ON THE HADAMARD TYPE INEQUALITIES INVOLVING PRODUCT OF TWO CONVEX FUNCTIONS ON THE CO-ORDINATES

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Abstract. In this paper some Hadamard-type inequalities for product of convex funcitons of $2-$variables on the co-ordinates are given.

1. INTRODUCTION

The inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$, the set of real numbers, and $a,b\in I$ with $a<b$, is well known in the literature as Hadamard's inequality.

For some recent results related to this classic inequality, see [1], [8], [10], [11], and [13], where further references are given.

In [2], Hudzik and Maligranda considered, among others, the class of functions which are $s-$convex in the second sense. This class is defined as following:

Definition 1. *A function $f:[0,\infty)\to\mathbb{R}$ is said to be $s-$convex in the second sense if*

$$f(\lambda x+(1-\lambda)y)\leq\lambda^s f(x)+(1-\lambda)^s f(y)$$

holds for all $x,y\in[0,\infty),\lambda\in[0,1]$ and for some fixed $s\in(0,1]$.

The class of $s-$convex functions in the second sense is usually denoted with K_s^2. It is clear that if we choose $s=1$ we have ordinary convexity of functions defined on $[0,\infty)$.

In [14], Kırmacı et al., proved the following inequalities related to product of convex functions. They are given in the next theorems.

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R},a,b\in[0,\infty),a<b$, be functions such that g and fg are in $L^1([a,b])$, If f is convex and nonnegative on $[a,b]$, and if g is $s-$convex on $[a,b]$ for some fixed $s\in(0,1)$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b) \tag{1.2}$$

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where

$$M(a,b)=f(a)g(a)+f(b)g(b) \text{ and } N(a,b)=f(a)g(b)+f(b)g(a).$$

Theorem 2. *Let $f,g:[a,b]\to\mathbb{R}, a,b\in[0,\infty), a<b,$ be functions such that g and fg are in $L^1([a,b])$, If f is s_1-convex and g is s_2-convex on $[a,b]$ for some fixed $s_1,s_2\in(0,1)$, then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{1}{s_1+s_2+1}M(a,b)+B(s_1+1,s_2+1)N(a,b)\\ &= \frac{1}{s_1+s_2+1}\left[M(a,b)+s_1s_2\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}N(a,b)\right] \quad (1.3)\end{aligned}$$

Theorem 3. *Let $f,g:[a,b]\to\mathbb{R}, a,b\in[0,\infty), a<b,$ be functions such that g and fg are in $L^1([a,b])$, If f is convex and nonnegative on $[a,b]$, and if g is $s-$convex on $[a,b]$ for some fixed $s\in(0,1)$, then*

$$\begin{aligned}&2^s f(\frac{a+b}{2})g(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)g(x)dx \quad (1.4)\\ &\leq \frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b)\end{aligned}$$

For similar results, see the papers [2], [12].

In [11], Dragomir defined convex functions on the co-ordinates as following and proved lemma 1 related to this definiton:

Definition 2. *Let us consider the bidimensional interval $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A function $f:\Delta\to\mathbb{R}$ is convex on Δ if the following inequality*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Lemma 1. *Every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates, but converse is not general true.*

In [11], Dragomir established the following inequalities:

Theorem 4. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned}&f(\frac{a+b}{2},\frac{c+d}{2})\\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \quad (1.5)\\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\end{aligned}$$

Similar results, refinements and generalizations can be found in [3], [5], [6], [7] and [9].

In [7], M. Alomari and M. Darus defined $s-$convexity on Δ with the folllowing definition:

Definition 3. *Consider the bidimensional interval* $\Delta := [a,b] \times [c,d]$ *in* $[0,\infty)^2$ *with* $a < b$ *and* $c < d$. *The mapping* $f : \Delta \to \mathbb{R}$ *is* $s-$*convex on* Δ *if*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x,y) + (1-\lambda)^s f(z,w)$$

holds for all $(x,y),(z,w) \in \Delta$ *with* $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$.

