A posteriori error estimates for a finite element approximation of transmission problems with sign changing coefficients

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Abstract

We perform the a posteriori error analysis of residual type of a transmission problem with sign changing coefficients. According to [6] if the contrast is large enough, the continuous problem can be transformed into a coercive one. We further show that a similar property holds for the discrete problem for any regular meshes, extending the framework from [6]. The reliability and efficiency of the proposed estimator is confirmed by some numerical tests.

Key Words A posteriori estimator, non positive definite diffusion problems.
AMS (MOS) subject classification 65N30; 65N15, 65N50,

1 Introduction

Recent years have witnessed a growing interest in the study of diffusion problems with a sign changing coefficient. These problems appear in several areas of physics, for example in electromagnetism [12, 15, 16, 18, 19]. Thus some mathematical investigations have been performed and concern existence results [7, 19] and numerical approximations by the finite element methods [19, 4, 5, 6], with some a priori error analyses. But for such problems the regularity of the solution may be poor and/or unknown and consequently an a posteriori error analysis would be more appropriate. This analysis is the aim of the present paper.

For continuous Galerkin finite element methods, there now exists a large amount of literature on a posteriori error estimations for (positive definite) problems in mechanics or electromagnetism. Usually locally defined a posteriori error estimators are designed. We refer the reader to the monographs [2, 3, 17, 21] for a good overview on this topic.

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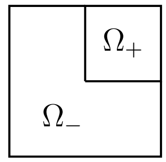

Figure 1: The domain Ω

In contrast to the recent paper [6] we will not use quasi-uniform meshes that are not realistic for an a posteriori error analysis. That is why we improve their finite element analysis in order to allow only regular meshes in Ciarlet's sense [8].

The paper is structured as follows: We recall in Section 2 the "diffusion" problem and the technique from [6] that allows to establish its well-posedness for sufficiently large contrast. In Section 3, we prove that the discrete approximation is well-posed by introducing an ad-hoc discrete lifting operator. The a posteriori error analysis is performed in Section 4, where upper and lower bounds are obtained. Finally in Section 5 some numerical tests are presented that confirm the reliability and efficiency of our estimator.

Let us finish this introduction with some notations used in the remainder of the paper: On D, the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$. The usual norm and semi-norm of $H^s(D)$ ($s \geq 0$) are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. In the case $D = \Omega$, the index Ω will be omitted. Finally, the notations $a \lesssim b$ and $a \sim b$ mean the existence of positive constants C_1 and C_2, which are independent of the mesh size and of the considered quantities a and b such that $a \leq C_2 b$ and $C_1 b \leq a \leq C_2 b$, respectively. In other words, the constants may depend on the aspect ratio of the mesh and the diffusion coefficient (see below).

2 The boundary value problem

Let Ω be a bounded open domain of $\mathbb{R}^2$ with boundary Γ. We suppose that Ω is split up into two sub-domains Ω_+ and Ω_- with a Lipschitz boundary that we suppose to be polygonal in such a way that

$$\bar{\Omega} = \bar{\Omega}_+ \cup \bar{\Omega}_-, \quad \Omega_+ \cap \Omega_- = \emptyset,$$

see Figure 1 for an example.
We now assume that the diffusion coefficient a belongs to $L^\infty(\Omega)$ and is positive (resp. negative) on Ω_+ (resp. Ω_-). Namely there exists $\epsilon_0 > 0$ such that

$$a(x) \geq \epsilon_0, \text{ for a. e. } x \in \Omega_+, \tag{1}$$

$$a(x) \leq -\epsilon_0, \text{ for a. e. } x \in \Omega_-. \tag{2}$$

In this situation we consider the following second order boundary value problem with Dirichlet boundary conditions:

$$\left\{ \begin{array}{rcll} -\operatorname{div}(a \nabla u) & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma. \end{array} \right. \tag{3}$$

The variational formulation of (3) involves the bilinear form

$$B(u, v) = \int_\Omega a\nabla u \cdot \nabla v$$

and the Hilbert space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}.$$

Due to the lack of coercivity of B on $H_0^1(\Omega)$ (see [7, 5, 6]), this problem does not fit into a standard framework. In [5, 6], the proposed approach is to use a bijective and continuous linear mapping $\mathbb{T}$ from $H_0^1(\Omega)$ into itself that allows to come back to the coercive framework. Namely these authors assume that $B(u, \mathbb{T}v)$ is coercive in the sense that there exists $\alpha > 0$ such that

$$B(u, \mathbb{T}u) \geq \alpha \|u\|_{1,\Omega}^2 \quad \forall u \in H_0^1(\Omega). \tag{4}$$

