfrak{g})$ is large. We consider first Lie algebras with involutions.

Corollary 4.5. *Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ be a Lie algebra and $\theta \in Aut(\mathfrak{g})$ such that $\theta^2 = id$ (θ is an involution), then $\theta^2(x) - x = 0 \in \mathcal{Z}(\mathfrak{g})$, $\forall x \in \mathfrak{g}$. Thus $(T_0^*(\mathfrak{g})_\Omega, [\ ,\ ]_\Omega, \Omega, B_\Omega)$ is a RQH-Lie algebra where $\Omega = \alpha + {}^t\alpha$.*

Example 4.6. *Recall that considering an involution on a Lie algebra $\mathfrak{g}$ is equivalent to have a $\mathbb{Z}_2$-graduation on $\mathfrak{g}$. The Lie algebras with involutions are called symmetric (see [17] [38]).*
It is well known that starting from a Lie algebra one may construct a symmetric Lie algebra in the following way :
Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra, we consider the Lie algebra $(\mathfrak{L}, [\cdot, \cdot]_{\mathfrak{L}})$ where $\mathfrak{L} = \mathfrak{g} \times \mathfrak{g}$ and the bracket defined by for all $x, y, x', y' \in \mathfrak{g}$ by $[(x,y), (x', y')]_{\mathfrak{L}} := ([x, x'], [y, y'])$.
It is easy to check that the map $\theta : \begin{array}{c} \mathfrak{L} \to \mathfrak{L} \\ (x, y) \mapsto (y, x) \end{array}$ is an automorphism of $\mathfrak{L}$.
Then the trivial T^*-extension of $\mathfrak{L}$ has $\Omega = \theta + {}^t\theta$ as a symmetric automorphism with respect to B_0. Moreover, Ω is an involution. According to corollary 4.5, we have $(T_0^*(\mathfrak{L})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$ is a quadratic Hom-Lie algebra.

Example 4.7. *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a semisimple Lie algebra with an involution θ. Recall that the classification of semisimple Lie algebras with involutions could be found in [28]. The Killing form $\mathcal{K}$ of $\mathfrak{g}$ is nondegenerate and θ is symmetric with respect to $\mathcal{K}$. Then $(\mathfrak{g}_\theta, [\ ,\ ]_\theta, \theta, \mathcal{K}_\theta)$ is a RQH-Lie algebra.*
For example, let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{K})$ (with $n \geq 2$). The linear map $\theta : \mathfrak{g} \to \mathfrak{g}$ defined for all $x \in \mathfrak{g}$ by $\theta(x) = -{}^tx$ is an involution automorphism. Therefore we endow $(\mathfrak{sl}_n(\mathbb{K}))_\theta$ with a RQH-Lie algebra structure.

Example 4.8. *We show an example of Lie algebra $\mathfrak{g}$ with automorphisms α which are not involutions and satisfying $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$.*
Let $\mathfrak{g}$ be a finite-dimensional vector space and $\mathfrak{A} = \{x_0, \cdots, x_n\}$, ($n \geq 1$) be a basis of $\mathfrak{g}$. We define on $\mathfrak{g}$ a structure of Lie algebra by

$$[x_0, x_i] = x_{i+1}, \quad \forall i \in \{1, \cdots, n-1\}$$

The others brackets are defined obtained by skewsymmetry or equal to zero.
This Lie algebra is nilpotent and called filiform Lie algebra, its center is the subvector space generated by $< x_n >$.
The endomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ defined by

$$
\begin{aligned}
\alpha(x_0) &= x_0 + \lambda x_n, \quad \text{where } \lambda \in \mathbb{K} \setminus \{0\}, \\
\alpha(x_i) &= x_i, \quad \forall i \in \{1, \cdots, n\}
\end{aligned}
$$

is an automorphism of $\mathfrak{g}$ satisfying

$$
\begin{aligned}
\alpha^2(x_0) &= x_0 + 2\lambda x_n, \\
\alpha^2(x_i) &= x_i, \quad \forall i \in \{1, \cdots, n\}
\end{aligned}
$$

Therefore $(\alpha^2 - id)(\mathfrak{g}) \subseteq \mathcal{Z}(\mathfrak{g})$ and $\alpha^2 \neq id$. According to Theorem 4.4, $(T_0^*(\mathfrak{L})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$, where $\Omega = \alpha + {}^t\alpha$, is a RQH-Lie algebra.

Example 4.9 (Nonabelian 2-nilpotent Lie algebras). *Let $\mathfrak{g} = V \oplus \mathcal{Z}(\mathfrak{g})$, where $V \neq \{0\}$ is a subspace of the vector space $\mathfrak{g}$ with $[V, V] = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathcal{Z}(\mathfrak{g})$.*
Let $\lambda : \mathfrak{g} \to \mathcal{Z}(\mathfrak{g})$ be a nontrivial linear map and $\alpha : \mathfrak{g} \to \mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ defined by

$$\alpha(v + z) := v + \lambda(v) + z \quad \forall v \in V \quad \forall z \in \mathcal{Z}(\mathfrak{g}).$$

We have $\alpha([v + z, v' + z']) = \alpha([v, v']) = [v, v']$ since $[v, v'] \in \mathcal{Z}(\mathfrak{g})$. Also $[\alpha(v + z), \alpha(v' + z')]) = [v, v']$. Therefore, the map α is an injective Lie algebra morphism. Thus α is an automorphism of $\mathfrak{g}$.
Moreover, if $v \in \mathfrak{g}$ and $z \in \mathcal{Z}(\mathfrak{g})$, we have

$$\begin{aligned}
(\alpha^2 - id)(v + z) &= \alpha^2(v + z) - (v + z) \\
&= \alpha(v + \lambda(v) + z) - (v + z) \\
&= v + 2\lambda(v) + z - v - z \\
&= 2\lambda(v).
\end{aligned}$$

Then $\alpha^2 - id \neq 0$ and $Im(\alpha^2 - id) \subseteq \mathcal{Z}(\mathfrak{g})$. It follows that $(T_0^*(\mathfrak{g})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$, where $\Omega = \alpha +^t \alpha$, is a RQH-Lie algebra.
It is clear that $T_0^*(\mathfrak{g})_\Omega$ is 2-nilpotente. It 's also a quadratic Lie algebra.

Proposition 4.10. *Let $\mathcal{A}$ be a commutative associative algebra and $\mathfrak{g}$ be a Lie algebra.*
If $\mathcal{A}$ has an automorphism θ such that $Im(\theta^2 - id) \subseteq Ann(\mathcal{A})$, where $Ann(\mathcal{A})$ denotes the annihilator of $\mathcal{A}$, then the endomorphism $\widetilde{\theta} := id_\mathfrak{g} \otimes \theta$ of $\mathfrak{g} \otimes \mathcal{A}$ is an automorphism of the Lie algebra $(\mathfrak{g} \otimes \mathcal{A}, [\ ,\ ])$, where $[x \otimes a, y \otimes b] := [x, y]_\mathfrak{g} \otimes ab$ for all $x, y \in \mathfrak{g}$ and $a, b \in \mathcal{A}$. In addition, $Im(\widetilde{\theta}^2 - id_{\mathfrak{g} \otimes \mathcal{A}}) \subseteq \mathcal{Z}(\mathfrak{g} \otimes \mathcal{A})$. Then $(T_0^*(\mathfrak{g} \otimes \mathcal{A})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$ is a RQH-Lie algebra.
Moreover, if $\theta^2 \neq id_\mathcal{A}$ then $\widetilde{\theta}^2 \neq id_{\mathfrak{g} \otimes \mathcal{A}}$.

Proof. It follows from direct calculation and Theorem 4.4. □

Example 4.11. *Let $\mathcal{A}$ be the vector space spanned by $\{e, f, h, t\}$ on which we define a commutative associative multiplication by*

$$ee = f, \quad ef = fe = h, \quad eh = he = t, \quad ff = t.$$

We consider the algebra morphism $\theta : \mathcal{A} \to \mathcal{A}$ defined by

$$\theta(e) = e + qt, \quad \theta(f) = f, \quad \theta(h) = h, \quad \theta(t) = t.$$

where $q \in \mathbb{K} \setminus \{0\}$. It is easy to check that $Im(\theta^2 - id) \subseteq Ann(\mathcal{A})$ and $\theta^2 \neq id$.
Let $\mathfrak{g}$ be a Lie algebra (for example $\mathfrak{g} = \mathfrak{sl}(2)$), then $\mathfrak{g} \otimes \mathcal{A}$ is a Lie algebra with $\widetilde{\theta} = id_\mathfrak{g} \otimes \theta \in Aut(\mathfrak{g} \otimes \mathcal{A})$ satisfying $Im(\widetilde{\theta}^2 - id_{\mathfrak{g} \otimes \mathcal{A}}) \subseteq \mathcal{Z}(\mathfrak{g} \otimes \mathcal{A})$. Then $(T_0^*(\mathfrak{g} \otimes \mathcal{A})_\Omega, [\ ,\ ]_\Omega, \Omega, (B_0)_\Omega)$ is a RQH-Lie algebra.

5. Connection between quadratic Lie algebras and Hom-Lie algebras

We establish a connection between some classes of Lie algebras (resp. quadratic Lie algebras) and classes of Hom-Lie algebras (resp. quadratic Hom-Lie algebras).

5.1. **Lie algebras with involutive automorphisms.**

Theorem 5.1. *There exists a biunivoque correspondence between the class of Lie algebras (resp. quadratic Lie algebras) admitting involutive automorphisms (resp. symmetric involutive automorphisms) and the class of Hom-Lie algebras (resp. quadratic Hom-Lie algebras) where twist maps are involutive automorphisms (resp. symmetric involutive automorphisms)*

Proof. Let $(\mathfrak{g}, [\ ,\ ])$ be a symmetric Lie algebra with θ an involutive automorphism of $\mathfrak{g}$.
Then, according to Theorem 1.9, $(g_\theta, [\ ,\ ]_\theta, \theta)$ is a Hom-Lie algebra where θ is an involutive automorphism of $\mathfrak{g}_\theta$.
Moreover, if $\mathfrak{g}$ has an invariant scalar product B such that θ is symmetric with respect to B, we have seen that

(5.1) $$B_\theta : \begin{array}{c} \mathfrak{g}_\theta \times \mathfrak{g}_\theta \to \mathbb{K} \\ (x, y) \mapsto B_\theta(x, y) := B(\theta(x), y) \end{array}$$

defines a quadratic structure on $\mathfrak{g}_\theta$.

Conversely, let $(H, [\ ,\ ]_H, \theta)$ be a Hom-Lie algebra where θ is an involutive automorphism of H.

