Spin-spin correlation functions of the q-VBS state of an integer spin model

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Abstract

We consider the valence-bond-solid ground state of the q-deformed higher-spin AKLT model (q-VBS state). We investigate the eigenvalues and eigenvectors of a matrix (G matrix), which is constructed from the matrix product representation of the q-VBS state. We compute the longitudinal and transverse spin-spin correlation functions, and determine the correlation amplitudes and correlation lengths for real q.

1 Introduction

In one-dimensional quantum systems, a completely different behavior for the integer spin chains from the half-integer spin chains was predicted by Haldane [1, 2]. The antiferromagnetic isotopric spin-1 model introduced by Affleck, Kennedy, Lieb and Tasaki (AKLT model) [3], whose ground state can be exactly calculated, has been a useful toy model to validate Haldane's prediction of the massive behavior for integer spin chains. Moreover, it lead to a deeper understanding for integer spin chains such as the discovery of the special type of long-range order [4, 5].

The AKLT model has been generalized to higher-spin models, anisotropic models, etc [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The Hamiltonians are essentially linear combinations of projection operators with nonnegative coefficients, and their ground states are called valence-bond-solid (VBS) state.

There are largely three types of representations for the ground state which are equivalent to each other: the Schwinger boson representation, the spin coherent representation and the matrix product representation. For isotropic higher-spin models, the spin-spin correlation functions [18] and the entanglement entropy [19, 20] have been calculated by utilizing the spin coherent representation and the properties of Legendre polynomials. For the q-deformed spin-1 model, spin-spin correlation functions were evaluated [7, 8, 9] from the matrix product representation.

In this paper, we consider the ground state of a q-deformed higher-integer-spin model which was constructed recently in [22] (q-VBS state). From its matrix product representation, we analyze one and two point functions of the q-VBS ground state for real

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q. We notice that a matrix, which is constructed from the matrix product representation, plays a fundamental role in computing correlation functions, especially spin-spin two point correlation functions. Investigating the structure of the matrix in detail, we obtain its eigenvalues and eigenvectors. Utilizing the results, we determine the correlation amplitudes and correlation lengths of the longitudinal and transverse spin-spin correlation functions.

This paper is organized as follows. In the next section, we briefly review the quantum group $U_q(su(2))$, and investigate the finite dimensional highest weight representation in terms of Schwinger bosons. In Section 3, we precisely define the higher-spin generalization of the q-deformed AKLT model on an L-site chain, and rigorously derive its q-VBS ground state in a matrix product form. The squared norm of the state will be written in terms of the trace of the L-th power of a matrix G, which plays an important role in this paper. In section 4, we obtain the eigenvalues and eigenvectors of G. Utilizing them, we compute one and two point functions in Section 5. Especially, we determine the correlation amplitudes and correlation lengths of the longitudinal and transverse spin-spin correlation functions. Section 6 is devoted to the conclusion of this paper.

2 The quantum group $U_q(su(2))$

We introduce several notations. Let us define the q-integer, q-factorial and q-binomial coefficient for $N \in \mathbb{Z}_{\geq 0}$ as

$$[N] = \frac{q^N - q^{-N}}{q - q^{-1}}, \quad [N]! = \begin{cases} \prod_{I=1}^{N} [I] & N \in \mathbb{N}, \\ 1 & N = 0, \end{cases}$$
$$\begin{bmatrix} N \\ K \end{bmatrix} = \begin{cases} \frac{[N]!}{[K]![N-K]!} & K = 0, \dots, N, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

respectively.

The quantum group $U_q(su(2))$ [23, 24] is defined by generators X^+, X^- and H with relations

$$\left[X^+, X^-\right] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad \left[H, X^{\pm}\right] = \pm 2X^{\pm}. \tag{2.2}$$

The comultiplication is given by

$$\Delta\left(X^{\pm}\right) = X^{\pm} \otimes q^{H/2} + q^{-H/2} \otimes X^{\pm}, \quad \Delta(H) = H \otimes \mathrm{Id} + \mathrm{Id} \otimes H. \tag{2.3}$$

$U_q(su(2))$ has the Schwinger boson representation, where the generators are realized as

$$X^+ = a^\dagger b, \quad X^- = b^\dagger a, \quad H = N_a - N_b, \tag{2.4}$$

with q-bosons a and b satisfying

$$aa^\dagger - qa^\dagger a = q^{-N_a}, \quad bb^\dagger - qb^\dagger b = q^{-N_b}, \tag{2.5}$$

$$[N_a, a] = -a, \quad [N_a, a^\dagger] = a^\dagger, \quad [N_b, b] = -b, \quad [N_b, b^\dagger] = b^\dagger. \tag{2.6}$$

We denote the space where $(2j+1)$-dimensional highest weight representation of $U_q(su(2))$ is realized by V_j. The basis of V_j is given by

$$|j;m\rangle = \frac{(a^\dagger)^{j+m}(b^\dagger)^{j-m}}{\sqrt{[j+m]![j-m]!}}|\mathrm{vac}\rangle, \quad (m=-j,\ldots,j). \tag{2.7}$$

The Weyl representation which we describe below, is an equivalent representation to the Schwinger boson representation, and is efficient for practical calculation. Let us denote the q-bosons a and b acting on the α-th site as a_α and b_α. The Weyl representation is to represent $a_\alpha^\dagger$, $b_\alpha^\dagger$, a_α and b_α on the space of polynomials $\mathbb{C}[x_\alpha, y_\alpha]$ as

$$a_\alpha^\dagger = x_\alpha, \quad b_\alpha^\dagger = y_\alpha, \quad a_\alpha = \frac{1}{x_\alpha}\frac{D_q^{x_\alpha} - D_{q^{-1}}^{x_\alpha}}{q-q^{-1}}, \quad b_\alpha = \frac{1}{y_\alpha}\frac{D_q^{y_\alpha} - D_{q^{-1}}^{y_\alpha}}{q-q^{-1}}, \tag{2.8}$$

where

$$D_p^{x_\alpha} f(x_\alpha, y_\alpha) = f(px_\alpha, y_\alpha), \quad D_p^{y_\alpha} f(x_\alpha, y_\alpha) = f(x_\alpha, py_\alpha). \tag{2.9}$$

The generators of $U_q(su(2))$ are now represented as

$$X_\alpha^+ = \frac{x_\alpha}{y_\alpha}\frac{D_q^{y_\alpha} - D_{q^{-1}}^{y_\alpha}}{q-q^{-1}}, \; X_\alpha^- = \frac{y_\alpha}{x_\alpha}\frac{D_q^{x_\alpha} - D_{q^{-1}}^{x_\alpha}}{q-q^{-1}}, \; q^{H_\alpha} = D_q^{x_\alpha} D_{q^{-1}}^{y_\alpha}. \tag{2.10}$$

