On sets of directions and angles determined by subsets of $\mathbb{R}^d$

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ABSTRACT. Given $E \subset \mathbb{R}^d$, $d \geq 2$, define $\mathcal{D}(E) \equiv \left\{ \frac{x-y}{|x-y|} : x, y \in E \right\} \subset S^{d-1}$, the set of directions determined by E. We prove that if the Hausdorff dimension of E is greater than $d-1$, then $\sigma(\mathcal{D}(E)) > 0$, where σ denotes the surface measure on S^{d-1}. This result is sharp since the conclusion fails to hold if E is a $(d-1)$-dimensional hyper-plane. This result can be viewed as a continuous analog of a recent result of Pach, Pinchasi, and Sharir ([24], [25]) on angles determined by finite subsets of $\mathbb{R}^d$. We also discuss the case when the Hausdorff dimension of E is precisely $d-1$, where some interesting counter-examples were previously obtained by Simon and Solomyak ([26]) in the planar case. Also define $\mathcal{A}(E) = \{\theta(x,y,z) : x, y, z \in E\}$, where $\theta(x,y,z)$ is the angle between $x-y$ and $y-z$. We use the techniques developed to handle the problem of directions and results on distance sets previously obtained by Wolff and Erdogan to prove that if the Hasudorff dimension of E is greater than $\frac{d-1}{2} + \frac{1}{3}$, then the Lebesgue measure of $\mathcal{A}(E)$ is positive. This result can be viewed as a continuous analog of a recent result of Apfelbaum and Sharir ([1]).

At the end of this paper we show that our continuous results can be used to recover and in some case improve the exponents for the corresponding results in the discrete setting for large classes of finite point sets. In particular, we prove that a finite point set $P \subset \mathbb{R}^d$, $d \geq 3$, satisfying a certain discrete energy condition (Definition 4.1), determines $\gtrapprox \#P$ distinct directions and $\gtrapprox (\#P)^{\frac{6}{3d-1}}$ distinct angles. In two dimensions, the lower bound on the number of angles is $\gtrapprox \#P$.

1. Introduction

A large class of Erdős type problem in geometric combinatorics asks whether a large set of points in Euclidean space determines a suitably large sets of geometric relations or objects. For example, the classical Erdős distance problem asks whether N points in $\mathbb{R}^d$, $d \geq 2$, determines $\gtrapprox N^{\frac{2}{d}}$ distinct distances, where here, and throughout, $X \lessapprox Y$, with the controlling parameter N means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon N^\epsilon Y$. See, for example [2], [18], [21], [23], [27] and the references contained therein for a thorough description of these types of problems and recent results.

Continuous variants of Erdős type geometric problems have also received much attention in recent decades. Perhaps the best known of these is the Falconer distance problem, which asks whether the Lebesgue measure of the distance set $\{|x-y| : x, y \in E\}$ is positive, provided that the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$. See [6] and [28] for the best currently known results on this problem. See also [4], [16], [19] and [20]. Also see [5] for the closely related problem on finite point configurations. For related problems under the assumption of positive Lebesgue density, see, for example, [3], [9] and [30].

In this paper we study the sets of directions and angles determined by subsets of the Euclidean space. In the discrete setting the problem of angles was studied in recent years by Pach, Pinchasi, and Sharir. See [24] and [25]. In the latter paper they prove that if P is a set of n points in $\mathbb{R}^3$, not all in a common plane, then the pairs of points of P determine at least $2n-5$ distinct directions if n is odd and at least $2n-7$ distinct directions if n is even. Our main result can be viewed as a continuous variant of this result where finite point sets are replaced by infinite sets of a given Hausdorff dimension. An explicit quantitative connection between our main result on directions (Theorem 1.5 below) and the work of Pach, Pinchasi, and Sharir is made in Section 4 below. We show that a finite set P, satisfying the $(d-1+\epsilon)$-adaptability assumption (see Definition 4.1 below), determines $\gtrapprox \#P$ distinct directions. In dimensions two and three, this result is weaker than then the result of Pach, Pinchasi, and Sharir described above. However, in dimensions four and higher, our result gives, to the best of our knowledge, the only known bounds.

In the finite field setting, the problem of directions was previously studied by the first listed author and Hannah Morgan. See [14] and the references contained therein.

The problem of angles in the discrete setting was studied in recent years by Apfelbaum and Sharir ([1]). They prove that in three dimensions, a single angle does not arise more than $O(n^{\frac{7}{3}})$ times among n points. This implies, by the pigeon-hole principle, that the set of n points in the three dimensions determines at least $\Omega(n^{\frac{2}{3}})$ distinct angles. In four dimensions, they prove that a single angle, different from $\frac{\pi}{2}$, does not arise more than $O(n^{\frac{5}{2}})$, up to the inverse Ackerman function, times among n points. Similarly, this implies that any set of n points in four dimensions determines at least $\Omega(n^{\frac{1}{2}})$ distinct directions. The explicit connection between our main result on angles (Theorem 1.6 below) and the work of Apfelbaum and Sharir is made in Section 4 below. We prove in the plane that a finite set P, satisfying the s-adaptability assumption, determines $\gtrapprox \#P$ distinct angles in two dimensions and $\gtrapprox (\#P)^{\frac{6}{3d-1}}$ distinct angles in dimension three and higher. In dimensions three and four this gives a slight improvement over the result of Apfelbaum and Sharir, albeit restricted to s-adaptable sets. In dimension five and higher, our result appears to be entirely new.

Definition 1.1. Given $E \subset \mathbb{R}^d$, $d \geq 2$, define

$$\mathcal{D}(E) = \left\{ \frac{x-y}{|x-y|} : x, y \in E \right\} \subset S^{d-1},$$

the set of directions determined by E.

Definition 1.2. Given $E \subset \mathbb{R}^d$, $d \geq 2$, define

$$\mathcal{A}(E) = \{\theta(x,y,z) : x,y,z \in E\},$$

where θ is the angle between $x-y$ and $y-z$, measured counter-clockwise.

Definition 1.3. Let $E \subset S^{d-1}$, $d \geq 3$. We say that γ_d is an *acceptable spherical Falconer exponent* if the Lebesgue measure of $\Delta(E) = \{|x-y| : x, y \in E\}$ is positive whenever the Hausdorff dimension of E is greater than γ_d.

Remark 1.4. The result due to Wolff in two dimensions ([28]) and Erdogan in higher dimensions ([6]) says that if the Hausdorff dimension of $E \subset \mathbb{R}^d$. $d \geq 2$, is greater than $\frac{d}{2} + \frac{1}{3}$, then the Lebesgue measure of the set of distances is positive. One can check that Erdogan's proof carries over to the case of the sphere. Namely, if the Hausdorff dimension of $E \subset S^{d-1}$, $d \geq 3$, is greater than $\frac{d-1}{2} + \frac{1}{3}$, then the Lebesgue measure of the set of distances is positive. In particular, any $s > \frac{d-1}{2} + \frac{1}{3}$ is an acceptable spherical Falconer exponent.

Our main results are the following.

Theorem 1.5. *Let $E \subset \mathbb{R}^d$, $d \geq 2$, of Hausdorff dimension greater than $d-1$. Then*

$$\sigma(\mathcal{D}(E)) > 0, \tag{1.1}$$

where σ denotes the Lebesgue measure on S^{d-1}.

Theorem 1.6. *Suppose that γ_d is an acceptable spherical Falconer exponent, and $E \subset \mathbb{R}^d$, with $d \geq 3$. Then if the Hausdorff dimension of E is greater than γ_d, the one-dimensional Lebesgue measure of $\mathcal{A}(E)$ is positive.*

In two dimensions, the assumption that the Hausdorff dimension of E is greater than one guarantees that the Lebesgue measure of $\mathcal{A}(E)$ is positive.