Hence given $f \in L^2(\Omega)$, by the Lax-Milgram theorem the problem

$$B(u, \mathbb{T}v) = \int_\Omega f\mathbb{T}v \quad \forall v \in H_0^1(\Omega), \tag{5}$$

has a unique solution $u \in H_0^1(\Omega)$. Since $\mathbb{T}$ is an isomorphism, the original problem

$$B(u, v) = \int_\Omega fv \quad \forall v \in H_0^1(\Omega), \tag{6}$$

has also a unique solution $u \in H_0^1(\Omega)$.

In [6], the mapping $\mathbb{T}$ is built by using a trace lifting operator $\mathcal{R}$ from $H_{00}^{1/2}(\Sigma)$ into $H_-^1(\Omega_-)$, where $\Sigma = \partial\Omega_- \cap \partial\Omega_+$ is the interface between Ω_- and Ω_+,

$$H_\pm^1(\Omega_\pm) = \{u \in H^1(\Omega_\pm) : u = 0 \text{ on } \partial\Omega_\pm \setminus \Sigma\},$$

and

$$H_{00}^{1/2}(\Sigma) = \{u_{|\Sigma} : u \in H_-^1(\Omega_-)\} = \{u_{|\Sigma} : u \in H_+^1(\Omega_+)\}$$

is the space of the restrictions to Σ of functions in $H_-^1(\Omega_-)$ (or in $H_+^1(\Omega_+)$). This last space may be equipped with the norms

$$\|p\|_{1/2,\pm} = \inf_{\substack{u \in H_\pm^1(\Omega_\pm) \\ p = u_{|\Sigma}}} |u|_{1,\Omega_\pm}.$$

With the help of such a lifting, a possible mapping $\mathbb{T}$ is given by (see [6])

$$\mathbb{T}v = \begin{cases} v_+ \text{ in } \Omega_+, \\ -v_- + 2\mathcal{R}(v_{+|\Sigma}) \text{ in } \Omega_-, \end{cases}$$

where $v_\pm$ denotes the restriction of v to $\Omega_\pm$. With this choice, it is shown in Proposition 3.1 of [6] that (4) holds if

$$K_\mathcal{R} = \sup_{\substack{v \in H_+^1(\Omega_+) \\ v \neq 0}} \frac{|B_-(\mathcal{R}(v_{|\Sigma}), \mathcal{R}(v_{|\Sigma}))|}{B_+(v, v)} < 1, \tag{7}$$

where $B_\pm(u,v) = \displaystyle\int_{\Omega_\pm} a\nabla u \cdot \nabla v$.

For concrete applications, one can make the following particular choice for $\mathcal{R}$, that we denote by $\mathcal{R}_p$: for any $\varphi \in H^{1/2}_{00}(\Sigma)$ we define $\mathcal{R}_p(\varphi) = w$ as the unique solution $w \in H^1_-(\Omega_-)$ of

$$\Delta w = 0 \text{ in } \Omega_-, \quad w = \varphi \text{ on } \Sigma.$$

With this choice, one obtains that $K_{\mathcal{R}_p} < 1$ if the contrast

$$\frac{\min_{\Omega_-} |a|}{\max_{\Omega_+} a}$$

is large enough, we refer to Section 3 of [6] for more details.

Remark 2.1 *Note that in [5, 6] the authors consider sub-domains Ω_+ and Ω_- with a pseudo-Lipschitz boundary. However the previous arguments from [6] (shortly summarized above) are not valid in this case since the space $H^1(\Omega_+)$ equipped with the norm $|\cdot|_{1,\Omega_+}$ is not complete.*

