TILTING MUTATION OF BRAUER TREE ALGEBRAS

TAKUMA AIHARA

Abstract. We define tilting mutations of symmetric algebras as the endomorphism algebras of Okuyama-Rickard complexes. For Brauer tree algebras, we give an explicit description of the change of Brauer trees under mutation.

In representation theory of algebras, derived and stable equivalences play a crucial role. Rickard [Ri] proved that derived equivalent classes of Brauer tree algebras are determined by the same numerical invariants. Okuyama and Rickard [Ok, Ri2] introduced tilting mutation for tilting complexes in the case of symmetric algebras, and their tilting mutation plays a crucial role for the study of Broué's abelian defect group conjecture. In this paper, we will show tilting mutations of Brauer trees using their tilting mutation.

1. Main theorem

In this paper, let A be a finite dimensional k-algebra for an algebraically closed field k and we assume that A is basic and indecomposable as A-A-bimodules. Let $\{e_1, e_2, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A and put $E = \{1, 2, \cdots, n\}$. For any $i \in E$, we set $P_i = e_iA$ and $S_i = P_i/\mathrm{rad}P_i$.

We denote by mod-A the category of finitely generated right A-modules, by proj-A the full subcategory of mod-A consisting of finitely generated projective right A-modules, by $\underline{\text{mod}}$-A the stable module category of mod-A and by $\mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ the homotopy category of bounded complexes over proj-A.

In [Ok], Okuyama constructed a tilting complex induced by a subset of E for a symmetric algebra. Let us start with the following definition.

Definition 1.1. Let E_0 be a subset of E and put $e = \sum_{i \in E_0} e_i$. For any $i \in E$, we set a complex by

$$T_i = \begin{cases} \quad (0\text{th}) \qquad\quad (1\text{st}) & \\ \quad P_i \longrightarrow 0 & (i \in E_0) \\ \quad Q_i \xrightarrow[\pi_i]{} P_i & (i \notin E_0) \end{cases}$$

where $Q_i \xrightarrow{\pi_i} P_i$ is a minimal projective presentation of e_iA/e_iAeA. Now we define $T = \oplus_{i \in E} T_i$ and call it the *Okuyama-Rickard complex with respect to* E_0.

Note that this is a special case of silting mutation defined in [AI].

We have the following observation.

Proposition 1.2. [Ok, Proposition 1.1] *If A is symmetric, then any Okuyama-Rickard complex T is a tilting complex. In particular $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T)$ is derived equivalent to A.*

Note that an Okuyama-Rickard complex is not necessarily a tilting complex if we drop the assumption that A is symmetric.

Definition 1.3. Let B be a finite dimensional k-algebra. For any $i \in E$, we say that B is the *mutation of A with respect to i* and write $A \xrightarrow{i} B$ or $B = \mu_i(A)$ if B is the endomorphism algebra of the Okuyama-Rickard complex with respect to $E_0 = E \backslash \{i\}$.

The aim of this paper is to give an explicit description of the change of Brauer trees under mutation. Let us recall the definitions of Brauer trees and Brauer tree algebras

Definition 1.4. [Alp, GR] A *Brauer graph G* is a finite connected graph, together with the following data:

(i) There exists a cyclic ordering of the edges adjacent to each vertex, usually described by the clockwise ordering given by a fixed planar representation of G;
(ii) For each vertex v, there exists a positive integer m_v assigned to v, called the *multiplicity*. We call a vertex v *exceptional* if $m_v > 1$.

A *Brauer tree G* is a Brauer graph which is a tree and having at most one exceptional vertex.

A *Brauer tree algebra A* is a basic algebra given by a Brauer tree G as follows:

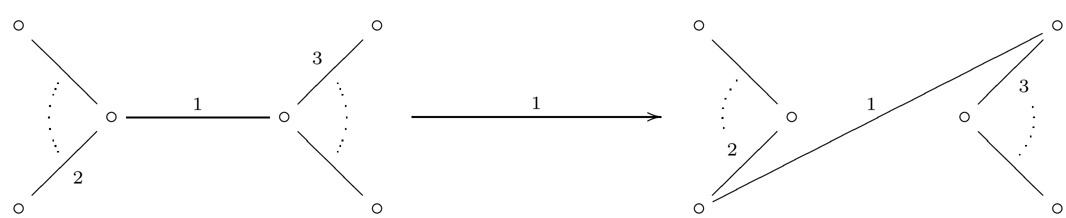
(i) There exists a one-to-one correspondence between the simple A-modules S_i and the edges i of G;
(ii) For any edges i of G, the projective indecomposable A-module P_i has $\mathrm{soc}(P_i) \simeq P_i/\mathrm{rad}(P_i)$ and $\mathrm{rad}(P_i)/\mathrm{soc}(P_i)$ is the direct sum of two uniserial modules whose composition factors are, for the cyclic ordering $(i, i_1, \cdots, i_a, i)$ of the edges adjacent to a vertex v, $S_{i_1}, \cdots, S_{i_a}, S_i, S_{i_1}, \cdots, S_{i_a}$ (from the top to the socle) where S_i appears $m_v - 1$ times.

Any Brauer tree algebra is a symmetric algebra which is uniquely determined by the Brauer tree up to isomorphism.

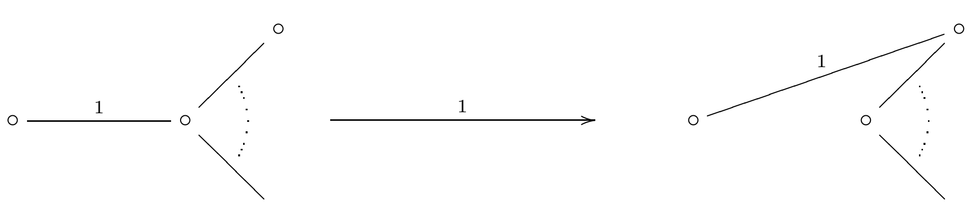
Now we state main theorem in this paper.

Theorem 1.5. *Let A be a Brauer tree algebra and B the mutation of A with respect to the edge* 1*. Then B is a Brauer tree algebra, and the Brauer tree of B is given by the following rule, where the multiplicities of vertices do not change:*

(1) *the case of which the edge* 1 *is not at the end;*



(2) *the case of which the edge* 1 *is at the end;*



2. Proof of main theorem

We define an $(n \times n)$-matrix C^A as $C^A_{ij} = \dim_k \mathrm{Hom}_A(P_i, P_j)$ for any $i, j \in E$, called the *Cartan matrix* of A. Note that if A is symmetric, then we have $C^A_{ij} = C^A_{ji}$ for any $i, j \in E$.

We have the following property.

Lemma 2.1. *Let A be a Brauer tree algebra. Then the following hold:*

(1) *If the Brauer tree of A is given, then the Cartan matrix C^A of A is determined as follows:*

$$C^A_{ij} = \begin{cases} (\text{the sum of the multiplicities of the ends of the edge } i) \text{ if } i = j; \\ (\text{the multiplicity of } v) \text{ if } i \neq j \text{ and the edges } i \text{ and } j \text{ have the common vertex } v; \\ 0 \text{ otherwise.} \end{cases}$$

(2) *If the Cartan matrix C^A of A and $\dim_k \mathrm{Ext}^1_A(S_i, S_j)$ for any $i, j \in E$ are given, then we can determine the Brauer tree of A explicitly.*

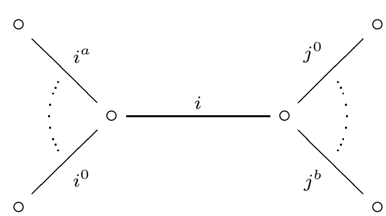
Proof. The assertion (1) can be checked easily. We show the assertion (2). To prove the assertion (2), we shall give the cyclic ordering of the edges. Fix $i \in E$. We define a subset I of E by $I = \{j \in E \mid C^A_{ij} \neq 0\}$. Since G is a Brauer tree, we have a disjoint union $I = \{i\} \cup I_0 \cup I_1$ satisfying the following:

$$\text{If } i_0 \in I_0 \text{ and } i_1 \in I_1, \text{ then } C^A_{i_0 i_1} = 0.$$

Since G is a Brauer tree, for any $j \in I_0$ there uniquely exists $j' \in \{i\} \cup I_0$ such that $\mathrm{Ext}^1_A(S_j, S_{j'}) \neq 0$. Therefore there exist sequences

$$i = i^0, i^1, \cdots, i^a, i^{a+1} = i \text{ in } \{i\} \cup I_0$$
$$i = j^0, j^1, \cdots, j^b, j^{b+1} = i \text{ in } \{i\} \cup I_1$$

such that $\mathrm{Ext}^1_A(S_{i^x}, S_{i^{x+1}}) \neq 0$ for any $0 \leq x \leq a$ and $\mathrm{Ext}^1_A(S_{j^y}, S_{j^{y+1}}) \neq 0$ for any $0 \leq y \leq b$. Hence we can explicitly determine the Brauer tree of A



□

The following result was proved by [Ri].