The tensor product of two irreducible representations has the Clebsch-Gordan decomposition

$$V_S \otimes V_S = \bigoplus_{J=0}^{2S} V_J, \tag{2.11}$$

$$|S;m_1\rangle \otimes |S;m_2\rangle = \sum_{J=0}^{2S} \begin{bmatrix} S & S & J \\ m_1 & m_2 & m_1+m_2 \end{bmatrix} |J;m_1+m_2\rangle, \tag{2.12}$$

where

$$\begin{aligned}
\begin{bmatrix} S_1 & S_2 & J \\ m_1 & m_2 & m \end{bmatrix} &= \delta_{m_1+m_2,m}(-1)^{S_1-m_1} q^{m_1(m_1+m_2+1)+\{S_2(S_2+1)-S_1(S_1+1)-J(J+1)\}/2} \\
\times &\sqrt{\frac{[J+m]![J-m]![S_1-m_1]![S_2-m_2]![S_1+S_2-J]![2J+1]}{[S_1+m_1]![S_2+m_2]![S_1-S_2+J]![S_2-S_1+J]![S_1+S_2+J+1]!}} \\
\times &\sum_{z=\mathrm{Max}(0,-S_1-m_1,J-S_2-m_1)}^{\mathrm{Min}(J-m,S_1-m_1,S_2+J-m_1)} \frac{(-q^{m+J+1})^z [S_1+m_1+z]![S_2+J-m_1-z]!}{[z]![J-m-z]![S_1-m_1-z]![S_2-J+m_1+z]!},
\end{aligned} \tag{2.13}$$

is the q-analog of the Clebsch-Gordan coefficient [21].[1] This coefficient is compatible with the inverse of the decomposition (2.12)

$$|J;m\rangle = \sum_{m_1+m_2=m} \begin{bmatrix} S & S & J \\ m_1 & m_2 & m_1+m_2 \end{bmatrix} |S;m_1\rangle \otimes |S;m_2\rangle. \tag{2.14}$$

[1] Note that the factor $q^{m_1 m_2/2}$ is missing in [21].

For later purpose, we will also investigate the Clebsch-Gordan decomposition of $U_q(su(2))$ in terms of the Schwinger boson or the Weyl representation. Utilizing

$$\Delta X^{\pm}_{\alpha\beta} = X^{\pm}_{\alpha} \otimes q^{H_\beta/2} + q^{-H_\alpha/2} \otimes X^{\pm}_{\beta}, \tag{2.15}$$

one can show that the highest weight vector $v_J \in V_J$ $(\Delta X^+ v_J = 0)$ acting on the α-th and β-th site is given by

$$v_J = (x_\alpha x_\beta)^J \prod_{\nu=1}^{2S-J} (x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha). \tag{2.16}$$

Moreover, we can show the following:

Proposition 2.1.

$$\begin{aligned}\left(\Delta X^{-}_{\alpha\beta}\right)^n v_J =& (x_\alpha x_\beta)^{J-n} q^{nS} [n]! \sum_{\mu=0}^{n} q^{-2\mu S} \begin{bmatrix} J \\ \mu \end{bmatrix} \begin{bmatrix} J \\ n-\mu \end{bmatrix} (x_\alpha y_\beta)^\mu (x_\beta y_\alpha)^{n-\mu} \\ &\times \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha\right).\end{aligned} \tag{2.17}$$

A proof of this proposition is given in Appendix A.

Remark 2.2. Let $n \geq 2J+1$. Noting

$$\begin{bmatrix} J \\ \mu \end{bmatrix} \begin{bmatrix} J \\ n-\mu \end{bmatrix} = 0, \tag{2.18}$$

for $0 \leq \mu \leq n$, one can see that

$$\left(\Delta X^{-}_{\alpha\beta}\right)^n v_J = 0. \tag{2.19}$$

3 q-VBS state

The model we treat in this paper is an anisotropic integer spin-S Hamiltonian on an L-site chain with the periodic boundary condition

$$\mathcal{H} = \sum_{k\in\mathbb{Z}_L} \sum_{J=S+1}^{2S} C_J(k,k+1) \left(\pi_J\right)_{k,k+1}, \tag{3.1}$$

where $C_J(k,k+1) > 0$, and $(\pi_J)_{k,k+1}$, which acts on the k-th and $(k+1)$-th sites, is the $U_q(su(2))$ projection operator from $V_S \otimes V_S$ to V_J as

$$\begin{aligned}\pi_J = & \sum_{m_1,m_2,m_1',m_2'=0}^{S} \begin{bmatrix} S & S & J \\ m_1 & m_2 & m_1+m_2 \end{bmatrix} \begin{bmatrix} S & S & J \\ m_1' & m_2' & m_1'+m_2' \end{bmatrix} \\ & \times \delta_{m_1+m_2, m_1'+m_2'} |S;m_1'\rangle\langle S;m_1| \otimes |S;m_2'\rangle\langle S;m_2|.\end{aligned} \tag{3.2}$$

The nonnegativity

$$\langle\psi|\,(\pi_J)_{k,k+1}\,|\psi\rangle \geq 0 \quad \text{(for any vector } |\psi\rangle\text{)} \tag{3.3}$$

implies that all the eigenvalues of $\mathcal{H}$ are nonnegative. (Of course, $\langle\psi|$ is the Hermitian conjugate of $|\psi\rangle$.) Moreover, we will see that the energy of the ground state $|\Psi\rangle$ is zero:

$$\mathcal{H}|\Psi\rangle = 0. \tag{3.4}$$

Since we set $C_J(k, k+1) > 0$, we find that (3.4) is equivalent to

$$(\pi_J)_{k,k+1}\,|\Psi\rangle = 0 \quad (\forall k \in \mathbb{Z}_L,\ \forall J \in \{S+1, \ldots, 2S\})\,, \tag{3.5}$$

noting the nonnegativity (3.3). From Proposition 2.1, one observes that any vector in $\bigoplus_{0\leq J\leq S} V_J \subset V_S \otimes V_S$ of the k-th and $(k+1)$-th sites has the form

$$\sum_{0\leq A,B\leq S} C_{AB}\, x_k^A y_k^{S-A} x_{k+1}^B y_{k+1}^{S-B} \prod_{m=1}^{S} (q^m x_k y_{k+1} - q^{-m} y_k x_{k+1}), \tag{3.6}$$

where C_{AB} does not depend on x_k, y_k, x_{k+1} or y_{k+1}. Thus, the condition (3.5) imposes the restriction that $|\Psi\rangle$ has the form

$$|\Psi\rangle = P\left(\{x_k\}_{k\in\mathbb{Z}_L}, \{y_k\}_{k\in\mathbb{Z}_L}\right) \prod_{k\in\mathbb{Z}_L} \prod_{m=1}^{S} (q^m x_k y_{k+1} - q^{-m} y_k x_{k+1}) \tag{3.7}$$

with some polynomials P such that this form is consistent with (3.6) for $\forall k \in \mathbb{Z}_L$. The unique choice of P with such consistency is a constant (which can be set to be 1), and we achieve the unique ground state

$$|\Psi\rangle = \prod_{k\in\mathbb{Z}_L} \prod_{m=1}^{S} (q^m x_k y_{k+1} - q^{-m} y_k x_{k+1}). \tag{3.8}$$

In the Schwinger boson representation, we have

$$|\Psi\rangle = \prod_{k\in\mathbb{Z}_L} \prod_{m=1}^{S} (q^m a_k^\dagger b_{k+1}^\dagger - q^{-m} b_k^\dagger a_{k+1}^\dagger)|\text{vac}\rangle, \tag{3.9}$$

which is a generalization of the $q = 1$ case [6]. Note that each site have the correct spin value: $N_k|\Psi\rangle = S|\Psi\rangle$ $(k \in \mathbb{Z}_L)$ where $N_k := (N_{a_k} + N_{b_k})/2$. Our ground state is a q-deformation of the valence-bond-solid (VBS) state, which we call q-VBS state, see figure 1.