3 The discrete approximated problem

Here we consider the following standard Galerkin approximation of our continuous problem. We consider a triangulation $\mathcal{T}$ of Ω, that is a "partition" of Ω made of triangles T (closed subsets of $\bar{\Omega}$) whose edges are denoted by e. We assume that this triangulation is regular, i.e., for any element T, the ratio h_T/ρ_T is bounded by a constant $\sigma > 0$ independent of T and of the mesh size $h = \max_{T\in\mathcal{T}} h_T$, where h_T is the diameter of T and ρ_T the diameter of its largest inscribed ball. We further assume that $\mathcal{T}$ is conforming with the partition of Ω, i.e., each triangle is assumed to be either included into $\bar{\Omega}_+$ or into $\bar{\Omega}_-$. With each edge e of the triangulation, we denote by h_e its length and n_e a unit normal vector (whose orientation can be arbitrary chosen) and the so-called patch $\omega_e = \cup_{e\subset T} T$, the union of triangles having e as edge. We similarly associate with each vertex x, a patch $\omega_x = \cup_{x\in T} T$. For a triangle T, n_T stands for the outer unit normal vector of T. $\mathcal{E}$ (resp. $\mathcal{N}$) represents the set of edges (resp. vertices) of the triangulation. In the sequel, we need to distinguish between edges (or vertices) included into Ω or into Γ, in other words, we set

$$\begin{aligned}
\mathcal{E}_{int} &= \{e \in \mathcal{E} : e \subset \Omega\},\\
\mathcal{E}_{\Gamma} &= \{e \in \mathcal{E} : e \subset \Gamma\},\\
\mathcal{N}_{int} &= \{x \in \mathcal{N} : x \in \Omega\}.
\end{aligned}$$

Problem (6) is approximated by the continuous finite element space:

$$V_h = \left\{ v_h \in H^1_0(\Omega) : v_{h|T} \in \mathbb{P}_\ell(T),\, \forall T \in \mathcal{T} \right\}, \tag{8}$$

where ℓ is a fixed positive integer and the space $\mathbb{P}_\ell(T)$ consists of polynomials of degree at most ℓ.
The Galerkin approximation of problem (6) reads now: Find $u_h \in V_h$, such that

$$B(u_h, v_h) = \int_\Omega f v_h \quad \forall v_h \in V_h. \tag{9}$$

Since there is no reason that the bilinear form would be coercive on V_h, as in [6] we need to use a discrete mapping $\mathbb{T}_h$ from V_h into itself defined by (see [6])

$$\mathbb{T}_h v_h = \begin{cases} v_{h+} \text{ in } \Omega_+, \\ -v_{h-} + 2\mathcal{R}_h(v_{h+|\Sigma}) \text{ in } \Omega_-, \end{cases}$$

where $\mathcal{R}_h$ is a discrete version of the operator $\mathcal{R}$. Here contrary to [6] and in order to avoid the use of quasi-uniform meshes (meaningless in an a posteriori error analysis), we take

$$\mathcal{R}_h = I_h \mathcal{R}, \tag{10}$$

where I_h is a sort of Clément interpolation operator [9] and $\mathcal{R}$ is any trace lifting operator from $H^{1/2}_{00}(\Sigma)$ into $H^1_-(\Omega_-)$ (see the previous section). More precisely for $\varphi_h \in H_h(\Sigma) = \{v_{h|\Sigma} : v_h \in V_h\}$, we set

$$I_h \mathcal{R}(\varphi_h) = \sum_{x \in \mathcal{N}_-} \alpha_x \lambda_x,$$

where $\mathcal{N}_- = \mathcal{N}_{int} \cap \bar{\Omega}_-$, λ_x is the standard hat function (defined by $\lambda_x \in V_h$ and satisfying $\lambda_x(y) = \delta_{xy}$) and $\alpha_x \in \mathbb{R}$ are defined by

$$\alpha_x = \begin{cases} |\omega_x|^{-1} \int_{\omega_x} \mathcal{R}(\varphi_h) & \text{if } x \in \mathcal{N}_{int} \cap \Omega_-, \\ \varphi_h(x) & \text{if } x \in \mathcal{N}_{int} \cap \Sigma, \end{cases}$$

where we recall that ω_x is the patch associated with x, which is simply the support of λ_x. Note that I_h coincides with the Clément interpolation operator I_{Cl} for the nodes in Ω_- and only differs on the nodes on Σ. Indeed let us recall the definition of $I_{\mathrm{Cl}}\mathcal{R}(\varphi_h)$ (defined in a Scott-Zhang manner [20] for the points belonging to Σ):

$$I_{\mathrm{Cl}} \mathcal{R}(\varphi_h) = \sum_{x \in \mathcal{N}_-} \beta_x \lambda_x$$

with

$$\beta_x = \begin{cases} |\omega_x|^{-1} \int_{\omega_x} \mathcal{R}(\varphi_h) & \text{if } x \in \mathcal{N}_{int} \cap \Omega_-, \\ |e_x|^{-1} \int_{e_x} \mathcal{R}(\varphi_h) d\sigma & \text{if } x \in \mathcal{N}_{int} \cap \Sigma \text{ with } e_x = \omega_x \cap \Sigma. \end{cases}$$