Theorem 2.2. [Ri, Theorem 4.2] *For any Brauer tree algebras A, there exists a tilting complex P in $\mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ such that the endomorphism algebra $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(P)$ of P is a Brauer tree algebra for a star with exceptional vertex in the center. In particular, up to derived equivalence, a Brauer tree algebra is determined by the number of the edges and the multiplicity of the Brauer tree.*

We show the following easy observation.

Proposition 2.3. *Let A and B be derived equivalent symmetric k-algebras. If A is a Brauer tree algebra, then so is B.*

Proof. By Theorem 2.2, A is derived equivalent to a Brauer tree algebra C for a star with the exceptional vertex in the center. This implies that B is stable equivalent to C. Note that C is a symmetric Nakayama algebra. Hence the assertion follows from [ARS, X, Theorem 3.14]. □

We also need the following result.

Lemma 2.4. [Ok, Lemma 2.1] *Let E_0 be a subset of E and put $e = \sum_{i\in E_0} e_i$. Suppose that A is symmetric and T is the Okuyama-Rickard complex with respect to E_0. Now the endomorphism algebra $B = \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T)$ of T is stable equivalent to A and we denote the stable equivalence by $F : \underline{\mathrm{mod}}\text{-}A \overset{\sim}{\to} \underline{\mathrm{mod}}\text{-}B$. Then the following hold:*

(1) *If $i \notin E_0$, then $F(\Omega(S_i))$ is a simple B-module;*
(2) *If $i \in E_0$, then $F(\Omega(X_i))$ is a simple B-module where an A-module X_i satisfies the following conditions:*
   (*i*) $\mathrm{top}(X_i) \simeq S_i$;
   (*ii*) $\mathrm{soc}(X_i)e \simeq \mathrm{soc}(X_i)$;
   (*iii*) $\Omega(X_i)e \simeq S_i$.

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let T be the Okuyama-Rickard complex with respect to $E_0 = \{2, 3, \cdots, n\}$. Then T is defined as the direct sum of the following complexes:

$$\begin{array}{llll} & \text{(0th)} & \text{(1st)} & \\ T_1 : & P_2 \oplus P_3 \longrightarrow & P_1 & \\ T_i : & P_i \longrightarrow & 0 & (i \neq 1) \end{array}$$

(If the edge 1 is at the end, then replace the above first complex with $P_2 \to P_1$ or $P_3 \to P_1$.) Put $B = \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T)$. Since T is tilting by Proposition 1.2, B is a Brauer tree algebra by Proposition 2.3. To determine the Brauer tree of B, we shall calculate the Cartan matrix of B and the extensions among simple B-modules.

Let C^A, C^B be Cartan matrices of A, B. We calculate C^B_{ij}. For any $i \in E$, we denote by P^B_i a projective indecomposable B-module corresponding to T_i.

(i) We show $C^B_{ij} = C^A_{ij}$ for any $i \neq 1$ and $j \neq 1$. Assume $i \neq 1$ and $j \neq 1$. We can calculate as follows:

$$\begin{aligned} C^B_{ij} &= \dim_k \mathrm{Hom}_B(P^B_i, P^B_j) \\ &= \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T_i, T_j) \\ &= \dim_k \mathrm{Hom}_A(P_i, P_j) \\ &= C^A_{ij}. \end{aligned}$$

Assume $i \notin \{1, 2, 3\}$.

(ii) We show $C^B_{1i} = 0$ if $C^A_{1i} \neq 0$ and $C^A_{2i} \neq 0$. Assume $C^A_{1i} \neq 0$ and $C^A_{2i} \neq 0$. Then we have $C^A_{1i} = C^A_{2i}$ and $C^A_{3i} = 0$. Therefore we obtain an equation

$$\begin{aligned} C^B_{1i} &= \dim_k \mathrm{Hom}_B(P^B_1, P^B_i) \\ &= \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T_1, T_i) \\ &= \dim_k \mathrm{Hom}_A(P_2, P_i) + \dim_k \mathrm{Hom}_A(P_3, P_i) - \dim_k \mathrm{Hom}_A(P_1, P_i) \\ &= C^A_{2i} + C^A_{3i} - C^A_{1i} \\ &= 0. \end{aligned}$$

Similary, we have $C^B_{1i} = 0$ if $C^A_{1i} \neq 0$ and $C^A_{3i} \neq 0$.

