Estimation of the spectral measure of multivariate regularly varying distributions

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Abstract

In the paper, the estimator for the spectral measure of multivariate stable distributions introduced by Davydov and co-workers are extended to the regularly varying distributions. The sampling method is modified to optimize the rate of convergence of estimator. An estimator of the total mass of spectral measure is proposed. The consistency and the asymptotic normality of estimators are proved.

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1 Introduction

A random $\mathbb{R}^d$-valued vector X has a regularly varying distribution with characteristic exponent $\alpha > 0$ if there exists a finite measure σ in the unit sphere $S^{d-1} = \{ x \mid \|x\| = 1, x \in \mathbb{R}^d \}$ such that $\forall B \in \mathcal{B}(S^{d-1})$ with $\sigma(\partial B) = 0$,

$$\lim_{x \to \infty} \frac{x^\alpha}{L(x)} \mathbf{P} \left\{ \frac{X}{\|X\|} \in B, \|X\| > x \right\} = \sigma(B), \qquad (1.1)$$

where L is a slowly varying function, i.e., $\frac{L(\lambda x)}{L(x)} \to 1$ as $x \to \infty$, $\forall \lambda > 0$. Here the notation $\|\cdot\|$ denotes Euclidean norm. The measure σ is called *spectral measure*, and α is called simply *tail index*. The unit sphere S^{d-1} will be simply denoted by S. The fact that X has a regularly varying distribution with tail index α and spectral measure σ will be noted later by "$X \in \mathrm{RV}(\alpha, \sigma)$". Regular variation condition appears frequently in the studies of the limit theorem for normalized sums of i.i.d. random terms, see e.g. [23] and [14], and the extreme value theory, see e.g. [19]. Regular variation is necessary and sufficient conditions for a random $\mathbb{R}^d$-valued vector belongs to the domain of attraction of a strictly α-stable distribution, if $\alpha \in (0, 2)$, see e.g. [1].

There are various characterizations of the property $X \in \mathrm{RV}(\alpha, \sigma)$ (see e.g. [15]). We give here an equivalent definition.

Definition 1.1. *A random $\mathbb{R}^d$-valued vector $X \in RV(\alpha, \sigma)$ if there exists a slowly varying function $\tilde{L}$ such that for all $r > 0$ and $B \in \mathcal{B}(S)$ with $\sigma(\partial B) = 0$*

$$\lim_{n \to \infty} n \mathbf{P} \left\{ \frac{X}{\|X\|} \in B, \|X\| > r b_n \right\} = \sigma(B) r^{-\alpha}, \qquad (1.2)$$

where $b_n = n^{1/\alpha} \tilde{L}(n)$.

It is well known that in $\mathbb{R}^d$ the convergence (1.2) are equivalent to the convergence in distribution of binomial point processes $\beta_n = \sum_{k=1}^{n} \delta_{X_k/b_n}$ to a Poisson point process $\pi_{\alpha,\sigma}$ whose intensity measure has a particular form [19]. This result is generalized to random elements in an abstract cone [6]. Moreover, in a cone which possesses the sub-invariant norm, the convergence (1.2) with $\alpha \in (0, 1)$ implies that X belongs to the domain of attraction of a strictly α-stable distribution (see Th. 4.7 [6]).

We are interested in the problem of estimation of the tail index α and the spectral measure σ of a regularly varying distribution. By using the relation between the stable distributions and the point processes, Davydov and co-workers (see [7], [8] and [17]) proposed a method to estimate the tail index α and the normalized spectral measure σ of the stable distributions in $\mathbb{R}^d$. The objective of this work is to extend this method to the multivariate regularly varying distribution.

Suppose that we have a sample $\xi_1, \xi_2, \ldots, \xi_N$, taken from a regularly varying distribution in $\mathbb{R}^d$ with unknown tail index α and unknown spectral measure σ.

We divide the sample into n groups $G_{m,1}, \ldots, G_{m,n}$, each group containing m random vectors. In practice, we choose

$$n = [N^r], r \in (0,1), \quad \text{and then} \quad m = [N/n], \tag{1.3}$$

where $[a]$ stands for the integer part of a number $a > 0$. As N tends to infinity, we have $nm \sim N$. Let

$$M_{m,i}^{(1)} = \max\{\|\xi\| \mid \xi \in G_{m,i}\}, \ i = 1, \ldots, n, \tag{1.4}$$

that is, $M_{m,i}^{(1)}$ denote the largest norm in the group $G_{m,i}$. Let $\xi_{m,i} = \xi_j = \xi_{j(m,i)}$ where the index $j(m,i)$ is such that

$$\|\xi_{j(m,i)}\| = M_{m,i}^{(1)}. \tag{1.5}$$

We set

$$M_{m,i}^{(2)} = \max\{\|\xi\| \mid \xi \in G_{m,i}\backslash\{\xi_{m,i}\}\}, \ i = 1, \ldots, n, \tag{1.6}$$

that is, $M_{m,i}^{(2)}$ denote the second largest norm in the same group. Let us denote

$$\varkappa_{m,i} = \frac{M_{m,i}^{(2)}}{M_{m,i}^{(1)}}, \quad S_n = \sum_{i=1}^{n} \varkappa_{m,i}$$

and

$$\hat{\alpha}_N = \frac{S_n}{n - S_n}.$$

The regular variation condition (1.1) implies

$$\mathbf{P}\{\|\xi\| > x\} = x^{-\alpha}\sigma(S)L(x) + o(x^{-\alpha}L(x)) \quad \text{as} \quad x \to \infty. \tag{1.7}$$

In the following we will need the stronger relation : for sufficiently large x and for some $\beta > \alpha$

$$\mathbf{P}\{\|\xi\| > x\} = C_1 x^{-\alpha} + C_2 x^{-\beta} + o(x^{-\beta}). \tag{1.8}$$

Under the regular variation assumption and the second-order asymptotic relation (1.8), the consistency and the asymptotic normality of the estimator $\hat{\alpha}_N$ were proved firstly for $n = m = [\sqrt{N}]$ in [8] and then for more general setting (1.3) in [17]. We resume these results in the following theorem.

Theorem A. *([17]) Let $\xi, \xi_1, \ldots, \xi_N$ be i.i.d. random $\mathbb{R}^d$-valued vectors with a distribution satisfying (1.7) and let the numbers n and m satisfy the relation (1.3), then*

$$\frac{1}{n} S_n \xrightarrow[N \to \infty]{a.s.} \frac{\alpha}{1 + \alpha}.$$

If the distribution of ξ satisfies (1.8) with $0 < \alpha < \beta \leq \infty$ and we choose

$$n = N^{2\zeta/(1+2\zeta)-\varepsilon}, \ m = N^{1/(1+2\zeta)+\varepsilon},$$

where $\zeta = (\beta - \alpha)/\alpha$ and $\varepsilon \to 0$ as $N \to \infty$, then as $N \to \infty$

$$\frac{\sqrt{n}\left(\frac{1}{n}S_n - \frac{\alpha}{\alpha + 1}\right)}{\left(\frac{1}{n}\sum_{i=1}^{n}\varkappa_{m,i}^2 - \left(\frac{1}{n}S_n\right)^2\right)^{1/2}} \Rightarrow \mathcal{N}(0,1).$$

This paper focus on the estimation of the spectral measure. In [8] an estimator of normalized spectral measure $\tilde{\sigma}(\cdot) = \sigma(\cdot)/\sigma(S)$ was proposed as follows. We set

$$\theta_{m,i} = \frac{\xi_{m,i}}{\|\xi_{m,i}\|}, \ i = 1, \ldots, n \tag{1.9}$$

where $\xi_{m,i}$ is defined by (1.5). Let us denote

$$\hat{\sigma}_N(\cdot) = \frac{1}{n}\sum_{i=1}^{n}\delta_{\theta_{m,i}}(\cdot). \tag{1.10}$$

Random vectors $\theta_{m,1}, \ldots, \theta_{m,n}$ are i.i.d. and it is proved in [8] that $\hat{\sigma}_N(\cdot)$ is consistent considering a fixed set B, that is, $\forall B \in \mathcal{B}(S)$ with $\sigma(\partial B) = 0$,

$$\hat{\sigma}_N(B) \xrightarrow{a.s.} \tilde{\sigma}(B).$$

The asymptotic normality for $\hat{\sigma}_N(B)$ was proved in [7]. All these relations were obtained under the assumption that $n = m$. Inspired by the work in [17] we modify the sampling method of regrouping and discuss the convergence rate of the estimator of spectral measure for the general setting (1.3). By finding a countable collection of the σ-continuity sets which is closed under the operation of finite intersection, we obtain $\hat{\sigma}_N \overset{a.s.}{\Rightarrow} \tilde{\sigma}$ as $N \to \infty$, where $\Rightarrow$ indicate convergence in distribution.