In [7], M. Alomari and M. Darus proved the following lemma:

Lemma 2. *Every* $s-$*convex mappings* $f : \Delta := [a,b] \times [c,d] \subset [0,\infty)^2 \to [0,\infty)$ *is* $s-$*convex on the co-ordinates, but converse is not general true.*

In [4], M. A. Latif and M. Alomari established Hadamard-type inequalities for product of two convex functions on the co-ordinates as follow:

Theorem 5. *Let* $f, g : \Delta := [a,b] \times [c,d] \subset \mathbb{R}^2 \to [0,\infty)$ *be convex functions on the co-ordinates on* Δ *with* $a < b$ *and* $c < d$. *Then*

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)\,dxdy \\ &\leq \frac{1}{9}L(a,b,c,d) + \frac{1}{18}M(a,b,c,d) + \frac{1}{36}N(a,b,c,d) \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} L(a,b,c,d) &= f(a,c)g(a,c) + f(b,c)g(b,c) + f(a,d)g(a,d) + f(b,d)g(b,d) \\ M(a,b,c,d) &= f(a,c)g(a,d) + f(a,d)g(a,c) + f(b,c)g(b,d) + f(b,d)g(b,c) \\ &\quad + f(b,c)g(a,c) + f(b,d)g(a,d) + f(a,c)g(b,c) + f(a,d)g(b,d) \\ N(a,b,c,d) &= f(b,c)g(a,d) + f(b,d)g(a,c) + f(a,c)g(b,d) + f(a,d)g(b,c) \end{aligned}$$

Theorem 6. *Let* $f, g : \Delta := [a,b] \times [c,d] \subset \mathbb{R}^2 \to [0,\infty)$ *be convex functions on the co-ordinates on* Δ *with* $a < b$ *and* $c < d$. *Then*

$$\begin{aligned} &4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)\,dxdy \\ &\quad + \frac{5}{36}L(a,b,c,d) + \frac{7}{36}M(a,b,c,d) + \frac{2}{9}N(a,b,c,d) \end{aligned} \tag{1.7}$$

where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ *as in (1.6).*

The main purpose of this paper is to establish new inequalities like (1.6) and (1.7), but now for convex functions and $s-$convex functions of $2-$variables on the co-ordinates.

2. Main Results

Theorem 7. *Let* $f : \Delta := [a,b] \times [c,d] \subset [0,\infty)^2 \to [0,\infty)$ *be convex function on the co-ordinates and* $g : \Delta := [a,b] \times [c,d] \subset [0,\infty)^2 \to [0,\infty)$ *be* $s-$*convex function on the co-ordinates with* $a < b, c < d$ *and* $f_x(y)g_x(y)$, $f_y(x)g_y(x) \in L_1[\Delta]$ *for some* *fixed* $s \in (0,1)$. *Then one has the inequality:*

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \tag{2.1}$$
$$\leq \frac{1}{(s+2)^2}L(a,b,c,d)+\frac{1}{(s+1)(s+2)^2}M(a,b,c,d)$$
$$+\frac{1}{(s+1)^2(s+2)^2}N(a,b,c,d)$$

where

$$\begin{aligned}
L(a,b,c,d) &= \frac{1}{(s+2)^2}\left([f(a,c)g(a,c)+f(b,c)g(b,c)]+[f(a,d)g(a,d)+f(b,d)g(b,d)]\right)\\
M(a,b,c,d) &= \frac{1}{(s+1)(s+2)^2}\left([f(a,c)g(b,c)+f(b,c)g(a,c)]+[f(a,d)g(b,d)+f(b,d)g(a,d)]\right)\\
&\quad+\frac{1}{(s+1)(s+2)^2}\left([f(a,c)g(a,d)+f(b,c)g(b,d)]+[f(a,d)g(a,c)+f(b,d)g(b,c)]\right)\\
N(a,b,c,d) &= \frac{1}{(s+1)^2(s+2)^2}\left([f(a,c)g(b,d)+f(b,c)g(a,d)]+[f(a,d)g(b,c)+f(b,d)g(a,c)]\right)
\end{aligned}$$