The Schwinger boson representation of the ground state (3.9) can be transformed into the following equivalent form called the matrix product representation [22], which generalizes the $q = 1$ [25] or $S = 1$ [7] case. Noting (2.7), we have

$$|\Psi\rangle = \mathrm{Tr}[g_1 \star g_2 \star \cdots \star g_{L-1} \star g_L], \tag{3.10}$$

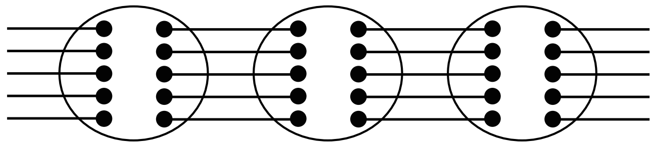

Figure 1: Conceptual figure of the q-VBS state. Each line is a q-deformed valence bond, and the circle ◯ represents the q-symmetrization of spin-1/2 particles • at each site.

where g_k is an $(S+1)\times(S+1)$ vector-valued matrix acting on the k-th site whose element is given by

$$\begin{aligned} g_k(i,i') &= (-1)^{S-i} q^{(i+i'-S)(S+1)/2} \sqrt{\begin{bmatrix} S \\ i \end{bmatrix} \begin{bmatrix} S \\ i' \end{bmatrix} [S-i+i']![S+i-i']!}\ |S; i'-i\rangle_k \\ &=: h_{ii'} |S; i'-i\rangle_k, \qquad (0 \le i, i' \le S). \end{aligned} \tag{3.11}$$

The symbol $\star$ for two $(S+1)\times(S+1)$ vector-valued matrices

$$x = \begin{pmatrix} |x_{00}\rangle & \cdots & |x_{0S}\rangle \\ \vdots & \ddots & \vdots \\ |x_{S0}\rangle & \cdots & |x_{SS}\rangle \end{pmatrix}, \quad y = \begin{pmatrix} |y_{00}\rangle & \cdots & |y_{0S}\rangle \\ \vdots & \ddots & \vdots \\ |y_{S0}\rangle & \cdots & |y_{SS}\rangle \end{pmatrix}, \tag{3.12}$$

is defined by

$$x \star y = \begin{pmatrix} \sum_{u=0}^S |x_{0u}\rangle \otimes |y_{u0}\rangle & \cdots & \sum_{u=0}^S |x_{0u}\rangle \otimes |y_{uS}\rangle \\ \vdots & \ddots & \vdots \\ \sum_{u=0}^S |x_{Su}\rangle \otimes |y_{u0}\rangle & \cdots & \sum_{u=0}^S |x_{Su}\rangle \otimes |y_{uS}\rangle \end{pmatrix}, \tag{3.13}$$

which is apparently an additive operation.

For example, for $S=2$,

$$g_k = \begin{pmatrix} h_{00}|2;0\rangle_k & h_{01}|2;1\rangle_k & h_{02}|2;2\rangle_k \\ h_{10}|2;-1\rangle_k & h_{11}|2;0\rangle_k & h_{12}|2;1\rangle_k \\ h_{20}|2;-2\rangle_k & h_{21}|2;-1\rangle_k & h_{22}|2;0\rangle_k \end{pmatrix}, \tag{3.14}$$

and the product in the form (3.10) is calculated as

$$\begin{aligned} & g_1 \star \cdots \star g_L \\ &= \begin{pmatrix} (g_1 \star \cdots \star g_L)(0,0) & (g_1 \star \cdots \star g_L)(0,1) & (g_1 \star \cdots \star g_L)(0,2) \\ (g_1 \star \cdots \star g_L)(1,0) & (g_1 \star \cdots \star g_L)(1,1) & (g_1 \star \cdots \star g_L)(1,2) \\ (g_1 \star \cdots \star g_L)(2,0) & (g_1 \star \cdots \star g_L)(2,1) & (g_1 \star \cdots \star g_L)(2,2) \end{pmatrix}, \end{aligned} \tag{3.15}$$

with

$$\begin{aligned} & (g_1 \star \cdots \star g_L)(i,i') \\ &= \sum_{i_k=0,1,2} h_{ii_2} h_{i_2 i_3} \cdots h_{i_{L-1} i_L} h_{i_L i'} \\ &\qquad \times |2; i_2 - i\rangle_1 \otimes |2; i_3 - i_2\rangle_2 \otimes \cdots \otimes |2; i_L - i_{L-1}\rangle_{L-1} \otimes |2; i' - i_L\rangle_L. \end{aligned} \tag{3.16}$$

Then the matrix product ground state (3.10) is

$$\begin{aligned}
&(g_1 \star \cdots \star g_L)(0,0) + (g_1 \star \cdots \star g_L)(1,1) + (g_1 \star \cdots \star g_L)(2,2) \\
&= \sum_{i_k=0,1,2} h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{L-1} i_L} h_{i_L i_1} \\
&\quad \times |2; i_2 - i_1\rangle_1 \otimes |2; i_3 - i_2\rangle_2 \otimes \cdots \otimes |2; i_L - i_{L-1}\rangle_{L-1} \otimes |2; i_1 - i_L\rangle_L .
\end{aligned} \tag{3.17}$$

We define $g_k^\dagger$ by replacing each ket vector in the matrix g_k by its corresponding vectors:

$$g_k^\dagger(i,i') = h_{ii'}\ {}_k\langle S; i'-i|. \tag{3.18}$$

For example, for $S=2$,

$$g_k^\dagger = \begin{pmatrix} h_{00}\ {}_k\langle 2;0| & h_{01}\ {}_k\langle 2;1| & h_{02}\ {}_k\langle 2;2| \\ h_{10}\ {}_k\langle 2;-1| & h_{11}\ {}_k\langle 2;0| & h_{12}\ {}_k\langle 2;1| \\ h_{20}\ {}_k\langle 2;-2| & h_{21}\ {}_k\langle 2;-1| & h_{22}\ {}_k\langle 2;0| \end{pmatrix}. \tag{3.19}$$

Now we introduce "G matrix", which will play an important role in our study. Let us set an $(S+1)^2$ dimensional vector space W and its dual orthogonal space W^* as

$$W = \bigoplus_{0\le a,b\le S} \mathbb{C}|a,b\rangle\!\rangle, \quad W^* = \bigoplus_{0\le a,b\le S} \mathbb{C}\langle\!\langle a,b|. \tag{3.20}$$

Here, $\{|a,b\rangle\!\rangle \mid a,b=0,\ldots,S\}$ ($\{\langle\!\langle a,b| \mid a,b=0,\ldots,S\}$) is an orthonormal (dual orthonormal) basis. We define an $(S+1)^2 \times (S+1)^2$ matrix G acting on the space W as

$$G_{(a,b;c,d)} = \langle\!\langle a,b|G|c,d\rangle\!\rangle = g^\dagger(a,c)g(b,d), \tag{3.21}$$

or equivalently as

$$G = g^\dagger \otimes g. \tag{3.22}$$

We also introduce G_A for an operator A acting on the one-site vector space V_S as

$$(G_A)_{(a,b;c,d)} = \langle\!\langle a,b|G_A|c,d\rangle\!\rangle = g^\dagger(a,c)Ag(b,d). \tag{3.23}$$

Each element of the matrix G can be expressed explicitly as

$$G_{(a,b;c,d)} = \delta_{c-a,d-b} T_{abcd}, \tag{3.24}$$

where

$$\begin{aligned}
T_{abcd} &= h_{ac}h_{bd} = (-1)^{a+b} q^{(a+b+c+d-2S)(S+1)/2} \\
&\times \sqrt{\begin{bmatrix} S \\ a \end{bmatrix}\begin{bmatrix} S \\ b \end{bmatrix}\begin{bmatrix} S \\ c \end{bmatrix}\begin{bmatrix} S \\ d \end{bmatrix} [S-a+c]![S+a-c]![S-b+d]![S+b-d]!}\ .
\end{aligned} \tag{3.25}$$