Since $\hat{\sigma}_N$ in (1.10) gives the normalized spectral measure, it remains for us to estimate the total mass $\sigma(S)$. Note that the condition (1.1) implies

$$\lim_{x \to \infty} x^\alpha L(x)^{-1}\mathbf{P}\{\|\xi\| > x\} = \sigma(S)$$

and therefore the value of $\sigma(S)$ depends on the choice of slowly varying function $L(x)$. Estimation of this function is discussed in [20] and [21]. Here we assume that the random vector ξ satisfies the condition (1.1) with $L(x) = 1$. That means if $\alpha \in (0, 2)$, the law of ξ belongs to the normal domain of attraction of an α-stable distribution. Let us denote

$$q_{m,i} = \frac{M_{m,i}^{(1)}}{m^{1/\alpha}} \tag{1.11}$$

where $M_{m,i}^{(1)}$ is defined by (1.4). The proposed estimator is defined by

$$\widehat{\sigma(S)}_N = \left(\frac{1}{n\Gamma(1 - \frac{t}{\alpha})}\sum_{i=1}^{n}q_{m,i}^t\right)^{\frac{\alpha}{t}}, \ t > 0. \tag{1.12}$$

4

Example 1. We generated samples from univariate stable distribution with $\alpha = 1.75$, $\sigma(S) = 1$ and $\rho = \frac{\sigma(1) - \sigma(-1)}{\sigma(S)} = 0.5$. The sample size is $100,000$. We calculated the estimators $\hat{p} = \hat{p}_r$ as a function of r. This procedure was repeated 50 times on the independent sets of samples. Then we plotted $\{(1 - r, \bar{p}_r), 0 < r < 1\}$ where $\bar{p}_r$ is the mean of 50 estimated values $\hat{p}_r$. Since the estimator of total mass depends on α, we present here the simulated results of the parameters α and ρ in Figure 1. The horizontal line corresponds to the true value of the parameter p. It seems that both plots have the optimal value of $1 - r$ which is around 0.4. In fact by Theorem A and the fact that $\beta = 2\alpha$ for a stable random variable, the asymptotically optimal value of $1 - r$ is approximately $1/3 + \varepsilon$. A similar result for the estimator $\hat{\sigma}$ is given in Theorem 3.2.
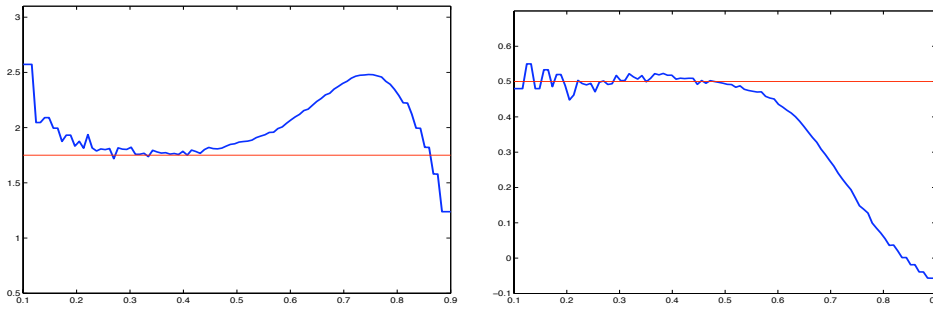

Figure 1: Estimation results of parameters α (left) and ρ (droite) in terms of r. The x-axis represent $1 - r$. Horizontal line represent the true value.

Example 2. Consider the bivariate strictly stable distribution with $\alpha = 0.75$, $\sigma(S) = 1$ and the density of spectral measure defined by $f(\theta) = \frac{1}{4}|\cos(2\theta)|$, $\theta \in (0, 2\pi)$. We calculated the estimators with $r = 0.5$. Simulated samples had $50,000$ data vectors. The estimated parameters are $\hat{\alpha} = 0.74$, $\widehat{\sigma(S)} = 0.99$. The confidence intervals with level 95% are respectively $(\hat{\alpha} - 0.07, \hat{\alpha} + 0.07)$ and $(\widehat{\sigma(S)} - 0.11, \widehat{\sigma(S)} + 0.13)$. The result of estimation of the cumulative distribution function (cdf) of spectral measure is shown in Figure 2.

The main results of the paper are contained in Section 2 and 3. The last section contains the proof of results presented in the previous sections.

2 The consistency of estimators

We assume that $\xi_1, \xi_2, \ldots, \xi_N$ are i.i.d. random $\mathbb{R}^d$- valued vectors with a regularly varying distribution. Our aim is to estimate the spectral measure σ from the sample. The estimators $\hat{\sigma}_N(\cdot)$ and $\widehat{\sigma(S)}_N$ are defined by (1.10) and (1.12). To establish consistency of these estimators we need two auxiliary results.
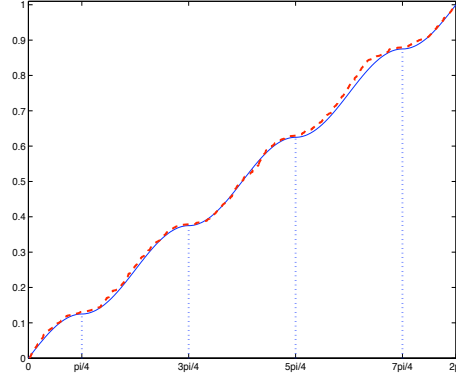
Figure 2: Estimates of the cdf of spectral measure in Example 2. The solid line is the exact cdf, the dashed line is the estimated cdf.

Let X be a $\mathbb{R}^d$-valued random vector, we denote $G(x) = \mathbf{P}\{\|X\| > x\}$. Let $Y_1, Y_2, \ldots$ be the random variables i.i.d. with distribution function $1 - G$ and $Y_{n,1}, Y_{n,2}, \cdots, Y_{n,n}, Y_{n,1} \geq Y_{n,2} \geq \cdots \geq Y_{n,n}$, the corresponding order statistics. Our first result extends a one-dimensional lemma in [13] (Lemma 1) to $d > 1$. It says that, for a random variable X satisfying (1.1) with $\sigma(S) = 1$, the vector $b_n^{-1}(Y_{n,1}, \ldots, Y_{n,n}, 0, 0, \ldots)$ converge in distribution to $(\Gamma_1^{-1/\alpha}, \Gamma_2^{-1/\alpha}, \ldots)$. The multivariate version of this lemma is as follows.

Lemma 2.1. *If $X \in RV(\alpha, \sigma)$, then the vector $b_n^{-1}(Y_{n,1}, \ldots, Y_{n,n}, 0, 0, \ldots)$ converge in distribution in $\mathbb{R}^\infty$ to $\sigma(S)^{1/\alpha}(\Gamma_1^{-1/\alpha}, \Gamma_2^{-1/\alpha}, \ldots)$, where $\Gamma_i = \sum\limits_{j=1}^{i} \lambda_j$ and $\lambda_1, \lambda_2, \ldots$ are i.i.d. random variables with a standard exponential distribution, i.e. $\mathbf{E}(\lambda_i) = 1$.*

The proof is given in Section 4.

The second result is a variant of the strong law of large numbers for a triangular array.

Proposition 2.2. *Let $\{X_{m,i}, 1 \leq i \leq n\}$ be i.i.d. real random variables for each m. Suppose that the indices n and m satisfy the following relations*

$$n \sim N^r, \quad m \sim N^{1-r} \quad as \quad N \to \infty \tag{2.1}$$

where $0 < r < 1$ is a constant and $N \in \mathbb{N}$. If there exists a real number $k > \frac{2}{r}$ and a constant $M > 0$ such that $\mathbf{E}|X_{m,1}|^k \leq M < \infty$, then

$$\frac{1}{n} \sum_{i=1}^{n} X_{m,i} - \mathbf{E}X_{m,1} \xrightarrow[N \to \infty]{a.s.} 0. \tag{2.2}$$

6

Remark 1. The convergence (2.2) holds if we replace the condition $\mathbf{E}|X_{m,1}|^k \leq M$, $k > \frac{2}{r}$ by a less restrict hypothesis $\sum\limits_{m=1}^{\infty} m^{-\frac{kr}{2(1-r)}} \mathbf{E}|X_{m,1}|^k < \infty$.