(iii) We show $C^B_{1i} \neq 0$ if $C^A_{1i} = 0$ and $C^A_{2i} \neq 0$. Assume $C^A_{1i} = 0$ and $C^A_{2i} \neq 0$. Then we have $C^A_{3i} = 0$. Therefore we obtain an equation

$$\begin{aligned} C^B_{1i} &= \dim_k \mathrm{Hom}_B(P^B_1, P^B_i) \\ &= \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T_1, T_i) \\ &= \dim_k \mathrm{Hom}_A(P_2, P_i) + \dim_k \mathrm{Hom}_A(P_3, P_i) - \dim_k \mathrm{Hom}_A(P_1, P_i) \\ &= C^A_{2i} + C^A_{3i} - C^A_{1i} \\ &= C^A_{2i}, \end{aligned}$$

which implies $C^B_{1i} \neq 0$. Similary, we have $C^B_{1i} \neq 0$ if $C^A_{1i} = 0$ and $C^A_{3i} \neq 0$.

(iv) We can easily show $C^B_{1i} = 0$ if $C^B_{1i} = C^B_{2i} = C^B_{3i} = 0$.

For any $i \in E$, we put $S^B_i = P^B_i / \mathrm{rad} P^B_i$. We calculate $\dim_k \mathrm{Ext}^1_B(S^B_i, S^B_j)$. We denote by $F : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}B$ the stable equivalence between A and B given by T.

(a) We show $\mathrm{Ext}^1_B(S^B_i, S^B_j) \simeq \mathrm{Ext}^1_A(S_i, S_j)$ for any $i, j \notin \{1,2,3\}$. Let $i \notin \{1,2,3\}$. Since $\mathrm{Ext}^1_A(S_1, S_i) = 0$, by Lemma 2.4 we have $F(S_i) \simeq S^B_i$. Therefore for any $i, j \notin \{1,2,3\}$, we have $\mathrm{Ext}^1_B(S^B_i, S^B_j) \simeq \mathrm{Ext}^1_A(S_i, S_j)$.

(b) We show $\mathrm{Ext}^1_B(S^B_2, S^B_1) \neq 0$. We define Y_2 as a maximal submodule of P_2 satisfying $(Y_2)e \simeq \mathrm{soc}(P_2)$ and put $X_2 = P_2/Y_2$. By Lemma 2.4, $F(\Omega(X_2))$ is isomorphic to a simple B-module S^B_2 and $F(\Omega(S_1))$ is isomorphic to a simple B-module S^B_1. Therefore we have an isomorphism

$$\begin{aligned} \mathrm{Ext}^1_B(S^B_2, S^B_1) &\simeq \mathrm{Ext}^1_A(X_2, S_1) \\ &\simeq \mathrm{Hom}_A(Y_2, S_1). \end{aligned}$$

Since $\mathrm{Ext}^1_A(S_1, S_2) \neq 0$, we have $\mathrm{Ext}^1_B(S^B_2, S^B_1) \neq 0$. Similary, we have $\mathrm{Ext}^1_B(S^B_3, S^B_1) \neq 0$.

(c) Let $(1, 2, i_1, \cdots, i_h, 1)$ be the cyclic ordering of the edges in the Brauer tree of A. We show that the Brauer tree of B has the cyclic ordering $(2, i_1, \cdots, i_h, 2)$ of the edges. By (i) and (ii), we have $C^B_{2i} \neq 0$ and $C^B_{1i} = 0$ for any $i \in \{i_1, \cdots, i_h\}$. If $j \in E$ satisfies $C^B_{ji} \neq 0$ for any $i \in \{i_1, \cdots, i_h\}$, then we have $j \in \{2, i_1, \cdots, i_h\}$. Therefore our assertion follows from (a). Similary, for the cyclic ordering $(1, 3, j_1, \cdots, j_\ell, 1)$ of the edges in the Brauer tree of A, the Brauer tree of B has the cyclic ordering $(3, j_1, \cdots, j_\ell, 3)$ of the edges.