Remark 1.7. Given a set $E \subset \mathbb{R}^d$, and a point $y \in E$, let $\mathcal{A}_y(E)$ denote the pinned angle set, defined as $\{\theta(x, y, z) : x, y, z \in E\}$. The proof of Theorem 1.6 actually guarantees the existence of a point, $y_0 \in E$, for which the Lebesgue measure of $\mathcal{A}_{y_0}(E)$ is positive.

Remark 1.8. The approaches to similar problems in geometric measure theory (see e.g. [**7**], [**6**], [**28**]) typically involve constructing a measure on a set under consideration, (in this case– directions), and then proving, using Fourier transform methods, that this measure is in L^2 or in L^∞. While our approach is also Fourier based, we prove that the measure of a ball centered at a point in the direction set equals a quantity comparable to ϵ^{d-1} plus an error. We then show that this error is $o(\epsilon^{d-1})$ in L^1 norm provided that the Hausdorff dimension of the underlying set E is greater than $d-1$. This, combined with the Lebesgue differentiation theorem, allows us to conclude that the set of directions has a positive Lebesgue measure on the sphere. This approach is quite reminiscent of the techniques used to study geometric combinatorics problems in the finite field setting. See, for example, ([**12**]) and the references contained therein.

Remark 1.9. The proof below gives us a bit more. It shows that if the Hausdorff dimension of E is greater than k, $1 \leq k \leq d-1$, then there exists $S^k \subset S^{d-1}$ such that $\sigma_k(\mathcal{D}(E)) > 0$, where σ_k is the Lebesgue measure on S^k.

Remark 1.10. It is not difficult to check that if E is a $(d-1)$-dimensional Lipschitz surface in $\mathbb{R}^d$, which is not contained in a $(d-1)$-dimensional plane, then $\sigma(\mathcal{D}(E)) > 0$. It is reasonable to conjecture that the same conclusion holds if E is merely a $(d-1)$-dimensional rectifiable subset of $\mathbb{R}^d$. We discuss the purely non-recitifiable case in the Subsection 1.1 below.

Remark 1.11. It is interesting to contrast this result with the Besicovitch-Kakeya conjecture (see e.g. [**29**] and the references contained therein), which says that any subset of $\mathbb{R}^d$, containing a unit line segment in every direction has Hausdorff dimension d. On the other hand, Theorem 1.5 says that Hausdorff dimension greater than $d-1$ is sufficient for the set to contain endpoints of a segment of some length pointing in the direction of a positive proportion of vectors in S^{d-1}.

1.1. Sharpness of the main results:

1.1.1. *Directions:* Theorem 1.5 cannot be improved in the following sense. Suppose that E is contained in a $(d-1)$-dimensional hyper-plane. Then $\sigma(\mathcal{D}(E)) = 0$. It follows that the conclusion of Theorem 1.5 does not in general hold if the Hausdorff dimension of E is less than or equal to $d-1$.

Another very different sharpness example comes from the theory of distance sets. Let E_q denote the $q^{-\frac{d}{s}}$-neighborhood of

$$q^{-1}\left(\mathbb{Z}^d \cap [0,q]^d\right),$$

where $\mathbb{Z}^d$ denote the standard integer lattice and $0 < s < d$. It is known that if q_i is a sequence of integers given by $q_1 = 2$, $q_{i+1} > q_i^i$, then the Hausdorff dimension of

$$E = \cap_i E_{q_i}$$

is s. See, for example, [7], [8]. Observe that

$$\sigma(\mathcal{D}(E_q)) \approx q^{-\frac{d(d-1)}{s}} \cdot q^d$$

since the number of lattice points in $[0,q]^d$, $d \geq 2$, with relatively prime coordinates is equal to

$$\frac{q^d}{\zeta(d)}(1+o(1)),$$

where $\zeta(t)$ is the Riemann zeta function. See, for example, [17]. It follows that

$$\sigma(\mathcal{D}(E_q)) \to 0 \text{ as } q \to \infty$$

if $s < d-1$. It follows that $\sigma(\mathcal{D}(E)) = 0$. This example does not rule out $s = d-1$ and one might reasonably conjecture, consistent in spirit with the result due to Pach, Pinchasi, and Sharir stated above, that if the Hausdorff dimension of E is equal to $d-1$, then (1.1) holds if and only E is not a subset of a single $(d-1)$-dimensional hyper-plane. This, however, is not true. A result due to Simon and Solomyak ([26] shows that for every self-similar set of Hausdorff dimension one satisfying an additional mild condition, the Lebesgue measure of $\mathcal{D}(E)$ is zero. In particular, if E is the four-cornered Cantor set known as the Garnett set (see e.g. [10]), then the Hausdorff dimension of E is *one* and the Lebesgue measure of $\mathcal{D}(E)$ is zero. It is not difficult to use Simon and Solomyak's result to construct a set of Hausdorff dimension $d-1$ in $\mathbb{R}^d$ that is not contained in the plane and the Lebesgue measure of $\mathcal{D}(E)$ is zero.

In the realm of rectifiable sets, we believe that Theorem 1.5 can be strengthened as follows.

Conjecture 1.12. *Let $E \subset \mathbb{R}^d$, $d \geq 2$, of Hausdorff dimension $d-1$. Suppose that E is rectifiable. Then (1.1) holds.*

1.1.2. *Angles:* There are fewer known sharpness examples for problems involving angles in $\mathbb{R}^d$. Suppose that the set E is contained in a straight line. Clearly, points in such a set can only determine the angles 0 and $\pi/2$. Thus, it is tempting to conjecture that for any set, E, with Hausdorff dimension strictly greater than one, the Lebesgue measure of $\mathcal{A}(E)$ should be positive.

1.2. Structure of the paper. Theorem 1.5 is proved in Section 2 below. The modifications of that proof which yield Theorem 1.6 are described in Section 3. In Section 4 we describe an explicit connection between the main results of the paper and the discrete problems, such as those studied by Apfelbaum, Pach, Pinchasi, and Sharir.

2. Proof of Theorem 1.5

Let s be the Hausdorff dimension of E. Now, although the set $\mathcal{D}(E)$ is a subset of the $(d-1)$-dimensional sphere, in the arguments to follow, it is convenient to work with sets of slopes of line segments defined by pairs of points in the set E. If p and q are two points, with coordinates $(p_1, p_2, ..., p_d)$ and $(q_1, q_2, ..., q_d)$, then we define the slope of the line segment between p and q as the $(d-1)$-tuple

$$\left\{\frac{p_1 - q_1}{p_d - q_d}, \frac{p_2 - q_2}{p_d - q_d}, ..., \frac{p_{d-1} - q_{d-1}}{p_d - q_d}\right\}.$$

It is not difficult to see that if the $(d-1)$-dimesional Lebesgue measure of the set of slopes determined by E is positive, then (1.1) holds. With a slight abuse of notation, we will refer to the set of slopes as $\mathcal{D}(E)$ as well. It is convenient to extract two subsets from E, separated from each other in at least one of the coordinates. To this end, we have the following construction.