The proof is given in Section 4.

We consider at first the estimator of the total mass of spectral measure.

Theorem 2.3. *Let $\xi_1, \ldots, \xi_N$ be i.i.d. $\mathbb{R}^d$-valued random vectors such that the condition (1.2) is satisfied with $b_n = n^{1/\alpha}$. If the condition (1.3) holds, then for $0 < t < \frac{\alpha r}{2}$,*

$$\widehat{\sigma(S)}_N - \sigma(S) \xrightarrow[N\to\infty]{a.s.} 0,$$

where $\widehat{\sigma(S)}_N$ is defined by (1.12).

Proof. It suffices to prove the following convergence

$$\frac{1}{n} \sum_{i=1}^{n} q_{m,i}^t - \Gamma\left(1 - \frac{t}{\alpha}\right) \sigma(S)^{\frac{t}{\alpha}} \xrightarrow[N\to\infty]{a.s.} 0. \tag{2.3}$$

It follows from Lemma 2.1 and the assumption $b_n = n^{1/\alpha}$ that for all i and $t > 0$

$$q_{m,i}^t \Rightarrow \frac{\sigma(S)^{t/\alpha}}{\Gamma_1^{t/\alpha}} \quad \text{as } m \to \infty. \tag{2.4}$$

Since $b_n = n^{1/\alpha}$, the condition (1.2) can be wrote as

$$\lim_{x\to\infty} x^{\alpha} \mathbf{P}\{\|\xi\| > x\} = \sigma(S).$$

Therefore there exist $\delta > 0$ and a constant $M > 0$ such that for $x > \delta$ we have

$$
\begin{aligned}
\mathbf{P}\{q_{m,i}^t \geq x\} &= \mathbf{P}\{M_{m,i}^{(1)} \geq m^{1/\alpha} x^{1/t}\} \\
&= 1 - (\mathbf{P}\{\|\xi\| \leq m^{1/\alpha} x^{1/t}\})^m \\
&\leq m\mathbf{P}\{\|\xi\| > m^{1/\alpha} x^{1/t}\} \\
&\leq Mx^{-\alpha/t}.
\end{aligned}
$$

Taking $x_0 > \delta$ and $C = x_0 + Mx_0^{1-\alpha/t}\frac{\alpha}{\alpha-t}$ we have for $0 < t < \alpha$

$$
\begin{aligned}
\mathbf{E}q_{m,i}^t &= \int_{\{x<x_0\}\cup\{x\geq x_0\}} x d\mathbf{P}_{q_{m,i}^t}(x) \\
&\leq x_0 + x_0\mathbf{P}\{q_{m,i}^t \geq x_0\} + \int_{x_0}^{\infty} \mathbf{P}\{q_{m,i}^t \geq x\}dx \\
&\leq x_0 + Mx_0^{1-\alpha/t} + \int_{x_0}^{\infty} Mx^{-\alpha/t}dx \\
&= C. \tag{2.5}
\end{aligned}
$$

Now we choose a real number $0 < \delta < 1$ and $1 < \delta' < \frac{1}{\delta}$. We set $t = \frac{\alpha r \delta}{2}$ and $k = \frac{2\delta'}{r}$. Since $0 < tk < \alpha$, it follows from (2.5) that there exists a constant C such that

$$\mathbf{E}q_{m,i}^{tk} \leq C \tag{2.6}$$

In combination (2.4) (2.6) and Proposition 2.2, we obtain the convergence (2.3).
□

Let us consider the estimator of the normalized spectral measure defined by (1.10). Note that the random vectors $\theta_{m,1}, \dots, \theta_{m,n}$ are i.i.d. in S. The following lemma shows the asymptotic property for each $\theta_{m,i}$, $i = 1, \dots, n$.

Lemma 2.4. *Let $\xi, \xi_1, \dots, \xi_N$ be i.i.d. $\mathbb{R}^d$-valued random vectors and $\xi \in RV(\alpha, \sigma)$. If $\theta_{m,i}$ is defined by (1.9), then*

$$\theta_{m,i} \Rightarrow \tilde{\sigma} \ as \ m \to \infty$$

for each i.

Proof. For all Borel set B in unite sphere S such that $\sigma(\partial B) = 0$, we have

$$
\begin{aligned}
\mathbf{P}\{\theta_{m,i} \in B\} &= \mathbf{P}\{\xi_{m,1}/\|\xi_{m,1}\| \in B\} \\
&= \sum_{k=1}^{m} \mathbf{P}\{\xi_{m,1}/\|\xi_{m,1}\| \in B, \xi_{m,1} = \xi_k\} \\
&= m\mathbf{P}\{\xi_m/\|\xi_m\| \in B, \xi_{m,1} = \xi_m\} \\
&= m\mathbf{P}\{\xi_m/\|\xi_m\| \in B, \|\xi_m\| \geq \|\xi_k\|, \forall k = 1, \dots, m-1\} \\
&= m\mathbf{P}\{\xi_m/\|\xi_m\| \in B, \|\xi_m\| \geq b_m \tau_{m-1}\} \\
&= \int m\mathbf{P}\{\xi_m/\|\xi_m\| \in B, \|\xi_m\| \geq b_m x\}\mathbf{P}_{\tau_{m-1}}(dx).
\end{aligned}
$$

where $\mathbf{P}_{\tau_{m-1}}$ is the distribution of $\tau_{m-1} = \max_{1 \leq k \leq m-1}(\|\xi_k\|b_m^{-1})$.

By (1.2) and Lemma 2.1, the last term converges to

$$\int \sigma(B)x^{-\alpha}\mathbf{P}_{\sigma(S)^{1/\alpha}\Gamma_1^{-1/\alpha}}(dx) = \frac{\sigma(B)}{\sigma(S)}\mathbf{E}(\Gamma_1) = \tilde{\sigma}(B).$$

□

Therefore for each Borel set B in unit sphere S such that $\sigma(\partial B) = 0$, we have

$$\mathbb{1}_B(\theta_{m,i}) \Rightarrow \mathbb{1}_B(\eta) \ as \ m \to \infty,$$

where η is a random vector with distribution $\tilde{\sigma}$. This yields

$$\mathbf{E}\mathbb{1}_B(\theta_{m,i}) \xrightarrow[m \to \infty]{} \tilde{\sigma}(B). \tag{2.7}$$

If there exists a constant $r > 0$ such that $n \sim N^r$, applying Proposition 2.2 for the triangular array $\{\mathbb{1}_B(\theta_{m,1}), \ldots, \mathbb{1}_B(\theta_{m,n})\}$, we have for each fixed set $B \in \mathcal{B}(S)$ with $\sigma(\partial B) = 0$,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_B(\theta_{m,i}) - \mathbf{E}\mathbb{1}_B(\theta_{m,1}) \xrightarrow[N \to \infty]{a.s.} 0. \tag{2.8}$$

Together (2.7) and (2.8) we have the following result.

Theorem 2.5. *Let $\xi, \xi_1, \ldots, \xi_N$ be i.i.d. $\mathbb{R}^d$-valued random vectors and $\xi \in RV(\alpha, \sigma)$. If $\hat{\sigma}_N(\cdot)$ is defined by (1.10) and the condition (1.3) holds, then $\forall B \in \mathcal{B}(S)$ with $\sigma(\partial B) = 0$,*

$$\hat{\sigma}_N(B) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_B(\theta_{m,i}) \xrightarrow[N \to \infty]{a.s.} \tilde{\sigma}(B).$$

The result is for a fixed set. A stronger convergence can be proved by an immediate application of the following proposition.

Proposition 2.6. *Let $(S, \mathcal{S})$ be a complete separable metric space. Let $\{\sigma_n\}$ be a sequence of random probability measures in S. If σ is a probability measure on $(S, \mathcal{S})$ such that for each set $B \in \mathcal{B}(S)$ with $\sigma(\partial B) = 0$ we have the convergence $\sigma_n(B) \overset{a.s.}{\to} \sigma(B)$, then*

$$\sigma_n \overset{a.s.}{\Rightarrow} \sigma \ \text{ as } n \to \infty.$$

The proof is given in Section 4.

Corollary 2.7. *Under the same assumption of Theorem 2.5, we have*

$$\hat{\sigma}_N \overset{a.s.}{\Rightarrow} \tilde{\sigma} \ \text{ as } N \to \infty.$$

3   The asymptotic normality of estimators

In this section we consider the asymptotic normality of the estimators $\widehat{\sigma(S)}_N$ and $\hat{\sigma}_N$.