(d) Let $(2, i'_1, \cdots, i'_{h'}, 2)$ $(i'_{h'} \neq 1)$ be the cyclic ordering of the edges in the Brauer tree of A. We show that the Brauer tree of B has the cyclic ordering $(2, 1, i'_1, \cdots, i'_{h'}, 2)$ of the edges. By (i) and (iii), we have $C^B_{2i} \neq 0$ and $C^B_{1i} \neq 0$ for any $i \in \{i'_1, \cdots, i'_{h'}\}$. If $j \in E$ satisfies $C^B_{ji} \neq 0$ for any $i \in \{i'_1, \cdots, i'_{h'}\}$, then we have $j \in \{2, 1, i'_1, \cdots, i'_{h'}\}$. By (b), the next edge of the edge 2 is the edge 1. Therefore our assertion follows from (a). Similary, for the cyclic ordering $(3, j'_1, \cdots, j'_{\ell'}, 3)$ $(j'_{\ell'} \neq 1)$ of the edges in the Brauer tree of A, the Brauer tree of B has the cyclic ordering $(3, 1, j'_1, \cdots, j'_{\ell'}, 3)$ of the edges.

Hence the assertion (1) and (2) follow from Lemma 2.1 and its proof.

We shall show that the position of the exceptional vertex does not change. Assume that the edge 1 does not have the exceptional vertex. By the above argument of the Cartan matrix of B, we have our assertion. Assume that the common vertex of the edges 1 and 2 of the Brauer tree of A is the exceptional vertex. Since $C^B_{22} = C^A_{22}$, the edge 2 of the Brauer tree of B must have the exceptional vertex. We have an equation

$$\begin{aligned} C_{12}^B &= \dim_k \mathrm{Hom}_B(P_1^B, P_2^B) \\ &= \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T_1, T_2) \\ &= \dim_k \mathrm{Hom}_A(P_2, P_2) + \dim_k \mathrm{Hom}_A(P_3, P_2) - \dim_k \mathrm{Hom}_A(P_1, P_2) \\ &= C_{22}^A + C_{32}^A - C_{12}^A \\ &= 1. \end{aligned}$$

This implies our assertion. Assume that the edge 1 of the Brauer tree of A has the exceptional vetex which is at the end. Since $C_{ij}^B = C_{ij}^A$ for any $i, j \in E_0$, the edge 1 of the Brauer tree of B must have the exceptional vertex and the edges except the edge 1 do not have it. This implies our assertion. □

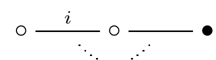
3. Applications of main theorem

In this section, we give some applications of Theorem 1.5.

We can prove the following stronger statement than Theorem 2.2.

Corollary 3.1. *Let A be a Brauer tree algebra. Then there exists a tilting complex $T \in \mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ of length 2 such that the Brauer tree of the endomorphism algebra of T is a star with exceptional vertex in the center.*

Proof. Take the edge i whose next edge has the exceptional vetex;



Let B be the mutation of A with respect to i. By Theorem 1.5, the edge i of the Brauer tree of B has the exceptional vertex. In applying this mutation to each edge at most once, we can mutate A to a Brauer tree algebra for a star with exceptional vertex in the center.

For any edges i, we denote by $F_i : \mathsf{K}^{\mathrm{b}}(\text{proj-}\mu_i(A)) \to \mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ the derived equivalence between A and $\mu_i(A)$ induced by the Okuyama-Rickard complex of A with respect to i. Take distinct edges $1, 2, \cdots, \ell$. Put $B_\ell = \mu_\ell \cdots \mu_1(A)$, $T_\ell = F_1 \cdots F_\ell(B_\ell)$ and $T_0 = A$. Then P_ℓ is a direct summand of $T_{\ell-1}$ and terms of $T_{\ell-1} \backslash P_\ell$ are 0 unless 0th and 1st. Since T_ℓ is an extension of $P_\ell[-1]$ and $T_{\ell-1} \backslash P_\ell$, we have that T_ℓ is of length 2. This implies the assertion. □

The corollary below is an immediate consequence of Corollary 3.1 and its proof.

Corollary 3.2. *Let A be a Brauer tree algebra. Any basic algebra which is derived equivalent to A is obtained from A by successive mutation.*

The corresponding statement is shown for representation-finite symmetric algebras in [A].