Each element of G_A for $A = S^z, S^+$ and S^-, which act on $|S;m\rangle$ as

$$S^z|S;m\rangle = m|S;m\rangle, \tag{3.26}$$
$$S^+|S;m\rangle = \sqrt{(S-m)(S+m+1)}|S;m+1\rangle, \tag{3.27}$$
$$S^-|S;m\rangle = \sqrt{(S+m)(S-m+1)}|S;m-1\rangle, \tag{3.28}$$

can be also expressed as

$$(G_{S^z})_{(a,b;c,d)} = \delta_{c-a,d-b}(d-b)T_{abcd}, \tag{3.29}$$
$$(G_{S^+})_{(a,b;c,d)} = \delta_{c-a,d-b+1}\sqrt{(S-d+b)(S+d-b+1)}T_{abcd}, \tag{3.30}$$
$$(G_{S^-})_{(a,b;c,d)} = \delta_{c-a,d-b-1}\sqrt{(S+d-b)(S-d+b+1)}T_{abcd}. \tag{3.31}$$

The squared norm of the ground state is calculated as

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= \mathrm{Tr}\left[g_1^\dagger \star \cdots \star g_L^\dagger\right] \mathrm{Tr}\left[g_1 \star \cdots \star g_L\right] \\
&= \mathrm{Tr}\left[\left(g_1^\dagger \star \cdots \star g_L^\dagger\right) \otimes \left(g_1 \star \cdots \star g_L\right)\right] \\
&= \mathrm{Tr}\left[\left(g_1^\dagger \otimes g_1\right) \star \cdots \star \left(g_L^\dagger \otimes g_L\right)\right] \\
&= \mathrm{Tr}\, G^L.
\end{aligned} \tag{3.32}$$

Note that the elements of $g_k^\dagger \otimes g_k = G$ are no longer vectors, and thus we can replace the symbol $\star$ by the usual product in the third line of (3.32). The one point function $\langle A\rangle$ of an operator A can be written in terms of G and G_A as

$$\langle A\rangle = \frac{\langle\Psi|A_1|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\mathrm{Tr}\left[g_1^\dagger \star \cdots \star g_L^\dagger\right] \mathrm{Tr}\left[A_1 g_1 \star g_2 \star \cdots \star g_L\right]}{\langle\Psi|\Psi\rangle} = \frac{\mathrm{Tr}\, G_A G^{L-1}}{\mathrm{Tr}\, G^L}, \tag{3.33}$$

where $A_k g_k$ is defined by $(A_k g_k)(i,i') = A_k(g_k(i,i'))$. In the same way, the two point function of A and B can also be written in terms of G, G_A and G_B as

$$\langle A_1 B_r\rangle = \left(\mathrm{Tr}\, G^L\right)^{-1} \mathrm{Tr}\, G_A G^{r-2} G_B G^{L-r}. \tag{3.34}$$

Investigating the eigenvalues and eigenvectors of the matrix G will be crucial for the analysis of correlation functions. In the next section, we study the G matrix in detail.

4 Spectral structure of the G matrix

In [22], we conjectured that the spectrum of G is given by

$$\lambda_\ell = (-1)^\ell \left([S]!\right)^2 \begin{bmatrix} 2S+1 \\ S-\ell \end{bmatrix}, \quad (\ell = 0, 1, \ldots, S), \tag{4.1}$$

where the degree of the degeneracy of each λ_ℓ is $2\ell+1$. One can easily find that

$$|\lambda_0| > |\lambda_1| > \cdots > |\lambda_S|. \tag{4.2}$$

In this section, we prove the conjecture by giving an exact form for the eigenvector corresponding to each eigenvalue.
First one observes that the G matrix has the following block diagonal structure:

$$G = \bigoplus_{-S \le j \le S} G^{(j)}, \quad G^{(j)} \in \mathrm{End} W_j, \tag{4.3}$$

$$W = \bigoplus_{-S \le j \le S} W_j, \quad W_j = \begin{cases} \displaystyle\bigoplus_{0 \le i \le S-j} \mathbb{C}|i, i+j\rangle\rangle & j \ge 0, \\ \displaystyle\bigoplus_{0 \le i \le S+j} \mathbb{C}|i-j, i\rangle\rangle & j < 0. \end{cases} \tag{4.4}$$

The size of each block $G^{(j)}$ is $(S-|j|+1) \times (S-|j|+1)$. Each element of $G^{(j)}$ is

$$\begin{aligned} \langle\langle a, a+j|G^{(j)}|c, c+j\rangle\rangle =& (-1)^j q^{(a+c+j-S)(S+1)} [S-a+c]! \, [S+a-c]! \\ & \times \sqrt{\begin{bmatrix} S \\ a \end{bmatrix} \begin{bmatrix} S \\ a+j \end{bmatrix} \begin{bmatrix} S \\ c \end{bmatrix} \begin{bmatrix} S \\ c+j \end{bmatrix}} . \end{aligned} \tag{4.5}$$

We construct intertwiners among the $2S+1$ block diagonal matrices $G^{(j)}$ $(j = -S, \ldots, S)$. This helps us to construct eigenvectors of each block diagonal matrix from another block with a smaller size. (The same idea was used in [26] to study the spectrum of a multi-species exclusion process). Let us define a family of linear operators $\{I_j\}_{-S \le j \le -1, 1 \le j \le S}$ as

$$I_j \in \mathrm{Hom}(W_j, W_{j-1}), \tag{4.6}$$

$$\langle\langle a, a+j-1|I_j|c, c+j\rangle\rangle = \begin{cases} q^{-a}\sqrt{\dfrac{[a+j]\,[S-a-j+1]}{[j]\,[S-j+1]}} & c = a, \\ -q^{1-a-j}\sqrt{\dfrac{[a]\,[S-a+1]}{[j]\,[S-j+1]}} & c = a-1, \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

for $1 \le j \le S$, and

$$I_j \in \mathrm{Hom}(W_j, W_{j+1}), \tag{4.8}$$

$$\langle\langle a-j-1, a|I_j|c-j, c\rangle\rangle = \begin{cases} q^{-a}\sqrt{\dfrac{[a-j]\,[S-a+j+1]}{[-j]\,[S+j+1]}} & c = a, \\ -q^{1-a+j}\sqrt{\dfrac{[a]\,[S-a+1]}{[-j]\,[S+j+1]}} & c = a-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

for $-S \le j \le -1$. By direct calculation, one finds

Proposition 4.1. *The matrix I_j enjoys the intertwining relation*

$$\begin{aligned} I_j G^{(j)} =& G^{(j-1)} I_j \quad \text{for } 1 \le j \le S, \\ I_j G^{(j)} =& G^{(j+1)} I_j \quad \text{for } -S \le j \le -1. \end{aligned} \tag{4.10}$$

With the use of Proposition 4.1, one can show the following:

Theorem 4.2. *Each block matrix $G^{(j)}$ has a simple (nondegenerated) spectrum*

$$\text{Spec } G^{(j)} = \{\lambda_\ell\}_{|j| \le \ell \le S}, \tag{4.11}$$

and the corresponding eigenvectors are given by

$$|\lambda_{|j|}\rangle\!\rangle_j = \begin{cases} \displaystyle\sum_{0 \le i \le S-\ell} q^{(\ell+1)i} \sqrt{\frac{[S-\ell]!\,[i+\ell]!\,[S-i]!}{[S]![\ell]![S-i-\ell]![i]!}} |i, i+\ell\rangle\!\rangle & j \ge 0, \\ \displaystyle\sum_{0 \le i \le S-\ell} q^{(\ell+1)i} \sqrt{\frac{[S-\ell]!\,[i+\ell]!\,[S-i]!}{[S]![\ell]![S-i-\ell]![i]!}} |i+\ell, i\rangle\!\rangle & j < 0, \end{cases} \tag{4.12}$$

for $\ell = |j|$, and

$$|\lambda_\ell\rangle\!\rangle_j = \begin{cases} I_{j+1}|\lambda_\ell\rangle\!\rangle_{j+1} = I_{j+1} I_{j+2} \cdots I_\ell |\lambda_\ell\rangle\!\rangle_\ell & j \ge 0, \\ I_{j-1}|\lambda_\ell\rangle\!\rangle_{j-1} = I_{j-1} I_{j-2} \cdots I_{-\ell} |\lambda_\ell\rangle\!\rangle_{-\ell} & j < 0. \end{cases} \tag{4.13}$$

for $|j| + 1 \le \ell \le S$.

Figure 2 is helpful to understand how the eigenvectors are constructed. We prove this theorem below for only $j \ge 0$ since one can show it for $j < 0$ in the same way.

Proof of Theorem 4.2. First, by direct calculation given below, we find that $G^{(j)}$ has an eigenvalue λ_j and its eigenvector is $|\lambda_j\rangle\!\rangle_j$ defined by (4.12). Each element of $G^{(j)}|\lambda_j\rangle\!\rangle_j$ is calculated as

$$\begin{aligned} &\langle\!\langle a, a+j | G^{(j)} | \lambda_j \rangle\!\rangle_j \\ &= \sum_{0 \le c \le S-j} (-1)^j q^{(a+c+j-S)(S+1)} [S-a+c]!\,[S+a-c]! \\ &\quad \times \sqrt{\begin{bmatrix} S \\ a \end{bmatrix} \begin{bmatrix} S \\ a+j \end{bmatrix} \begin{bmatrix} S \\ c \end{bmatrix} \begin{bmatrix} S \\ c+j \end{bmatrix}} q^{(j+1)c} \sqrt{\frac{[S-j]!\,[c+j]!\,[S-c]!}{[S]!\,[j]!\,[S-c-j]!\,[c]!}} \\ &= (-1)^j q^{(a+j-S-1)(S+1)-(j+1)} \sqrt{\begin{bmatrix} S \\ a \end{bmatrix} \begin{bmatrix} S \\ a+j \end{bmatrix} \frac{[S-j]!}{[S]!\,[j]!}} \\ &\quad \times [S]! \sum_{0 \le c \le S-j} q^{(c+1)(S+j+2)} \frac{[S-a+c]!\,[S+a-c]!}{[S-c-j]![c]!}. \end{aligned} \tag{4.14}$$

Using the formula

$$\sum_{0 \le k \le n} \begin{bmatrix} \alpha+n-k \\ n-k \end{bmatrix} \begin{bmatrix} \beta+k \\ k \end{bmatrix} q^{k(\alpha+\beta+2)} = \begin{bmatrix} \alpha+\beta+n+1 \\ n \end{bmatrix} q^{n(1+\beta)}, \tag{4.15}$$

$$
\begin{array}{llllllll}
W_S & |\lambda_S\rangle\!\rangle_S & & & & & & \\
\downarrow_{I_S} & \downarrow & & & & & & \\
W_{S-1} & |\lambda_S\rangle\!\rangle_{S-1} & |\lambda_{S-1}\rangle\!\rangle_{S-1} & & & & & \\
\downarrow_{I_{S-1}} & \downarrow & \downarrow & & & & & \\
W_{S-2} & |\lambda_S\rangle\!\rangle_{S-2} & |\lambda_{S-1}\rangle\!\rangle_{S-2} & |\lambda_{S-2}\rangle\!\rangle_{S-2} & & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & \\
W_2 & |\lambda_S\rangle\!\rangle_2 & |\lambda_{S-1}\rangle\!\rangle_2 & |\lambda_{S-2}\rangle\!\rangle_2 & \cdots & |\lambda_2\rangle\!\rangle_2 & & \\
\downarrow_{I_2} & \downarrow & \downarrow & \downarrow & & \downarrow & & \\
W_1 & |\lambda_S\rangle\!\rangle_1 & |\lambda_{S-1}\rangle\!\rangle_1 & |\lambda_{S-2}\rangle\!\rangle_1 & \cdots & |\lambda_2\rangle\!\rangle_1 & |\lambda_1\rangle\!\rangle_1 & \\
\downarrow_{I_1} & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\
W_0 & |\lambda_S\rangle\!\rangle_0 & |\lambda_{S-1}\rangle\!\rangle_0 & |\lambda_{S-2}\rangle\!\rangle_0 & \cdots & |\lambda_2\rangle\!\rangle_0 & |\lambda_1\rangle\!\rangle_0 & |\lambda_0\rangle\!\rangle_0 \\
\uparrow_{I_{-1}} & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \\
W_{-1} & |\lambda_S\rangle\!\rangle_{-1} & |\lambda_{S-1}\rangle\!\rangle_{-1} & |\lambda_{S-2}\rangle\!\rangle_{-1} & \cdots & |\lambda_2\rangle\!\rangle_{-1} & |\lambda_1\rangle\!\rangle_{-1} & \\
\uparrow_{I_{-2}} & \uparrow & \uparrow & \uparrow & & \uparrow & & \\
W_{-2} & |\lambda_S\rangle\!\rangle_{-2} & |\lambda_{S-1}\rangle\!\rangle_{-2} & |\lambda_{S-2}\rangle\!\rangle_{-2} & \cdots & |\lambda_2\rangle\!\rangle_{-2} & & \\
\vdots & \vdots & \vdots & \vdots & \iddots & & & \\
W_{-S+2} & |\lambda_S\rangle\!\rangle_{-S+2} & |\lambda_{S-1}\rangle\!\rangle_{-S+2} & |\lambda_{S-2}\rangle\!\rangle_{-S+2} & & & & \\
\uparrow_{I_{-S+1}} & \uparrow & \uparrow & & & & & \\
W_{-S+1} & |\lambda_S\rangle\!\rangle_{-S+1} & |\lambda_{S-1}\rangle\!\rangle_{-S+1} & & & & & \\
\uparrow_{I_{-S}} & \uparrow & & & & & & \\
W_{-S} & |\lambda_S\rangle\!\rangle_{-S} & & & & & &
\end{array}
$$

Figure 2: Structure of the eigenvectors of G (3.21).

we obtain

$$
\begin{aligned}
\langle\!\langle a, a+j | G^{(j)} | \lambda_j \rangle\!\rangle_j &= (-1)^j q^{(a+j-S-1)(S+1)-(j+1)} \sqrt{\begin{bmatrix} S \\ a \end{bmatrix} \begin{bmatrix} S \\ a+j \end{bmatrix} \frac{[S-j]!}{[S]!\,[j]!}} \\
&\quad \times [S]!\, q^{-Sj+a(j-S)+S^2+2S+2} \begin{bmatrix} 2S+1 \\ S-j \end{bmatrix} [S-a]!\,[a+j]! \\
&= (-1)^j \left([S]!\right)^2 \begin{bmatrix} 2S+1 \\ S-j \end{bmatrix} q^{(j+1)a} \sqrt{\frac{[S-j]!\,[a+j]!\,[S-a]!}{[S]!\,[j]!\,[S-a-j]!\,[a]!}} \\
&= \lambda_j \langle\!\langle a, a+j | \lambda_j \rangle\!\rangle_j .
\end{aligned} \tag{4.16}
$$

Note that the first element of $|\lambda_j\rangle\!\rangle_j$ is 1 by the definition (4.12): $\langle\!\langle 0, j | \lambda_j \rangle\!\rangle_j = 1$.

