This is a preprint of a paper whose final form has appeared in Commun. Frac. Calc. 1 (2010) 41–47.

Combined Delta-Nabla Sum Operator in Discrete Fractional Calculus

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Abstract

We introduce a more general discrete fractional operator, given by convex linear combination of the delta and nabla fractional sums. Fundamental properties of the new fractional operator are proved. As particular cases, results on delta and nabla discrete fractional calculus are obtained.

Keywords: Discrete fractional calculus; Delta and nabla operators; Convex linear combination.

MSC 2010: 39A12; 26A33.

1 Introduction

The main goal of this note is to introduce a new and more general fractional sum operator that unify and extend the discrete fractional operators used in fractional calculus. Looking to the literature of discrete fractional difference operators, two approaches are found (see, e.g., [1, 2]): one using the Δ point of view (sometimes called the forward fractional difference approach), another using the ∇ perspective (sometimes called the backward fractional difference approach). Here we introduce a new operator, making use of the symbol ${}_\gamma\diamondsuit$ (cf. Definition 3.1). When $\gamma = 1$ the ${}_\gamma\diamondsuit$ operator is reduced to the Δ one; when $\gamma = 0$ the ${}_\gamma\diamondsuit$ operator coincides with the corresponding ∇ fractional sum.

The work is organized as follows. In Section 2 we review the basic definitions of the discrete fractional calculus. Our results are then given in Section 3: we introduce the fractional diamond sum (Definition 3.1) and prove its main properties. We end with Section 4 of conclusions and future perspectives.

2 Preliminaries

Here we only give a very short introduction to the basic definitions in discrete fractional calculus. For more on the subject we refer the reader to [3–5].

We begin by introducing some notation used throughout. Let a be an arbitrary real number and $b=a+k$ for a certain $k\in\mathbb{N}$ with $k\geq 2$. Let $\mathbb{T}=\{a,a+1,\ldots,b\}$. According with [6], we define the factorial function

$$t^{(n)}=t(t-1)(t-2)\ldots(t-n+1), \quad n\in\mathbb{N}.$$

Also in agreement with the same authors [7], we define

$$t^{\overline{n}}=t(t+1)(t+2)\ldots(t+n-1), \quad n\in\mathbb{N},$$

and $t^{\overline{0}}=1$. Extending the two above definitions from an integer n to an arbitrary real number α, we have

$$t^{(\alpha)}=\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)} \quad \text{and} \quad t^{\overline{\alpha}}=\frac{\Gamma(t+\alpha)}{\Gamma(t)},$$

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where Γ is the Euler gamma function. Throughout the text we shall use the standard notations $\sigma(s)=s+1$ and $\rho(s)=s-1$ of the time scale calculus in $\mathbb{Z}$ [6, 7].

Definition 2.1 ([8])**.** The discrete delta fractional sum operator is defined by

$$(\Delta_a^{-\alpha} f)(t)=\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-\sigma(s))^{(\alpha-1)}f(s),$$

where $\alpha>0$. Here f is defined for $s=a \mod (1)$ and $\Delta_a^{-\alpha} f$ is defined for $t=(a+\alpha) \mod (1)$.

Remark 2.2. Given a real number a and $b=a+k$, $k\in\mathbb{N}$, $\sum_{s=a}^{b} g(s)=g(a)+g(a+1)+\cdots+g(b)$.

Remark 2.3. Let $\mathbb{N}_t=\{t,t+1,t+2,\ldots\}$. We note that $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.

Analogously to Definition 2.1, one considers the discrete nabla fractional sum operator:

Definition 2.4 ([1])**.** The discrete nabla fractional sum operator is defined by

$$(\nabla_a^{-\beta} f)(t)=\frac{1}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}f(s),$$

where $\beta>0$. Here f is defined for $s=a \mod (1)$ and $\nabla_a^{-\beta} f$ is defined for $t=a \mod (1)$.

Remark 2.5. Let $\mathbb{N}_a=\{a,a+1,a+2,\ldots\}$. The operator $\nabla_a^{-\beta}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$. The fact that f and $\nabla_a^{-\beta} f$ have the same domain, while f and $\Delta_a^{-\alpha} f$ do not, explains why some authors prefer the nabla approach.

The next result gives a relation between the delta fractional sum and the nabla fractional sum operators.