Using Theorem 1.5, we can also show the following result.

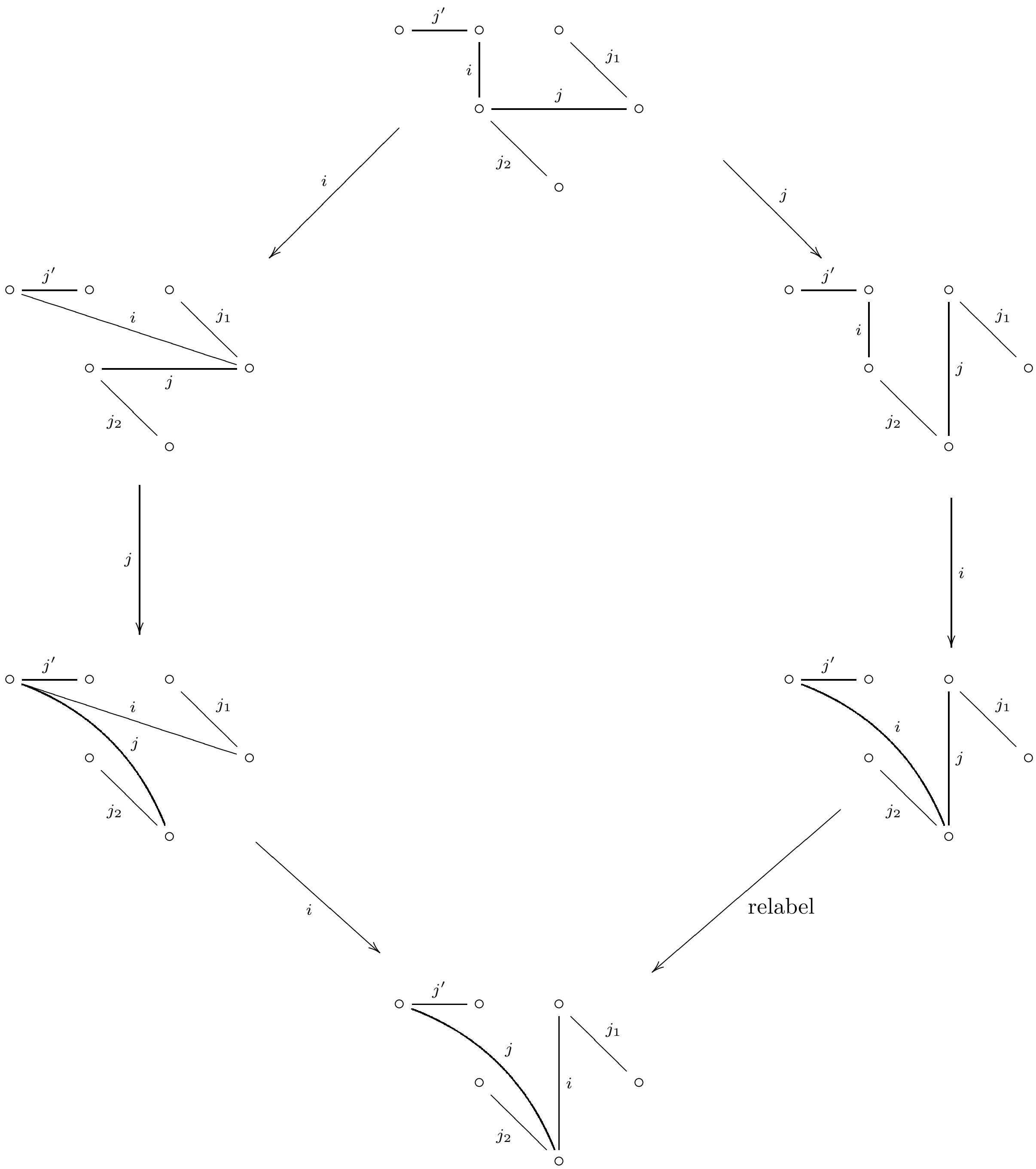
Corollary 3.3. *Let A be a Brauer tree algebra. Then for any edges i, j of the Brauer tree of A, we have the following relations:*

(1) $(\mu_i)^s(A) \simeq A$ *for some positive integer s;*
(2) $\mu_j \mu_i(A) \simeq \mu_i \mu_j(A)$ *if i and j are not mutually the next edges in the cyclic ordering;*
(3) $\mu_i \mu_j \mu_i(A) \simeq \mu_i \mu_j(A)$ *if j is the next edge of i in the cyclic ordering.*

Proof. (1) We denote by $\mathrm{Br}(n, m)$ the set of labeled Brauer trees which have n edges and multiplicity m of the exceptional vertex. We can regard mutation as group action to $\mathrm{Br}(n, m)$. Since $\mathrm{Br}(n, m)$ is a finite set, the order of mutation is finite.