The definition of I_h aims at ensuring that

$$I_h \mathcal{R}(\varphi_h) = \varphi_h \text{ on } \Sigma.$$

Let us now prove that $\mathcal{R}_h$ is uniformly bounded.

Theorem 3.1 *For all $h > 0$ and $\varphi_h \in H_h(\Sigma)$, one has*

$$|\mathcal{R}_h(\varphi_h)|_{1,\Omega_-} \lesssim \|\varphi_h\|_{1/2,-}.$$

Proof: For the sake of simplicity we make the proof in the case $\ell = 1$, the general case is treated in the same manner by using modified Clément interpolation operator.
Since $\mathcal{R}$ is bounded from $H_{00}^{1/2}(\Sigma)$ into $H_-^1(\Omega_-)$, one has

$$|\mathcal{R}(\varphi_h)|_{1,\Omega_-} \lesssim \|\varphi_h\|_{1/2,-}. \tag{11}$$

Hence it suffices to show that

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1,\Omega_-} \lesssim \|\varphi_h\|_{1/2,-}. \tag{12}$$

For that purpose, we distinguish the triangles T that have no nodes in $\mathcal{N}_{int} \cap \Sigma$ to the other ones:

1. If T has no nodes in $\mathcal{N}_{int} \cap \Sigma$, then $I_h\mathcal{R}(\varphi_h)$ coincides with $I_{\text{Cl}}\mathcal{R}(\varphi_h)$ on T and therefore by a standard property of the Clément interpolation operator, we have

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1,T} = |(I - I_{\text{Cl}})\mathcal{R}(\varphi_h)|_{1,T} \lesssim \|\mathcal{R}(\varphi_h)\|_{1,\omega_T}, \tag{13}$$

where the patch ω_T is given by $\omega_T = \bigcup_{T' \cap T \neq \emptyset} T'$.

2. If T has at least one node in $\mathcal{N}_{int} \cap \Sigma$, by the triangle inequality we may write

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1,T} \leq |(I - I_{\text{Cl}})\mathcal{R}(\varphi_h)|_{1,T} + |(I_{\text{Cl}} - I_h)\mathcal{R}(\varphi_h)|_{1,T}.$$

For the first term of this right-hand side we can still use (13) and therefore it remains to estimate the second term. For that one, we notice that

$$(I_{\text{Cl}} - I_h)\mathcal{R}(\varphi_h) = \sum_{x \in T \cap \Sigma} (\alpha_x - \beta_x)\lambda_x \text{ on } T.$$

Hence

$$|(I_{\text{Cl}} - I_h)\mathcal{R}(\varphi_h)|_{1,T} \lesssim \sum_{x \in T \cap \Sigma} |\alpha_x - \beta_x|.$$

Since $\mathcal{R}(\varphi_h) = \varphi_h$ on Σ and due to the definition of I_{Cl}, it follows that for $x \in T \cap \Sigma$,

$$|\alpha_x - \beta_x| = \left| \varphi_h(x) - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma \right|.$$

Since all norms are equivalent in finite dimensional spaces, we have for all $v_h \in \mathbb{P}_1(e_x)$,

$$|v_h(x)| \lesssim |e_x|^{-1/2} \|v_h\|_{e_x}. \tag{14}$$

Moreover,

$$|e_x|^{-1/2} \left\| \varphi_h - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma \right\|_{e_x} \lesssim |\varphi_h|_{1/2,e_x}, \tag{15}$$

where here $|\cdot|_{1/2,e_x}$ means the standard $H^{1/2}(e_x)$-seminorm. Thus Inequalities (14) with $v_h = \varphi_h - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma$ and (15) imply that

$$|\alpha_x - \beta_x| \lesssim |\varphi_h|_{1/2,e_x}.$$

All together we have shown that

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1,T} \lesssim \|\mathcal{R}(\varphi_h)\|_{1,\omega_T \cap \bar{\Omega}_-} + |\varphi_h|_{1/2,\omega_T \cap \Sigma}. \tag{16}$$