Theorem 3.1. *Suppose that random vector ξ satisfies the condition (1.8) with $\beta > \alpha + 1$, the condition (1.3) holds. If we choose*

$$r = \frac{3\alpha - 4(\beta - 1) + \sqrt{16(\beta - 1)^2 - 8\alpha(\beta - 1) - 7\alpha^2}}{2\alpha} - \varepsilon, \ \text{ if } \ \beta \leq \frac{11}{8}\alpha + 1,$$

$$r = \frac{1}{2} - \varepsilon, \ \text{ if } \ \beta > \frac{11}{8}\alpha + 1, \tag{3.1}$$

where ε is an arbitrarily small positive constant, then in the following two cases

$$a) \ 0 < t < \frac{\alpha r}{4} \wedge 1 \qquad \quad if \ \ \alpha + 1 < \beta \leq \frac{11}{8}\alpha + 1 \ \ or \ \ \beta \geq \frac{3}{2}\alpha + 1,$$

$$b) \ 0 < t < \frac{3\alpha + 2 - 2\beta}{2} \wedge 1 \ \ if \ \ \frac{11}{8}\alpha + 1 < \beta < \frac{3}{2}\alpha + 1, \tag{3.2}$$

we have, as $N \to \infty$,

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} q_{m,i}^{t} - \Gamma\left(1 - \frac{t}{\alpha}\right)\sigma(S)^{t/\alpha}\right)}{\left(\frac{1}{n}\sum_{i=1}^{n} q_{m,i}^{2t} - \left(\frac{1}{n}\sum_{i=1}^{n} q_{m,i}^{t}\right)^{2}\right)^{1/2}} \Rightarrow \mathcal{N}(0,1). \tag{3.3}$$

Remark 2. Fristedt, [9], proved an asymptotic expansion for the distribution of the norm of a strictly α-stable random vector in $\mathbb{R}^{d}$

$$G(x) = c_1 x^{-\alpha} + c_2 x^{-2\alpha} + O(x^{-3\alpha}), \text{ as } x \to \infty. \tag{3.4}$$

That means $\beta = 2\alpha$. Therefore the condition of this theorem is satisfied if $\alpha > 1$. If $\alpha > 8/5$ the rate of convergence of estimator in $\mathcal{L}_1$ is close to $N^{1/4}$.

The proof is given in Section 4.

Before considering the asymptotic normality of the estimator of normalized spectral measure, we present a *strong second-asymptotic relation* : $\forall B \in \mathcal{B}(S)$ with $\sigma(\partial B) = 0$,

$$\mathbf{P}\left\{\frac{\xi}{\|\xi\|} \in B, \|\xi\| > x\right\} = \sigma(B)x^{-\alpha} + Cx^{-\beta} + o(x^{-\beta}) \text{ as } x \to \infty, \tag{3.5}$$

where $\beta > \alpha > 0$. Note that this condition implies the conditions (1.1) and (1.8).

Let us denote $\tilde{\sigma}(B) = b$, $\mathbb{1}_B(\theta_{m,i}) = \eta_{m,i}, i = 1, 2, \ldots, n$. Then

$$Z_n = \hat{\sigma}_N(B) - \tilde{\sigma}(B) = \frac{1}{n}\sum_{i=1}^{n}(\eta_{m,i} - \tilde{\sigma}(B)),$$

$$\sqrt{n}Z_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\eta_{m,i} - \tilde{\sigma}(B)) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\eta_{m,i} - \mathbf{E}\eta_{m,1}) + \sqrt{n}(\mathbf{E}\eta_{m,1} - \tilde{\sigma}(B)),$$

$$U_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\eta_{m,i} - \mathbf{E}\eta_{m,1}), \ r_m = \mathbf{E}\eta_{m,1} - \tilde{\sigma}(B).$$

We set

$$T_N = \frac{\sqrt{n}Z_n}{\left(\frac{1}{n}\sum_{i=1}^{n}\eta_{m,i}^{2} - \left(\frac{1}{n}\sum_{i=1}^{n}\eta_{m,i}\right)^{2}\right)^{1/2}},$$

then the asymptotic property of $\hat{\sigma}_N$ can be described as follows.

Theorem 3.2. *Let $\xi, \xi_1, \ldots, \xi_N$ be i.i.d. $\mathbb{R}^d$-valued random vectors and the distribution of ξ satisfies the condition (3.5). If we choose*

$$n = N^{2\zeta/(1+2\zeta)-\varepsilon}, \quad m = N^{1/(1+2\zeta)+\varepsilon},$$

where $\zeta = \min(\frac{\beta-\alpha}{\alpha}, 1)$ and ε is an arbitrarily small positive constant, then

$$T_N \Rightarrow \mathcal{N}(0,1). \tag{3.6}$$

Remark 3. We can get also asymptotic normality for $\sqrt{n}Z_n$, but the variance of the limit normal law is $\sigma(B)(1-\sigma(B))$ which we are estimating.

Remark 4. If ξ is a strictly α-stable random vector in $\mathbb{R}^d$, by (3.4) we get $\beta = 2\alpha$, thus the asymptotically optimal value of n is approximately $N^{2/3}$. The rate of convergence of $\hat{\sigma}_N(B)$ in $\mathcal{L}_1$ is close to $N^{1/3}$.

The proof is given in Section 4.

4 Proofs

Preliminary remarks

We recall the definition of a regularly varying function. We say that L is *a regular varying function of index α at infinity (respectively at origin)* and we denote $L \in R_\alpha$ (respectively $L \in R_\alpha(0+)$) if

$$\frac{L(\lambda x)}{L(x)} \to x^\alpha, \text{ as } x \to \infty \ (x \to 0) \text{ for all } \lambda > 0.$$

Let X be $\mathbb{R}^d$-valued random vector satisfying the regular variation condition (1.2). We denote $G(x) = \mathbf{P}\{\|X\| > x\}$. Then

$$nG(b_n x) \to \sigma(S)x^{-\alpha}, \text{ as } n \to \infty, \text{ for all } x > 0. \tag{4.1}$$

For positive fixed x, we choose n the smallest integer such that $b_{n+1} > x$. Then $b_n \leq x < b_{n+1}$ and for a non-creasing function G we have

$$\frac{G(\lambda b_{n+1})}{G(b_n)} \leq \frac{G(\lambda x)}{G(x)} \leq \frac{G(\lambda b_n)}{G(b_{n+1})}, \quad \text{for all } \lambda > 0.$$

By (4.1) we have $nG(b_n) \to \sigma(S)$, then

$$\frac{G(\lambda x)}{G(x)} \to \lambda^{-\alpha}, \text{ as } x \to \infty \text{ for all } \lambda > 0.$$

We deduce that $G \in R_{-\alpha}$, which allow us write the following equivalence

$$G(x) \sim x^{-\alpha}L(x), \tag{4.2}$$

where $L(x)$ is a slowly varying function.

We recall a well known result on the asymptotic inverse of a regular varying function.

Theorem 4.1. *([3] Th. 1.5.12) Let $f \in R_\alpha$ with $\alpha > 0$, then $\exists g(x) \in R_{1/\alpha}$ such that the following relation holds*

$$f(g(x)) \sim g(f(x)) \sim x \text{ as } x \to \infty. \tag{4.3}$$

Here g (the asymptotic inverse of f) is defined uniquely up to asymptotic equivalence, and a version of g is

$$f^\leftarrow(x) = \inf\{y : f(y) \le x\}.$$

We denote

$$f(x) = \frac{1}{G(x)} \sim x^\alpha \frac{1}{L(x)}, \tag{4.4}$$

then $f(x) \in R_\alpha$ with $\alpha > 0$. By applying the previous theorem, we obtain the inverse $g(x)$ of $f(x)$ in the following form,

$$g(x) = x^{1/\alpha} L^\sharp(x)$$

where the slowly varying function $L^\sharp$ verifies the following relation

$$L(x)^{-1/\alpha} L^\sharp(f(x)) \to 1, \tag{4.5}$$

and

$$L(g(x))^{-1} L^\sharp(x)^\alpha \to 1, \ x \to \infty.$$

By (4.3) and (4.4) we have

$$G\left(g\left(\frac{1}{x}\right)\right) \sim x, \ x \to 0. \tag{4.6}$$

Defining the generalized inverse

$$G^{-1}(x) := \inf\{y : G(y) < x\},$$

we can prove that

$$G(G^{-1}(x)) \sim x, \ x \to 0. \tag{4.7}$$

For this we choose $\lambda > 1$, $A > 1$, $\delta \in (0, \infty)$, then by the theorem of Potter (Th. 1.5.6 [3] page 25) there exists u_0 such that

$$A^{-1}\lambda^{-\alpha-\delta} G(v) \le G(u) \le A\lambda^{\alpha+\delta} G(v), \ \forall v \in [\lambda^{-1}u, \lambda u], \ u \ge u_0.$$