Next, we show by induction that $G^{(j)}$ has eigenvalues λ_ℓ ($j \le \ell \le S$) and their corresponding eigenvectors are given by $|\lambda_\ell\rangle\!\rangle_j$ defined by (4.13). Suppose the theorem is true for $|\lambda_\ell\rangle\!\rangle_{j+1}, \ell = j+1, \dots, S$ ($j \ge 0$), that is to say that the block diagonal matrix $G^{(j+1)}$ has the eigenvalues λ_ℓ and their corresponding eigenvectors $|\lambda_\ell\rangle\!\rangle_{j+1}$ $\left(G^{(j+1)} |\lambda_\ell\rangle\!\rangle_{j+1} = \lambda_\ell |\lambda_\ell\rangle\!\rangle_{j+1} \text{ with } |\lambda_\ell\rangle\!\rangle_{j+1} \neq 0\right)$ for $\ell = j+1, \dots, S$. Additionally, suppose that the first element of each $|\lambda_\ell\rangle\!\rangle_{j+1}$ is 1. Using the intertwining relation (4.10), one finds $G^{(j)} I_{j+1}|\lambda_\ell\rangle\!\rangle_{j+1} = \lambda_\ell I_{j+1}|\lambda_\ell\rangle\!\rangle_{j+1}$. We also find that the first element of $I_{j+1}|\lambda_\ell\rangle\!\rangle_{j+1}$ is 1, and thus $I_{j+1}|\lambda_\ell\rangle\!\rangle_{j+1}$ is nonzero. Furthermore, thanks to $\ell_1 \neq \ell_2 \Rightarrow \lambda_{\ell_1} \neq \lambda_{\ell_2}$, the vectors $I_{j+1}|\lambda_\ell\rangle\!\rangle_{j+1}$ $(j+1 \leq \ell \leq S)$ are distinct (in other words, I_{j+1} is injective). We have already constructed the remaining eigenvector of $G^{(j)}$ explicitly, which is $|\lambda_j\rangle\!\rangle_j$ with its eigenvalue λ_j distinct from λ_ℓ $(j+1 \leq \ell \leq S)$. □

The conjecture for the eigenvalues of the G matrix that we exhibited in the beginning of this section follows as a simple corollary of Theorem 4.2. Moreover, we constructed their eigenvectors which are important for computing spin-spin correlation functions.

Proposition 4.3. *The squared norm of* $|\lambda_\ell\rangle\!\rangle_j$ *is*

$$
{}_j\langle\!\langle\lambda_\ell|\lambda_\ell\rangle\!\rangle_j = q^{S(|j|+1)-\ell(\ell+1)} \frac{[S+\ell+1]![\ell-|j|]![S-\ell]![|j|]!}{[S]![\ell+|j|]![S-|j|]![2\ell+1]}, \tag{4.17}
$$

where we denote the transpose of $|\lambda_\ell\rangle\!\rangle_j$ *by* ${}_j\langle\!\langle\lambda_\ell|$.

We prove this proposition only for $j \geq 0$.

Proof of Proposition 4.3. One can easily show that the product of intertwiners (which is also an intertwiner) has the following form by induction:

$$
\begin{aligned}
&\langle\!\langle a, a+j|I_{j+1}I_{j+2}\cdots I_{\ell+1}I_\ell|c, c+\ell\rangle\!\rangle \\
=&(-1)^{a-c} q^{cj-a\ell} \begin{bmatrix} \ell-j \\ a-c \end{bmatrix} \sqrt{\frac{[j]![S-\ell]![a]![S-c]![c+\ell]![S-(a+j)]!}{[\ell]![S-j]![c]![S-a]![a+j]![S-(c+\ell)]!}}.
\end{aligned} \tag{4.18}
$$

Then, ${}_j\langle\!\langle\lambda_\ell|\lambda_\ell\rangle\!\rangle_j = {}_\ell\langle\!\langle\lambda_\ell|\left(I_{j+1}\cdots I_\ell\right)^{\mathrm{T}} I_{j+1}\cdots I_\ell|\lambda_\ell\rangle\!\rangle_\ell$ is calculated as

$$
\begin{aligned}
{}_j\langle\!\langle\lambda_\ell|\lambda_\ell\rangle\!\rangle_j =& \frac{([S-\ell]!)^2 [j]!}{[S]!([\ell]!)^2 [S-j]!} \sum_{\substack{0\leq a\leq S-j \\ 0\leq i,i'\leq S}} (-1)^{i+i'} q^{(i+i')(\ell+j+1)-2a\ell} \\
&\times \begin{bmatrix} \ell-j \\ a-i \end{bmatrix} \begin{bmatrix} \ell-j \\ a-i' \end{bmatrix} \frac{[S-i]![i+\ell]![S-i']![i'+\ell]![a]![S-(a+j)]!}{[i]![S-(i+\ell)]![i']![S-(i'+\ell)]![S-a]![a+j]!}.
\end{aligned} \tag{4.19}
$$

The triple sum has the closed form

$$
q^{(j+1)S-\ell(\ell+1)} \frac{([\ell]!)^2 [\ell-j]![S+\ell+1]!}{[S-\ell]![j+\ell]![2\ell+1]}, \tag{4.20}
$$

which finishes the proof. □

5 Spin-spin correlation functions

In the last section, we investigated the eigenvalues and eigenvectors of the G matrix. By utilizing Theorem 4.2 and noting (4.2), the one point function $\langle A\rangle$ can be represented as

$$
\langle A\rangle = \lambda_0^{-1} \frac{{}_0\langle\!\langle\lambda_0|G_A|\lambda_0\rangle\!\rangle_0}{{}_0\langle\!\langle\lambda_0|\lambda_0\rangle\!\rangle_0} \tag{5.1}
$$

in the thermodynamic limit $L \to \infty$. As an application, we can calculate the probability of finding $S^z = m$ value as

$$\begin{aligned}\text{Prob}(S^z = m) =& \langle\, |S;m\rangle\langle S;m|\,\rangle \\ =& \frac{[S+m]![S-m]!}{[2S+1]!} \sum_{i=0}^{S} q^{(S+2)(2i-m-S)} \left[\begin{array}{c} S \\ i-m \end{array}\right] \left[\begin{array}{c} S \\ i \end{array}\right].\end{aligned} \tag{5.2}$$