We will untwist the Hom-Lie algebra structure by considering the vector space H and the bracket

$$(5.2) \qquad [\ ,\ ]: \begin{array}{c} H \times H \to H \\ (x, y) \mapsto [x, y] := [\theta(x), \theta(y)]_H \end{array}$$

Obviously the new bracket is bilinear and skewsymmetric. We show that it satisfies the Jacobi identity.

Indeed, for $x, y, z \in H$ we have

$$\begin{aligned}
[x, [y, z]] &= [\theta(x), \theta([y, z])]_H \\
&= [\theta(x), \theta([\theta(y), \theta(z)]_H)]_H \\
&= [\theta(x), [\theta^2(y), \theta^2(z)]_H)]_H \\
&= [\theta(x), [y, z]_H)]_H.
\end{aligned}$$

Thus

$$\circlearrowleft_{x,y,z} [x, [y, z]] = \circlearrowleft_{x,y,z} [\theta(x), [y, z]_H)]_H = 0.$$

Thus $(H, [\ ,\ ])$ is a Lie algebra.

Furthermore, for $x, y \in H$

$$\theta([x, y]) = \theta([\theta(x), \theta(y)]_H) = [\theta^2(x), \theta^2(y)]_H = [x, y]_H$$

and

$$[\theta(x), \theta(y)] = [\theta^2(x), \theta^2(y)]_H = [x, y]_H.$$

Then $\theta([x, y]) = [\theta(x), \theta(y)]$. Therefore θ is an involutive automorphism of the Lie algebra $(H, [\ ,\ ])$.

Also for $x, y \in H$

$$[x, y]_\theta := [\theta(x), \theta(y)] = [\theta^2(x), \theta^2(y)]_H = [x, y]_H.$$

Then $(H, [\ ,\ ]_\theta, \theta)$ is the Hom-Lie algebra $(H, [\ ,\ ]_H, \theta)$.

Now, let $(H, [\ ,\ ]_H, \theta, B)$ be a quadratic Hom-Lie algebra.

The bilinear form

$$(5.3) \qquad T: \begin{array}{c} H \times H \to \mathbb{K} \\ (x, y) \mapsto T(x, y) = B(\theta(x), y) \end{array}$$

is symmetric and nondegenerate.

Indeed, for Let $x, y, z \in H$, we have

$$\begin{aligned}
T([x, y], z) &= B(\theta([x, y]), z) \\
&= B(\theta[\theta(x), \theta(y)]_H, z) \\
&= B([x, y]_H, z) \\
&= B(x, [y, z]_H) \\
&= B(\theta(x), \theta([y, z]_H)) \qquad \theta \text{ is } B\text{-symmetric} \\
&= B(\theta(x), [\theta(y), \theta(z)]_H)) \\
&= B(\theta(x), [y, z])) \\
&= T(x, [y, z]).
\end{aligned}$$

Then T is invariant. In the other hand,

$$T(\theta(x), y) = B(x, y) = B(\theta(x), \theta(y)) = T(x, \theta(y)).$$

That is θ is symmetric with respect to T.

Therefore $(H, [\ ,\ ], T)$ is a quadratic Lie algebra and $(H, [\ ,\ ]_\theta, \theta, T_\theta)$ is an IQH-Lie algebra. $\qquad \square$

5.2. **Quadratic Hom-Lie algebras and Centroid's elements.** Now we discuss the connection between Hom-Lie algebras where the twist map is in the centroid and quadratic Lie algebras. Let $(\mathfrak{g}, [\,,\,], B)$ be a quadratic Lie algebra and $\theta \in Cent(\mathfrak{g})$ such that θ is invertible and symmetric with respect to B. We set

$$Cent_S(\mathfrak{g}) = \{\theta \in Cent(\mathfrak{g}) : \theta \text{ symmetric with respect to } B\}.$$

We consider

$$(5.4) \qquad\qquad B_\theta : \begin{array}{c} \mathfrak{g} \times \mathfrak{g} \to \mathbb{K} \\ (x,y) \mapsto B_\theta(x,y) := B(\theta(x), y) \end{array}$$

We have

- B_θ is symmetric.
- B_θ is nondegenerate.
- B_θ is invariant, indeed

$$\begin{aligned}
B_\theta(\{x,y\}, z) = B_\theta([\theta(x), y], z) &= B(\theta([\theta(x), y]), z) \\
&= B([\theta(x), y], \theta(z)) = B(\theta(x), [y, \theta(z)]) \\
&= B(\theta(x), [\theta(y), z]) = B(\theta(x), \{y, z\}) \\
&= B_\theta(x, \{y, z\}).
\end{aligned}$$

Also, we have

$$B_\theta(\theta(x), y) = B(\theta^2(x), y) = B(\theta(x), \theta(y)) = B_\theta(x, \theta(y)).$$

Then $(\mathfrak{g}, \{\,,\,\}, \theta, B_\theta)$ is a quadratic Hom-Lie algebra.

Notice that B_θ is also an invariant scalar product of the Lie algebra $\mathfrak{g}$.

We have also that $(\mathfrak{g}, [\,,\,]_\theta, \theta, B_\theta)$ is a quadratic Hom-Lie algebra. indeed

$$\begin{aligned}
B_\theta([x,y]_\theta, z) = B_\theta([\theta(x), \theta(y)], z) &= B(\theta([\theta(x), \theta(y)]), z) \\
&= B([\theta(x), \theta(y)], \theta(z)) = B(\theta(x), [\theta(y), \theta(z)]) \\
&= B(\theta(x), [y, z]_\theta) \\
&= B_\theta(x, [y, z]_\theta).
\end{aligned}$$

Observe that

$$\theta([x, y]_\theta) = \theta[\theta(x), \theta(y)] = [\theta^2(x), \theta(y)] = [\theta(x), y]_\theta.$$
$$\theta(\{x, y\}) = \theta[\theta(x), y] = [\theta^2(x), y] = \{\theta(x), y\}.$$

We may say that $\theta \in Cent(\mathfrak{g}, \{\,,\,\})$ and $\theta \in Cent(\mathfrak{g}, [\,,\,]_\theta)$.

Conversely, let $(\mathfrak{g}, [\,,\,], \alpha)$ be a Hom-Lie algebra such that $\alpha \in Cent(\mathfrak{g}, [\,,\,], \alpha)$.

We define a new bracket as $\{x, y\} := [\alpha(x), y]$.. Then $(\mathfrak{g}, \{\,,\,\}))$ is a Lie algebra. Indeed

- The bracket is skewsymmetric.
- We have

$$\begin{aligned}
\{x, \{y, z\}\} &= [\alpha(x), [\alpha(y), z]], \\
\{y, \{z, x\}\} &= [\alpha(y), [\alpha(z), x] = [\alpha^2(y), [z, x]], \\
\{z, \{x, y\}\} &= [\alpha(z), [\alpha(x), y]] = [\alpha(z), [x, \alpha(y)]].
\end{aligned}$$

Then

$$\circlearrowleft_{x,y,z} \{x, \{y, z\}\} = [\alpha(x), [\alpha(y), z]] + [\alpha^2(y), [z, x]] + [\alpha(z), [x, \alpha(y)]] = 0.$$

We may define another which gives rise to also a Lie algebra by $[x, y]_\alpha := [\alpha(x), \alpha(y)]$.. Indeed the bracket is skewsymmetric and we have

$$\begin{aligned}
[x, [y, z]_\alpha]_\alpha &= [\alpha(x), \alpha([\alpha(y), \alpha(z)])] = [\alpha(x), [\alpha^2(y), \alpha(z)]] = [\alpha^2(x), [\alpha(y), \alpha(z)]], \\
[y, [z, x]_\alpha]_\alpha &= [\alpha(y), \alpha([\alpha(z), \alpha(x)])] = [\alpha(y), [\alpha^2(z), \alpha(x)]] = [\alpha^2(y), [\alpha(z), \alpha(x)]], \\
[z, [x, y]_\alpha]_\alpha &= [\alpha(z), \alpha([\alpha(x), \alpha(y)])] = [\alpha(z), [\alpha^2(x), \alpha(y)]] = [\alpha^2(z), [\alpha(x), \alpha(y)]].
\end{aligned}$$

Therefore
$$[\alpha^2(x), [\alpha(y), \alpha(z)]] + [\alpha^2(y), [\alpha(z), \alpha(x)]] + [\alpha^2(z), [\alpha(x), \alpha(y)]] = 0.$$
Now if there is an invariant scalar product B on $(\mathfrak{g}, [\ , \ ])$ and assume that α is invertible and symmetric with respect to B. Consider the bilinear form B_α defined by $B_\alpha(x, y) = B(\alpha(x), y)$. We have

$$
\begin{aligned}
B_\alpha(\{x, y\}, z) &= B(\alpha(\{x, y\})]), z) \\
&= B(\alpha([\alpha(x), y], z) \\
&= B(\alpha(x), [y, \alpha(z)]) \\
&= B(\alpha(x), [\alpha(y), z]) \\
&= B_\alpha(x, \{y, z\}).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
B_\alpha([x, y]_\alpha, z) &= B(\alpha([\alpha(x), \alpha(y)]), z) \\
&= B([\alpha(x), \alpha(y)], \alpha(z)) \\
&= B(\alpha(x), [\alpha(y), \alpha(z)]) \\
&= B(\alpha(x), [y, z]_\alpha) \\
&= B_\alpha(x, [y, z]_\alpha).
\end{aligned}
$$

Therefore $(\mathfrak{g}, \{ \ , \ \}, B_\alpha)$ and $(\mathfrak{g}, [\ , \ ]_\alpha, B_\alpha)$ are quadratic Lie algebras. Hence, we have the following theorem:

Theorem 5.2. *There exists a biunivoque correspondence between the class of Lie algebras (resp. quadratic Lie algebras) admitting an element in the centroid (resp. symmetric invertible element in the centroid) and the class of Hom-Lie algebras (resp. quadratic Hom-Lie algebras) where twist map is in the centroid (resp. symmetric invertible element in the centroid).*

An interesting case is when the element θ of the centroid is not of the form $\theta = k \ id_\mathfrak{g}$ where $k \in \mathbb{K}$. One may replace the class of quadratic Lie algebras admitting a symmetric invertible element θ in the centroid, such that $\theta \neq k \ id_\mathfrak{g}$ where $k \in \mathbb{K}$, by the class of quadratic Lie algebras of quadratic dimension larger than 2, i.e. there exist B and B' two invariant scalar products on $\mathfrak{g}$ such that $\nexists \lambda \in \mathbb{K}$ such that $B = \lambda B'$. For quadratic dimension see [5, 8, 35].