Proof. Since f is co-ordinated convex and g is co-ordinated $s-$convex, from Lemma 1 and Lemma 2, the partial mappings

$$\begin{aligned}
f_y &: [a,b]\to[0,\infty), f_y(x)=f(x,y)\\
f_x &: [c,d]\to[0,\infty), f_x(y)=f(x,y)
\end{aligned}$$

and

$$\begin{aligned}
g_y &: [a,b]\to[0,\infty), g_y(x)=g(x,y)\\
g_x &: [c,d]\to[0,\infty), g_x(y)=g(x,y)
\end{aligned}$$

are convex on $[a,b]$ and $[c,d]$, where $x\in[a,b]$, $y\in[c,d]$.

Using (1.2), we can write

$$\begin{aligned}
\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy &\leq \frac{1}{s+2}\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right]\\
&\quad+\frac{1}{(s+1)(s+2)}\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right]
\end{aligned}$$

That is

$$\begin{aligned}
\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy &\leq \frac{1}{s+2}\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]\\
&\quad+\frac{1}{(s+1)(s+2)}\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right]
\end{aligned}$$

Dividing both sides $(b-a)$ and integrating over $[a,b]$, we get

$$\begin{aligned}(2.2)\qquad & \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\ \leq\ & \frac{1}{s+2}\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right] \\ & +\frac{1}{(s+1)(s+2)}\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right]\end{aligned}$$

By applying (1.2) to each term of right hand side of above inequality, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx \quad\leq\quad & \frac{1}{s+2}\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right] \\ & +\frac{1}{(s+1)(s+2)}\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \quad\leq\quad & \frac{1}{s+2}\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right] \\ & +\frac{1}{(s+1)(s+2)}\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx \quad\leq\quad & \frac{1}{s+2}\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right] \\ & +\frac{1}{(s+1)(s+2)}\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \quad\leq\quad & \frac{1}{s+2}\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right] \\ & +\frac{1}{(s+1)(s+2)}\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right]\end{aligned}$$

Using these inequalities in (2.2), (2.1) is proved, that is

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
\leq\ & \frac{1}{(s+2)^2}\left([f(a,c)g(a,c)+f(b,c)g(b,c)]+[f(a,d)g(a,d)+f(b,d)g(b,d)]\right) \\
&+\frac{1}{(s+1)(s+2)^2}\left([f(a,c)g(b,c)+f(b,c)g(a,c)]+[f(a,d)g(b,d)+f(b,d)g(a,d)]\right) \\
&+\frac{1}{(s+1)(s+2)^2}\left([f(a,c)g(a,d)+f(b,c)g(b,d)]+[f(a,d)g(a,c)+f(b,d)g(b,c)]\right) \\
&+\frac{1}{(s+1)^2(s+2)^2}\left([f(a,c)g(b,d)+f(b,c)g(a,d)]+[f(a,d)g(b,c)+f(b,d)g(a,c)]\right)
\end{aligned}$$

We can find the same result using by $f_y(x)g_y(x)$. □

Remark 1. *In (2.1), if we choose $s=1$, (1.6) is obtained.*

Remark 2. *In (2.1), if we choose $s=1$ and $f(x)=1$ which is convex, we get the second inequality in (1.5) :*

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d g(x,y)dxdy \leq \frac{(g(a,c)+g(b,c)+g(a,d)+g(b,d))}{4}$$

In the next theorem we will also make use of the Beta function of Euler type, which is for $x,y>0$ defined as

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and the Gamma function is defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, \text{ for } x>0.$$