Lemma 2.1. *If μ is a Frostman probability measure on $E \subset \mathbb{R}^d$, with Hausdorff dimension greater than d$-$1, then there exist c_1, c_2 positive constants and E_1, E_2 subsets of E such that $\mu(E_j) \geq c_1 > 0$, for $j = 1, 2$ and*

$$\max_{1 \leq k \leq d}(\inf\{|x_k - y_k| : x \in E_1, y \in E_2\}) \geq c_2 > 0.$$

We will employ a stopping time argument. Define C_δ to be the constant in the Frostman condition,

$$\mu(B_r) \leq C_\delta r^{s-\delta}.$$

Let $[0,1]^d$ be the unit cube in $\mathbb{R}^d$, and subdivide it into 4^d smaller cubes of side-length $\frac{1}{4}$. Choose 2^d collections of subcubes such that no two cubes of the same collection touch each other. Then by the pigeon-hole principle, at least one of them has measure greater than or equal to $\frac{1}{2^d}$. If there are two cubes, Q_1, and Q_1', in the same collection, such that $\mu(Q_1), \mu(Q_1') \geq \frac{c}{2^d}$ for some $c > 0$, then we are done. If not, there exists a cube Q_1 of side-length $\frac{1}{4}$, so that $\mu(Q_1) \geq \frac{1}{2^d}$. Then we repeat the same procedure on the cube Q_1. Now, either we have two cubes, Q_2 and Q_2', with $\mu(Q_2)$, and $\mu(Q_2') \geq \frac{c}{2^{2d}}$, for some $c > 0$, which are in the same collection, or we do not. If we do not, then again, there must be a cube, Q_2, with side length $\frac{1}{4^2}$, so that $\mu(Q_2) \geq \frac{1}{2^{2d}}$. We can repeat this process, and at each stage check for two cubes, from the same collection, with the requisite measure. Let the integer n depend on C_δ. If we fail to find two such cubes at the n-th iteration, we obtain a cube Q_n of side-length $\frac{1}{4^n}$ for which $\mu(Q) \geq \frac{1}{2^{dn}}$. By the Frostman measure condition, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$\frac{1}{2^{dn}} \leq \mu(Q) \leq C_\delta \frac{1}{4^{n(s-\delta)}},$$

which is true if $n \leq \frac{\log_2(C_\delta)}{(2s-2\delta-d)}$. So, picking $n > \frac{\log_2(C_\delta)}{(2s-2\delta-d)}$, it is only true whenever $s < \frac{d}{2} + \delta$, and since $s > d-1 > \delta + \frac{d}{2}$, for sufficiently small δ, we have a contradiction.

Let x and y be points in $\mathbb{R}^d$ with coordinates $(x_1, x_2, ..., x_d)$ and $(y_1, y_2, ..., y_d)$. Apply Lemma 2.1 to E. Without loss of generality, let the sets E_1 and E_2 be separated in the d-th coordinate.

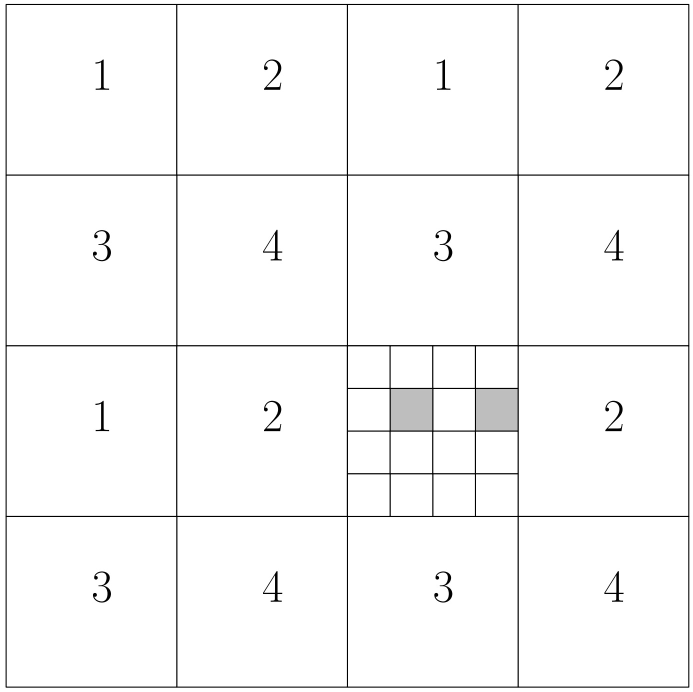

FIGURE 1. The first decomposition into four collections of four cubes each is shown with a 1 in the cubes of the first collection, a 2 in the cubes of the second collection, etc... In this case, the first decomposition was not enough, and a positive proportion of the mass was in the lower-right cube of the first collection. After the second iteration, there are two shaded boxes, representing E_1 and E_2.

Let μ_1 and μ_2 be restrictions of μ to the sets E_1 and E_2, respectively. Let $t = (t_1, t_2, \dots, t_{d-1})$. For slopes $t \in [\frac{1}{2}, 1]^{d-1}$, define $\nu_\epsilon(t)$ to be the quantity

$$\mu \times \mu \left\{ (x,y) \in E_1 \times E_2 : t_1 - \epsilon \leq \frac{x_1 - y_1}{x_d - y_d} \leq t_1 + \epsilon, ..., t_{d-1} - \epsilon \leq \frac{x_{d-1} - y_{d-1}}{x_d - y_d} \leq t_{d-1} + \epsilon \right\}.$$

Since $x_d - y_d$ is guaranteed to be more than c_2 by Lemma 2.1, we can multiply each inequality through by the denominator to get that

$$\begin{aligned} \nu_\epsilon(t) \approx \mu_1 \times \mu_2 \{ (x,y) \in E_1 \times E_2 : (x_d - y_d)t_1 - \epsilon \leq x_1 - y_1 \leq (x_d - y_d)t_1 + \epsilon, ..., \\ (x_d - y_d)t_{d-1} - \epsilon \leq x_{d-1} - y_{d-1} \leq (x_d - y_d)t_{d-1} + \epsilon \}. \end{aligned}$$

Our plan is to show that $\nu_\epsilon(t)$ is comparable to a quantity which can be written as the sum of two terms. We shall prove that the first term is bounded from below by a constant multiple of ϵ^{d-1}. Then we will show that the integral of the second second term in $t_1, \dots, t_{d-1}$ is much smaller than ϵ^{d-1}. We then conclude $\nu_\epsilon(t) > 0$ for a subset of $\left[\frac{1}{2}, 1\right]^{d-1}$ of positive $(d-1)$-dimensional Lebesgue measure. As we note above, this implies that $\sigma(\mathcal{D}(E)) > 0$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, even bump function, whose support is contained in the set $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ such that $\widehat{\psi}(0) = 1$. We have

$$\nu_\epsilon(t) \approx \int\int \psi\left(\frac{(x_1-y_1)-t_1(x_d-y_d)}{\epsilon}\right)\psi\left(\frac{(x_2-y_2)-t_2(x_d-y_d)}{\epsilon}\right)\cdots$$
$$\psi\left(\frac{(x_{d-1}-y_{d-1})-t_{d-1}(x_d-y_d)}{\epsilon}\right)d\mu_1(x)d\mu_2(y).$$