Corollary 5.3. *There exists a one to one correspondence between the class of quadratic Lie algebra of quadratic dimension larger than 2 and the class of quadratic Hom-Lie algebras where the twist map is invertible and in the centroid such that it is not a multiple of $id_\mathfrak{g}$.*

Consider a quadratic Lie algebras of quadratic dimension larger than 2, then there exists B and B' two invariant scalar products on $\mathfrak{g}$ such that it doesn't exist $\lambda \in \mathbb{K}$ such that $B = \lambda B'$. Since B is nondegenerate then there exist $\theta \in End(\mathfrak{g})$ such that $B'(x, y) = B(\theta(x), y)$ and $\theta \neq k \ id_\mathfrak{g}$ where $k \in \mathbb{K}$. Observe that the invariance of B and B' induces that θ is in the centroid of $\mathfrak{g}$. Indeed for any $x, y, z \in \mathfrak{g}$ the identity $B'([x, y], z) = B'(x, [y, z])$ is equivalent to $B(\theta([x, y]), z) = B(\theta(x), [y, z]) = B([\theta(x), y], z)$. It leads to $\theta([x, y]) = [\theta(x), y]$ since B is non degenerate. Hence B and B' are nondegenerate implies that θ is invertible.
Notice that $B'(x, y) = B'(y, x)$ implies $B(\theta(x), y) = B(\theta(y), x) = B(x, \theta(y))$. Conversely when we start with a Hom-Lie algebra where the twist map is invertible and in the centroid, we have constructed in the proof of the previous theorem two quadratic structures on this algebra.

6. Simple and Semisimple Hom-Lie algebras

In this section we give some observations about simple and semisimple Hom-Lie algebras. In particular, we study simple and semisimple Hom-Lie algebras with involution. Simple involutive Hom-Lie algebras will appear in the last section when we will discuss a structure theorem of IQH-Lie algebra.
A Hom-Lie algebra is said to be simple if it has no non-trivial ideals and is not abelian. A Hom-Lie algebra $\mathfrak{g}$ is called semisimple if its radical is zero (a radical is the maximal solvable ideal). Equivalently,

$\mathfrak{g}$ is semisimple if it does not contain any non-zero abelian ideals. In particular, a simple Hom-Lie algebra is semisimple. In [23], the authors studied when a finite-dimensional semi-simple Lie algebra admits non-trivial hom-Lie algebra structures.

Proposition 6.1. *Let* $(\mathfrak{g}, [\cdot, \cdot])$ *be a Lie algebra and* $\alpha \in Aut(\mathfrak{g})$. *If* $\mathfrak{g}$ *is simple then* $\mathfrak{g}_\alpha$ *obtained by composition method is simple.*

Proof. Indeed, let I be an ideal in $\mathfrak{g}_\alpha$, that is $[\mathfrak{g}_\alpha, I] \subseteq I$ and $\alpha(I) \subseteq I$. Then, $\forall x \in \mathfrak{g}$ and $\forall a \in I$ we have $[\alpha(x), \alpha(a)] \in I$. Therefore I is an ideal of $\mathfrak{g}$ because $\alpha(I) = I$. Hence, $I = \{0\}$ or $I = \mathfrak{g} = \mathfrak{g}_\alpha$.
Since $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]_\alpha \neq \{0\}$. Thus, $\mathfrak{g}_\alpha$ is a simple Hom-Lie algebra. $\square$

Hence the previous proposition provides a way to construct simple Hom-Lie algebras, see example 4.7. We have proved in Proposition 6.1 that a simple Lie algebra with an automorphism (in particular an involution) gives rise to a simple Hom-Lie algebra using composition method. The converse is discussed further according to Proposition 1.10.
Let $(\mathfrak{g}, [\ ,\ ], \theta)$ be a simple involutive Hom-Lie algebra. We denote by $\rho(x)$, for $x \in \mathfrak{g}$, the linear map $\rho(x) : \mathfrak{g} \to \mathfrak{g}$ defined for $y \in \mathfrak{g}$ by $\rho(x)(y) = [x, y]$.
We also set B to be the map $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined for $x, y \in \mathfrak{g}$ by $B(x, y) = tr(\rho(x)\rho(y))$. Obviously B is bilinear and symmetric. We show now that it is also invariant. Let $x, y, z \in \mathfrak{g}$ then

$$\begin{aligned}
\rho([x,y])(z) &= [[x,y],z] = -[z,[x,y]] \\
&= -[\theta(\theta(z)), [x,y]] = [\theta(x), [y, \theta(z)]] + [\theta(y), [\theta(z), x]] \\
&= \theta([x, [\theta(y), z]]) + \theta([y, [z, \theta(x)]]) \\
&= (\theta \circ \rho(x) \circ \rho(\theta(y)) - \theta \circ \rho(y) \circ \rho(\theta(x)))(z).
\end{aligned}$$

In addition we have $\rho(\theta(x))(z) = [\theta(x), z] = \theta([x, \theta(z)]) = \theta \circ \rho(x) \circ \theta(z)$ which leads for any $x \in \mathfrak{g}$ to

$$\rho(\theta(x)) = \theta \circ \rho(x) \circ \theta.$$

Since for any $x, y \in \mathfrak{g}$ we have

$$\rho([x,y]) = \theta \circ \rho(x) \circ \rho(\theta(y)) - \theta \circ \rho(y) \circ \rho(\theta(x)),$$

it follows for any $x, y, z \in \mathfrak{g}$

$$\begin{aligned}
\rho([x,y])\rho(z) &= (\theta \circ \rho(x) \circ \rho(\theta(y)) - \theta \circ \rho(y) \circ \rho(\theta(x)))\rho(z), \\
&= \theta \circ \rho(x) \circ \theta \circ \rho(y) \circ \theta \circ \rho(z) - \theta \circ \rho(y) \circ \theta \circ \rho(x) \circ \theta \circ \rho(z).
\end{aligned}$$

Therefore

$$\begin{aligned}
tr(\rho([x,y])\rho(z)) &= tr(\theta \circ \rho(x) \circ \theta \circ \rho(y) \circ \theta \circ \rho(z)) - tr(\theta \circ \rho(y) \circ \theta \circ \rho(x) \circ \theta \circ \rho(z)), \\
&= tr(\theta \circ \rho(y) \circ \theta \circ \rho(z) \circ \theta \circ \rho(x)) - tr(\theta \circ \rho(z) \circ \theta \circ \rho(y) \circ \theta \circ \rho(x)), \\
&= tr((\theta \circ \rho(y) \circ \theta \circ \rho(z) \circ \theta - \theta \circ \rho(z) \circ \theta \circ \rho(y) \circ \theta) \circ \rho(x)), \\
&= tr(\rho([y,z]) \circ \rho(x)), \\
&= tr(\rho(x) \circ \rho([y,z])).
\end{aligned}$$

Which proves that $B([x,y], z) = B(x, [y, z])$, that is B is invariant.
Let us consider $B_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined for any $x, y \in \mathfrak{g}$ by $B_\theta(x, y) = B(\theta(x), y)$.
We have for $x, y, z \in \mathfrak{g}$

$$\begin{aligned}
\rho(\theta(x))\rho(y)(z) &= [\theta(x), [y, z]] = [x, \theta([y, z])]_\theta, \\
&= [\theta^2(x), [\theta(y), \theta(z)]]_\theta, \\
&= [\theta^2(x), [\theta^2(y), \theta^2(z)]_\theta]_\theta, \\
&= [x, [y, z]_\theta]_\theta.
\end{aligned}$$

Hence $\rho(\theta(x))\rho(y) = ad_{\mathfrak{g}_\theta}(x) \circ ad_{\mathfrak{g}_\theta}(y)$, where $\mathfrak{g}_\theta$ is the Lie algebra associated to $(\mathfrak{g}, [\ ,\ ], \theta)$.
We conclude that B_θ is the Killing form of the Lie algebra $\mathfrak{g}_\theta$.
Then $B_\theta = \mathcal{K}$ and for $x, y \in \mathfrak{g}$, $B(x, y) = \mathcal{K}(\theta(x), y) = \mathcal{K}_\theta(x, y)$.

Theorem 6.2. *Let* $(\mathfrak{g}, [\ ,\ ], \theta)$ *be a simple involutive Hom-Lie algebra then* $g_\theta = (\mathfrak{g}, [\ ,\ ]_\theta)$ *is either a simple Lie algebra or a semisimple Lie algebra. Moreover in the second case it decomposes into* $S \oplus \theta(S)$ *where* S *is a simple ideal.*
In addition the form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, *defined for* $x, y \in \mathfrak{g}$ *by* $B(x, y) = tr(\rho(x)\rho(y))$ *where* $\rho(x) = [x, \cdot]$, *defines a quadratic structure on* $(\mathfrak{g}, [\ ,\ ], \theta)$ *and* $B(x, y) = \mathcal{K}(\theta(x), y)$, *where* $\mathcal{K}$ *is the Killing form of the Lie algebra* $g_\theta = (\mathfrak{g}, [\ ,\ ]_\theta)$.
Conversely, if $(\mathfrak{g}, [\ ])$ *is a simple Lie algebra and* $\theta \in Aut(\mathfrak{g})$ *is an involution then* $(\mathfrak{g}, [\ ,\ ], \theta)$ *is a simple Hom-Lie algebra.*