Taking the sum of the square of (13) and of (16), we obtain that

$$|(I - I_h)\mathcal{R}(\varphi_h)|^2_{1,\Omega_-} \lesssim \|\mathcal{R}(\varphi_h)\|^2_{1,\Omega_-} + |\varphi_h|^2_{1/2,\Sigma}.$$

We conclude thanks to (11) and to the fact that

$$|\varphi_h|_{1/2,\Sigma} \lesssim \|\varphi_h\|_{1/2,-}.$$

■

This Theorem and Proposition 4.2 of [6] allow to conclude that (9) has a unique solution provided that (7) holds, in particular if the contrast is large enough.
Note that the advantage of our construction of $\mathcal{R}_h$ is that we no more need the quasi-uniform property of the meshes imposed in [6].

4 The a posteriori error analysis

Error estimators can be constructed in many different ways as, for example, using residual type error estimators which measure locally the jump of the discrete flux [21]. A different method, based on equilibrated fluxes, consists in solving local Neumann boundary value problems [2] or in using Raviart-Thomas interpolant [1, 10, 13, 14]. Here since the coercivity constant is not explicitly known, we chose the simplest approach of residual type.

The residual estimators are denoted by

$$\eta_R^2 = \sum_{T \in \mathcal{T}} \eta_{R,T}^2, \quad \eta_J^2 = \sum_{T \in \mathcal{T}} \eta_{J,T}^2, \tag{17}$$

where the indicators $\eta_{R,T}$ and $\eta_{J,T}$ are defined by

$$\begin{aligned} \eta_{R,T} &= h_T \|f_T + \operatorname{div}(a\nabla u_h)\|_T, \\ \eta_{J,T} &= \sum_{e \in \mathcal{E}_{int}: e \subset T} h_e^{1/2} \|[\![a\nabla u_h \cdot n_e]\!]\|_e, \end{aligned}$$

when f_T is an approximation of f, for instance

$$f_T = |T|^{-1} \int_T f.$$

Note that $\eta_{R,T}^2$ is meaningful if $a_{|T} \in W^{1,1}(T)$, for all $T \in \mathcal{T}$.

4.1 Upper bound

Theorem 4.1 *Assume that $a \in L^\infty(\Omega)$ satisfies (1)-(2) and that $a_{|T} \in W^{1,1}(T)$, for all $T \in \mathcal{T}$. Assume further that (7) holds. Let $u \in H_0^1(\Omega)$ be the unique solution of Problem (6) and let u_h be its Galerkin approximation, i.e. $u_h \in V_h$ a solution of (9). Then one has*

$$\|\nabla(u-u_h)\| \lesssim \eta_R + \eta_J + \mathrm{osc}(f), \tag{18}$$

where

$$\mathrm{osc}(f) = \left(\sum_{T\in\mathcal{T}} h_T^2 \|f - f_T\|^2\right)^{\frac{1}{2}}.$$

Proof: By the coerciveness assumption (4), we may write

$$\|\nabla(u-u_h)\|^2 \lesssim B(u-u_h, \mathbb{T}(u-u_h)). \tag{19}$$

But we notice that the Galerkin relation

$$B(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$$

holds. Hence by taking $v_h = I_{Cl}\mathbb{T}(u-u_h)$, (19) may be written

$$\|\nabla(u-u_h)\|^2 \lesssim B(u-u_h, (I-I_{Cl})\mathbb{T}(u-u_h)). \tag{20}$$

Now we apply standard arguments, see for instance [21]. Namely applying element-wise Green's formula and writing for shortness $w = (I-I_{Cl})\mathbb{T}(u-u_h)$, we get

$$\begin{aligned}\|\nabla(u-u_h)\|^2 \lesssim & -\sum_{T\in\mathcal{T}} \int_T \mathrm{div}\,(a\nabla(u-u_h))w \\ & + \sum_{e\in\mathcal{E}_{int}} \int_e [\![a\nabla(u-u_h)\cdot n]\!] w\, d\sigma,\end{aligned}$$

reminding that $w = 0$ on Γ. By Cauchy-Schwarz's inequality we directly obtain

$$\begin{aligned}\|\nabla(u-u_h)\|^2 \lesssim & \sum_{T\in\mathcal{T}} \|f + \mathrm{div}\,(a\nabla u_h)\|_T \|w\|_T \\ & + \sum_{e\in\mathcal{E}_{int}} \|[\![a\nabla u_h \cdot n]\!]\|_e \|w\|_e.\end{aligned}$$