Theorem 8. *Let $f:\Delta:=[a,b]\times[c,d]\subset[0,\infty)^2\to[0,\infty)$ be s_1-convex function on the co-ordinates and $g:\Delta:=[a,b]\times[c,d]\subset[0,\infty)^2\to[0,\infty)$ be s_2-convex functions on the co-ordinates with $a<b, c<d$ and $f_x(y)g_x(y)$, $f_y(x)g_y(x)\in L_1[\Delta]$*

for some fixed $s_1, s_2 \in (0,1)$. *Then one has the inequality:*

$$\begin{aligned}
(2.3)\qquad & \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
\leq\ & \frac{1}{(s_1+s_2+1)^2}L(a,b,c,d)+\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}M(a,b,c,d) \\
& +[B(s_1+1,s_2+1)]^2 N(a,b,c,d) \\
=\ & \frac{1}{(s_1+s_2+1)^2}\left[L(a,b,c,d)+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}M(a,b,c,d)\right. \\
& \left.+\left[\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}\right]^2 N(a,b,c,d)\right]
\end{aligned}$$

where

$$\begin{aligned}
L(a,b,c,d) &= [f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)] \\
M(a,b,c,d) &= [f(a,c)g(b,c)+f(b,c)g(a,c)+f(a,d)g(b,d)+f(b,d)g(a,d)] \\
&\quad +[f(a,c)g(a,d)+f(b,c)g(b,d)+f(a,d)g(a,c)+f(b,d)g(b,c)] \\
N(a,b,c,d) &= [f(a,c)g(b,d)+f(b,c)g(a,d)+f(a,d)g(b,c)+f(b,d)g(a,c)]
\end{aligned}$$

Proof. Since f is co-ordinated s_1-convex and g is co-ordinated s_2-convex, from Lemma 2, the partial mappings

$$\begin{aligned}
f_y &: [a,b]\to[0,\infty), f_y(x)=f(x,y) \\
f_x &: [c,d]\to[0,\infty), f_x(y)=f(x,y)
\end{aligned}$$

and

$$\begin{aligned}
g_y &: [a,b]\to[0,\infty), g_y(x)=g(x,y) \\
g_x &: [c,d]\to[0,\infty), g_x(y)=g(x,y)
\end{aligned}$$

are convex on $[a,b]$ and $[c,d]$, where $x\in[a,b]$, $y\in[c,d]$.

Using (1.3), we get

$$\begin{aligned}
\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy &\leq \frac{1}{s_1+s_2+1}[f_x(c)g_x(c)+f_x(d)g_x(d)] \\
&\quad +B(s_1+1,s_2+1)[f_x(c)g_x(d)+f_x(d)g_x(c)]
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy &\leq \frac{1}{s_1+s_2+1}[f(x,c)g(x,c)+f(x,d)g(x,d)] \\
&\quad +B(s_1+1,s_2+1)[f(x,c)g(x,d)+f(x,d)g(x,c)]
\end{aligned}$$

Dividing both sides of the above inequality $(b-a)$ and integrating over $[a,b]$, we have

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \tag{2.4}$$
$$\leq \frac{1}{s_1+s_2+1}\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right]$$
$$+B(s_1+1,s_2+1)\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right]$$

By applying (1.3) to right side of (2.4), and we proceed similarly as in the proof of Theorem 7, we can write

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy$$
$$\leq \frac{1}{(s_1+s_2+1)^2}\left[f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)\right]$$
$$+\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}\left[f(a,c)g(b,c)+f(b,c)g(a,c)+f(a,d)g(b,d)+f(b,d)g(a,d)\right]$$
$$+\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}\left[f(a,c)g(a,d)+f(b,c)g(b,d)+f(a,d)g(a,c)+f(b,d)g(b,c)\right]$$
$$+\left[B(s_1+1,s_2+1)\right]^2\left[f(a,c)g(b,d)+f(b,c)g(a,d)+f(a,d)g(b,c)+f(b,d)g(a,c)\right]$$

That is;

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy$$
$$\leq \frac{1}{(s_1+s_2+1)^2}L(a,b,c,d)+\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}M(a,b,c,d)$$
$$+\left[B(s_1+1,s_2+1)\right]^2N(a,b,c,d)$$
$$= \frac{1}{(s_1+s_2+1)^2}\left[L(a,b,c,d)+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}M(a,b,c,d)\right.$$
$$\left.+\left[\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}\right]^2N(a,b,c,d)\right]$$