Proof. Assume that g_θ is not simple, then g_θ contains a minimal ideal S. Notice that g_θ is equal to $\mathfrak{g}$ as a linear space, we refer by $\mathfrak{g}$ to the Hom-Lie algebra and by $\mathfrak{g}_\theta$ for the corresponding Lie algebra by composition. Recall that I is a minimal ideal of g_θ if $I \neq \{0\}$, $I \neq \mathfrak{g}_\theta$ and if J is another ideal of g_θ such that $J \subseteq I$ then $J = \{0\}$ or $J = I$.
We do have $[g_\theta, S]_\theta$ is an ideal of g_θ such that $[g_\theta, S]_\theta \subseteq S$. Since S is minimal then $[g_\theta, S]_\theta = \{0\}$ or $[g_\theta, S]_\theta = S$.
If $[g_\theta, S]_\theta = \{0\}$ then $[\theta(g_\theta), \theta(S)] = \{0\}$ which implies $[\mathfrak{g}, \theta(S)] = \{0\}$ and $\theta(S) \subseteq \mathcal{Z}(g)$. Hence $\mathcal{Z}(g) \neq \{0\}$ since $\theta(S) \neq \{0\}$. This contradicts the fact that $(\mathfrak{g}, [\ ,\ ], \theta)$ is a simple Hom-Lie algebra.
Consequently we have $[g_\theta, S]_\theta = S$. It follows $[\mathfrak{g}, \theta(S)] = S$, then $\theta([\mathfrak{g}, \theta(S)]) = \theta(S)$. Thus $[\theta(\mathfrak{g}), \theta^2(S)]) = [\mathfrak{g}, S] = \theta(S)$. In addition, $\theta(S + \theta(S)) = \theta(S) + \theta^2(S) = \theta(S) + S$.
Then $S + \theta(S)$ is an ideal of $(\mathfrak{g}, [\ ,\ ], \theta)$ and $S + \theta(S) \neq \{0\}$. Therefore $\mathfrak{g} = S + \theta(S)$.
We show that we have a direct sum. Since θ is an automorphism of $\mathfrak{g}_\theta$ then $\theta(S)$ is an ideal of $\mathfrak{g}_\theta$.Thus $S \bigcap \theta(S) = \{0\}$ or $S \bigcap \theta(S) = S$ since S is a minimal ideal in g_θ. Assume $S \bigcap \theta(S) = S$, then $S \subseteq \theta(S)$ which leads to $S = \theta(S)$ since θ is bijective.
We do have $[\mathfrak{g}, S] = \theta([\theta(\mathfrak{g}), \theta(S)] = \theta([\mathfrak{g}, S]_\theta) \subseteq \theta(S) = S$ and $\theta(S) = \theta^2(S) = S$. Then S is an ideal of $(\mathfrak{g}, [\ ,\ ], \theta)$. Thus $S = \mathfrak{g}$ which contradicts the assumption $S \neq \mathfrak{g}$. Hence $S \bigcap \theta(S) = \{0\}$ and $\mathfrak{g} = S \oplus \theta(S)$. Consequently, we do have $\mathfrak{g}_\theta = S \oplus \theta(S)$. Indeed θ is an automorphism of $\mathfrak{g}$ then θ is an automorphism of $\mathfrak{g}_\theta$. Therefore $\theta(S)$ is an ideal of $\mathfrak{g}$. We do have $[\mathfrak{g}_\theta, S]_\theta = S$ then $[S \oplus \theta(S), S]_\theta = S$ which implies $[S, S]_\theta = S$ since $[\theta(S), S]_\theta = \{0\}$.
Therefore S is a simple ideal because S is a minimal ideal such that $[S, S]_\theta = S$.
It follows that $\mathfrak{g}_\theta$ is a semisimple Lie algebra because S and $\theta(S)$ are simple ideals of $\mathfrak{g}_\theta$. We may view $\mathfrak{g}_\theta$ as a $\mathbb{Z}_2$-graded simple Lie algebra.
We have also that $\mathcal{K}$ is nondegenerate and consequently B is non degenerate since $B(x, y) = \mathcal{K}(\theta(x), y)$ for $x, y \in \mathfrak{g}$. $\square$

We have proved that if $(\mathfrak{g}, [\ ,\ ])$ is a simple Lie algebra with involution θ then $(\mathfrak{g}_\theta, [\ ,\ ]_\theta, \theta)$ is a simple Hom-Lie algebra. We have also proved that if $(\mathfrak{g}, [\ ,\ ], \theta)$ is a simple Hom-Lie algebra then the Lie algebra $(\mathfrak{g}_\theta, [\ ,\ ]_\theta)$ is either simple or $\mathfrak{g}_\theta = S \oplus \theta(S)$, where S is a simple ideal of the Lie algebra $(\mathfrak{g}_\theta, [\ ,\ ]_\theta)$, (in particular $(\mathfrak{g}_\theta, [\ ,\ ]_\theta)$ is a semisimple Lie algebra.

Proposition 6.3. *Let* $(\mathfrak{g}, [\ ,\ ])$ *is a semisimple Lie algebra different from* $\{0\}$ *with an involution* θ *such that* $\mathfrak{g} = S + \theta(S)$, *where* S *is a simple ideal of* $(\mathfrak{g}, [\ ,\ ])$. *Then the Hom-Lie algebra* $(\mathfrak{g}_\theta, [\ ,\ ]_\theta, \theta)$ *is simple.*

Proof. Let I be an ideal of $\mathfrak{g}_\theta$ such that $I \neq \{0\}$. Since $\theta(\mathfrak{g}) = \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_\theta$ then $[\mathfrak{g}, \theta(I)] \subseteq I$ which implies that $[\mathfrak{g}, I] \subseteq I$. It follows that I is an ideal of $\mathfrak{g}$. Consequently $I = S$ or $I = \theta(S)$ or $I = \mathfrak{g}$. Since $\theta(I) = I$ then $I \neq S$ and $I \neq \theta(S)$, therefore $I = S + \theta(S) = \mathfrak{g}$. Moreover $\mathfrak{g}$ is semisimple then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ which implies $[\mathfrak{g}_\theta, \mathfrak{g}_\theta]_\theta = \mathfrak{g}_\theta$.Then $(\mathfrak{g}_\theta, [\ ,\ ]_\theta, \theta)$ is a simple Hom-Lie algebra. $\square$

More generally, let $(\mathfrak{g}, [\ ,\ ], \theta)$ be an involutive Hom-Lie algebra. We consider the associated Lie algebra $\mathfrak{g}_\theta = (\mathfrak{g}, [\ ,\ ]_\theta)$ where $[x, y]_\theta = [\theta(x), \theta(y)]$, $\forall x, y \in \mathfrak{g}$. Let $\mathfrak{R}(\mathfrak{g})$ be the solvable radical of $\mathfrak{g}_\theta$. By Taft's Theorem (see [37]), there exists a Levi's component $\mathfrak{s}$ of $\mathfrak{g}_\theta$ invariant by θ. It is clear that $\theta(\mathfrak{R}(\mathfrak{g})) = \mathfrak{R}(\mathfrak{g})$. The component $\mathfrak{s}$ is a semisimple Lie algebra then it may be written $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ where $\{\mathfrak{s}\}_{1 \leq i \leq n}$ is the set of the simple ideals of $\mathfrak{s}$. The fact $\theta(\mathfrak{s}) \subseteq \mathfrak{s}$ implies that $\theta_{|\mathfrak{s}}$ is an involutive automorphism of $\mathfrak{s}$. Then for $i \in \{1, \cdots, n\}$ we have $\theta(\mathfrak{s}_i)$ is a simple ideal of the Lie algebra $\mathfrak{s}$. Therefore for any $i \in \{1, \cdots, n\}$ there exist a unique $j_i \in \{1, \cdots, n\}$ such that $\theta(\mathfrak{s}_i) = \mathfrak{s}_{j_i}$. If $i = j_i$ then $\theta(\mathfrak{s}_i) = \mathfrak{s}_i$ and in

this case $(\mathfrak{s}_i, [\ ,\ ]_{\theta_{|\mathfrak{s}_i \times \mathfrak{s}_i}})$ is a simple Lie algebra with $\theta_i = \theta_{|\mathfrak{s}_i}$ is an involutive automorphism of $\mathfrak{s}_i$. We can then assume that there exist $m, p \in \mathbb{N}$ such that $m + 2p = n$ and

$$\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m \oplus (\mathfrak{s}_{m+1} \oplus \theta(\mathfrak{s}_{m+1})) \oplus \cdots \oplus (\mathfrak{s}_{m+p} \oplus \theta(\mathfrak{s}_{m+p}))$$

where, for $i \in \{1, \cdots, m + p\}$, $\mathfrak{s}_i$ are simple ideals of $\mathfrak{s}$ and for $i \in \{1, \cdots, m\}$ we have $\theta(\mathfrak{s}_i) = \mathfrak{s}_i$.
Let us set, for $i \in \{1, \cdots, m\}$, $\mathfrak{S}_i = \mathfrak{s}_i$ and for $i \in \{1, \cdots, p\}$, $\mathfrak{S}_{m+i} = \mathfrak{s}_{m+i} \oplus \theta(\mathfrak{s}_{m+i})$. Then, for $i \in \{1, \cdots, m\}$, $\mathfrak{S}_i$ is a simple ideal of $\mathfrak{s}$ that is invariant by θ. Also, for $i \in \{1, \cdots, p\}$, $\mathfrak{S}_{m+i}$ is a semisimple ideal of $\mathfrak{s}$ that is invariant by θ.
Since, for $x, y \in \mathfrak{g}$, $[x, y] = [\theta(x), \theta(y)]_\theta$ then it is clear that

- $\mathfrak{R}(\mathfrak{g})$ is a solvable ideal of the Hom-Lie algebra $(\mathfrak{g}, [\ ,\ ], \theta)$.
- $\mathfrak{s}$ is a subalgebra of the Hom-Lie algebra $(\mathfrak{g}, [\ ,\ ], \theta)$.
- For $i \in \{1, \cdots, m + p\}$
  (1) $\mathfrak{S}_i$ is a simple ideal of the Hom-Lie algebra $(\mathfrak{s}, [\ ,\ ]_{\theta_{|\mathfrak{s} \times \mathfrak{s}}}, \theta_{|\mathfrak{s}})$ is a simple
  (2) $\mathfrak{S}_i$ is a simple subalgebra of the Hom-Lie algebra $(\mathfrak{g}, [\ ,\ ], \theta)$.

Proposition 6.4. $\mathfrak{R}(\mathfrak{g})$ *is the greatest solvable ideal of the Hom-Lie algebra* $(\mathfrak{g}, [\ ,\ ], \theta)$.

Proof. Indeed, if I is a solvable ideal of $(\mathfrak{g}, [\ ,\ ], \theta)$, then I is a solvable ideal of the Lie algebra $\mathfrak{g}_\theta$, because $\theta(I) = I$. Consequently $I \subseteq \mathfrak{R}(\mathfrak{g})$. $\square$

Corollary 6.5. *Let* $(\mathfrak{g}, [\ ,\ ], \theta)$ *be an involutive Hom-Lie algebra then*

$$\mathfrak{g} = \mathfrak{S}_1 \oplus \cdots \mathfrak{S}_{m+p} \oplus \mathfrak{R}(\mathfrak{g})$$

where

- *for $i \in \{1, \cdots, m + p\}$, $\mathfrak{S}_i$ is a simple subalgebra of* $(\mathfrak{g}, [\ ,\ ], \theta)$,
- $\mathfrak{R}(\mathfrak{g})$ *is the greatest solvable ideal of* $(\mathfrak{g}, [\ ,\ ], \theta)$.

7. DOUBLE EXTENSION THEOREMS OF HOM-LIE ALGEBRAS

The fundamental result on quadratic Lie algebras in [34] leads to constructing and characterizing quadratic Lie algebras using double extension. While T^*-extension concept introduced in [9] concerns nonassociative algebras with nondegenerate associative symmetric bilinear form.
The following theorem extends the double extension by one-dimensional Lie algebras to Hom-Lie algebras case.