By standard interpolation error estimates, we get

$$\begin{aligned}\|\nabla(u-u_h)\|^2 \lesssim & \Big(\sum_{T\in\mathcal{T}} h_T^2 \|f + \mathrm{div}\,(a\nabla u_h)\|_T^2 \\ & + \sum_{e\in\mathcal{E}_{int}} h_e \|[\![a\nabla u_h \cdot n]\!]\|_e^2\Big)^{1/2} |\mathbb{T}(u-u_h)|_{1,\Omega}.\end{aligned}$$

Since $\mathbb{T}$ is an isomorphism, we conclude that

$$\|\nabla(u-u_h)\| \lesssim \Big(\sum_{T\in\mathcal{T}} h_T^2 \|f + \operatorname{div}(a\nabla u_h)\|_T^2 + \sum_{e\in\mathcal{E}_{int}} h_e \|[\![a\nabla u_h \cdot n]\!]\|_e^2 \Big)^{1/2}.$$

This leads to the conclusion due to the triangle inequality. ■

4.2 Lower bound

The lower bound is fully standard since by a careful reading of the proof of Proposition 1.5 of [21], we see that it does not use the positiveness of the diffusion coefficient a. Hence we can state the

Theorem 4.2 *Let the assumptions of Theorems 4.1 be satisfied. Assume furthermore that* $a_{|T}$ *is constant for all* $T \in \mathcal{T}$*. Then for each element* $T \in \mathcal{T}$ *the following estimate holds*

$$\eta_{R,T} + \eta_{J,T} \lesssim |u-u_h|_{1,\omega_T} + \operatorname{osc}(f,\omega_T),$$

where

$$\operatorname{osc}(f,\omega_T)^2 = \sum_{T'\subset\omega_T} h_{T'}^2 \|f - f_T'\|_{T'}^2.$$

5 Numerical results

5.1 The polynomial solution

In order to illustrate our theoretical predictions, this first numerical test consists in validating our computations on a simple case, using an uniform refinement process. Let Ω be the square $(-1,1)^2$, $\Omega_+ = (0,1)\times(-1,1)$ and $\Omega_- = (-1,0)\times(-1,1)$. We assume that $a=1$ on Ω_+ and $a=\mu<0$ on Ω_-. In such a situation we can take

$$\mathcal{R}(v_+)(x,y) = v_+(-x,y) \quad \forall (x,y)\in\Omega_-.$$

With this choice we see that

$$K_{\mathcal{R}} = |\mu|,$$

and therefore for $|\mu|<1$, (4) holds and Problem (6) has a unique solution. We further easily check that the corresponding mapping $\mathbb{T}$ is an isomorphism since $(\mathbb{T})^2 = \mathbb{T}$. Similarly by exchanging the role of Ω_+ and Ω_-, (4) will also hold if $|\mu|>1$.
Now we take as exact solution

$$u(x,y) = \mu x(x+1)(x-1)(y+1)(y-1) \quad \forall (x,y)\in\Omega_+,$$
$$u(x,y) = x(x+1)(x-1)(y+1)(y-1) \quad \forall (x,y)\in\Omega_-,$$

f being fixed accordingly.

Let us recall that u_h is the finite element solution, and set $e_{L^2}(u_h) = \|u - u_h\|$ and $e_{H^1}(u_h) = \|u - u_h\|_1$ the L^2 and H^1 errors. Moreover let us define $\eta(u_h) = \eta_R + \eta_J$ the estimator and CV_{L^2} (resp. CV_{H^1}) as the experimental convergence rate of the error $e_{L^2}(u_h)$ (resp. $e_{H^1}(u_h)$) with respect to the mesh size defined by $DoF^{-1/2}$, where the number of degrees of freedom is DoF, computed from one line of the table to the following one. Computations are performed with $\mu = -3$ using a global mesh refinement process from an initial cartesian grid. First, it can be seen from Table 1 that the convergence rate of the H^1 error norm is equal to one, as theoretically expected (see [6]). Furthermore the convergence rate of the L^2 error norm is 2, which is a consequence of the Aubin-Nitsche trick and regularity results for Problem (3). Finally, the reliability of the estimator is ensured since the ratio in the last column (the so-called effectivity index), converges towards a constant close to 6.5.