which completes the proof. □

Remark 3. *In (2.3) if we choose $s_1=s_2=1$, (2.3) reduces to (1.6).*

Theorem 9. *Let $f:\Delta:=[a,b]\times[c,d]\subset[0,\infty)^2\to[0,\infty)$ be convex function on the co-ordinates and $g:\Delta:=[a,b]\times[c,d]\subset[0,\infty)^2\to[0,\infty)$ be $s-$convex function on the co-ordinates with $a<b, c<d$ and $f_x(y)g_x(y),\ f_y(x)g_y(x)\in L_1[\Delta]$ for some*

fixed $s \in (0,1)$. *Then one has the inequality:*

$$
\begin{aligned}
(2.5)\qquad & 2^{2s+1} f(\frac{a+b}{2},\frac{c+d}{2})g(\frac{a+b}{2},\frac{c+d}{2}) \\
\leq\ & \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
& +\frac{5}{(s+1)(s+2)^2}L(a,b,c,d)+\frac{2s^2+6s+6}{(s+1)^2(s+2)^2}M(a,b,c,d) \\
& +\frac{2s+6}{(s+1)(s+2)^2}N(a,b,c,d)
\end{aligned}
$$

Proof. Since f is co-ordinated convex and g is co-ordinated $s-$convex, from Lemma 1 and Lemma 2, the partial mappings

$$
\begin{aligned}
(2.6)\qquad f_y &: [a,b]\to[0,\infty), f_y(x)=f(x,y) \\
f_x &: [c,d]\to[0,\infty), f_x(y)=f(x,y)
\end{aligned}
$$

and

$$
\begin{aligned}
g_y &: [a,b]\to[0,\infty), g_y(x)=g(x,y) \\
g_x &: [c,d]\to[0,\infty), g_x(y)=g(x,y)
\end{aligned}
$$

are convex on $[a,b]$ and $[c,d]$, where $x\in[a,b]$, $y\in[c,d]$.

Using (1.4) and multiplying both sides of the inequalities by 2^s, we get

$$
\begin{aligned}
(2.7)\qquad & 2^{2s} f(\frac{a+b}{2},\frac{c+d}{2})g(\frac{a+b}{2},\frac{c+d}{2}) \\
& -\frac{2^s}{b-a}\int_a^b f(x,\frac{c+d}{2})g(x,\frac{c+d}{2})dx \\
\leq\ & \frac{2^s}{(s+1)(s+2)}\left[f(a,\frac{c+d}{2})g(a,\frac{c+d}{2})+f(b,\frac{c+d}{2})g(b,\frac{c+d}{2})\right] \\
& +\frac{2^s}{s+2}\left[f(a,\frac{c+d}{2})g(b,\frac{c+d}{2})+f(b,\frac{c+d}{2})g(a,\frac{c+d}{2})\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(2.8)\qquad & 2^{2s} f(\frac{a+b}{2},\frac{c+d}{2})g(\frac{a+b}{2},\frac{c+d}{2}) \\
& -\frac{2^s}{d-c}\int_c^d f(\frac{a+b}{2},y)g(\frac{a+b}{2},y)dy \\
\leq\ & \frac{2^s}{(s+1)(s+2)}\left[f(\frac{a+b}{2},c)g(\frac{a+b}{2},c)+f(\frac{a+b}{2},d)g(\frac{a+b}{2},d)\right] \\
& +\frac{2^s}{s+2}\left[f(\frac{a+b}{2},c)g(\frac{a+b}{2},d)+f(\frac{a+b}{2},d)g(\frac{a+b}{2},c)\right]
\end{aligned}
$$