Theorem 7.1. *Let* $(V, [\ ,\ ]_V, \alpha_V, B_V)$ *be a quadratic multiplicative Hom-Lie algebra and* $\delta : V \to V$ *be a linear map. Set* $x_0 \in V$, $\lambda, \lambda_0 \in \mathbb{K}$. *We denote, for* $x \in V$, $\rho_V(x) := [x, \cdot]_V$ *and the bracket of two linear maps stands for the commutator on* $End(V)$.
If hold the following conditions

(7.1) $\alpha_V \circ \delta \circ \alpha_V - \lambda\delta = \rho_V(x_0)$,

(7.2) $[\alpha_V, \delta^2] = \rho_V(\delta(x_0))$,

(7.3) $\forall x, y \in V, \quad (\lambda\delta + \rho_V(x_0))([x, y]_V) = [\delta(x), \alpha_V(y)]_V + [\alpha_V(x), \delta(y)]_V$,

then the vector space $\mathfrak{g} := \mathbb{K}b \oplus V \oplus \mathbb{K}e$, *where* $\mathbb{K}b$ *and* $\mathbb{K}e$ *are one-dimensional vector spaces, is a multiplicative Hom-Lie algebra with the bracket* $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *defined by*

$$[x, y] = [x, y]_V + B_V(\delta(x), y) \quad \forall x, y \in V$$
$$[b, x] = \delta(x) \quad \forall x \in V$$
$$[e, z] = 0 \quad \forall z \in \mathfrak{g}$$

and the linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ *defined by*

$$\alpha(x) = \alpha_V(x) + B_V(x_0, x)e \quad \forall x \in V$$
$$\alpha(b) = \lambda b + x_0 + \lambda_0 e$$
$$\alpha(e) = \lambda e$$

In addition the symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ *defined by*

$$B(x, y) = B_V(x, y) \quad \forall x, y \in V$$
$$B(b, e) = 1$$
$$B(x, b) = B(x, e) = 0 \quad \forall x \in V$$
$$B(b, b) = B(e, e) = 0$$

is an invariant scalar product on $(\mathfrak{g}, [\ ,\ ], \alpha)$.
In the particular case, when α_V *is invertible then* α *is invertible if and only if* $\lambda \neq 0$ *and when* α_V *is an involution, then* α *is an involution if and only if*

(7.4) $$\lambda \in \{-1, 1\}, \quad \alpha(x_0) = -\lambda x_0 \quad and \quad \lambda_0 = -\frac{1}{2\lambda} B_V(x_0, x_0).$$

Proof. One shows that $\mathfrak{g}$ is a Hom-Lie algebra and B an invariant scalar product by straightforward calculations.

We assume now that α is invertible. Then $\lambda \neq 0$. Let $x \in V$ such that $\alpha_V(x) = 0$. It follows that $\alpha(x) = B_V(x_0, x)e$ which implies $\alpha(x) = \lambda^{-1} B_V(x_0, x)\alpha(e)$. Then $x = \lambda^{-1} B_V(x_0, x)e$. Thus $\lambda^{-1} B_V(x_0, x) = 0$, then $B_V(x_0, x) = 0$. Likewise $x = 0$. Hence α is invertible.

Conversely, assume that α_V is invertible and $\lambda \neq 0$. Let $x \in \mathfrak{g}$ such that $\alpha(x) = 0$. The element x in $\mathfrak{g}$ may be written $x = rb + y + se$ where $r, s \in \mathbb{K}$ and $y \in V$. Then

$$0 = \alpha(x) = r(\lambda b + x_0 + \lambda_0 e) + (\alpha_V(y) + B_V(x_0, y)e) + s\lambda e$$
$$= (\lambda r)b + (\alpha_V(y) + rx_0) + (r\lambda_0 + B(x_0, y) + s\lambda)e.$$

It follows

$$\lambda r = 0$$
$$\alpha_V(y) = -rx_0$$
$$r\lambda_0 + B_V(x_0, y) + sh = 0.$$

The solution of the system is $r = 0$, $\alpha_V(y) = 0$, $B_V(x_0, y)e) = -s\lambda$. Then $r = 0$, $y = 0$, $s = 0$, which gives $x = 0$. Then we conclude that α is invertible.

Now, assume that α is an involution. Then $b = \alpha^2(b)$ is equivalent to

$$b = \lambda\alpha(b) + \alpha(x_0) + \lambda\alpha(e),$$
$$= \lambda(\lambda b + x_0 + \lambda_0 e) + \alpha_V(x_0) + B(x_0, x_0)e + \lambda_0\lambda e$$

By identification, it follows

$$1 - \lambda^2 = 0$$
$$\{ \quad \alpha_V(x_0) = -\lambda x_0$$
$$B(x_0, x_0) = -2\lambda\lambda_0.$$

Therefore, $\lambda = \pm 1$ which are the eigenvalues of the involution, $\alpha_V(x_0) = -\lambda x_0$ and $\lambda_0 = -\frac{1}{2\lambda} B_V(x_0, x_0)$. Let $x \in V$. The identity $\alpha^2(x) = x$ may be written

$$\alpha(\alpha_V(x) + B(x_0, x)e) = x$$
$$\alpha(\alpha_V(x)) + B(x_0, x)\alpha(e) = x$$
$$\alpha^2(x) + B(x_0, \alpha_V(x))e + \lambda B(x_0, x)e = x$$

Then its equivalent to $\alpha_V^2(x) = x$ and $\alpha_V(x_0) = -\lambda x_0$. This ends the proof that α is an involution. Moreover, we have proved that α is an involution is equivalent to

$$\alpha_V^2 = id_V, \quad \lambda^2 = \pm 1, \quad \lambda_0 = -\frac{1}{2\lambda} B_V(x_0, x_0).$$

$\square$

Definition 7.2. The quadratic multiplicative Hom-Lie algebra $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ constructed in Theorem 7.1 is called a double extension of $(V, [\ ,\ ]_V, \alpha_V, B_V)$ (by the one-dimensional Lie algebra) by means $(\delta, x_0, \lambda, \lambda_0)$.

It is called involutive double extension when α_V is an involution and the condition (7.4) satisfied and its called regular double extension when α_V is invertible and $\lambda \neq 0$.

We provide in the following the converse of the previous Theorem.

Theorem 7.3. *Let $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ be an irreducible quadratic multiplicative Hom-Lie algebra such that $dim\mathfrak{g} > 1$.*
If $\mathcal{Z}(\mathfrak{g}) \neq \{0\}$, then $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ is a double extension of a quadratic Hom-Lie algebra $(V, [\ ,\ ]_V, \alpha_V, B_V)$ (by a one-dimensional Lie algebra) such that $dim V = dim\mathfrak{g} - 2$.
If in addition α is an involution then the IQH-Lie algebra $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ is an involutive double extension of an a IQH-Lie algebra $(V, [\ ,\ ]_V, \alpha_V, B_V)$ and if α is invertible then the RQH-Lie algebra $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ is a regular double extension of a RQH-Lie algebra $(V, [\ ,\ ]_V, \alpha_V, B_V)$.

Proof. Let $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ be an irreducible quadratic Hom-Lie algebra such that $dim\mathfrak{g} > 1$. Assume that $\mathcal{Z}(\mathfrak{g}) \neq \{0\}$ and $\alpha(\mathcal{Z}(\mathfrak{g})) \subseteq \mathcal{Z}(\mathfrak{g})$. Then there exists $\lambda \in \mathbb{K}$ ($\mathbb{K}$ algebraically closed) and $e \in \mathcal{Z}(\mathfrak{g}) \setminus \{0\}$ such that $\alpha(e) = \lambda e$. We have $B(e, e) = 0$ because $\mathbb{K}e$ is an ideal of $\mathfrak{g}$, $\mathfrak{g}$ is irreducible and $dim\mathfrak{g} > 1$. Since B is nondegenerate then there exists $b \in \mathfrak{g}$ such that $B(e, b) = 1$ and $B(b, b) = 0$.
The vector space $A = \mathbb{K}e \oplus \mathbb{K}b$ is nondegenerate (i.e. $B_{|A \times A}$ is nondegenerate), then $\mathfrak{g} = \mathcal{A} \oplus \mathcal{A}^\perp$. Set $V = \mathcal{A}^\perp$, then $B_{|V \times V}$ is nondegenerate and $(\mathbb{K}e)^\perp = \mathbb{K}e \oplus V$.
Then $\mathfrak{g} = \mathbb{K}e \oplus V \oplus \mathbb{K}b$ with $B(e, e) = B(b, b) = 0$, $B(e, b) = 1$, $B(V, e) = B(V, b) = \{0\}$ and $B_{|V \times V}$ is nondegenerate.
There exist a linear map $\alpha_V : V \to V$ and a linear form $f : V \to \mathbb{K}$ such that $\alpha(x) = f(x)e + \alpha_V(x)$, $\forall x \in V$, because $(\mathbb{K}e)^\perp$ is an ideal of $\mathfrak{g}$ implies $\alpha((\mathbb{K}e)^\perp) \subseteq (\mathbb{K}e)^\perp$.
There exist $\lambda_0 \in \mathbb{K}$, $\gamma_0 \in \mathbb{K}$, $x_0 \in W$ such that $\alpha(b) = \lambda_0 e + x_0 + \gamma_0 b$.
The fact that $(\mathbb{K}e)^\perp$ is ideal implies that

    (1) there exist a bilinear map $[\ ,\ ]_V : V \times V \to V$ and a bilinear form $\varphi : V \times V \to \mathbb{K}$ such that

$$(7.5) \qquad\qquad [x, y] = \varphi(x, y)e + [x, y]_V, \quad \forall x, y \in V.$$

    (2) there exist a linear map $\delta : V \to V$ and a linear form $h : V \to \mathbb{K}$ such that

$$(7.6) \qquad\qquad [b, y] = \delta(x) + h(x)e, \quad \forall x \in V.$$

Let $x, y, z \in V$, we have $[\alpha(x), [y, z]] = [\alpha(x), [y, z]_V]$ because $e \in \mathcal{Z}(\mathfrak{g})$. Then

$$[\alpha(x), [y, z]] = [\alpha_V(x), [y, z]_V] = [\alpha_V(x), [y, z]_V]_V + \varphi(\alpha_V(x), [y, z]_V)e.$$

Therefore the Hom-Jacobi identity $\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0$ implies that

$$(7.7) \qquad\qquad \circlearrowleft_{x,y,z} [\alpha_V(x), [y, z]_V]_V = 0,$$

and

$$(7.8) \qquad\qquad \circlearrowleft_{x,y,z} \varphi(\alpha_V(x), [y, z]_V) = 0.$$

Also the skewsymmetry $[\ ,\ ]$ is equivalent for $x, y \in V$ to

$$(7.9) \qquad\qquad\qquad [x, y]_V = -[y, x]_V,$$
$$(7.10) \qquad\qquad\qquad \varphi(x, y) = -\varphi(y, x).$$

Thus (7.7) and (7.9) show that $(V, [\ ,\ ]_V, \alpha_V)$ is a Hom-Lie algebra.