By Fourier inversion, this quantity equals

$$\int\int\int\cdots\int \widehat{\psi}(\lambda_1)\widehat{\psi}(\lambda_2)\ldots\widehat{\psi}(\lambda_{d-1})e^{\frac{2\pi i}{\epsilon}\lambda_1((x_1-y_1)-t_1(x_d-y_d))}e^{\frac{2\pi i}{\epsilon}\lambda_2((x_2-y_2)-t_2(x_d-y_d))}\cdots$$
$$e^{\frac{2\pi i}{\epsilon}\lambda_{d-1}((x_{d-1}-y_{d-1})-t_{d-1}(x_d-y_d))}d\lambda_1 d\lambda_2\ldots d\lambda_{d-1}d\mu_1(x)d\mu_2(y)$$
$$=\int\int\cdots\int \widehat{\psi}(\lambda_1)\widehat{\psi}(\lambda_2)\ldots\widehat{\psi}(\lambda_{d-1})\widehat{\mu_1}\left(-\frac{\lambda_1}{\epsilon},-\frac{\lambda_2}{\epsilon},\ldots,\left(\frac{t_1\lambda_1+t_2\lambda_2+\ldots t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right)$$
$$\overline{\widehat{\mu_2}}\left(-\frac{\lambda_1}{\epsilon},-\frac{\lambda_2}{\epsilon},\ldots,\left(\frac{t_1\lambda_1+t_2\lambda_2+\ldots t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right)d\lambda_1 d\lambda_2\ldots d\lambda_{d-1}\equiv I.$$

We handle the integral above by splitting it into two integrals, where the domains of integration are $[0, c_0\epsilon]^{d-1}$ and $F_\epsilon \equiv [0,1]^{d-1} \setminus [0, c_0\epsilon]^{d-1}$, respectively.

$$I = \int_0^{c_0\epsilon}\int_0^{c_0\epsilon}\cdots\int_0^{c_0\epsilon} d\lambda_1 d\lambda_2\ldots d\lambda_{d-1} + \int_{F_\epsilon} d\lambda_1 d\lambda_2\ldots d\lambda_{d-1} \equiv M_\epsilon(t) + R_\epsilon(t).$$

From the first integral, $M_\epsilon(t)$, we obtain the main term and an error term. We will bound $|M_\epsilon(t)|$ from below by a constant multiple of ϵ^{d-1}. The second integral will be denoted by $R_\epsilon(t)$ for which we will show the bound $||R_\epsilon(t)||_{L^1} = o(\epsilon^s)$.

Observe that $\widehat{\psi}(0) = 1$ and $\widehat{\mu_j}(0,0,\ldots,0) > c_1$, so we add and subtract the appropriate constants and have that $M_\epsilon(t)$ is equal to

$$\int_0^{c_0\epsilon}\int_0^{c_0\epsilon}\cdots\int_0^{c_0\epsilon}\widehat{\mu_1}(0,0,\ldots,0)\overline{\widehat{\mu_2}}(0,0,\ldots,0)d\lambda_1 d\lambda_2\ldots d\lambda_{d-1} + M'_\epsilon(t),$$

where $M'_\epsilon(t)$ is the error term, whose modulus, since $|\widehat{\mu_j}| \leq 1$, for $j = 1, 2$, is bounded from above by

$$\sum_{1\le i\le d-1}\int_{[0,c_0\epsilon]^{d-1}} |\widehat{\psi}(\lambda_i)-\widehat{\psi}(0)|d\lambda_1 d\lambda_2 ... d\lambda_{d-1}+$$
$$\sum_{\substack{1\le i,j\le d-1\\ i\neq j}}\int_{[0,c_0\epsilon]^{d-1}} |\widehat{\psi}(\lambda_i)-\widehat{\psi}(0)||\widehat{\psi}(\lambda_j)-\widehat{\psi}(0)|d\lambda_1 d\lambda_2 ... d\lambda_{d-1}+\cdots+$$
$$\int_{[0,c_0\epsilon]^{d-1}}\prod_{1\le k\le d-1}|\widehat{\psi}(\lambda_k)-\widehat{\psi}(0)|d\lambda_1 d\lambda_2 ... d\lambda_{d-1}+$$
$$\int_{[0,c_0\epsilon]^{d-1}} |f(\lambda_1,...,\lambda_{d-1})-f(0,0,...,0)|\,|g(\lambda_1,...,\lambda_{d-1})-g(0,0,...,0)|\,d\lambda_1 d\lambda_2 ... d\lambda_{d-1}+$$
$$\int_{[0,c_0\epsilon]^{d-1}} |f(0,0,...,0)|\,|g(\lambda_1,...,\lambda_{d-1})-g(0,0,...,0)|\,d\lambda_1 d\lambda_2 ... d\lambda_{d-1}+$$
$$\int_{[0,c_0\epsilon]^{d-1}} |f(\lambda_1,...,\lambda_{d-1})-f(0,0,...,0)|\,|g(0,0,...,0)|d\lambda_1 d\lambda_2 ... d\lambda_{d-1},$$

where

$$f(\lambda_1,...,\lambda_{d-1})\equiv\widehat{\mu_1}\left(-\frac{\lambda_1}{\epsilon},-\frac{\lambda_2}{\epsilon},...,\left(\frac{t_1\lambda_1+t_2\lambda_2+...t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right)$$
$$g(\lambda_1,...,\lambda_{d-1})\equiv\overline{\widehat{\mu_2}}\left(-\frac{\lambda_1}{\epsilon},-\frac{\lambda_2}{\epsilon},...,\left(\frac{t_1\lambda_1+t_2\lambda_2+...t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right).$$

Since $\widehat{\psi}$, $\widehat{\mu_1}$, and $\widehat{\mu_2}$ are uniformly continuous, for every $\eta>0$, there exists $\delta>0$ and $c_0(\delta)>0$, with $c_0\epsilon<\delta$, such that

$$|M'_\epsilon(t)|\le\left((d-1)\eta+\binom{d-1}{2}\eta^2+...+\eta^{d-1}+2\eta+\eta^2\right)(c_0\epsilon)^{d-1},$$

where an appropriate choice of $\eta>0$ guarantees that

$$\theta\equiv c_0^{d-1}\left((d-1)\eta+\binom{d-1}{2}\eta^2+...+\eta^{d-1}+\eta^2+2\eta\right)<\frac{1}{10^6}.$$

So, $|M'_\epsilon(t)|<\theta\epsilon^{d-1}$, for some $\theta<\frac{1}{10^6}$.

Now, by the definition of $\mu_j, j=1,2$, we obtain that

$$\int_0^{c_0\epsilon}\int_0^{c_0\epsilon}...\int_0^{c_0\epsilon}\widehat{\mu_1}(0,0,...,0)\overline{\widehat{\mu_2}}(0,0,...,0)d\lambda_1 d\lambda_2 ... d\lambda_{d-1}\ge c_0^{d-1}c_1^2\epsilon^{d-1},$$

which gives us the desired lower bound for the main term, namely

$$|M_\epsilon(t)|\gtrsim\epsilon^{d-1}.$$

Now we will bound the L^1 norm of the quantity $R_\epsilon(t)$. Recall, $F_\epsilon = [0,1]^{d-1} \setminus [0, c_0\epsilon]^{d-1}$.

$$\int\limits_{[\frac{1}{2},1]^{d-1}} |R_\epsilon(t)| dt_1 ... dt_{d-1} \leq \int\limits_{[\frac{1}{2},1]^{d-1}} \int\limits_{F_\epsilon} |\widehat{\psi}(\lambda_1)\widehat{\psi}(\lambda_2)...\widehat{\psi}(\lambda_{d-1})|$$
$$\left|\widehat{\mu_1}\left(-\frac{\lambda_1}{\epsilon}, -\frac{\lambda_2}{\epsilon}, ..., \left(\frac{t_1\lambda_1 + t_2\lambda_2 + ...t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right)\right|$$
$$\left|\overline{\widehat{\mu_2}}\left(-\frac{\lambda_1}{\epsilon}, -\frac{\lambda_2}{\epsilon}, ..., \left(\frac{t_1\lambda_1 + t_2\lambda_2 + ...t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right)\right| d\lambda_1 d\lambda_2 ... d\lambda_{d-1} dt_1 ... dt_{d-1}.$$