Lemma 2.6 ([1])**.** *Let $0\le m-1<\nu\le m$, where m denotes an integer. Let a be a positive integer, and $y(t)$ be defined on $t\in\mathbb{N}_a=\{a,a+1,a+2,\ldots\}$. The following statement holds: $(\Delta_a^{-\nu} y)(t+\nu)=(\nabla_a^{-\nu} y)(t)$, $t\in\mathbb{N}_a$.*

3 Main Results

We introduce a general discrete diamond-gamma fractional sum operator by using a convex combination of the delta and nabla fractional sum operators.

Definition 3.1. The diamond-γ fractional operator of order (α,β) is given, when applied to a function f at point t, by

$$\left({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} f\right)(t)=\gamma\left(\Delta_a^{-\alpha} f\right)(t+\alpha)+(1-\gamma)\left(\nabla_a^{-\beta} f\right)(t),$$

where $\alpha>0$, $\beta>0$, and $\gamma\in[0,1]$. Here, both f and ${}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} f$ are defined for $t=a \mod (1)$.

Remark 3.2. Similarly to the nabla fractional operator, our operator ${}_{\gamma}\diamondsuit_a^{-\alpha,-\beta}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$, $\mathbb{N}_a=\{a,a+1,a+2,\ldots\}$ for a a given real number.

Remark 3.3. The new diamond fractional operator of Definition 3.1 gives, as particular cases, the operator of Definition 2.1 for $\gamma=1$,

$$\left({}_{1}\diamondsuit_a^{-\alpha,-\beta} f\right)(t)=\left(\Delta_a^{-\alpha} f\right)(t+\alpha),\quad t\equiv a \mod (1),$$

and the operator of Definition 2.4 for $\gamma=0$,

$$\left({}_{0}\diamondsuit_a^{-\alpha,-\beta} f\right)(t)=\left(\nabla_a^{-\beta} f\right)(t),\quad t\equiv a \mod (1).$$

The next theorems give important properties of the new, more general, discrete fractional operator ${}_{\gamma}\diamondsuit_a^{-\alpha,-\beta}$.

Theorem 3.4. *Let f and g be real functions defined on $\mathbb{N}_a$, $\mathbb{N}_a=\{a,a+1,a+2,\ldots\}$ for a a given real number. The following equality holds:*

$$\left({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta}(f+g)\right)(t)=\left({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} f\right)(t)+\left({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} g\right)(t).$$

Proof. The intended equality follows from the definition of diamond-γ fractional sum of order (α,β):

$$\begin{aligned}({}_\gamma\diamondsuit_a^{-\alpha,-\beta}(f+g))(t) &= \gamma(\Delta_a^{-\alpha}(f+g))(t+\alpha)+(1-\gamma)(\nabla_a^{-\beta}(f+g))(t)\\ &= \frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)}(f(s)+g(s))+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}(f(s)+g(s))\\ &= \frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)}f(s)+\frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)}g(s)\\ &\quad+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}f(s)+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}g(s)\\ &= \left[\frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)}f(s)+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}f(s)\right]\\ &\quad+\left[\frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)}g(s)+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}g(s)\right]\\ &= ({}_\gamma\diamondsuit_a^{-\alpha,-\beta}f)(t)+({}_\gamma\diamondsuit_a^{-\alpha,-\beta}g)(t).\end{aligned}$$

□

Theorem 3.5. *Let $f(t)=k$ on $\mathbb{N}_a$, k a constant. The following equality holds:*

$$({}_\gamma\diamondsuit_a^{-\alpha,-\beta}f)(t)=\gamma\frac{\Gamma(t-a+1+\alpha)k}{\Gamma(\alpha+1)\Gamma(t-a+1)}+(1-\gamma)\frac{\Gamma(t-a+1+\beta)k}{\Gamma(\beta+1)\Gamma(t-a+1)}.$$

Proof. By definition of diamond-γ fractional sum of order (α,β), we have

$$\begin{aligned}({}_\gamma\diamondsuit_a^{-\alpha,-\beta}k)(t) &= \gamma(\Delta_a^{-\alpha}k)(t+\alpha)+(1-\gamma)(\nabla_a^{-\beta}k)(t)=\frac{\gamma}{\Gamma(\alpha)}\sum_{s=0}^{t}k(t+\alpha-\sigma(s))^{(\alpha-1)}+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=0}^{t}k(t-\rho(s))^{\overline{\beta-1}}\\ &= \gamma\frac{\Gamma(t-a+1+\alpha)}{\alpha\Gamma(\alpha)\Gamma(t-a+1)}k+(1-\gamma)\frac{\Gamma(t-a+1+\beta)}{\beta\Gamma(\beta)\Gamma(t-a+1)}k=\gamma\frac{\Gamma(t-a+1+\alpha)}{\Gamma(\alpha+1)\Gamma(t-a+1)}k+(1-\gamma)\frac{\Gamma(t-a+1+\beta)}{\Gamma(\beta+1)\Gamma(t-a+1)}k.\end{aligned}$$