Let $x, y \in \mathcal{Z}$

$$
\begin{aligned}
0 &= [\alpha(b), [x, y]] + [\alpha(x), [y, b]] + [\alpha(y), [b, x]] \\
&= [x_0, [x, y]] + \lambda[b, [x, y]] + [\alpha_V(x), [y, b]] + [\alpha_V(y), [b, x]] \\
&= [x_0, [x, y]_V] + \lambda[b, [x, y]_V] - [\alpha_V(x), \delta(y)] + [\alpha_V(y), \delta(x)] \\
&= [x_0, [x, y]_V]_V + \varphi(x_0, [x, y]_V)e + \lambda\delta([x, y]_V) + \lambda h([x, y]_V)e \\
&\quad + [\alpha_V(x), \delta(y)]_V + \varphi(\alpha_V(x), \delta(y))e + [\alpha_V(y), \delta(x)]_V + \varphi(\alpha_V(y), \delta(x))e.
\end{aligned}
$$

Therefore

(7.11) $\qquad \varphi(x_0, [x, y]_V) + \lambda h([x, y]_V) - \varphi(\alpha_V(x), \delta(y)) + \varphi(\alpha_V(y), \delta(x)) = 0,$

(7.12) $\qquad [x_0, [x, y]_V]_V + \lambda\delta([x, y]_V) - [\alpha_V(x), \delta(y)]_V + [\alpha_V(y), \delta(x)]_V = 0.$

Let $x, y \in V$

$$
\varphi(x, y) = B([x, y], b) = B(b, [x, y]) = B([b, x], y) = B(\delta(x), y),
$$

because $B(e, y) = 0$. Then $[x, y] = [x, y]_V + B(\delta(x), y)e$, that is (7.5).

Let $x \in V$,

$$
B([b, x], b) = -B(x, [b, b]) = 0.
$$

In the other hand $B([b, x], b) = h(x)$ because $B(V, b) = \{0\}$ and $B(e, b) = 1$. Then $h(x) = 0$. We conclude that $[B, x] = \delta(x) \in V$ i.e. (7.6).

Let $y \in V$, $B(\alpha(b), y) = B(b, \alpha(y))$, which is equivalent to $B(x_0, y) = B(b, \alpha_V(y) + f(y)e) = f(y)$. Then $f(y) = B(x_0, y)$. It follows $\alpha(y) = \alpha_V(y) + B(x_0, y)e$.

Also $B(\alpha(b), e) = B(b, \alpha(e))$ is equivalent to $\gamma_0 = B(b, \lambda e) = \lambda$. Then

$$
\alpha(b) = \lambda_0 e + x_0 + \lambda b = \lambda b + x_0 + \lambda_0 e.
$$

Now (7.11) is equivalent to

(7.13) $\qquad B(\delta(x_0), [x, y]_V) - B(\delta(\alpha_V(x)), \delta(y)) + B(\delta(\alpha_V(y)), \delta(x)) = 0,$

because $h = 0$ and $\varphi(x, y) = B(\delta(x), y), \forall x, y \in V$.

Let $x, y, z \in V$

$$
\begin{aligned}
B([x, y], z) &= B([x, y]_V, z) = B_V([x, y]_V, z), \\
B(x, [y, z]) &= B(x, [y, z]_V) = B_V(x, [y, z]_V).
\end{aligned}
$$

Then B_V is invariant.

In addition $B(\alpha(x), y) = B(x, \alpha(y))$ implies $B_V(\alpha_V(x), y) = B_V(x, \alpha_V(y))$, that is α_V is B_V-symmetric.

It is obvious that B_V is symmetric.

We have

$$
\begin{aligned}
\alpha([x, y]) &= \alpha([x, y]_V + B(\delta(x), y)e) \\
&= \alpha_V([x, y]_V) + B(x_0, [x, y]_V)e + \lambda B(\delta(x), y)e \\
&= \alpha_V([x, y]_V) + B_V(x_0, [x, y]_V)e + B(\lambda\delta(x), y)e.
\end{aligned}
$$

In the other hand

$$
[\alpha(x), \alpha(y)] = [\alpha_V(x), \alpha_V(y)]_V + B(\delta(\alpha_V(x)), \alpha_V(y))e.
$$

Then

(7.14) $\qquad\qquad\qquad \alpha_V([x, y]_V) = [\alpha_V(x), \alpha_V(y)].$

Also

$$
B_V(x_0, [x, y]_V) + B(\lambda\delta(x), y) = B(\delta(\alpha_V(x), \alpha_V(y))
$$

which is equivalent to

$$
B_V([x_0, x]_V, y) + B(\lambda\delta(x), y) = B(\alpha_V \circ \delta \circ \alpha_V(x), y),
$$

and to

(7.15) $$B_V([x_0, x]_V + \lambda\delta(x) - \alpha_V \circ \delta \circ \alpha_V(x), y) = 0.$$

Since B is nondegenerate, then (7.15) leads to

$$[x_0, x]_V + \lambda\delta(x) - \alpha_V \circ \delta \circ \alpha_V(x) = 0.$$

By setting $\rho_V(x_0) = [x_0, \cdot]_V$, the previous identity may be written

(7.16) $$\alpha_V \circ \delta \circ \alpha_V - \lambda\delta(x) = \rho_V(x_0).$$

Then we have proved that $(V, [\ ,\ ]_V, \alpha_V, B_V)$ is quadratic Hom-Lie algebra. Also there exists $\delta \in End(V)$ such that the condition 7.16 is satisfied. Hence

$$[x, y] = [x, y]_V + B_V(\delta(x), y) \quad \forall x, y \in V$$
$$[b, x] = \delta(x) \quad \forall x \in V$$

and the linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\alpha(x) = \alpha_V(x) + B_V(x_0, x)e \quad \forall x \in V$$
$$\alpha(e) = \lambda e$$
$$\alpha(b) = \lambda b + x_0 + \lambda_0 e.$$

Let $x, y \in V$, we have $B([b, x], y) = -B(x, [b, y])$ which is equivalent to $B(\delta(x), y) = -B(x, \delta(y))$, then to $B_V(\delta(x), y) = -B(x, \delta(y))$. It means that δ is skewsymmetric with respect to B.

The condition (7.13) is equivalent to $B_V(\delta(x_0), [x, y]_V) - B - V(\delta \circ \alpha_V(x), \delta(y)) + B_V(\delta \circ \alpha_V(y), \delta(x)) = 0$ then $B_V([\delta(x_0), x], y) + B(\delta^2 \circ \alpha_V(x), y) - B_V(y, \alpha_V \circ \delta^2(x)) = 0$. It is equivalent to $[\delta(x_0), x] + (\delta^2 \circ \alpha_V - \alpha_V \circ \delta^2)(x) = 0$, thus to

(7.17) $$\delta^2 \circ \alpha_V - \alpha_V \circ \delta^2 = -[\delta(x_0), \cdot].$$

Recall that we have denoted $[\delta(x_0), \cdot]$ by $\rho(\delta(x_0))$.

Let $x, y \in V$. We have

$$\circlearrowleft_{b,x,y} [\alpha(b), [x, y]] = [\lambda b + x_0, [x, y]_V] - [\alpha_V(x), \delta(y)] + [\alpha_V(y), \delta(x)]$$
$$= \lambda\delta([x, y]_V) + [x_0, [x, y]_V]_V + B(\delta(x_0), [x, y]_V)e - [\alpha_V(x), \delta(y)]_V$$
$$- B(\delta(\alpha_V(x)), \delta(y))e + [\alpha_V(y), \delta(x)]_V + B(\delta(\alpha_V(y)), \delta(x))e = 0$$

which leads to

$$\lambda\delta([x, y]_V) + [x_0, [x, y]_V] - [\alpha_V(x), \delta(y)]_V + [\alpha_V(y), \delta(x)]_V = 0$$

and

$$B([\delta(x_0), x] + \delta^2 \circ \alpha_V(x) - \alpha_V \circ \delta^2(x), y) = 0.$$

These two identities are equivalent to

$$\lambda\delta([x, y]_V) = [\delta(x), \alpha_V(y)]_V + [\alpha_V(x), \delta(y)]_V - [x_0, [x, y]_V]$$

which is equivalent to (7.13) and

$$\delta^2 \circ \alpha_V - \alpha_V \circ \delta^2 = [\delta(x_0), \cdot],$$

which is equivalent to (7.17). The identity (**??**) may be written

(7.18) $$(\lambda\delta + [x_0, \cdot]_V)([x, y]_V) = [\delta(x), \alpha_V(y)]_V + [\alpha_V(x), \delta(y)]_V.$$

$\circlearrowleft_{b,b,x} [\alpha(b), [b, x]] = 0$ is satisfied.

We have proved that $(V, [\ ,\ ]_V, \alpha_V, B_V)$ is quadratic multiplicative Hom-Lie algebra. Also it is clear that using identities (7.16),(7.17),(7.18) the Hom-Lie algebra $\mathfrak{g}$ is a double extension of this Hom-Lie algebra by means of $(\delta, x_0, \lambda, \lambda_0)$. $\square$

Theorem 7.3 and the Lemma 3.9 lead to

Proposition 7.4. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ *be a RQH-Lie algebra. Then it is obtained from a centerless IQH-Lie algebra by a finite sequence of regular double extensions by the one dimensional Lie algebra or/and orthogonal direct sum of RQH-Lie algebras.*

Corollary 7.5. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ *be an IQH-Lie algebra. Then it is obtained from centerless IQH-Lie algebras by a finite sequence of involutive double extensions by the one dimensional Lie algebra or/and orthogonal direct sum of IQH-Lie algebras.*

Remark 7.6. The Proposition 7.4 reduces the study of IQH-Lie algebras to centerless IQH-Lie algebras. Hence, the problem of Quadratic Hom-Lie algebras reduces to study two classes

(1) Quadratic Hom-Lie algebras $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ with α nilpotent,
(2) centerless IQH-Lie algebras.

We study in the following the double extension of IQH-algebra by an involutive Hom-Lie algebra of any dimension.