We take x small enough such that $G^{-1}(x) \ge u_0$, then by the definition of G^{-1} there exists $y \in [\lambda^{-1} G^{-1}(x), G^{-1}(x)]$ such that $G(y) \ge x$, and there exists $y' \in [G^{-1}(x), \lambda G^{-1}(x)]$ such that $G(y') < x$. Taking $G^{-1}(x)$ for u, y and y' for v, we get

$$A^{-1}\lambda^{-\alpha-\delta} G(y) \le G(G^{-1}(x)) \le A\lambda^{\alpha+\delta} G(y').$$

Hence $\limsup$ and $\liminf$ of $G(G^{-1}(x))/x$ are between $A\lambda^{\alpha+\delta}$ and $A^{-1}\lambda^{-\alpha-\delta}$ as $x \to \infty$. Taking $A, \lambda \downarrow 1$, we have $G(G^{-1}(x))/x \to 1$.

The relations (4.6) and (4.7) give immediately

$$G^{-1}(x) \sim g(1/x), \ x \to 0.$$

Thus we have the equivalent expression of the inverse of $G(x)$:

$$G^{-1}(x) \sim x^{-1/\alpha} L^{\sharp}(1/x) \in R_{-1/\alpha}(0+). \tag{4.8}$$

Lemma 4.2. *Let X be a $\mathbb{R}^d$-valued random vector, $G(x) = \mathbf{P}\{\|X\| > x\}$. If $X \in RV(\alpha, \sigma)$, then for each $i = 1, 2, \ldots$,*

$$b_n^{-1} G^{-1}\left(\frac{\Gamma_i}{\Gamma_{n+1}}\right) \xrightarrow[n \to \infty]{} \sigma(S)^{1/\alpha} \Gamma_i^{-1/\alpha} \ \textit{with probability 1.} \tag{4.9}$$

Proof. We recall (4.1) $nG(b_n x) \to \sigma(S)x^{-\alpha}$ which implies

$$G(x_n) \sim \frac{\sigma(S)}{n} \left(\frac{b_n}{x_n}\right)^{\alpha}, \ n \to \infty$$

where $x_n = b_n x$, $x_n \to \infty$, as $n \to \infty$. By replacing the left term in the previous formula by (4.2), we get an equivalent expression of b_n in terms of $L(x)$:

$$b_n \sim \left(\frac{nL(x_n)}{\sigma(S)}\right)^{1/\alpha}, \ n \to \infty. \tag{4.10}$$

Considering (4.8), we have an equivalent expression with probability 1 for each i,

$$G^{-1}\left(\frac{\Gamma_i}{\Gamma_{n+1}}\right) \sim \left(\frac{\Gamma_i}{\Gamma_{n+1}}\right)^{-1/\alpha} L^{\sharp}\left(\frac{\Gamma_{n+1}}{\Gamma_i}\right), \ n \to \infty \tag{4.11}$$

where $L^{\sharp}$ satisfies (4.5) which means

$$L(x_n)^{-1/\alpha} L^{\sharp}\left(\frac{\Gamma_{n+1}}{\Gamma_i}\right) \to 1, \ n \to \infty. \tag{4.12}$$

Collecting (4.10)-(4.12) we deduce that with probability 1

$$b_n^{-1} G^{-1}\left(\frac{\Gamma_i}{\Gamma_{n+1}}\right) \sim \sigma(S)^{1/\alpha} \left(\frac{\Gamma_{n+1}}{n\Gamma_i}\right)^{1/\alpha} L(x_n)^{-1/\alpha} L^{\sharp}\left(\frac{\Gamma_{n+1}}{\Gamma_i}\right) \sim \left(\frac{\sigma(S)}{\Gamma_i}\right)^{1/\alpha},$$

as $n \to \infty$, since $\Gamma_{n+1}/n \xrightarrow[n \to \infty]{a.s.} 1$; the lemma is proved. $\square$

Proof of Lemma 2.1: It is well known that (see, e.g. [4] Section 13.6)

$$(Y_{n,1}, \ldots, Y_{n,n}) \overset{\mathcal{L}}{=} \left(G^{-1}\left(\frac{\Gamma_1}{\Gamma_{n+1}}\right), \ldots, G^{-1}\left(\frac{\Gamma_n}{\Gamma_{n+1}}\right)\right). \tag{4.13}$$

The lemma follows from (4.9) and (4.13). $\square$

Proof of Proposition 2.2: Denote $Y_{m,i} = X_{m,i} - \mathbf{E}X_{m,1}$, then the random variables $\{Y_{m,i}, 1 \leq i \leq n\}$ are centered and i.i.d.. We have

$$\mathbf{E}|Y_{m,1}|^k \leq \mathbf{E}(|X_{m,1}|+|\mathbf{E}X_{m,1}|)^k \leq \mathbf{E}(2^{k-1}(|X_{m,1}|^k+|\mathbf{E}X_{m,1}|^k)) \leq 2^k\mathbf{E}|X_{m,1}|^k.$$

It is well known (see [22]) that for $k \geq 2$ we have

$$\mathbf{E}\left|\sum_{i=1}^n Y_{m,i}\right|^k \leq c(k)n^{k/2}\mathbf{E}|Y_{m,1}|^k,$$

where $c(k)$ is a positive constant depending only on k. It follows from the condition (2.1) that there exists a constant $C > 0$ such that $n \geq CN^r$. Hence for all $\varepsilon > 0$ we have

$$\mathbf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n Y_{m,i}\right| > \varepsilon\right\} \leq \frac{\mathbf{E}\left|\sum_{i=1}^n Y_{m,i}\right|^k}{n^k\varepsilon^k} \leq \frac{2^k c(k)\mathbf{E}|X_{m,1}|^k}{C^{\frac{k}{2}}N^{\frac{kr}{2}}\varepsilon^k} = \frac{c_0}{N^{\frac{kr}{2}}\varepsilon^k},$$

where $c_0 = 2^k c(k)M/C^{\frac{k}{2}}$. Since $\frac{kr}{2} > 1$, we can find a small enough positive number ε' such that $\frac{kr}{2} - \varepsilon' > 1$. Taking $\varepsilon = \varepsilon_N = N^{-\frac{\varepsilon'}{k}}$ and applying the Borel-Cantelli lemma, we have that with probability 1 and for N large enough

$$\left|\frac{1}{n}\sum_{i=1}^n X_{m,i} - \mathbf{E}X_{m,1}\right| \leq N^{-\frac{\varepsilon'}{k}},$$

the proposition is proved. $\qquad\square$

Proof of Proposition 2.6: We denote the collection of all σ-continuity sets by

$$\mathcal{D}_\sigma = \{B \mid B \in \mathcal{B}(S),\ \sigma(\partial B) = 0\}.$$

Since space S is separable, there exists a countable dense set in S, denoted by

$$W = \{x_1, x_2, \ldots\},\ x_i \in S,\ i = 1, 2, \ldots.$$

We denote the open ball with centre x_i in W and radius r by

$$V(x_i, r) = \{x \mid x \in S, \|x - x_i\| < r\}.$$

Since for each $x_i \in W$ the boundaries $\partial\{V(x_i, r)\} \subset \{x \mid \|x - x_i\| = r\}$ are disjoints for different r, at most a countable number of them can have positive σ-measure. Therefore, there exists a sequence of positive numbers $r_k^i \downarrow 0$ as $k \to \infty$ for each x_i such that

$$\mathbb{L}_i = \{V(x_i, r_k^i), k = 1, 2, \ldots\} \subset \mathcal{D}_\sigma.$$

The collection $\mathbb{L} = \bigcup\limits_{x_i \in W} \mathbb{L}_i$ is countable. It is clear that for each $x_i \in W$, the collection $\mathbb{L}_i$ is a local base at point x_i for la topology $\mathcal{S}$. Since W is dense in S, $\mathbb{L}$ is a base of $\mathcal{S}$. The σ-algebra generated by $\mathbb{L}$, denoted by $\sigma(\mathbb{L})$, is the Borel-field $\mathcal{B}(S)$.