By applying Cauchy-Schwarz, the square of the expression above is

$$\lesssim \int\limits_{[\frac{1}{2},1]^{d-1}} \int\limits_{F_\epsilon} \left|\widehat{\psi}(\lambda_1)\widehat{\psi}(\lambda_2)...\widehat{\psi}(\lambda_{d-1})\right| \psi(t_1)\psi(t_2)...\psi(t_{d-1})$$
$$\left|\widehat{\mu_1}\left(-\frac{\lambda_1}{\epsilon}, -\frac{\lambda_2}{\epsilon}, ..., \left(\frac{t_1\lambda_1 + t_2\lambda_2 + ...t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right)\right|^2 d\lambda_1 ... d\lambda_{d-1} dt_1 ... dt_{d-1}$$
$$\int\limits_{[\frac{1}{2},1]^{d-1}} \int\limits_{F_\epsilon} \left|\widehat{\psi}(\lambda_1)\widehat{\psi}(\lambda_2)...\widehat{\psi}(\lambda_{d-1})\right| \psi(t_1)\psi(t_2)...\psi(t_{d-1})$$
$$\left|\overline{\widehat{\mu_2}}\left(-\frac{\lambda_1}{\epsilon}, -\frac{\lambda_2}{\epsilon}, ..., \left(\frac{t_1\lambda_1 + t_2\lambda_2 + ...t_{d-1}\lambda_{d-1}}{\epsilon}\right)\right)\right|^2 d\lambda_1 ... d\lambda_{d-1} dt_1 ... dt_{d-1}$$
$$= A \cdot B,$$

where A is the first integral, and B is the second. We will break each of these integrals up into integrals over regions which make up the whole region.

$$A = \int\limits_{[\frac{1}{2},1]^{d-1}} \int\limits_{F_\epsilon} = \underbrace{\int\limits_{[\frac{1}{2},1]^{d-1}} \int\limits_{[c_0\epsilon,1]^{d-1}}}_{A_1} + \underbrace{\int\limits_{[\frac{1}{2},1]^{d-1}} \int\limits_{c_0\epsilon}^{1} \int\limits_{0}^{c_0\epsilon} \cdots \int\limits_{c_0\epsilon}^{1}}_{A_2} + \underbrace{\int\limits_{[\frac{1}{2},1]^{d-1}} \int\limits_{0}^{c_0\epsilon} \int\limits_{0}^{c_0\epsilon} \cdots \int\limits_{c_0\epsilon}^{1}}_{A_3} + ...$$

We will now estimate the A_j. The estimates of the corresponding B_j are identical. Notice that in each integral A_j, there will always be at least one variable whose range goes from $c_0\epsilon$ to $\frac{1}{\epsilon}$. In what follows, we will treat λ_1 as the variable with this range. If λ_1 does not have this range, we can alter the change of variables to suit that particular integral. Now, in both integrals, we will make a change of variables.

$$z_1 = \sum_{k=1}^{d-1} t_k\lambda_k, z_2 = t_2, z_3 = t_3, ..., z_{d-1} = t_{d-1},$$

$$w_1 = -\lambda_1, w_2 = -\lambda_2, ..., w_{d-3} = -\lambda_{d-3}.$$

The last two assignments depend on the dimension. If d is even, let $w_{d-2} = -\lambda_{d-2}$ and $w_{d-1} = -\lambda_{d-1}$. If d is odd, let $w_{d-2} = -\lambda_{d-1}$ and $w_{d-1} = -\lambda_{d-2}$. This change of variables will have Jacobian determinant of $-\lambda_1$, or w_1. Here, we will illustrate the estimation of the integral A_1. The other pieces can be estimated in a similar fashion.

$$A_1 = \int_{[\frac{1}{2},1]^{d-1}} \int_{[c_0\epsilon,1]^{d-1}} \left|\widehat{\psi}(-w_1)\widehat{\psi}(-w_2)...\widehat{\psi}(-w_{d-1})\right| \left|\widehat{\mu_1}\left(\frac{w_1}{\epsilon}, \frac{w_2}{\epsilon}, ..., \frac{w_{d-1}}{\epsilon}, \frac{z_1}{\epsilon}\right)\right|^2$$
$$\psi\left(\frac{z_1 + z_2w_2 + ... + z_{d-1}w_{d-1}}{w_1}\right)\psi(z_2)...\psi(z_{d-1})\frac{dw_1}{w_1}dw_2...dw_{d-1}dz_1...dz_{d-1}.$$

Now we make another change of variables, $u_j = \frac{w_j}{\epsilon}$ and $u_{j+d-1} = \frac{z_j}{\epsilon}$, for $1 \le j \le d-1$, which gives us

$$A_1 = \epsilon^{2d-2} \int_{[\frac{1}{2\epsilon},\frac{1}{\epsilon}]^{d-1}} \int_{[c_0,\frac{1}{\epsilon}]^{d-1}} \left|\frac{\widehat{\psi}(-\epsilon u_1)}{\epsilon u_1}\widehat{\psi}(-\epsilon u_2)...\widehat{\psi}(-\epsilon u_{d-1})\right| \left|\widehat{\mu_1}\left(u_1, u_2, ..., u_{d-1}, u_d\right)\right|^2$$
$$du_1du_2...du_{d-1}\psi\left(\frac{u_d + \epsilon(u_2u_{d+1} + ... + u_{2(d-2)}u_{d-1})}{u_1}\right)du_d\psi(\epsilon u_{d+1})du_{d+1}...\psi(\epsilon u_{2(d-1)})du_{2(d-1)}$$
$$\lesssim \epsilon^{2d-2}\cdot\epsilon^{-(d-2)}\epsilon^{s-d} = \epsilon^{s-(d-1)}\cdot\epsilon^{d-1}.$$

For the last inequality we used that the energy integral is bounded by $C\epsilon^{s-d}$ along with the fact that for $u_1 > c_0$,

$$\left|\widehat{\psi}(\epsilon u_1)\right| \le C_n \sum_{m\ge 0} 2^{-nm}\phi(2^{-m}\epsilon u_1), \tag{2.1}$$

where ϕ is a smooth cut-off with the additional property $\phi^{(k)}(0) = 0$, for any $k \in \mathbb{N}$.

In the term corresponding to $m = 0$, we have

$$\frac{\phi(\epsilon u_1)}{u_1} = \epsilon\frac{\phi(\epsilon u_1)}{\epsilon u_1} = \epsilon\cdot\widetilde{\phi}(\epsilon u_1),$$

where $\widetilde{\phi}(\epsilon u_1)$ is, once again, a smooth cut-off function due to the fact that ϕ and its derivatives vanish at the origin. This allows us to handle the term corresponding to $m = 0$, while for $m \ge 1$, we only have to adjust the power n to obtain a decaying factor which allows our series to converge. The same trick applies to $\widehat{\psi}$ for any of the first d variables without the need to be bounded from below, while for the remaining variables we just use that $\|\psi\|_{L^1} = 1$. The other A_j are estimated the same way, so now we have a bound on the L^1 norm of $R_\epsilon(t)$,

$$\int |R_\epsilon(t)|\,dt \lesssim \epsilon^s = o\left(\epsilon^{d-1}\right).$$

One can repeat the argument above for $\nu_\epsilon^\delta(t) = (\nu_\epsilon * \rho_\delta)(t)$, where

$$\rho_\delta(x) = \frac{1}{\delta^{d-1}}\rho\left(\frac{x}{\delta}\right),$$

$\delta > 0$, and $\rho \in C_0^\infty\left(\mathbb{R}^{d-1}\right)$ is a mollifier. We will obtain the same bounds independently of δ. Let $M_\epsilon^\delta(t)$ and $R_\epsilon^\delta(t)$ denote the main and the remainder term of $\nu_\epsilon^\delta(t)$, respectively. Let also $\mathcal{I}_{d-1} \equiv \left[\frac{1}{2}, 1\right]^{d-1}$. What is proved is that

$$\left|M_\epsilon^\delta(t)\right| \gtrsim \epsilon^{d-1},$$

while

$$\int \left|R_\epsilon^\delta(t)\right| dt = o(\epsilon^{d-1}).$$

Hence, by Chebyshev's inequality,

$$\left|\left\{t \in \mathcal{I}_{d-1} : \left|R_\epsilon^\delta(t)\right| \geq \frac{C}{2}\epsilon^{d-1}\right\}\right| \leq \frac{2}{C\epsilon^{d-1}} \int \left|R_\epsilon^\delta(t)\right| dt \lesssim \epsilon^{s-(d-1)}. \tag{2.2}$$