Theorem 7.7 (Involutive double extension Theorem). *Let* $(V, [\ ,\ ]_V, \alpha_V, B_V)$ *be an IQH-Lie algebra and* $(A, [\ ,\ ]_A, \alpha_A)$ *be an involutive Hom-Lie algebra.*
Let $\phi : \begin{array}{c} A \to End(V) \\ a \to \phi(a) \end{array}$ *be a representation of* A *on* (V, α_A) *such that*

$$(7.19) \qquad \alpha_V \circ \phi(a)([x,y]_V) = [\phi(a) \circ \alpha_V(x), y]_V + [x, \phi(a) \circ \alpha_V(y)]_V \quad \forall x, y \in V,$$

$$(7.20) \qquad \phi(a) \circ \alpha_A(a) = \alpha_V \circ \phi(a) \circ \alpha_V \quad \cdot \forall a \in A,$$

$$(7.21) \qquad B_V(\phi(a)(x), y) = -B_V(x, \phi(a)(y) \quad \forall x, y \in V \quad \forall a \in A.$$

Let $\psi : \begin{array}{c} V \times V \to A^* \\ (x, y) \to \psi(x, y) \end{array}$ *defined by* $\forall x, y \in V \ \forall a \in A$ *by* $\psi(x, y)(a) = B_V(\phi(a)(x), y)$.
Then the vector space $\mathfrak{g} := A \oplus V \oplus A^*$ *endowed with the bracket* $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *defined for* $f, f' \in A^*$, $a, a' in A$ *and* $v, v' in V$ *by*

$$[f + v + a, f' + v' + a'] = \widetilde{\mathrm{ad}}_A(a)(f') - \widetilde{\mathrm{ad}}_A(a')(f) + \psi(v, v') + [v, v']_V + [a, a']_A$$

and the linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ *defined by*

$$\alpha(f + v + a) = {}^t \alpha_A(a) + \alpha_V(f) + \alpha_A(a)$$

is an involutive Hom-Lie algebra.
In addition, the bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ *defined by*

$$B_\gamma(f + v + a, f' + v' + a') = B_V(a, a') + f(w') + f'(w) + \gamma(a, a')$$

where γ *is an invariant symmetric bilinear form on* $(A, [\ ,\ ], \alpha_A)$ *such that* $\gamma(\alpha(a), a') = \gamma(a, \alpha(a'))$, $\forall a, a' \in A$, *is a quadratic structure on* $\mathfrak{g}, [\ ,\ ], \alpha)$.
The IQH-Lie algebra $\mathfrak{g}, [\ ,\ ], \alpha, B_\gamma)$ *is called the involutive double extension of* $(V, [\ ,\ ]_V, \alpha_V, B_V)$ *by* $(A, [\ ,\ ]_A, \alpha_A)$ *by means of* (ϕ, γ).

Proof. The proof is similar to the proof in [6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We pursue the characterization of IQH-Lie algebras.

Theorem 7.8. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ *be an irreducible IQH-Lie algebra such that* $\mathcal{Z}(\mathfrak{g}) = \{0\}$. *Then it is either an involutive simple Hom-Lie algebra or it is obtained by an involutive double extension of an IQH-Lie algebra* $(V, [\ ,\ ]_V, \alpha_V, B_V)$ *by an involutive simple Hom-Lie algebra.*

Proof. Let $(\mathfrak{g}, [\ ,\ ], \alpha, Q)$ be an irreducible involutive Hom-Lie algebra such that $\mathfrak{g}$ is not simple, $dim \mathfrak{g} > 1$ and $\mathcal{Z}(\mathfrak{g}) = \{0\}$. Then $\mathfrak{g} \neq \mathfrak{R}(\mathfrak{g})$ and $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n \oplus \mathfrak{R}(\mathfrak{g})$ where $\mathfrak{s}_i$ are simple Hom-subalgebras (Corollary 6.5), $J = \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_n \oplus \mathfrak{R}(\mathfrak{g})$ is a maximal ideal of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s}_1 \oplus J$. Set $A = \mathfrak{s}_1$ which is a subalgebra of $(\mathfrak{g}, [\ ,\ ], \alpha)$. The fact that J is a maximal ideal implies that $I := J^\perp$ is a minimal ideal of $\mathfrak{g}$. Also $I \bigcap I^\perp$ is an ideal of $\mathfrak{g}$ contained in I. Then $I \bigcap I^\perp = \{0\}$ or $I \bigcap I^\perp = I$, i.e. $I \subset I^\perp = J$. Since $\mathfrak{g}$ is irreducible then it implies that $I \subset J$, consequently $B(I, I) = \{0\}$, i.e. I is isotropy.

It is clear that $B_{|(I\oplus A)\times(I\oplus A)}$ is nondegenerate. It follows that $\mathfrak{g} = (I\oplus A)\oplus(I\oplus A)^{\perp}$. We set $V := (I\oplus A)^{\perp}$, which is a vector subspace of $\mathfrak{g}$. Consequently, $J = I\oplus V$ and $B_{V\times V}$ is nondegenerate. Since $[I,I^{\perp}] = \{0\}$, then $[I,I] = \{0\}$ because $I\subset J = I^{\perp}$ and $[I,V] = \{0\}$.

Let $x,y\in V$, then $[x,y] = [x,y]_V + \varphi(x,y)$, where $[\ ,\ ]_V : V\times V\to V$ and $\varphi : V\times V\to I$ are bilinear maps.

Let $x\in V$ and $a\in A$, then $[a,x] = \phi(a)(x) + \varphi'(a,x)$ where $\varphi' : A\times V\to I$ is a bilinear map and $\phi(a)\in End(V)$.

We denote, for $a,a'\in A$, the bracket $[a,a']$ by $[a,a']_A$.

Let $x,y\in V$ and $a\in A$, we have $B([x,y],a) = B(\varphi(x,y),a)$ and $B(x,[y,a]) = -B(x,[a,y]) = -B(x,\phi(a)y)$, then

$$B(\varphi(x,y),a) = -B(x,\phi(a)y).$$

Also $B([a,x],y) = -B(x,[a,y])$ is equivalent to

$$B(\phi(a)x,y) = -B(x,\phi(a)y).$$

Therefore we have

$$B(\varphi(x,y),a) = -B(x,\phi(a)y) = B(\phi(a)x,y).$$

For $x\in V$ we have $\alpha(x)\in J = I^{\perp}$ which implies $\alpha(x) = \alpha_V(x) + \gamma(x)$ where $\alpha_V(x)\in V$ and $\gamma(x)\in I$. Let $x\in V$ and $a\in A$, then $B(\alpha(x),a) = B(x,\alpha(a))$ implies $B(\gamma(x),a) = B(x,\alpha(a)) = 0$ since $B(V,A) = \{0\}$. Therefore $\gamma(x) = 0$. Thus $\alpha(V)\subseteq V$ and we denote in the sequel $\alpha_{|V}$ by α_V.

Let $x\in V$ and $a,b\in A$, then $B([a,x],b) = B(\varphi'(a,x),b)$ and on the other hand $B([a,x],b) = -B(x,[a,b]) = 0$ (because $B(V,A) = \{0\}$). Therefore $B(\varphi'(a,x),b) = 0$ for any $b\in A$. Then $\varphi'(a,x) = 0$. Thus

$$[a,x] = \phi(a)x, \quad \forall a\in A, \forall x\in V.$$

Let $x,y\in V$, then $[x,y] = -[y,x]$ implies

$$[x,y]_V = -[y,x]_V,$$
$$\varphi(x,y) = -\varphi(y,x).$$

We show that $[\ ,\ ]_V$ and φ satisfy the Hom-Jacobi condition. Let $x,y,z\in V$, we have

$$[\alpha(x),[y,z]] = [\alpha(x),[y,z]_V] \quad \text{because } [V,I] = \{0\}$$
$$= [\alpha(x),[y,z]_V]_V + \varphi(\alpha(x),[y,z]_V)$$
$$= [\alpha_V(x),[y,z]_V]_V + \varphi(\alpha_V(x),[y,z]_V).$$

Then

$$0 = \circlearrowleft_{x,y,z}[\alpha(x),[y,z]] = \circlearrowleft_{x,y,z}[\alpha_V(x),[y,z]_V]_V + \circlearrowleft_{x,y,z}\varphi(\alpha_V(x),[y,z]_V)$$

Since the first summation is in V and the second in I then we obtain

(7.22) $$\circlearrowleft_{x,y,z}[\alpha_V(x),[y,z]_V]_V = 0,$$

(7.23) $$\circlearrowleft_{x,y,z}\varphi(\alpha_V(x),[y,z]_V) = 0.$$

Also $\alpha([x,y]) = [\alpha(x),\alpha(y)] = [\alpha_V(x),\alpha_V(y)]$ implies $\alpha_V([x,y]_V) + \alpha(\varphi(x,y)) = [\alpha_V(x),\alpha_V(y)]_V + \varphi(\alpha_V(x),\alpha_V(y))$. Therefore

$$\alpha_V([x,y]_V) = [\alpha_V(x),\alpha_V(y)]_V$$
$$\alpha(\varphi(x,y)) = \varphi(\alpha_V(x),\alpha_V(y)).$$

It is obvious that $\alpha^2 = id$ induces $\alpha_V^2 = id$.

The previous calculations leads to the fact that $(V,[\ ,\ ]_V,\alpha_V)$ is an involutive Hom-Lie algebra.