□

Corollary 3.6. *Let $f(t)\equiv k$ for a certain constant k. Then,*

$$(\Delta_a^{-\alpha}f)(t+\alpha)=\frac{\Gamma(t-a+1+\alpha)}{\Gamma(\alpha+1)\Gamma(t-a+1)}k. \tag{1}$$

Proof. The result follows from Theorem 3.5 choosing $\gamma=1$ and recalling that $({}_1\diamondsuit_a^{-\alpha,-\beta}k)(t)=(\Delta_a^{-\alpha}k)(t+\alpha)$. □

Remark 3.7. In the particular case when $a=0$, equality (1) coincides with the result of [8, Sect. 5].

The fractional nabla result analogous to Corollary 3.6 is easily obtained:

Corollary 3.8. *If k is a constant, then*

$$(\nabla_a^{-\beta}k)(t)=\frac{\Gamma(t-a+1+\beta)}{\Gamma(\beta+1)\Gamma(t-a+1)}k.$$

Proof. The result follows from Theorem 3.5 choosing $\gamma=0$ and recalling that $({}_0\diamondsuit_a^{-\alpha,-\beta}k)(t)=(\nabla_a^{-\beta}k)(t)$. □

Theorem 3.9. *Let f be a real valued function and α_1, α_2, β_1, $\beta_2>0$. Then,*

$$\left({}_\gamma\diamondsuit_a^{-\alpha_1,-\beta_1}\left({}_\gamma\diamondsuit_a^{-\alpha_2,-\beta_2}f\right)\right)(t)=\gamma\left({}_\gamma\diamondsuit_a^{-(\alpha_1+\alpha_2),-(\beta_1+\alpha_2)}f\right)(t)+(1-\gamma)\left({}_\gamma\diamondsuit_a^{-(\alpha_1+\beta_2),-(\beta_1+\beta_2)}f\right)(t).$$

Proof. Direct calculations show the intended relation:

$$\begin{aligned}
&({}_\gamma\diamondsuit_a^{-\alpha_1,-\beta_1}({}_\gamma\diamondsuit_a^{-\alpha_2,-\beta_2}f))(t)=\gamma(\Delta_a^{-\alpha_1}({}_\gamma\diamondsuit_a^{-\alpha_2,-\beta_2}f))(t+\alpha_1)+(1-\gamma)(\nabla_a^{-\beta_1}({}_\gamma\diamondsuit_a^{-\alpha_2,-\beta_2}f))(t)\\
&\quad=\gamma^2(\Delta_a^{-\alpha_1}(\Delta_a^{-\alpha_2}f))(t+\alpha_1+\alpha_2)+\gamma(1-\gamma)\left(\Delta_a^{-\alpha_1}\left(\nabla_a^{-\beta_2}f\right)\right)(t+\alpha_1)+(1-\gamma)\gamma\left(\nabla_a^{-\beta_1}\left(\Delta_a^{-\alpha_2}f\right)\right)(t+\alpha_2)\\
&\qquad+(1-\gamma)^2\left(\nabla_a^{-\beta_1}\left(\nabla_a^{-\beta_2}f\right)\right)(t)\\
&\quad=\gamma^2\left(\Delta_a^{-(\alpha_1+\alpha_2)}f\right)(t+\alpha_1+\alpha_2)+\gamma(1-\gamma)\left(\Delta_a^{-\alpha_1}\left(\Delta_a^{-\beta_2}f\right)\right)(t+\alpha_1+\beta_2)\\
&\qquad+(1-\gamma)\gamma\left(\nabla_a^{-\beta_1}\left(\nabla_a^{-\alpha_2}f\right)\right)(t)+(1-\gamma)^2\left(\nabla_a^{-(\beta_1+\beta_2)}f\right)(t)\\
&\quad=\gamma^2\left(\Delta_a^{-(\alpha_1+\alpha_2)}f\right)(t+\alpha_1+\alpha_2)+\gamma(1-\gamma)\left(\Delta_a^{-(\alpha_1+\beta_2)}f\right)(t+\alpha_1+\beta_2)\\
&\qquad+(1-\gamma)\gamma\left(\nabla_a^{-(\beta_1+\alpha_2)}f\right)(t)+(1-\gamma)^2\left(\nabla_a^{-(\beta_1+\beta_2)}f\right)(t)\\
&\quad=\gamma\left[\gamma\left(\Delta_a^{-(\alpha_1+\alpha_2)}f\right)(t+\alpha_1+\alpha_2)+(1-\gamma)\left(\nabla_a^{-(\beta_1+\alpha_2)}f\right)(t)\right]\\
&\qquad+(1-\gamma)\left[\gamma\left(\Delta_a^{-(\alpha_1+\beta_2)}f\right)(t+\alpha_1+\beta_2)+(1-\gamma)\left(\nabla_a^{-(\beta_1+\beta_2)}f\right)(t)\right].
\end{aligned}$$