Set

$$U_\epsilon = \left\{t \in \mathcal{I}_{d-1} : |R_\epsilon^\delta(t)| \geq \frac{C}{2}\epsilon^{d-1}\right\},$$

and by (2.2), observe that $|U_\epsilon| \lesssim \epsilon^{s-(d-1)}$. This indicates that for any fixed $\epsilon > 0$,

$$\left|\{t \in \mathcal{I}_{d-1} : \nu_\epsilon^\delta(t) \geq 2C\epsilon^{d-1}\}\right| = |\mathcal{I}_{d-1} \setminus U_\epsilon| \gtrsim \frac{1}{2^{d-1}} - \epsilon^{s-(d-1)}.$$

Applying the Lebesgue differentiation theorem to $\nu_{\epsilon_j}^\delta(t)$, for the subsequence $\epsilon_j = 2^{-j}, j = 1, 2, ...$, we have that $\nu^\delta(t) > C > 0$, for almost every $t \in \mathcal{I}_{d-1} \setminus \bigcup_{j=N}^{\infty} U_j$, where N is a large positive integer such that

$$\left|\mathcal{I}_{d-1} \setminus \bigcup_{j=N}^{\infty} U_j\right| \geq \frac{1}{2^{d-1}} - \sum_{j>N} \frac{1}{2^{(s-(d-1))j}} > \frac{1}{2^d}.$$

Finally, since $supp(\nu) \subset supp(\nu^\delta) \subset \overline{B(0,\delta) + supp(\nu)}$, for any $\delta > 0$, taking $\delta \to 0$ concludes the proof of Theorem 1.5.

Remark 2.2. The above argument shows that for any $t \in \mathcal{I}_{d-1} \setminus \bigcup_{j=N}^{\infty} U_j$, we have that

$$\nu(B(t,r)) \approx r^{d-1}.$$

Remark 2.3. Observe that despite the fact that we integrate in all t_j's, it is enough to integrate in only one of them and obtain

$$\int_{1/2}^{1} |R_\epsilon(t_1, ..., t_{j-1}, t_j, t_{j+1}, ..., t_{d-1})| \, dt_j \lesssim \epsilon^s,$$

uniformly in $t_1, ..t_{j-1}, t_{j+1}, ..., t_{d-1}$.

3. Proof of Theorem 1.6

The proof of the two-dimensional case follows instantly from the proof of Theorem 1.5 above. We now confine our attention to the higher dimensional case.

Again, we consider a set $E \subset \mathbb{R}^d$ with Hausdorff dimension s. First, we notice, by Remark 1.9, that we can prove a statement analogous to Theorem 1.5 for any integral dimension, k. However, when the acceptable spherical Falconer exponent, γ_d, is not an integer, we need to argue that we can get a similar result on directions for $s > \gamma_d$. This will give us a more general result on the direction set of E, namely, the following claim.

Claim 3.1. Let $E \subset \mathbb{R}^d, d \geq 3$ with Hausdorff dimension greater than s_0, then the Hausdorff dimension of $\mathcal{D}(E)$ is at least s_0.

After proving this, we need to strengthen it slightly, by showing that there exists a fixed $y \in E$, for which a similar argument holds for the *pinned* direction set, $\mathcal{D}_y(E) = \left\{ \frac{x-y}{|x-y|} : x \in E \right\} \subset S^{d-1}$.

Claim 3.2. Let $E \subset \mathbb{R}^d, d \geq 2$ with Hausdorff dimension greater than s_0, then there exists a $y \in E$ for which the Hausdorff dimension of $\mathcal{D}_y(E)$ is at least s_0.

Once we prove Claim 3.1 and Claim 3.2, Theorem 1.6 follows. We have

$$dim_{\mathcal{H}} \left\{ \frac{x-y}{|x-y|} : x \in E \right\} \geq s_0,$$

for some $y \in E$. It follows that if s_0 is an acceptable spherical Falconer exponent, by Definition 1.3, we obtain that

$$\mathcal{L}^1 \left(\Delta \left(\mathcal{D}_y(E) \right) \right) > 0. \tag{3.1}$$

Now, we observe that there exists a one-to-one correspondence between the arcs determined by two different points x and $x' \in E$, which lie on the sphere of radius 1, centered at $y \in E$, and the angles $\theta(x, y, x')$. Thus, by (3.1), since the Lebesgue measure of all the different arclengths is positive, we have that the Lebesgue measure of all the different angles is also positive, i.e.

$$\mathcal{L}^1 \left(\mathcal{A}(E) \right) > 0.$$

3.1. Proof of Claim 3.1. Notice first that if $d = 2$, the pinned result, Claim 3.2, is enough. If $d \geq 3$, observe that for any positive integer $k < d - 1$, we can argue as in proof of Theorem 1.1, using $C_0 \epsilon^{\frac{k}{d-1}}$ instead of $C_0 \epsilon$ when we split the integral into the main part and the remainder. Then the main term is comparable to ϵ^k, while for the remainder we have

$$\int_{1/2}^{1} \left| R_\epsilon \left(t_1, ..., t_{j-1}, t_j, t_{j+1}, ..., t_{d-1} \right) \right| dt_j \lesssim \epsilon^s,$$

uniformly $t_1, ..t_{j-1}, t_{j+1}, ..., t_{d-1}$, as we did before. The main difference lies on the application of the Lebesgue differentiation Theorem, where in this case, it is applied in k variables. So, what we get is

$$|U_\epsilon| \lesssim \epsilon^{s-k},$$

where

$$\mathcal{I}_k = \left[\frac{1}{2}, 1 \right]^k,$$

and

$$U_\epsilon = \left\{ t \in \mathcal{I}_k : |R_\epsilon^\delta(t)| \geq \frac{C}{2} \epsilon^k \right\}.$$