The Hom-Lie algebra $(V,[\ ,\ ]_V,\alpha_V)$ is an IQH-Lie algebra by setting $B_V = B_{|V\times V}$. Indeed for any $x,y,z\in V$ we have

$$B([x,y]_V,z) = B([x,y],z) = B(x,[y,z]) = B(x,[y,z]_V)$$

and

$$B(\alpha_V(x),y) = B(\alpha(x),y) = B(x,\alpha(y)) = B(x,\alpha_V(y)).$$

Using the identity (7.23) we have for any $a \in A$ and $x, y, z \in V$

$$B(\circlearrowleft_{x,y,z} \varphi(\alpha_V(x), [y,z]_V), a) = 0$$
$$\Longleftrightarrow \circlearrowleft_{x,y,z} B(\varphi(\alpha_V(x), [y,z]_V), a) = 0$$
$$\Longleftrightarrow \circlearrowleft_{x,y,z} B(\alpha_V(x), \phi(a)([y,z]_V)) = 0$$

then we have

(7.24) $$\circlearrowleft_{x,y,z} B(\phi(a)(\alpha_V(x)), [y,z]_V) = 0$$

Expanding the identity (7.24) and using the fact that $(V, [\ ,\ ]_V, \alpha_V, B_V)$ is an IQH-Lie algebra we obtain

$$B_V(\phi(a)(\alpha_V(x)), [y,z]_V) + B_V(\phi(a)(\alpha_V(y)), [z,x]_V) + B_V(\phi(a)(\alpha_V(z)), [x,y]_V) = 0$$

which is equivalent to

$$B_V([\phi(a) \circ \alpha_V(x), y]_V + [x, \phi(a) \circ \alpha_V(y)] - \alpha_V \circ \phi(a)[x,y]_V, z) = 0.$$

Then we have

(7.25) $$\alpha_V \circ \phi(a)[x,y]_V = [\phi(a) \circ \alpha_V(x), y]_V + [x, \phi(a) \circ \alpha_V(y)]_V.$$

Let $a \in A$ and $x, y \in V$, the Hom-Jacobi identity

$$[\alpha(a), [x,y]] + [\alpha(x), [y,a]] + [\alpha(y), [a,x]] = 0$$

may be written

$$[\alpha(a), [x,y]_V] + [\alpha(a), \varphi(x,y)] - [\alpha(x), \phi(a)y]_V - \varphi(\alpha(x), \phi(a)y) + [\alpha(y), \phi(a)x]_V + \varphi(\alpha(y), \phi(a)x) = 0.$$

It's equivalent to

$$\phi(\alpha(a))([x,y]_V) + [\alpha(a), \varphi(x,y)] - [\alpha(x), \phi(a)y]_V - \varphi(\alpha(x), \phi(a)y) + [\alpha(y), \phi(a)x]_V + \varphi(\alpha(y), \phi(a)x) = 0.$$

By gathering the elements in V and elements in I, the previous identity leads to

(7.26) $$\phi(\alpha(a))([x,y]_V) = [\alpha(x), \phi(a)y]_V + [\phi(a)x, \alpha(y)]_V,$$

(7.27) $$[\alpha(a), \varphi(x,y)] = \varphi(\alpha(x), \phi(a)y) + \varphi(\phi(a)x, \alpha(y)).$$

The identity (7.26) leads for any $b \in A$ to

$$B(\phi(\alpha(a))([x,y]_V), b) = B([\alpha(x), \phi(a)y]_V, b) + B([\phi(a)x, \alpha(y)]_V, b)$$
$$\Leftrightarrow B(\varphi(x,y), [b, \alpha(a)]) = B(\phi(b)(\alpha(x)), \phi(a)y) + B(\phi(b)\phi(a)(x), \alpha(y))$$
$$\Leftrightarrow B(\phi([b, \alpha(a)])(x), y) = -B(\phi(a)\phi(b)(\alpha(x)), y) + B(\alpha(\phi(b)\phi(a)(x)), y).$$

Hence

$$\phi([b, \alpha(a)])(x) = -\phi(a)\phi(b)\alpha + \alpha(\phi(b)\phi(a),$$

which may be written

(7.28) $$\phi([\alpha(a), b]) = \phi(a)\phi(b)\alpha - \alpha\phi(b)\phi(a).$$

Let $a, b \in A$ and $x \in V$, the Hom-Jacobi identity

$$[\alpha(a), [b,x]] + [\alpha(b), [x,a]] + [\alpha(x), [a,b]] = 0$$

may be written

$$\phi(\alpha(a))\phi(b)(x) - \phi(\alpha(b))\phi(a)(x) - \phi([a,b])(\alpha(x)) = 0.$$

Therefore we have

(7.29) $$\phi([a,b]) \circ \alpha = \phi(\alpha(a))\phi(b) - \phi(\alpha(b))\phi(a).$$

In fact the identities (7.28) and (7.29) are equivalent.
Let $a \in A$ and $x \in V$, the identity $\alpha([a,x]) = [\alpha(a), \alpha(x)]$ may be written $\alpha(\phi(a)(x)) = \phi(\alpha(a))(\alpha(x))$ and also $\alpha_V(\phi(a)(x)) = \phi(\alpha_A(a))(\alpha_V(x))$ where $\alpha_A = \alpha_{|A}$. Then we have

$$\phi \circ \alpha_A = \alpha_V \phi(a)\alpha_V^{-1} = \alpha_V \phi(a)\alpha_V.$$

Now, we evaluate $\phi(\alpha(a))[x, y]_V$ for $a \in A$ and $x, y \in V$. We have $\phi(\alpha(a))[x, y]_V = \alpha_V \circ (\alpha_V \circ \phi(\alpha(a))[x, y]_V$ since α_V is a involution. By applying the identity (7.25) we obtain

$$\begin{aligned} \phi(\alpha(a))[x, y]_V \quad &= \alpha_V ([\phi(\alpha(a)) \circ \alpha_V(x), y]_V + [x, \phi(\alpha(a)) \circ \alpha_V(y)]_V) \\ &= [\alpha_V \circ \phi(\alpha(a)) \circ \alpha_V(x), \alpha_V(y)]_V + [\alpha_V(x), \alpha_V \circ \phi(\alpha(a)) \circ \alpha_V(y)]_V \\ &= [\alpha(x), \phi(a)y]_V + [\phi(a)x, \alpha(y)]_V \end{aligned}$$

Therefore we proved that the identity (7.25) implies (7.26). In an other hand we have

$$\begin{aligned} \alpha_V \phi(a)[x, y]_V \quad &= \alpha_V (\alpha_V \phi(\alpha(a)) \alpha_V)[x, y]_V \\ &= \phi(\alpha(a)) \alpha_V[x, y]_V = \phi(\alpha(a))[\alpha_V(x), \alpha_V(y)]_V \\ &= [x, \phi(a)(\alpha_V(y))]_V + [\phi(a)(\alpha_V(x)), y]_V. \end{aligned}$$

Therefore we proved the converse. Hence the identities (7.25) and (7.26) are equivalent.
We set for $a \in A$ and $i \in I$, $[a, i] := a \cdot i$. The Hom-Jacobi condition $[\alpha(a), [b, i]] + [\alpha(b), [i, a]] + [\alpha(i), [a, b]] = 0$ leads to

$$(7.30) \qquad\qquad \alpha(a) \cdot (b \cdot i) - \alpha(b) \cdot (a \cdot i) = [a, b] \cdot (\alpha(i))$$

which is equivalent to (7.27).

Denote $l: \begin{array}{c} A \to End(I) \\ a \mapsto l(a) \end{array}$ where $l(a): \begin{array}{c} I \to I \\ i \mapsto l(a)(i) = [a, i] = a \cdot i \end{array}$.

The identity (7.30) may be written for $a, b \in A$

$$(7.31) \qquad\qquad l([a, b]) \circ \alpha_I = l(\alpha_A(a))l(b) - l(\alpha_A(b))l(a).$$

Notice that

$$(7.32) \qquad\qquad l \circ \alpha_A(a) = \alpha_I \circ l(a) \circ \alpha_I$$

since $l(\alpha_A(a))(i) = [\alpha_A(a), i] = \alpha_I[a, \alpha_I(i)]$.
Also we have for $i \in I$ and $a, b, c \in A$

$$B([a, b], i) = B(a, [b, i]) = B(a, l(b)(i))$$

and

$$\begin{aligned} B(c, [\alpha(a), [b, i]] + [\alpha(b), [i, a]] + [\alpha(i), [a, b]] &= B([b, [\alpha(a), c]] - [a, [\alpha(b), c]] + \alpha([[a, b], c]), i) \\ &= B([b, [\alpha(a), c]] + [a, [c, \alpha(b)]] + \alpha([c, [b, a]]), i) \\ &= B(\alpha([\alpha(b), [a, \alpha(c)]] + [\alpha(a), [\alpha(c), b]] + [c, [b, a]]), i) \\ &= 0 \end{aligned}$$

since $c = \alpha(\alpha(c))$.

It is clear that $\nabla: \begin{array}{c} I \to A^* \\ i \mapsto \nabla(i) = B(i, \cdot) \end{array}$ is an isomorphism of vector spaces.

Denote $\alpha_A^*: \begin{array}{c} A^* \to A^* \\ f \mapsto \alpha_A^*(f) = f \circ \alpha_A \end{array}$ i.e. $\alpha_A^* = {}^t \alpha_A$.

Let $i \in I$ and $a \in A$, we have $\nabla(\alpha_I(i)(a) = B(\alpha_I(i), a) = B(i, \alpha_A(a)) = \nabla(i) \circ \alpha_A(a)$. Then

$$(7.33) \qquad\qquad \alpha^*(\nabla(i)) = \nabla(i) \circ \alpha_A = \nabla(\alpha_I(i).$$

Also $B(a \cdot i, b) = B([a, i], b) = -B(i, [a, b])$ that is

$$(7.34) \qquad\qquad \nabla(a \cdot i)(b) = -\nabla(i)([a, b]).$$

Thus ∇ is an isomorphism of the A-modules I and A^*.
Recall that if $(\mathfrak{g}, [\ ,\ ], \alpha)$ is a Hom-Lie algebra then $(\mathfrak{g}^*, \widetilde{ad}, \widetilde{\alpha})$ is a representation if and only if for $x, y, z \in \mathfrak{g}$

$$(7.35) \qquad\qquad \alpha([[x, y], z]) = [y, [\alpha(x), z]] + [x, [z, \alpha(y)]].$$

In our case α is an involution which leads to α_A is also an involution. Then the identity (7.35) is satisfied which implies that $(A^*, \widetilde{ad_A}, \alpha_A^* =^t \alpha_A)$ is a representation of $(A, [\ ,\ ]_A, \alpha_A)$ (i.e. A^* is an A-module).

$$\phi : \begin{array}{c} V \times V \to I \to A^* \\ (x,y) \mapsto \varphi(x,y) \mapsto B(\varphi(x,y), \cdot) \end{array}$$

Consider $\Gamma : \mathfrak{g} = A \oplus V \oplus I \to A \oplus V \oplus A^\star$ defined by $\Gamma(a + v + i) = a + v + \nabla(i)$. It is easy to show that it is a Hom-Lie algebras morphism. Notice that $A \oplus V \oplus A^\star$ is a double extension of V by A. $\qquad\square$

Let us denote by $\mathcal{U}$ the set incorporating the trivial Hom-Lie algebra $\{0\}$, the one-dimensional Hom-Lie algebra and simple involutive Hom-Lie algebras.

Theorem 7.9. *Let $(\mathfrak{g}, [\ ,\ ], \alpha, B)$ be an IQH-Lie algebra such that $\mathcal{Z}(\mathfrak{g}) = \{0\}$. Then it is either an element of $\mathfrak{g}$ or it is obtained from a finite number of element of $\mathcal{U}$ by a sequence of involutive double extension by the one-dimensional Hom-Lie algebra and/or double extension by a simple involutive Hom-Lie algebra and/or orthogonal direct sum of IQH-Lie algebras.*

According to Theorem 6.2 and Proposition 6.3 simple involutive Hom-Lie algebras are determined by simple or semisimple Lie algebras with involution.

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