Now we expand $\mathbb{L}$ by adding the finite intersections of members of $\mathbb{L}$, we denote

$$\mathcal{L} = \mathbb{L} \cup \left\{ \bigcap_{i \in I} V_i \ \middle|\ V_i \in \mathbb{L}, I \subset \mathbb{N}, \ \mathrm{card}(I) < \infty \right\}.$$

It is clear that $\mathcal{L}$ is still countable and $\sigma(\mathcal{L}) = \mathcal{B}(S)$, moreover $\mathcal{L} \subset \mathcal{D}_\sigma$. Since $\sigma_n(B) \stackrel{a.s.}{\to} \sigma(B)$ for all $B \in \mathcal{B}(S)$ and $\sigma(\partial B) = 0$, then $\forall V \in \mathcal{L}$, $\exists \Lambda_V \subset \Omega$ and $\mathbf{P}(\Lambda_V) = 0$, such that $\forall \omega \in \Lambda_V^{\complement}$ we have

$$\sigma_n(\omega, V) \to \sigma(V). \tag{4.14}$$

If we denote $\Lambda = \bigcup\limits_{V \in \mathcal{L}} \Lambda_V$, then $\mathbf{P}(\Lambda) = 0$. Moreover $\forall \omega \in \Lambda^{\complement}$ we have always the convergence (4.14) for all $V \in \mathcal{L}$. The collection $\mathcal{L}$ is closed under the operation of finite intersection. By Theorem 2.2 in [2] (page 14) we have $\sigma_n \Rightarrow \sigma$, $\forall \omega \in \Lambda^{\complement}$, which implies $\sigma_n \stackrel{a.s.}{\Rightarrow} \sigma$. $\qquad\square$

Proof of Theorem 3.1: We set $X_{m,i} = q_{m,i}^t$, $\mu_m = \mathbf{E}q_{m,i}^t$, $\sigma_m^2 = \mathrm{Var}(X_{m,i})$ and $r_m = \mu_m - \Gamma(1 - t/\alpha)\sigma(S)^{t/\alpha}$. Therefore the convergence (3.3) holds if we have the following three relations :

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_{m,i} - \mu_m) \Rightarrow \mathcal{N}(0, \sigma^2) \tag{4.15}$$

$$\frac{1}{n} \sum_{i=1}^{n} X_{m,i}^2 - \left(\frac{1}{n} \sum_{i=1}^{n} X_{m,i} \right)^2 \xrightarrow[N \to \infty]{P} \sigma^2. \tag{4.16}$$

and

$$\sqrt{n} r_m \to 0 \tag{4.17}$$

By the similar method to (2.6) we can prove the moments $\{\mathbf{E}X_{m,i}^4\}$ are uniformly bounded. Hence we have the following convergence

$$\sigma_m^2 := \mathbf{E}X_{m,i}^2 - (\mathbf{E}X_{m,i})^2 \to \sigma^2 = \sigma(S)^{2t/\alpha}(\Gamma(1 - 2t/\alpha) - (\Gamma(1 - t/\alpha))^2), \tag{4.18}$$

and the Lindeberg's condition, i.e. for all $\varepsilon > 0$

$$
\begin{aligned}
&\lim_{n \to \infty} \frac{1}{n\sigma_m^2} \sum_{i=1}^{n} \int_{\{|X_{m,i} - \mu_m| > \varepsilon\sqrt{n}\sigma_m\}} (X_{m,i} - \mu_m)^2 d\mathbf{P} \\
=\ &\lim_{n \to \infty} \frac{1}{\sigma_m^2} \int_{\{|X_{m,1} - \mu_m| > \varepsilon\sqrt{n}\sigma_m\}} (X_{m,1} - \mu_m)^2 d\mathbf{P} \\
\leq\ &\lim_{n \to \infty} \frac{1}{\sigma_m^2} (\mathbf{E}(X_{m,1} - \mu_m)^4)^{1/2} (\mathbf{P}\{|X_{m,1} - \mu_m| > \varepsilon\sqrt{n}\sigma_m\})^{1/2} \\
=\ &0.
\end{aligned}
$$

The convergence (4.15) follows from the central limit theorem applied to triangular array $\{X_{m,i}, 1 \leq i \leq n\}$. By Proposition 2.2 and inequality (2.5), if $0 < t < \frac{\alpha r}{4}$ we have the following convergences

$$\frac{1}{n} \sum_{i=1}^{n} X_{m,i}^2 - \mathbf{E}X_{m,i}^2 \xrightarrow[N\to\infty]{a.s.} 0$$

and

$$\frac{1}{n} \sum_{i=1}^{n} X_{m,i} - \mu_m \xrightarrow[N\to\infty]{a.s.} 0.$$

It follows

$$\frac{1}{n} \sum_{i=1}^{n} X_{m,i}^2 - \left(\frac{1}{n} \sum_{i=1}^{n} X_{m,i}\right)^2 - (\mathbf{E}X_{m,i}^2 - \mu_m^2) \xrightarrow[N\to\infty]{a.s.} 0,$$

considering (4.18) we obtain (4.16). Now it remains to verify(4.17).

Lemma 4.3. *If ξ satisfies the condition (1.8) with $\beta > \alpha + 1$ and $0 < t < 1$, then*

$$|r_m| \leq Cm^{-\zeta} \tag{4.19}$$

where $\zeta = \min(\frac{1}{2}, \frac{t+\beta-\alpha-1}{\alpha})$ and C depending only on c_1, c_2, α, β and t.

Proof. By relation (1.8) if x is sufficiently large we have

$$G(x) = c_1 x^{-\alpha} + c_2 x^{-\beta} + o(x^{-\beta}),$$

where $c_1 = \sigma(S)$. It is possible to write the inverse function for small value of t,

$$G^{-1}(t) = \sigma(S)^{1/\alpha} t^{-1/\alpha} + bt^s + O(t^{s+(\beta-\alpha)/\alpha})$$

with $b = \alpha^{-1}c_2\sigma(S)^{(1-\beta)/\alpha}$ et $s = (\beta - \alpha - 1)/\alpha$. It follows that for small $\delta > 0$ and $0 < t < \delta$

$$G^{-1}(t) - \sigma(S)^{1/\alpha} t^{-1/\alpha} = t^s(b + O(t^{(\beta-\alpha)/\alpha})).$$

We choose δ such that $|O(t^{(\beta-\alpha)/\alpha})| \leq |b|$ (this gives us $\delta = o(|b|^{\frac{\alpha}{\beta-\alpha}})$), then we can write

$$|G^{-1}(t) - \sigma(S)^{1/\alpha} t^{-1/\alpha}| \leq 2|b|t^s. \tag{4.20}$$

Let

$$R_+^{m+1} = \{\bar{x} = (x_1, \ldots, x_{m+1}) : x_i \geq 0, i = 1, \ldots, m+1\},$$

$$\Sigma_m = x_1 + \cdots + x_m,$$

$$A = \left\{\bar{x} \in R_+^{m+1} : \frac{x_1}{\Sigma_{m+1}} \geq \delta\right\}, A^{\complement} = R_+^{m+1} \backslash A,$$