We apply Lebesgue differentiation Theorem for the subsequence $\epsilon_j = 2^{-j}$, and then if we set

$$\mathcal{J}_k \equiv \mathcal{I}_k \setminus \bigcup_{j > N} U_j,$$

we can choose a large positive integer N, such that

$$|\mathcal{J}_k| > \frac{3}{4} \frac{1}{2^k}. \tag{3.2}$$

Let $\mu^\alpha = G_\alpha * \mu$, where G_α is the kernel of the Bessel potential with the property

$$\widehat{G_\alpha}(x) = \frac{1}{(1+4\pi^2|x|^2)^{\alpha/2}}.$$

After we associate the quantity $\nu_\epsilon^\alpha(t)$ with the cross product $\mu^\alpha \times \mu^\alpha$, and pick $\alpha_0 = \frac{k-s_0}{2}$ and $\alpha_1 = \frac{(k+1)-s_0}{2}$, the argument above and Remark 2.3 show that

$$\nu^{\delta,\alpha_j}(B(t,r)) \approx r^{k+j},$$

for every $t \in \mathcal{J}_{k+j}$, and $j = 0, 1$, where

$$\nu^{\delta,\alpha_j}(t) = (\nu^{\alpha_j} * \rho_\delta)(t).$$

Let $\mathcal{A}_0 \equiv \mathcal{J}_k \times \left[\frac{1}{2}, 1\right]^{d-1-k}$ and $\mathcal{A}_1 \equiv \mathcal{J}_{k+1} \times \left[\frac{1}{2}, 1\right]^{d-1-(k+1)}$. By (3.2),

$$|\mathcal{A}_j| > \frac{3}{4}\frac{1}{2^{d-1}},$$

for $j = 0, 1$. It is clear that there exists a set $\mathcal{O} \subset \mathcal{A}_0 \bigcap \mathcal{A}_1$, such that

$$|\mathcal{O}| > \frac{1}{2}\frac{1}{2^{d-1}}.$$

We restrict ourselves slightly by replacing the measure ν by the new measure $\nu \cdot \chi_{\mathcal{O}}$, which, for simplicity, we will keep denoting by ν. An interpolation argument for the energy integral will prove our claim. Let

$$F(\lambda) \equiv I_{\gamma(\lambda)}\left(\nu^{\delta,\alpha(\lambda)}\right) = \int |\widehat{\nu^{\delta,\alpha(\lambda)}}(\xi)|^2 |\xi|^{-(d-1)+\gamma(\lambda)}\, d\xi,$$

for $0 \le \lambda \le 1$, where

$$\alpha(\lambda) = (1-\lambda)\alpha_0 + \lambda\alpha_1,$$

$$\gamma(\lambda) = 2\alpha(\lambda) + s_0 - \beta,$$

for $\beta > 0$, and

$$\nu^{\delta,\alpha(\lambda)}(t) = (\nu^{\alpha(\lambda)} * \rho_\delta)(t).$$

It is enough to prove that $F(0)$ and $F(1)$ are finite. Then, by three-lines lemma, for $\lambda_0 = s_0 - k$,

$$F(\lambda_0) = I_{s_0-\beta}\left(\nu^\delta\right) < \infty,$$

which in turn, since β can be arbitrarily small, after taking the limit as $\delta \to 0$, proves our result.

To this end, we will prove $F(0) < \infty$, while the other case can be proved similarly.

$$\begin{aligned} F(0) &= \int_{|\xi|<1} |\widehat{\nu^{\delta,\alpha_0}}(\xi)|^2 |\xi|^{-(d-1+\beta-k)}\, d\xi + \sum_{j=0}^{\infty} \int_{|\xi|\approx 2^j} |\widehat{\nu^{\delta,\alpha_0}}(\xi)|^2 |\xi|^{-(d-1+\beta-k)}\, d\xi \\ &= \mathbf{I} + \sum_{j=0}^{\infty} \mathbf{I}_j. \end{aligned}$$

It can be easily shown that $\mathbf{I} < \infty$, since $\nu^{\alpha_0} \le 1$. Now,

$$\begin{aligned}\mathbf{I}_j &\lesssim 2^{-j(d-1+\beta-k)} \iiint \psi\left(\frac{\xi}{2^j}\right) e^{-2\pi(x-y)\xi} \nu^{\delta,\alpha_0}(x)\nu^{\delta,\alpha_0}(y)dxdyd\xi \\ &= 2^{-j(\beta-k)} \iint \widehat{\psi}\left(2^j(x-y)\right) \nu^{\delta,\alpha_0}(x)\nu^{\delta,\alpha_0}(y)dxdy,\end{aligned}$$

and since $\widehat{\psi}$ is a rapidly decreasing function we can replace it by

$$c_n \sum_{m>0} 2^{-nm} \chi_{B(0,2^{-m})}\left(2^j(x-y)\right).$$

So,

$$\mathbf{I}_j \lesssim c_n 2^{-j(\beta-k)} \sum_{m>0} 2^{-nm} 2^{k(m-j)} \lesssim 2^{-\beta j},$$

and hence, $F(0)$ is finite, which concludes the proof of Claim 3.1.

3.2. Proof of Claim 3.2. Here, we indicate how the above proofs would be modified to get the pinned result. Initially, we will apply Lemma 2.1, as before, to get two subsets, E_1 and E_2, which have measure comparable to that of E, and which are separated in at least one dimension. Without loss of generality, we suppose that these sets are separated in the d-th dimension. For every $y \in E_2$, we work as above to obtain the following quantity

$$\begin{aligned}\nu_\epsilon(t,y) \approx \int \int \psi\left(\frac{(x_1-y_1)-t_1(x_d-y_d)}{\epsilon}\right) \psi\left(\frac{(x_2-y_2)-t_2(x_d-y_d)}{\epsilon}\right) \cdots \\ \psi\left(\frac{(x_{d-1}-y_{d-1})-t_{d-1}(x_d-y_d)}{\epsilon}\right) d\mu_1(x).\end{aligned}$$

Again, we write this quantity as an integral which will be analogous to the integral in (2). Since y is fixed, we do not integrate with respect to μ_2. When we apply Fourier inversion to the integral defining $\nu(t,y)$, instead of getting $\widehat{\widehat{\mu_2}}$, we get exponentials in terms of the coordinates of y.

In the same way we split the earlier integral, we split this one into $M_\epsilon(t,y)$ and $R_\epsilon(t,y)$. In the first piece, all of the λ_j are smaller than $c_0\epsilon$, where c_0 is, again, a constant which will be chosen later. In the second piece, in each region of integration, at least one of the λ_j is larger than $c_0\epsilon$. As mentioned above, we will bound the first piece from below by a constant multiple of ϵ^{d-1}, and bound the $t_1, \ldots, t_{d-1}$ integral of the second piece from above by a quantity much smaller than ϵ^{d-1}.

Notice that the lower bound on the main term will follow by the same method as in the unpinned case. The only difference here is the exponentials in y_j. However, these are analytic, so we can use a similar argument.

For the upper bound on the error term, we want to show that there exists a $y \in E_2$ for which $|R_\epsilon(t,y)| = o(\epsilon^{d-1})$, which would guarantee that $\nu_\epsilon(t,y) \approx \epsilon^{d-1}$. We argue by contradiction. Suppose that for every $y \in E_2$, we have that $\nu_\epsilon(t,y) = o(\epsilon^{d-1})$, and we have that

$$|R_\epsilon(t,y)| \gtrsim \epsilon^{d-1}. \tag{3.3}$$

Then, if we integrate (3.3) in t, we get that

$$\int |R_\epsilon(t,y)|dt \gtrsim \epsilon^{d-1},$$

since $\nu_\epsilon(t,y) = o(\epsilon^{d-1})$. However, if we integrate again in y, we get that

$$\int \int |R_\epsilon(t,y)| dt d\mu_2(y) \gtrsim \epsilon^{d-1},$$

which contradicts the analogous bound in the proof of Theorem 1.5. Therefore, there must exist a y for which the bound $|R_\epsilon(t,y)| = o(\epsilon^{d-1})$.

With this final bound on $|R_\epsilon(t,y)|$ and arguing as in the proof of Claim 3.1, we are done.

4. Some connections between continuous and discrete aspects of the problem at hand

In this section we appeal to a conversion mechanism developed in [**13**], [**11**], and [**15**], to deduce a Pach-Pinchasi-Sharir type result from Theorem 1.5. In the aforementioned papers, the conversion mechanism was used in the context of distance sets. However, as we shall see below, the idea is quite flexible and lends itself to a variety of applications.