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\dfrac{\eta(u_h)}{e_{H^1}(u_h)}$
1	289	2.37E-02		5.33E-01		6.70
2	1089	5.95E-03	2.08	2.67E-01	1.04	6.59
3	4225	1.49E-03	2.04	1.34E-01	1.02	6.53
4	16641	3.73E-04	2.02	6.68E-02	1.01	6.49
5	32761	1.89E-04	2.01	4.75E-02	1.01	6.48
6	90601	6.79E-05	2.01	2.85E-02	1.00	6.47
7	251001	2.45E-05	2.00	1.71E-02	1.00	6.47

Table 1: The polynomial solution with $\mu = -3$ (uniform refinement).

5.2 A singular solution

Here we analyze an example introduced in [7] and precise some results from [7]. The domain $\Omega = (-1, 1)^2$ is decomposed into two sub-domains $\Omega_+ = (0, 1) \times (0, 1)$, and $\Omega_- = \Omega \setminus \bar{\Omega}_+$, see Figure 1. As before we take $a = 1$ on Ω_+ and $a = \mu < 0$ on Ω_-. According to Section 3 of [7], Problem (6) has a singularity S at $(0, 0)$ if $\mu < -3$ or if $\mu \in (-1/3, 0)$ given in polar coordinates by

$$\begin{aligned} S_+(r, \theta) &= r^\lambda (c_1 \sin(\lambda\theta) + c_2 \sin(\lambda(\frac{\pi}{2} - \theta))) \qquad \text{for } 0 < \theta < \frac{\pi}{2}, \\ S_-(r, \theta) &= r^\lambda (d_1 \sin(\lambda(\theta - \frac{\pi}{2}) + d_2 \sin(\lambda(2\pi - \theta))) \qquad \text{for } \frac{\pi}{2} < \theta < 2\pi, \end{aligned}$$

where $\lambda \in (0, 1)$ is given by

$$\lambda = \frac{2}{\pi} \arccos\left(\frac{1 - \mu}{2|1 + \mu|}\right),$$

and the constants c_1, c_2, d_1, d_2 are appropriately defined.
Now we show using the arguments of Section 2 that for $-\frac{1}{3} < \mu < 0$ and $\mu < -3$, the assumption (4) holds. As before we define

$$\mathcal{R}(v_+)(x,y) = \begin{cases} v_+(-x,y) & \forall (x,y) \in (-1,0)\times(0,1), \\ v_+(-x,-y) & \forall (x,y) \in (-1,0)\times(-1,0), \\ v_+(x,-y) & \forall (x,y) \in (0,1)\times(-1,0). \end{cases}$$

This extension defines an element of $H^1_-(\Omega_-)$ such that

$$\mathcal{R}(v_+) = v_+ \quad \text{on } \Sigma.$$

Moreover with this choice we have

$$\sup_{\substack{v\in H^1_+(\Omega_+)\\ v\neq 0}} \frac{|B_-(\mathcal{R}(v),\mathcal{R}(v))|}{B_+(v,v)} = 3|\mu|,$$

and therefore for

$$3|\mu| < 1,$$

we deduce that (4) holds.
To exchange the role of Ω_+ and Ω_- we define the following extension from Ω_- to Ω_+: for $v_- \in H^1_-(\Omega_-)$, let

$$\mathcal{R}(v_-)(x,y) = v_-(-x,y) + v_-(x,-y) - v_-(-x,-y) \quad \forall (x,y) \in \Omega_+.$$

We readily check that it defines an element of $H^1_+(\Omega_+)$ such that

$$\mathcal{R}(v_-) = v_- \quad \text{on } \Sigma.$$

Moreover with this choice we have (using the estimate $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ valid for all real numbers a, b, c)

$$\sup_{v\in H^1_-(\Omega_-), v\neq 0} \frac{B_+(\mathcal{R}(v),\mathcal{R}(v))}{|B_-(v,v)|} \leq 3/|\mu|,$$

and therefore for

$$3/|\mu| < 1,$$

we deduce that (4) holds.