Now, by addition (2.7) and (2.8), we get

$$
\begin{aligned}
(2.9)\qquad & 2^{2s+1}f(\frac{a+b}{2},\frac{c+d}{2})g(\frac{a+b}{2},\frac{c+d}{2}) \\
& -\frac{2^s}{b-a}\int_a^b f(x,\frac{c+d}{2})g(x,\frac{c+d}{2})dx-\frac{2^s}{d-c}\int_c^d f(\frac{a+b}{2},y)g(\frac{a+b}{2},y)dy \\
\leq\ & \frac{1}{(s+1)(s+2)}\left[2^s f(a,\frac{c+d}{2})g(a,\frac{c+d}{2})+2^s f(b,\frac{c+d}{2})g(b,\frac{c+d}{2})\right] \\
& +\frac{1}{s+2}\left[2^s f(a,\frac{c+d}{2})g(b,\frac{c+d}{2})+2^s f(b,\frac{c+d}{2})g(a,\frac{c+d}{2})\right] \\
& +\frac{1}{(s+1)(s+2)}\left[2^s f(\frac{a+b}{2},c)g(\frac{a+b}{2},c)+2^s f(\frac{a+b}{2},d)g(\frac{a+b}{2},d)\right] \\
& +\frac{1}{s+2}\left[2^s f(\frac{a+b}{2},c)g(\frac{a+b}{2},d)+2^s f(\frac{a+b}{2},d)g(\frac{a+b}{2},c)\right]
\end{aligned}
$$

Applying (1.4) to each term of right hand side of the above inequality, we have

$$
\begin{aligned}
& 2^s f(a,\frac{c+d}{2})g(a,\frac{c+d}{2}) \\
\leq\ & \frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(a,c)+f(a,d)g(a,d)\right] \\
& +\frac{1}{s+2}\left[f(a,c)g(a,d)+f(a,d)g(a,c)\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2^s f(b,\frac{c+d}{2})g(b,\frac{c+d}{2}) \\
\leq\ & \frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy+\frac{1}{(s+1)(s+2)}\left[f(b,c)g(b,c)+f(b,d)g(b,d)\right] \\
& +\frac{1}{s+2}\left[f(b,c)g(b,d)+f(b,d)g(b,c)\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2^s f(a,\frac{c+d}{2})g(b,\frac{c+d}{2}) \\
\leq\ & \frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(b,c)+f(a,d)g(b,d)\right] \\
& +\frac{1}{s+2}\left[f(a,c)g(b,d)+f(a,d)g(b,c)\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2^s f(b,\frac{c+d}{2})g(a,\frac{c+d}{2}) \\
\leq\ & \frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy+\frac{1}{(s+1)(s+2)}\left[f(b,c)g(a,c)+f(b,d)g(a,d)\right] \\
& +\frac{1}{s+2}\left[f(b,c)g(a,d)+f(b,d)g(a,c)\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2^s f(\frac{a+b}{2},c)g(\frac{a+b}{2},c) \\
\leq\ & \frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right] \\
& +\frac{1}{s+2}\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2^s f(\frac{a+b}{2},d)g(\frac{a+b}{2},d) \\
\leq\ & \frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx+\frac{1}{(s+1)(s+2)}\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right] \\
& +\frac{1}{s+2}\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2^s f(\frac{a+b}{2},c)g(\frac{a+b}{2},d) \\
\leq\ & \frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right] \\
& +\frac{1}{s+2}\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2^s f(\frac{a+b}{2},d)g(\frac{a+b}{2},c) \\
\leq\ & \frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx+\frac{1}{(s+1)(s+2)}\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right] \\
& +\frac{1}{s+2}\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right]
\end{aligned}
$$

Using these inequalities in (2.9), we have

$$\begin{aligned}
(2.10)\quad & 2^{2s+1}f(\frac{a+b}{2},\frac{c+d}{2})g(\frac{a+b}{2},\frac{c+d}{2}) \\
& -\frac{2^s}{b-a}\int_a^b f(x,\frac{c+d}{2})g(x,\frac{c+d}{2})dx-\frac{2^s}{d-c}\int_c^d f(\frac{a+b}{2},y)g(\frac{a+b}{2},y)dy \\
\leq\ & \frac{1}{(s+1)(s+2)}\frac{1}{(d-c)}\left[\int_c^d f(a,y)g(a,y)dy+\int_c^d f(b,y)g(b,y)dy\right] \\
& +\frac{1}{(s+2)}\frac{1}{(d-c)}\left[\int_c^d f(a,y)g(b,y)dy+\int_c^d f(b,y)g(a,y)dy\right] \\
& +\frac{1}{(s+1)(s+2)}\frac{1}{(b-a)}\left[\int_a^b f(x,c)g(x,c)dx+\int_a^b f(x,d)g(x,d)dx\right] \\
& +\frac{1}{(s+2)}\frac{1}{(b-a)}\left[\int_a^b f(x,c)g(x,d)dx+\int_a^b f(x,d)g(x,c)dx\right] \\
& +\frac{2}{(s+1)^2(s+2)^2}L(a,b,c,d)+\frac{2}{(s+1)\,(s+2)^2}M(a,b,c,d) \\
& +\frac{2}{(s+2)^2}N(a,b,c,d)
\end{aligned}$$