(2) This assertion can be checked easily from Theorem 1.5.

(3) Let j' be the next edge of i being not j and j_1, j_2 the next edges of j. By Theorem 1.5, we have a mutation of the Brauer tree of A

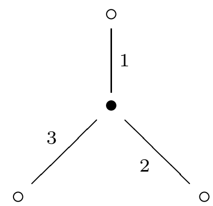


Therefore we have $\mu_i\mu_j\mu_i(A) \simeq \mu_i\mu_j(A)$.

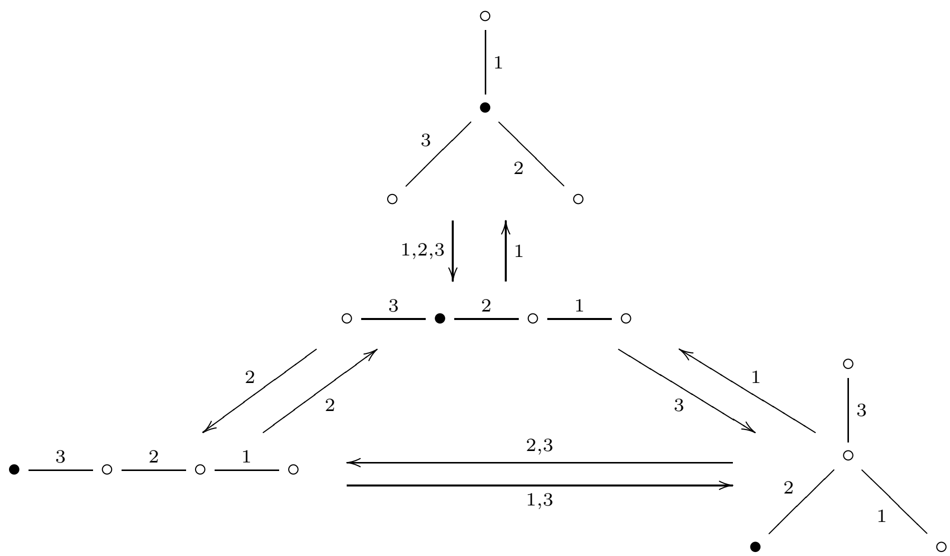
□

We close this paper by giving an example of Theorem 1.5.

Example 3.4. Let A be a Brauer tree algebra with the following Brauer tree:



where the vertex $\bullet$ is the exceptional vertex. Then we have the following diagram of mutations of A up to Morita equivalence:



where for each Brauer tree, we relabel the edges. These are all finite dimensional k-algebras which are derived equivalent to A, up to Morita equivalence.

References

[A] T. Aihara, *Silting mutation for self-injective algebras.* in preparation.
[AI] T. Aihara, O. Iyama, *Silting mutation in triangulated categories.* in preparation.
[Alp] J. L. Alperin, *Local Representation Theory.* Cambrige Univ. Press, Cambridge, 1986.
[ARS] M. Auslander, I. Reiten, S. O. Smalϕ, *Representation theory of Artin algebras.* Cambridge Stud. Adv. Math., vol. 36, Cambridge University Press, Cambridge, 1995.
[GR] P. Gabriel, Ch. Riedtmann, *Group representations without groups.* Comment. Math. Helv. **54**, 240–287, 1979.
[Ok] T. Okuyama, *Some examples of derived equivalent blocks of finite groups.* preprint, 1998.
[Ri] J. Rickard, *Derived categories and stable equivalence.* J. Pure. Appl. Alg. **61**, 303–317, 1989.
[Ri2] J. Rickard, *Equivalences of derived categories for symmetric algebras.* J. Alg. **257** 460–481, 2002.

Division of Mathematical Science and Physics, Graduate School of Science and Technology, Chiba University, Yayoi-cho, Chiba 263-8522, Japan
E-mail address: taihara@math.s.chiba-u.ac.jp