For this second test, we take as exact solution the singular function $u(x,y) = S(x,y)$ for $\mu = -5$ and $\mu = -100$, non-homogeneous Dirichlet boundary conditions on Γ are fixed accordingly. First, with uniform meshes, we obtain the expected convergence rate of order λ (resp. 2λ) for the H^1 (resp. L^2) error norm, see Tables 2 and 3. There, for sufficiently fine meshes, we may notice that the effectivity index varies between 1 and 0.6 for $\mu = -5$ or between 9 and 6 for $\mu = -100$. From these results we can say that the effectivity index

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	289	1.60E-02		2.84E-01		2.57
2	1089	8.66E-03	0.93	2.10E-01	0.45	1.94
3	4225	4.63E-03	0.92	1.55E-01	0.45	1.46
4	16641	2.47E-03	0.92	1.13E-01	0.45	1.09
5	32761	1.80E-03	0.92	9.69E-02	0.46	0.95
6	90601	1.13E-03	0.92	7.68E-02	0.46	0.76
7	251001	7.08E-04	0.92	6.08E-02	0.46	0.61

Table 2: The singular solution, $\mu = -5$, $\lambda \approx 0.46$ (uniform refinement).

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	289	6.12E03		1.54E-01		18.77
2	1089	2.59E-03	1.29	9.91E-02	0.66	15.04
3	4225	1.08E-03	1.29	6.35E-02	0.66	12.06
4	16641	4.46E-04	1.29	4.04E-02	0.66	9.66
5	32761	2.88E-04	1.29	3.24E-02	0.66	8.65
6	90601	1.49E-04	1.30	2.32E-02	0.66	7.33
7	251001	7.66E-05	1.30	1.66E-02	0.66	6.21

Table 3: The singular solution, $\mu = -100$, $\lambda \approx 0.66$ (uniform refinement).

depends on μ, this is confirmed by the numerical results obtained by an adaptive algorithm (see below).

Secondly, an adaptive mesh refinement strategy is used based on the estimator $\eta_T = \eta_{R,T} + \eta_{J,T}$, the marking procedure

$$\eta_T > 0.5 \max_{T'} \eta_{T'}$$

and a standard refinement procedure with a limitation on the minimal angle.

For $\mu = -5$ (resp. $\mu = -100$), Table 4 (resp. 5) displays the same quantitative results as before. There we see that the effectivity index is around 3 (resp. 34), which is quite satisfactory and comparable with results from [11, 14]. As before and in these references we notice that it deteriorates as the contrast becomes larger. On these tables we also remark a convergence order of 0.76 (resp. 1) in the H^1-norm and mainly the double in the L^2-norm. This yields better orders of convergence as for uniform meshes as expected, the case $\mu = -5$ giving less accurate results due to the high singular behavior of the solution (a similar phenomenon occurs in [11] for instance).

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	81	2.92E-02		3.79E-01		3.39
5	432	3.49E-03	2.54	1.40E-01	1.19	4.18
7	1672	1.25E-03	1.52	8.04E-02	0.82	4.07
10	5136	4.26E-04	1.92	4.90E-02	0.88	3.63
13	20588	1.64E-04	1.37	3.14E-02	0.64	3.32
18	80793	5.50E-05	1.60	1.80E-02	0.81	3.23
24	272923	2.39E-05	1.37	1.17E-02	0.71	2.5

Table 4: The singular solution, $\mu = -5$, $\lambda \approx 0.46$ (local refinement).

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	81	1.41E-02		2.35E-01		23.59
4	363	1.93E-03	2.65	8.77E-02	1.31	34.86
7	1566	4.94E-04	1.86	4.31E-02	0.97	33.10
11	5981	1.23E-04	2.07	2.15E-02	1.04	33.17
16	25452	2.98E-05	1.96	1.05E-02	0.99	34.65
24	106827	7.36E-06	1.95	5.23E-03	0.97	33.89

Table 5: The singular solution, $\mu = -100$, $\lambda \approx 0.66$ (local refinement).

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