□

Remark 3.10. If $\gamma=0$, then $({}_\gamma\diamondsuit_a^{-\alpha_1,-\beta_1}({}_\gamma\diamondsuit_a^{-\alpha_2,-\beta_2}f))(t)=\left(\nabla_a^{-(\beta_1+\beta_2)}f\right)(t)$.

Remark 3.11. If $\gamma=1$, then $({}_\gamma\diamondsuit_a^{-\alpha_1,-\beta_1}({}_\gamma\diamondsuit_a^{-\alpha_2,-\beta_2}f))(t)=\left(\Delta_a^{-(\alpha_1+\alpha_2)}f\right)(t+\alpha_1+\alpha_2)$.

Remark 3.12. If $\alpha_1=\alpha_2=\alpha$ and $\beta_1=\beta_2=\beta$, then $({}_\gamma\diamondsuit_a^{-\alpha,-\beta}({}_\gamma\diamondsuit_a^{-\alpha,-\beta}f))(t)=({}_\gamma\diamondsuit_a^{-\alpha,-\beta}f)(t)$.

We now prove a general Leibniz formula.

Theorem 3.13 (Leibniz formula)**.** *Let f and g be real valued functions, $0<\alpha,\ \beta<1$. For all t such that $t=a$* mod (1)*, the following equality holds:*

$$\begin{aligned}
\left({}_\gamma\diamondsuit_a^{-\alpha,-\beta}(fg)\right)(t)=\gamma\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left[\left(\nabla^k g\right)(t)\right]\cdot\left[\left(\Delta_a^{-(\alpha+k)}f\right)(t+\alpha+k)\right]&\\
+(1-\gamma)\sum_{k=0}^{\infty}\binom{-\beta}{k}\left[\left(\nabla^k g\right)(t)\right]\left[\left(\Delta_a^{-(\beta+k)}f\right)(t+\beta+k)\right],&\quad (2)
\end{aligned}$$

where

$$\binom{u}{v}=\frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}.$$

Proof. By definition of the diamond fractional sum,

$$\begin{aligned}
\left({}_\gamma\diamondsuit_a^{-\alpha,-\beta}(fg)\right)(t)&=\gamma\left(\Delta_a^{-\alpha}(fg)\right)(t+\alpha)+(1-\gamma)\left(\nabla_a^{-\beta}(fg)\right)(t)\\
&=\frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)}f(s)g(s)+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}f(s)g(s).
\end{aligned}$$

By Taylor's expansion of $g(s)$ [9],

$$g(s)=\sum_{k=0}^{\infty}\frac{(s-t)^{\overline{k}}}{k!}(\nabla^k g)(t)=\sum_{k=0}^{\infty}(-1)^k\frac{(t-s)^{(k)}}{k!}(\nabla^k g)(t).$$

Substituting the Taylor series of $g(s)$ at t,

$$\begin{aligned}
\left({}_\gamma\diamondsuit_a^{-\alpha,-\beta}(fg)\right)(t)=\frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)}f(s)\left[\sum_{k=0}^{\infty}(-1)^k(t-s)^{(k)}\frac{(\nabla^k g)(t)}{k!}\right]&\\
+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}}f(s)\left[\sum_{k=0}^{\infty}(-1)^k(t-s)^{(k)}\frac{(\nabla^k g)(t)}{k!}\right].&
\end{aligned}$$

Since

$$\begin{aligned}
(t+\alpha-\sigma(s))^{(\alpha-1)}(t-s)^{(k)}&=(t+\alpha-\sigma(s))^{(\alpha+k+1)},\\
(t-\rho(s))^{\overline{\beta-1}}(t-s)^{(k)}&=(t+\beta-\sigma(s))^{(\beta+k+1)},
\end{aligned}$$

and $\sum_{s=t-k+1}^{t}(t-s)^{(k)}=0$, we have

$$\left({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta}(fg)\right)(t)=\frac{\gamma}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(-1)^k\frac{(\nabla^k g)(t)}{k!}\sum_{s=a}^{t-k}(t+\alpha-\sigma(s))^{(\alpha+k-1)}f(s)$$
$$+\frac{1-\gamma}{\Gamma(\beta)}\sum_{k=0}^{\infty}(-1)^k\frac{(\nabla^k g)(t)}{k!}\sum_{s=a}^{t-k}(t+\beta-\sigma(s))^{(\beta+k-1)}f(s).$$