The two point function (3.34) can be also represented as

$$\langle A_1 B_r \rangle = \sum_{\ell=0}^{S} \lambda_\ell^{-2} \left(\frac{\lambda_\ell}{\lambda_0}\right)^r \sum_{j=-\ell}^{\ell} \frac{{}_0\langle\!\langle \lambda_0 | G_A | \lambda_\ell \rangle\!\rangle_{jj} \langle\!\langle \lambda_\ell | G_B | \lambda_0 \rangle\!\rangle_0}{{}_0\langle\!\langle \lambda_0 | \lambda_0 \rangle\!\rangle_{0j} \langle\!\langle \lambda_\ell | \lambda_\ell \rangle\!\rangle_j}, \tag{5.3}$$

in the thermodynamic limit. Inserting (3.29), (3.30), (3.31), (4.1), (4.12), (4.13) and (4.17) into (5.3), one finds the large-distance ($r \to \infty$) behaviors of the spin-spin correlation functions $\langle S_1^z S_r^z \rangle$ and $\langle S_1^+ S_r^- \rangle$ are

$$\langle S_1^z S_r^z \rangle = -\frac{[3][S+2]}{q^{2S-2}[S]([2S+1]!)^2} ({}_0\langle\!\langle \lambda_1 | G_{S^z} | \lambda_0 \rangle\!\rangle_0)^2 \left(-\frac{[S]}{[S+2]}\right)^r, \tag{5.4}$$

$$\langle S_1^+ S_r^- \rangle = -\frac{[2][3][S+2]}{q^{3S-2}([2S+1]![S])^2} ({}_{-1}\langle\!\langle \lambda_1 | G_{S^-} | \lambda_0 \rangle\!\rangle_0)^2 \left(-\frac{[S]}{[S+2]}\right)^r, \tag{5.5}$$

where

$$\begin{aligned}{}_0\langle\!\langle \lambda_1 | G_{S^z} | \lambda_0 \rangle\!\rangle_0 =& \frac{q^{-S^2-S-1}}{q^S - q^{-S}} \sum_{i,i'=0}^{S} (i-i') q^{(S+2)(i+i')} \\ & \times \{q^{S+1} + q^{-S-1} - (q+q^{-1}) q^{2i'-S}\} [S+i-i']! [S+i'-i]! \left[\begin{array}{c} S \\ i \end{array}\right] \left[\begin{array}{c} S \\ i' \end{array}\right],\end{aligned} \tag{5.6}$$

$$\begin{aligned}{}_{-1}\langle\!\langle \lambda_1 | G_{S^-} | \lambda_0 \rangle\!\rangle_0 =& {}_0\langle\!\langle \lambda_0 | G_{S^+} | \lambda_1 \rangle\!\rangle_{-1} \\ =& -q^{-S^2-S/2+1/2} \sum_{i=0}^{S} \sum_{i'=0}^{S-1} q^{(S+2)i+(S+3)i'} \sqrt{\left[\begin{array}{c} S \\ i'+1 \end{array}\right] \left[\begin{array}{c} S \\ i' \end{array}\right]} \\ & \times \sqrt{(S+i-i')[S+i-i'](S-i+i'+1)[S-i+i'+1]} \\ & \times \sqrt{[i'+1][S-i'][S]^{-1}} [S+i'-i]! [S+i-i'-1]! \left[\begin{array}{c} S \\ i \end{array}\right].\end{aligned} \tag{5.7}$$

Note that the terms with $(j, \ell) = (0, 1)$ and $(-1, 1)$ in (5.3) dominate the large-distance behaviors of $\langle S_1^z S_r^z \rangle$ and $\langle S_1^+ S_r^- \rangle$, respectively, since

$${}_0\langle\!\langle \lambda_0 | G_{S^z} | \lambda_0 \rangle\!\rangle_0 = {}_1\langle\!\langle \lambda_1 | G_{S^z} | \lambda_0 \rangle\!\rangle_0 = {}_{-1}\langle\!\langle \lambda_1 | G_{S^z} | \lambda_0 \rangle\!\rangle_0 = 0, \tag{5.8}$$

$${}_0\langle\!\langle \lambda_0 | G_{S^-} | \lambda_0 \rangle\!\rangle_0 = {}_1\langle\!\langle \lambda_1 | G_{S^-} | \lambda_0 \rangle\!\rangle_0 = {}_0\langle\!\langle \lambda_1 | G_{S^-} | \lambda_0 \rangle\!\rangle_0 = 0. \tag{5.9}$$

Both $\langle S_1^z S_r^z \rangle$ and $\langle S_1^+ S_r^- \rangle$ exhibit exponential decay with correlation length

$$\zeta = \left(\ln \frac{[S+2]}{[S]}\right)^{-1}, \tag{5.10}$$

generalizing the results for $q = 1$ [18] or $S = 1$ [7] case.

6 Conclusion

In this paper, we investigated one and two point functions of the q-VBS ground state of an integer spin model (the q-deformed higher-spin AKLT model). The formulation of correlation functions by use of the matrix product representation of the ground state shows that the structure of a matrix, which we call G matrix, plays an important role. We obtained the eigenvalues and eigenvectors of the G matrix with the help of constructing intertwiners connecting different block diagonal matrices of G. Then we calculated the spin-spin correlation functions by use of the eigenvalues and eigenvectors of the G matrix, and determined the correlation amplitudes and correlation lengths of the longitudinal and transverse spin-spin correlation functions.

It is interesting to investigate other types of correlation functions. For example, the entanglement entropy, which is defined in terms of the reduced density matrix, is a typical quantification of the entanglement of quantum systems. It is intriguing to calculate the entanglement entropy for the q-deformed model and observe the change from the isotropic point [19, 27, 28] (see also [29, 30] for other VBS states).

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A Proof of Proposition 2.1

We prove Proposition 2.1 by induction. Suppose that Proposition 2.1 holds for $\left(\Delta X^-_{\alpha\beta}\right)^n v_J$. We calculate the four terms of the action of

$$\begin{aligned}\Delta X^-_{\alpha\beta} =& \frac{1}{q-q^{-1}}\frac{y_\alpha}{x_\alpha}D^{x_\alpha}_q \otimes D^{x_\beta}_{\sqrt{q}}D^{y_\beta}_{1/\sqrt{q}} - \frac{1}{q-q^{-1}}\frac{y_\alpha}{x_\alpha}D^{x_\alpha}_{q^{-1}} \otimes D^{x_\beta}_{\sqrt{q}}D^{y_\beta}_{1/\sqrt{q}} \\ &+ \frac{1}{q-q^{-1}}D^{x_\alpha}_{1/\sqrt{q}}D^{y_\alpha}_{\sqrt{q}} \otimes \frac{y_\beta}{x_\beta}D^{x_\beta}_q - \frac{1}{q-q^{-1}}D^{x_\alpha}_{1/\sqrt{q}}D^{y_\alpha}_{\sqrt{q}} \otimes \frac{y_\beta}{x_\beta}D^{x_\beta}_{q^{-1}},\end{aligned} \tag{A.1}$$

on $\left(\Delta X^-_{\alpha\beta}\right)^n v_J$, separately.