16

where δ is chosen for that (4.20) holds. By its definition and the relation (4.13) the random variable $q_{m,i} = M_{m,i}^{(1)}/m^{1/\alpha}$ have the distribution

$$G^{-1}\left(\frac{\Gamma_1}{\Gamma_{m+1}}\right)\Big/m^{1/\alpha}$$

then

$$|r_m| = \left| \mathbf{E}\left(G^{-1}\left(\frac{\Gamma_1}{\Gamma_{m+1}}\right)\right)^t\Big/m^{t/\alpha} - \mathbf{E}\left(\frac{\sigma(S)}{\Gamma_1}\right)^{t/\alpha} \right| \le I_1 + I_2 + I_3 + I_4, \quad (4.21)$$

where

$$I_1 = \int_A m^{-t/\alpha}\left|\left(G^{-1}\left(\frac{x_1}{\Sigma_{m+1}}\right)\right)^t\right|\exp(-\Sigma_{m+1})d\bar{x},$$

$$I_2 = \int_{A^\complement} m^{-t/\alpha}\left|\left(G^{-1}\left(\frac{x_1}{\Sigma_{m+1}}\right)\right)^t - \left(\frac{x_1}{\Sigma_{m+1}}\right)^{-t/\alpha}\sigma(S)^{t/\alpha}\right|\exp(-\Sigma_{m+1})d\bar{x},$$

$$I_3 = \int_{A^\complement} m^{-t/\alpha}\sigma(S)^{t/\alpha}x_1^{-t/\alpha}\left|\Sigma_{m+1}^{t/\alpha} - m^{t/\alpha}\right|\exp(-\Sigma_{m+1})d\bar{x},$$

and

$$I_4 = \int_A \sigma(S)^{t/\alpha}x_1^{-t/\alpha}\exp(-\Sigma_{m+1})d\bar{x}.$$

Since in A we have $G^{-1}(x_1/\Sigma_{m+1}) \le G^{-1}(\delta)$, for all $\tau_1 > 0$ (which will be chosen later)

$$I_1 \le m^{-t/\alpha}(G^{-1}(\delta))^t\mathbf{P}\left\{\frac{\Gamma_1}{\Gamma_{m+1}} \ge \delta\right\} \le m^{-t/\alpha}C_1\mathbf{E}\left(\frac{\Gamma_1}{\Gamma_{m+1}}\right)^{\tau_1} \quad (4.22)$$

where $C_1 = (G^{-1}(\delta))^t\delta^{-\tau_1}$. In a similar way we estimate I_4, for all $\tau_2 > 0$,

$$I_4 \le \sigma(S)^{t/\alpha}(\mathbf{E}\Gamma_1^{-2t/\alpha})^{1/2}\left(\mathbf{P}\left\{\frac{\Gamma_1}{\Gamma_{m+1}} \ge \delta\right\}\right)^{1/2} \le C_2\mathbf{E}\left(\frac{\Gamma_1}{\Gamma_{m+1}}\right)^{\tau_2} \quad (4.23)$$

where $C_2 = \sigma(S)^{t/\alpha}(\Gamma(1 - 2t/\alpha))^{1/2}\delta^{-\tau_2}$. The function $f(x) = x^t, 0 < t < 1$ is Lipschitz continuous on $[0, \delta)$. Considering (4.20) we have

$$\begin{aligned}
I_2 &\le \int_{A^\complement} m^{-t/\alpha}C_3\left|G^{-1}\left(\frac{x_1}{\Sigma_{m+1}}\right) - \left(\frac{x_1}{\Sigma_{m+1}}\right)^{-1/\alpha}\sigma(S)^{1/\alpha}\right|\exp(-\Sigma_{m+1})d\bar{x} \\
&\le 2|b|C_3m^{-t/\alpha}\int_{A^\complement}\left(\frac{x_1}{\Sigma_{m+1}}\right)^s\exp(-\Sigma_{m+1})d\bar{x} \\
&\le 2|b|C_3m^{-t/\alpha}\mathbf{E}\left(\frac{\Gamma_1}{\Gamma_{m+1}}\right)^s \quad (4.24)
\end{aligned}$$

where $C_3 = t(\min(G^{-1}(\delta), \delta^{-1/\alpha}\sigma(S)^{1/\alpha}))^{t-1}$. It is well known (see for example [4]) that the m-dimensional random vector $\left(\dfrac{\Gamma_1}{\Gamma_{m+1}}, \ldots, \dfrac{\Gamma_m}{\Gamma_{m+1}}\right)$ has the same

17

density as the order statistics vector of random variables uniformly distributed on $[0, 1)$. In particular the random variable $\dfrac{\Gamma_1}{\Gamma_{m+1}}$ has the following density (suppose that $m \geq 2$)

$$g(x) = \begin{cases} m(1-x)^{m-1}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\mathbf{E}\left(\frac{\Gamma_1}{\Gamma_{m+1}}\right)^s = m \int_0^1 x^s (1-x)^{m-1} dx = mB(s+1, m) \leq 4\Gamma(s+1)m^{-s}. \quad (4.25)$$

By Cauchy Schwarz's inequality we have

$$I_3 \leq m^{-t/\alpha} \sigma(S)^{t/\alpha} (\mathbf{E}\Gamma_1^{-2t/\alpha})^{1/2} (\mathbf{E}(\Gamma_m^{t/\alpha} - m^{t/\alpha})^2)^{1/2}. \quad (4.26)$$

It remains to evaluate

$$\mathbf{E}(\Gamma_m^{t/\alpha} - m^{t/\alpha})^2 = \frac{\Gamma(m+1+2t/\alpha)}{\Gamma(m+1)} - 2m^{t/\alpha} \frac{\Gamma(m+1+t/\alpha)}{\Gamma(m+1)} + m^{2t/\alpha}.$$

Using the approximation of Gamma-function

$$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + o\left(\frac{1}{x^2}\right)\right), \quad x \to \infty,$$

we obtain

$$\mathbf{E}(\Gamma_m^{t/\alpha} - m^{t/\alpha})^2 = A_m(m+1+2t/\alpha)^{2t/\alpha} H_1 - 2B_m m^{t/\alpha}(m+1+t/\alpha)^{t/\alpha} H_2 + m^{2t/\alpha}, \quad (4.27)$$

where

$$A_m = \left(1 + \frac{2t/\alpha}{m+1}\right)^{m+1} \left(1 + \frac{2t/\alpha}{m+1}\right)^{-1/2} e^{-2t/\alpha},$$

$$B_m = \left(1 + \frac{t/\alpha}{m+1}\right)^{m+1} \left(1 + \frac{t/\alpha}{m+1}\right)^{-1/2} e^{-t/\alpha},$$

$$H_1 = 1 - \frac{t}{6\alpha m^2} + o\left(\frac{1}{m^2}\right) \quad \text{et} \quad H_2 = 1 - \frac{t}{12\alpha m^2} + o\left(\frac{1}{m^2}\right). \quad (4.28)$$

By a simple calculation we obtain the following expansion

$$A_m = 1 - \left(\frac{t}{\alpha} + \frac{2t^2}{\alpha^2}\right) \frac{1}{m+1} + o\left(\frac{1}{m}\right), \quad (4.29)$$

$$B_m = 1 - \left(\frac{t}{2\alpha} + \frac{t^2}{2\alpha^2}\right) \frac{1}{m+1} + o\left(\frac{1}{m}\right). \quad (4.30)$$

Considering (4.27)-(4.30) we have

$$\mathbf{E}(\Gamma_m^{t/\alpha} - m^{t/\alpha})^2 = \frac{t^2}{\alpha^2} m^{2t/\alpha - 1} + o(m^{2t/\alpha - 1}).$$

18

Therefore it follows from (4.26) that there exists a positive constant C_4 such that

$$I_3 \leq C_4 m^{-1/2}. \tag{4.31}$$

Collecting estimates (4.21)-(4.25), (4.31) and choosing $\tau_1 = (\beta - \alpha - 1)/\alpha$ and $\tau_2 = (t + \beta - \alpha - 1)/\alpha$ in (4.22) and (4.23), we obtain the estimate (4.19) with $C = C(\alpha, \beta, c_1, c_2, t)$. The lemma is proved. $\square$

Having the estimate for r_m, and taking $n \sim N^r$ with $r < \frac{2\zeta}{1+2\zeta}$ we obtain $\sqrt{n} r_m \to 0$. By a simple calculation we have the relations (3.1) and (3.2). $\square$

Proof of Theorem 3.2: We denote $c^2 = b(1-b)$. In order to prove (3.6) it suffices to show the following relations,

$$U_n \xrightarrow[N \to \infty]{D} \mathcal{N}(0, c^2), \tag{4.32}$$

$$\sqrt{n} r_m \to 0, \tag{4.33}$$

$$\frac{1}{n} \sum_{i=1}^{n} \eta_{m,i}^2 - \left(\frac{1}{n} \sum_{i=1}^{n} \eta_{m,i} \right)^2 \xrightarrow[N \to \infty]{P} c^2. \tag{4.34}$$