Now by applying (1.4) to $2^s f(\frac{a+b}{2},y)g(\frac{a+b}{2},y)$, integrating over $[c,d]$, dividing both sides by $(d-c)$, we get

$$\begin{aligned}
(2.11)\quad & \frac{2^s}{(d-c)}\int_c^d f(\frac{a+b}{2},y)g(\frac{a+b}{2},y)dy \\
& -\frac{1}{(b-a)\,(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
\leq\ & \frac{1}{(s+1)(s+2)}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)g(b,y)dy\right] \\
& +\frac{1}{s+2}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(b,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)g(a,y)dy\right]
\end{aligned}$$

Similarly by applying (1.4) to $2^s f(x,\frac{c+d}{2})g(x,\frac{c+d}{2})$, integrating over $[a,b]$, dividing both sides by $(b-a)$, we get

$$\begin{aligned}
&\frac{2^s}{(b-a)}\int_a^b f(x,\frac{c+d}{2})g(x,\frac{c+d}{2})dx \\
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
\leq\ &\frac{1}{(s+1)(s+2)}\left[\frac{1}{(b-a)}\int_a^b f(x,c)g(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)g(x,d)dx\right] \\
&+\frac{1}{s+2}\left[\frac{1}{(b-a)}\int_a^b f(x,c)g(x,d)dx+\frac{1}{(b-a)}\int_a^b f(x,d)g(x,c)dx\right]
\end{aligned}\tag{2.12}$$

By addition (2.11) and (2.12), we have

$$\begin{aligned}
&\frac{2^s}{(d-c)}\int_c^d f(\frac{a+b}{2},y)g(\frac{a+b}{2},y)dy+\frac{2^s}{(b-a)}\int_a^b f(x,\frac{c+d}{2})g(x,\frac{c+d}{2})dx \\
&-\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
\leq\ &\frac{1}{(s+1)(s+2)}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)g(b,y)dy\right. \\
&\left.+\frac{1}{(b-a)}\int_a^b f(x,c)g(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)g(x,d)dx\right] \\
&+\frac{1}{s+2}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(b,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)g(a,y)dy\right. \\
&\left.+\frac{1}{(b-a)}\int_a^b f(x,c)g(x,d)dx+\frac{1}{(b-a)}\int_a^b f(x,d)g(x,c)dx\right]
\end{aligned}\tag{2.13}$$

From (2.10) and (2.13) and simplifying we get

$$\begin{aligned}
&2^{2s+1}f(\frac{a+b}{2},\frac{c+d}{2})g(\frac{a+b}{2},\frac{c+d}{2})\leq\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dx \\
&+\frac{4s+6}{(s+1)^2(s+2)^2}L(a,b,c,d)+\frac{2s^2+6s+6}{(s+1)^2(s+2)^2}M(a,b,c,d) \\
&+\frac{2s^2+8s+6}{(s+1)^2(s+2)^2}N(a,b,c,d)
\end{aligned}$$

□

Remark 4. *In (2.5), if we choose $s = 1$, we obtained (1.7).*

Remark 5. *In (2.5), if we choose $s = 1$ and $f(x) = 1$ which is convex, we have the following Hadamard-type inequality like (1.5)*

$$
\begin{aligned}
&4g(\frac{a+b}{2}, \frac{c+d}{2}) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y)dx \\
\leq \quad & \frac{3\left[g(a,c)+g(b,c)+g(a,d)+g(b,d)\right]}{4}
\end{aligned}
$$

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