$$\begin{aligned}&\left(\frac{y_\alpha}{x_\alpha}D^{x_\alpha}_q \otimes D^{x_\beta}_{\sqrt{q}}D^{y_\beta}_{1/\sqrt{q}}\right)\left(\Delta X^-_{\alpha\beta}\right)^n v_J \Big/ q^{(n+1)S}[n]!(x_\alpha x_\beta)^{J-(n+1)} \\ &= \sum_{\mu=0}^{n} q^{-2\mu S+J-n}\begin{bmatrix} J \\ \mu \end{bmatrix}\begin{bmatrix} J \\ n-\mu \end{bmatrix}(x_\alpha y_\beta)^\mu (x_\beta y_\alpha)^{n+1-\mu}\prod_{\nu=1}^{2S-J}\left(x_\alpha y_\beta - q^{2(\nu-S-1)}x_\beta y_\alpha\right),\end{aligned} \tag{A.2}$$

$$\left(\frac{y_\alpha}{x_\alpha} D^{x_\alpha}_{q^{-1}} \otimes D^{x_\beta}_{\sqrt{q}} D^{y_\beta}_{1/\sqrt{q}}\right) \left(\Delta X^-_{\alpha\beta}\right)^n v_J \Big/ q^{(n+1)S}[n]!(x_\alpha x_\beta)^{J-(n+1)} \tag{A.3}$$

$$= \sum_{\mu=0}^{n} q^{-2\mu S-2\mu-4S+J+n} \left[\begin{array}{c} J \\ \mu \end{array}\right] \left[\begin{array}{c} J \\ n-\mu \end{array}\right] (x_\alpha y_\beta)^\mu \, (x_\beta y_\alpha)^{n+1-\mu} \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S)} x_\beta y_\alpha\right),$$

$$\left(D^{x_\alpha}_{1/\sqrt{q}} D^{y_\alpha}_{\sqrt{q}} \otimes \frac{y_\beta}{x_\beta} D^{x_\beta}_{q}\right) \left(\Delta X^-_{\alpha\beta}\right)^n v_J \Big/ q^{(n+1)S}[n]!(x_\alpha x_\beta)^{J-(n+1)} \tag{A.4}$$

$$= \sum_{\mu=0}^{n} q^{-2\mu S-2\mu-2S+J+n} \left[\begin{array}{c} J \\ \mu \end{array}\right] \left[\begin{array}{c} J \\ n-\mu \end{array}\right] (x_\alpha y_\beta)^{\mu+1} \, (x_\beta y_\alpha)^{n-\mu} \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S)} x_\beta y_\alpha\right),$$

$$\left(D^{x_\alpha}_{1/\sqrt{q}} D^{y_\alpha}_{\sqrt{q}} \otimes \frac{y_\beta}{x_\beta} D^{x_\beta}_{q^{-1}}\right) \left(\Delta X^-_{\alpha\beta}\right)^n v_J \Big/ q^{(n+1)S}[n]!(x_\alpha x_\beta)^{J-(n+1)} \tag{A.5}$$

$$= \sum_{\mu=0}^{n} q^{-2\mu S-2S-J+n} \left[\begin{array}{c} J \\ \mu \end{array}\right] \left[\begin{array}{c} J \\ n-\mu \end{array}\right] (x_\alpha y_\beta)^{\mu+1} \, (x_\beta y_\alpha)^{n-\mu} \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha\right).$$

(A.4) − (A.3) gives

$$\begin{aligned}
&\sum_{\mu=0}^{n} q^{-2\mu S-2\mu-2S+J+n} \left[\begin{array}{c} J \\ \mu \end{array}\right] \left[\begin{array}{c} J \\ n-\mu \end{array}\right] (x_\alpha y_\beta)^{\mu} \, (x_\beta y_\alpha)^{n-\mu} \\
&\quad \times \left(x_\alpha y_\beta - q^{-2S} x_\beta y_\alpha\right) \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S)} x_\beta y_\alpha\right) \\
&= \sum_{\mu=0}^{n} q^{-2\mu S-2\mu-2S+J+n} \left[\begin{array}{c} J \\ \mu \end{array}\right] \left[\begin{array}{c} J \\ n-\mu \end{array}\right] (x_\alpha y_\beta)^{\mu} \, (x_\beta y_\alpha)^{n-\mu} \\
&\quad \times \left(x_\alpha y_\beta - q^{2(S-J)} x_\beta y_\alpha\right) \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha\right) \\
&= \sum_{\mu=0}^{n} q^{-2\mu S-2\mu-2S+J+n} \left[\begin{array}{c} J \\ \mu \end{array}\right] \left[\begin{array}{c} J \\ n-\mu \end{array}\right] (x_\alpha y_\beta)^{\mu+1} \, (x_\beta y_\alpha)^{n-\mu} \\
&\quad \times \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha\right) \\
&\quad - \sum_{\mu=0}^{n} q^{-2\mu S-2\mu-J+n} \left[\begin{array}{c} J \\ \mu \end{array}\right] \left[\begin{array}{c} J \\ n-\mu \end{array}\right] (x_\alpha y_\beta)^{\mu} \, (x_\beta y_\alpha)^{n+1-\mu} \\
&\quad \times \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha\right).
\end{aligned} \tag{A.6}$$

Dividing the first term of (A.6) − (A.5) by $q - q^{-1}$, we obtain

$$\begin{aligned}
&\sum_{\mu=0}^{n} q^{-2\mu S-\mu-2S+n}[J-\mu] \left[\begin{array}{c} J \\ \mu \end{array} \right] \left[\begin{array}{c} J \\ n-\mu \end{array} \right] (x_\alpha y_\beta)^{\mu+1} (x_\beta y_\alpha)^{n-\mu} \\
&\quad \times \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha \right) \\
=& \sum_{\mu=1}^{n+1} q^{-2\mu S-\mu+n+1}[J-\mu+1] \left[\begin{array}{c} J \\ \mu-1 \end{array} \right] \left[\begin{array}{c} J \\ n+1-\mu \end{array} \right] (x_\alpha y_\beta)^{\mu} (x_\beta y_\alpha)^{n+1-\mu} \\
&\quad \times \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha \right),
\end{aligned} \tag{A.7}$$

where we replaced $\mu \to \mu - 1$. Dividing (A.2) − the second term of (A.6) by $q - q^{-1}$, we obtain

$$\begin{aligned}
&\sum_{\mu=0}^{n} q^{-2\mu S-\mu}[J+\mu-n] \left[\begin{array}{c} J \\ \mu \end{array} \right] \left[\begin{array}{c} J \\ n-\mu \end{array} \right] (x_\alpha y_\beta)^{\mu} (x_\beta y_\alpha)^{n+1-\mu} \\
&\quad \times \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha \right).
\end{aligned} \tag{A.8}$$

Since $\left[\begin{array}{c} J \\ -1 \end{array} \right] = 0$, we can extend $\sum$ in (A.7) and (A.8) to $0 \leq \mu \leq n+1$. Finally we have

$$\begin{aligned}
\text{(A.7)} + \text{(A.8)} =& \sum_{\mu=0}^{n+1} q^{-2\mu S}[n+1] \left[\begin{array}{c} J \\ \mu \end{array} \right] \left[\begin{array}{c} J \\ n+1-\mu \end{array} \right] (x_\alpha y_\beta)^{\mu} (x_\beta y_\alpha)^{n+1-\mu} \\
&\quad \times \prod_{\nu=1}^{2S-J} \left(x_\alpha y_\beta - q^{2(\nu-S-1)} x_\beta y_\alpha \right),
\end{aligned} \tag{A.9}$$

and since

$$\begin{aligned}
\left(\Delta X_{\alpha\beta}^{-} \right)^{n+1} v_J \Big/ q^{(n+1)S}[n]!(x_\alpha x_\beta)^{J-(n+1)} &= \frac{\text{(A.2)} + \text{(A.4)} - \text{(A.3)} - \text{(A.5)}}{q - q^{-1}} \\
&= \frac{\text{(A.2)} + \text{(A.6)} - \text{(A.5)}}{q - q^{-1}} \\
&= \text{(A.7)} + \text{(A.8)} \\
&= \text{(A.9)},
\end{aligned} \tag{A.10}$$

Proposition 2.1 is true for $\left(\Delta X_{\alpha\beta}^{-} \right)^{n+1} v_J$.

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