Definition 4.1. Let P be a set of n points contained in $[0,1]^d$, $d \geq 2$. Define the measure

$$d\mu_P^s(x) = n^{-1} \cdot n^{\frac{d}{s}} \cdot \sum_{p \in P} \chi_{B_{n^{-\frac{1}{s}}}(p)}(x) dx,$$

where $\chi_{B_{n^{-\frac{1}{s}}}(p)}(x)$ is the characteristic function of the ball of radius $n^{-\frac{1}{s}}$ centered at p.

We say that P is s-adaptable if P is $n^{-\frac{1}{s}}$-separated and

$$I_s(\mu_P) = \int \int |x-y|^{-s} d\mu_P^s(x) d\mu_P^s(y) < \infty.$$

This is equivalent to the statement

$$n^{-2} \sum_{p \neq p' \in P} |p - p'|^{-s} \lesssim 1.$$

To put it simply, s-adaptability means that a discrete point set P can be thickened into a set which is uniformly s-dimensional in the sense that its energy integral of order s is finite. Unfortunately, it is shown in [**15**] that there exist finite point sets which are not s-adaptable for certain ranges of the parameter s. The point is that the notion of Hausdorff dimension is much more subtle than the simple "size" estimate. This turns out to be a serious obstruction to efforts to convert "continuous" results into "discrete analogs".

The first main result of this section is the following.

Theorem 4.2. *Suppose that for arbitrarily small $\epsilon > 0$, P is a $(d-1+\epsilon)$-adaptable set in $[0,1]^d$, $d \geq 2$, consisting of n points. Then*

$$\#\mathcal{D}(P) \gtrapprox n. \tag{4.1}$$

Moreover, there exists a subset, $\mathcal{D}'(P) \subset \mathcal{D}(P)$, such that $\#\mathcal{D}'(P) \geq \frac{1}{2^{d-1}} \#\mathcal{D}(P)$ and the elements in $\mathcal{D}'(P)$ are $n^{-\frac{d-1}{d-1+\epsilon}}$-separated.

Observe that in dimensions two and three, this result is much weaker than what is known, as we note in the introduction above. Another weakness of this result is that it only holds for s-adaptable sets. However, in dimensions four and higher, Theorem 4.2 appears to give a new result in the discrete setting.

To prove Theorem 4.2 thicken each point of P by $n^{-\frac{1}{s}}$, where $s > d-1$. Let E_P denote the resulting set. Then

$$\sigma(\mathcal{D}(E_P)) \lesssim n^{-\frac{d-1}{s}} \cdot \#\mathcal{D}(P).$$

By the adaptability assumption and the proof of Theorem 1.5, we see that

$$\#\mathcal{D}(P) \gtrapprox n^{\frac{d-1}{s}},$$

establishing (4.1) in view of the fact that we may take s arbitrarily close to $d-1$. Note that since Theorem 1.5 does not hold for $s = d-1$, we cannot replace (4.1) by

$$\#\mathcal{D}(P) \geq C\#P. \tag{4.2}$$

To see that there exists a subset of the direction set which is $n^{-\frac{d-1}{d-1+\epsilon}}$-separated, we recall that $\sigma(\mathcal{D}(E_P)) > 0$, so we can break $\mathcal{D}(E_P)$ up into pieces with Lebesgue measure $n^{-\frac{d-1}{d-1+\epsilon}}$, each of which contains a representative from $\mathcal{D}(P)$. Then by a simple pigeon-hole argument, we see that at least $\dfrac{1}{2^{d-1}}\#\mathcal{D}(P)$ of these must be separated.

The other main result of this section is the following.

Theorem 4.3. *Suppose that for arbitrarily small $\epsilon > 0$, P is a $(\gamma_d + \epsilon)$-adaptable set in $[0,1]^d$, $d \geq 3$, consisting of n points. Then*

$$\#\mathcal{A}(P) \gtrapprox n^{\frac{1}{\gamma_d}}. \tag{4.3}$$

Moreover, there exists a subset, $\mathcal{A}'(P) \subset \mathcal{A}(P)$, such that $\#\mathcal{A}'(P) \geq \frac{1}{2}\#\mathcal{A}(P)$ and the elements in $\mathcal{A}'(P)$ are $n^{-\frac{1}{\gamma_d+\epsilon}}$-separated.

In two dimensions, under the same assumptions, $\#\mathcal{A}(P) \gtrapprox n$.

This result follows by a conversion similar to that of Theorem 4.2. Again, we construct a new set, E_P, by thickening each point of P by $n^{-\frac{1}{s}}$, where $s \geq \gamma_d$. Since P is s-adaptable, we have a continuous subset of $\mathbb{R}^d$, with Hausdorff dimension s. Since $s > \gamma_d$, the proof of Theorem 1.6 guarantees that the Lebesgue measure of the set of angles determined by E_P is positive. However, the Lebesgue measure of E_P is also bounded from above by

$$\sigma(\mathcal{A}(E_P)) \lesssim n^{-\frac{1}{s}} \cdot \#\mathcal{A}(P).$$

Therefore,

$$\#\mathcal{A}(P) \gtrapprox n^{\frac{1}{s}},$$

which yields (4.3) as s can be as low as an acceptable spherical Falconer exponent, γ_d.

To see that there exists a subset of the angle set which is $n^{-\frac{1}{\gamma_d+\epsilon}}$-separated, we recall that $\sigma(\mathcal{A}(E_P)) > 0$, so we can break $\mathcal{A}(E_P)$ up into pieces with Lebesgue measure $n^{-\frac{1}{\gamma_d+\epsilon}}$, each of which contains a representative from $\mathcal{A}(P)$. Then by a simple pigeon-hole argument, we see that at least $\frac{1}{2}\#\mathcal{A}(P)$ of these must be separated.

Theorem 4.3 should be compared to the results of Pach and Sharir, in [22], Apfelbaum and Sharir, in [1]. In [22], it is shown that for a set of n points in $\mathbb{R}^2$, no angle can occur more than $cn^2 \log n$ times. Since there are about n^3 triples of points, this implies that there must be at least $cn/\log n$ distinct angles. In [1], it is shown that for a set of n points in $\mathbb{R}^3$, no angle can occur more than $cn^{7/3}$ times, which gives a lower bound of at least $cn^{2/3}$ distinct angles. They also show that for a set of n points in $\mathbb{R}^4$, no angle besides $\pi/2$ can occur more than $cn^{5/2}\beta(n)$ times, where $\beta(n)$ grows extremely slowly with respect to n. This means that there must be about $n^{1/2}(\beta(n))^{-1}$ distinct angles.

Our exponents are a bit stronger in dimensions three and higher, albeit in the context of s-adaptable sets. In three dimensions our result implies that an $(\frac{4}{3}+\epsilon)$-adaptable set P determines $\gtrapprox (\#P)^{\frac{3}{4}}$ distinct angles. In four dimensions, an $(\frac{11}{6}+\epsilon)$-adaptable set determines $\gtrapprox (\#P)^{\frac{6}{11}}$ distinct angles. In dimensions five and higher, the only known results appear to be the ones given by Theorem 4.3.

Remark 4.4. One should take note that the separation statements in these last two theorems are relatively unique to continuous techniques. Most of the standard discrete results say nothing about the separation of the distinct elements in a given set. For example, in [24], there is a sharp lower bound on the number of distinct directions determined by a set of points in $\mathbb{R}^3$, but there are no guarantees on the separation or distribution of these directions on S^2.

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