Since $0 < \mathbb{1}_B(\theta_{m,i}) \leq 1$, the moments $\mathbf{E}|\mathbb{1}_B(\theta_{m,i}) - \mathbf{E}\mathbb{1}_B(\theta_{m,i})|^k$ are uniformly bounded for all $k > 0$. The limit variance is

$$c_m^2 := \mathbf{E}(\eta_{m,1} - \mathbf{E}\eta_{m,1})^2 = \mathbf{E}\eta_{m,1}^2 - (\mathbf{E}\eta_{m,1})^2 = b + r_m - (b + r_m)^2 \to c^2,$$

if we have (4.33). We consider the Lindeberg's condition: for all $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n \to \infty} \frac{1}{n c_m^2} \sum_{i=1}^{n} \int_{\{|\mathbb{1}_B(\theta_{m,i}) - \mathbf{E}\mathbb{1}_B(\theta_{m,i})| > \varepsilon \sqrt{n} c_m\}} (\mathbb{1}_B(\theta_{m,i}) - \mathbf{E}\mathbb{1}_B(\theta_{m,i}))^2 d\mathbf{P} \\
=~ &\lim_{n \to \infty} \frac{1}{c_m^2} \int_{\{|\mathbb{1}_B(\theta_{m,1}) - \mathbf{E}\mathbb{1}_B(\theta_{m,1})| > \varepsilon \sqrt{n} c_m\}} (\mathbb{1}_B(\theta_{m,1}) - \mathbf{E}\mathbb{1}_B(\theta_{m,1}))^2 d\mathbf{P} \\
\leq~ &\lim_{n \to \infty} \frac{1}{c_m^2} (\mathbf{E}(\mathbb{1}_B(\theta_{m,1}) - \mathbf{E}\mathbb{1}_B(\theta_{m,1}))^4)^{1/2} (\mathbf{P}\{|\eta_{m,1} - \mathbf{E}\eta_{m,1}| > \varepsilon \sqrt{n} c_m\})^{1/2} \\
=~ &0. \tag{4.35}
\end{aligned}$$

The relation (4.32) follows from the application of central limit theorem for the triangular array $\{\mathbb{1}_B(\theta_{m,i}) - \mathbf{E}\mathbb{1}_B(\theta_{m,i})\}$.

It is easy to see that (4.34) follows from the application of Proposition 2.2 to triangular array $\{\eta_{m,i}^2\}$. It remains to establish the relation (4.33). In fact, we have $|r_m| \leq C \max(m^{-1}, m^{-(\beta-\alpha)/\alpha})$ from the following lemma. Taking an arbitrarily small positive constant ε and

$$n = N^{\frac{2\zeta}{1+2\zeta} - \varepsilon}, ~ m = N^{\frac{1}{1+2\zeta} + \varepsilon},$$

we get $\sqrt{n} r_m \to 0$, the theorem is proved. $\square$

19

Lemma 4.4. *If the condition (3.5) is satisfied, then*

$$|r_m| \le C \max(m^{-1}, m^{-(\beta-\alpha)/\alpha}). \tag{4.36}$$

Proof. In order to prove (4.36), we need to show that

$$\mathbf{P}\{\theta_{m,i} \in B\} = \sigma(B) + R_m, \tag{4.37}$$

with the remainder term $R_m = O(\max(m^{-1}, m^{-(\beta-\alpha)/\alpha}))$. Let us denote

$$G_m(x) = \mathbf{P}\left\{ \max_{1 \le i \le m-1} \|\xi_i\| \le x \right\}.$$

Using the definition of $\theta_{m,i}$, it is not difficult to see that

$$\mathbf{P}\{\theta_{m,i} \in B\} = m \int_0^\infty \mathbf{P}\left\{ \frac{\xi_1}{\|\xi_1\|} \in B, \|\xi_1\| > r \right\} G_m(dr).$$

Let $\widetilde{G}_m(x) = G_m(xm^{1/\alpha})$. Assumption (3.5) implies (we suppose that $\sigma(S) = 1$) that for large s,

$$\mathbf{P}\{\|\xi\| > s\} = s^{-\alpha} + Cs^{-\beta} + o(s^{-\beta}).$$

Therefore, it is easy to get the relation

$$\lim_{m \to \infty} \widetilde{G}_m(x) = G_0(x) = \begin{cases} \exp(-x^{-\alpha}), & x > 0, \\ 0, & x \le 0. \end{cases}$$

Using (3.5) and the fact that $\int_0^\infty y^{-\alpha} dG_0(y) = 1$, we have (4.37) with $R_m = \sum_{i=1}^4 R_{m,i}$, where

$$
\begin{aligned}
R_{m,1} &= m \int_0^s \mathbf{P}\left\{ \frac{\xi_1}{\|\xi_1\|} \in B, \|\xi_1\| > r \right\} dG_m(r), \\
R_{m,2} &= -\sigma(B) \int_o^{s'} y^{-\alpha} dG_0(y), \\
R_{m,3} &= \sigma(B) \int_{s'}^\infty y^{-\alpha} d(\widetilde{G}_m(y) - G_0(y)), \\
R_{m,4} &= Cm^{-(\beta-\alpha)/\alpha} \int_{s'}^\infty y^{-\beta} d\widetilde{G}_m(y).
\end{aligned}
$$

Here $s' = sm^{-1/\alpha}$ and we shall choose s later. It is easy to see that

$$R_{m,1} \le m(1 - \mathbf{P}\{\|\xi_1\| > s\})^m = m(1 - hm^{-1})^m \le me^{-\frac{1}{2}h},$$

where $h = h(m, s) = m\mathbf{P}\{\|\xi_1\| > s\} \ge \frac{1}{2}ms^{-\alpha}$. We have used (4) for the last inequality. Thus, if we choose

$$s = \left(\frac{m}{K \ln m} \right)^{1/\alpha},$$

with sufficiently large K, then we get

$$R_{m,1} = o(m^{-1}). \tag{4.38}$$

Simple calculations show that

$$R_{m,2} = o(m^{-1}). \tag{4.39}$$

The main remainder term is $R_{m,3}$ and to estimate it we must first estimate the difference $\widetilde{G}_m(y) - G_0(y)$. A rather simple expansion of logarithmic function gives the following estimates which are sufficient for our purposes.

Lemma 4.5. ([7] Lemma 2) Let $\xi_i, i \geq 1$ be i.i.d. random vectors satisfying (4). Then for $y > cm^{-1/\alpha}$

$$|\widetilde{G}_m(y) - G_0(y)| \leq C(\alpha, \beta) \exp(-y^{-\alpha})(m^{-(\beta-\alpha)/\alpha} y^{-\beta} + m^{-1} y^{-2\alpha}) \tag{4.40}$$

and

$$\sup_y |\widetilde{G}_m(y) - G_0(y)| = C(\alpha, \beta) \max(m^{-1}, m^{-(\beta-\alpha)/\alpha}).$$

Now we can estimate the term $R_{m,3}$. Integrating by parts, we get

$$|R_{m,3}| = \sigma(B)(R_{m,3}^{(1)} + R_{m,3}^{(2)}), \tag{4.41}$$

where

$$R_{m,3}^{(1)} = s'^{-\alpha} |\widetilde{G}_m(s') - G_0(s')|,$$

$$R_{m,3}^{(2)} = \alpha \int_{s'}^{\infty} |\widetilde{G}_m(y) - G_0(y)| y^{-\alpha-1} dy.$$

Since $s' = (K \ln m)^{-1/\alpha} > Cm^{-1/\alpha}$, we can use (4.40) to estimate both quantities $R_{m,3}^{(i)}, i = 1, 2$. After some simple calculations, we get

$$R_{m,3}^{(1)} = o(m^{-1}),$$

$$R_{m,3}^{(2)} \leq C(\alpha, \beta) \max(m^{-1}, m^{-(\beta-\alpha)/\alpha}).$$

In a similar way we estimate $R_{m,4}$:

$$R_{m,4} = Cm^{-(\beta-\alpha)/\alpha} \int_{s'}^{\infty} y^{-\beta} d\widetilde{G}_m(y) = Cm^{-(\beta-\alpha)/\alpha}(R_{m,4}^{(1)} + R_{m,4}^{(2)}), \tag{4.42}$$

where

$$R_{m,4}^{(1)} = \int_{s'}^{\infty} y^{-\beta} dG_0(y),$$

$$R_{m,4}^{(2)} = \int_{s}^{\infty} y^{-\beta} d(\widetilde{G}_m(y) - G_0(y)).$$

It is easy to see that

$$R_{m,4}^{(1)} \leq C(\alpha, \beta) \tag{4.43}$$

and $R_{m,4}^{(2)}$ can be estimated in a similar way to $R_{m,3}$:

$$R_{m,4}^{(2)} \leq C(\alpha, \beta) \max(m^{-1}, m^{-(\beta - \alpha)/\alpha}). \qquad (4.44)$$

Collecting (4.38), (4.39), and (4.41)-(4.44) we get (4.37). $\qquad\square$

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