Because

$$(-1)^k=\frac{\Gamma(-\alpha+1)\Gamma(\alpha)}{\Gamma(-\alpha+k+1)\Gamma(k+\alpha)}=\frac{\Gamma(-\beta+1)\Gamma(\beta)}{\Gamma(-\beta+k+1)\Gamma(k+\beta)}$$

and $k!=\Gamma(k+1)$, the above expression becomes

$$\begin{aligned}\left({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta}(fg)\right)(t)&=\frac{\gamma}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\nabla^k g)(t)\binom{-\alpha}{k}\cdot\left[\frac{1}{\Gamma(k+\alpha)}\sum_{s=a}^{t-k}(t+\alpha-\sigma(s))^{(\alpha+k-1)}f(s)\right]\\&\quad+\frac{1-\gamma}{\Gamma(\beta)}\sum_{k=0}^{\infty}(\nabla^k g)(t)\binom{-\beta}{k}\left[\frac{1}{\Gamma(k+\beta)}\sum_{s=a}^{t-k}(t+\beta-\sigma(s))^{(\beta+k-1)}f(s)\right]\\&=\gamma\sum_{k=0}^{\infty}\binom{-\alpha}{k}(\nabla^k g)(t)(\Delta_a^{-(\alpha+k)}f)(t+\alpha+k)\\&\quad+(1-\gamma)\sum_{k=0}^{\infty}\binom{-\beta}{k}(\nabla^k g)(t)(\Delta_a^{-(\beta+k)}f)(t+\beta+k).\end{aligned}$$

□

Remark 3.14. Choosing $\gamma=0$ in our Leibniz formula (2), we obtain that

$$(\nabla_a^{-\beta}(fg))(t)=\sum_{k=0}^{\infty}\binom{-\beta}{k}\left[(\nabla^k g)(t)\right]\left[(\Delta_a^{-(\beta+k)}f)(t+\beta+k)\right].$$

Remark 3.15. Choosing $\gamma=1$ in our Leibniz formula (2), we obtain that

$$(\Delta_a^{-\alpha}(fg))(t+\alpha)=\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left[(\nabla^k g)(t)\right]\left[(\Delta_a^{-(\alpha+k)}f)(t+\alpha+k)\right]. \tag{3}$$

As a particular case of (3), let $a=0$. Then, recalling Lemma 2.6, we obtain the Leibniz formulas of [10].

4 Conclusion

The discrete fractional calculus is a subject under strong current research (see, e.g., [5, 11–14] and references therein). Two versions of the discrete fractional calculus, the delta and the nabla, are now standard in the fractional theory. Motivated by the diamond-alpha dynamic derivative on time scales [15–17] and the fractional derivative of [18], we introduce here a combined diamond-gamma fractional sum of order (alpha, beta), as a linear combination of the delta and nabla fractional sum operators of order alpha and beta, respectively. The new operator interpolates between the delta and nabla cases, reducing to the standard fractional delta operator when $\gamma=1$ and to the fractional nabla sum when $\gamma=0$.

Using the discrete fractional diamond sum here proposed, one can now introduce the discrete fractional diamond difference in the usual way. It is our intention to generalize the new discrete diamond fractional operator to an arbitrary time scale $\mathbb{T}$ (i.e., to an arbitrary nonempty closed set of the real numbers). Another line of research, to be addressed elsewhere, consists to investigate the usefulness of modeling with fractional diamond equations and study corresponding fractional variational principles.

Acknowledgments

This work is part of the first author's Ph.D. project, carried out at the University of Aveiro under the framework of the Doctoral Programme *Mathematics and Applications* of Universities of Aveiro and Minho, and was partially presented during the *3rd Conference on Nonlinear Science and Complexity* (NSC10), Cankaya University, Ankara, 26-29 July, 2010. The financial support of the Polytechnic Institute of Viseu and *The Portuguese Foundation for Science and Technology* (FCT), through the "Programa de apoio à formação avançada de docentes do Ensino Superior Politécnico", Ph.D. fellowship SFRH/PROTEC/49730/2009, is here gratefully acknowledged. The authors were also supported by FCT through the *Center for Research and Development in Mathematics and Applications* (CIDMA).

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