Complete Ricci-flat metrics through a rescaled exhaustion

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Abstract

Typical existence result on Ricci-flat metrics is in manifolds of finite geometry, that is, on $F = \overline{F} - D$ where $\overline{F}$ is a compact Kähler manifold and D is a smooth divisor.

We view this existence problem from a different perspective. For a given complex manifold X, we take a suitable exhaustion $\{X_r\}_{r>0}$ admitting complete Kähler-Einstein metrics of negative Ricci. Taking a positive decreasing sequence $\{\lambda_r\}_{r>0}, \lim_{r\to\infty} \lambda_r = 0$, we rescale the metric so that g_r is the complete Kähler-Einstein metric in X_r of Ricci curvature $-\lambda_r$. The idea is to show the limiting metric $\lim_{r\to\infty} g_r$ does exist. If so, it is a Ricci-flat metric in X. Several examples: $X = \mathbb{C}^n$ and $X = TM$ where M is a compact rank-one symmetric space have been studied in this article.

The existence of complete Kähler-Einstein metrics of negative Ricci in bounded domains of holomorphy is well-known. Nevertheless, there is very few known for unbounded cases. In the last section we show the existence, through exhaustion, of such kind of metric in the unbounded domain $T^\pi H^n$.

0 Introduction

The goal of this paper is to looking for a way to construct a complete Ricci-flat metric.

By Yau's solving to the Calabi conjecture, complete Ricci-flat metrics have existed in any compact Kähler manifold with vanishing first Chern class. There is no general existence theorem for the non-compact case yet. The most general existence is due to Tian-Yau [T-Y] on manifolds F of finite geometry, i.e., $F = \overline{F} - D$ where $\overline{F}$ is a compact Kähler manifold and D is a smooth divisor. Perturbing a suitable chosen Kähler-Einstein metric near the divisor followed by the continuity method, they are able to conclude the existence of a Ricci-flat metric. Their convergence argument has

*This research is partially supported by NSC97-2628-M-001-023-MY3 of Taiwan. *2000 Mathematics Subject Classification.* 32Q20, 32Q25 heavily relied on Sobolev inequalities. For the choice of a background metric near D and the use of Sobolev inequalities, certain topological condition on M has been asked for. The general requirement is $0 < c_1(\overline{F}) = \alpha c_1(L_D), \alpha \geq 1$.

We will deal with this problem from a different point of view motivated by the following example. The complete Kähler-Einstein metric of Ricci $-(n+1)$ for the ball $B_r(0) \subset \mathbb{C}^n$ is given as $g_r = \sum_j \frac{1}{r^2-|z|^2} dz_j d\bar{z}_j - \sum_{i,j} \frac{\bar{z}_i z_j}{(r^2-|z|^2)^2} dz_i d\bar{z}_j$. The rescaled metric $r^2 g_r$ has Ricci curvature $-\frac{(n+1)}{r^2}$. As $r \to \infty$, the limit metric is then a Kähler-Einstein with vanishing Ricci curvature in $\mathbb{C}^n$ if it existed. Indeed, $\lim_{r\to\infty} r^2 g_r = \sum_j dz_j d\bar{z}_j$.

The second example is the tangent bundle TM of a compact symmetric space of rank-one. TM is exhausted by disk bundles $\{T^r M\}_{r>0}$, named as Grauert tubes when TM is equipped with the adapted complex structure. As a bounded smooth strictly pseudoconvex domain in the Kähler manifold TM, the existence of a complete Kähler-Einstein metric with negative Ricci in $T^r M$ is guaranteed. Furthermore, potential functions could be represented by ordinary differential equations as discussed in [K2]. For some suitably chosen decreasing positive numbers $\{\lambda_r\}_{r>0}$, $\lim_{r\to\infty} \lambda_r = 0$, we pick the complete Kähler-Einstein metric g_r with Ricci $-\lambda_r$ in $T^r M$ and let K_r be its Kähler potential uniquely determined by the corresponding ODE. Through some analysis in the ODEs, the family $\{K_r\}_{r>0}$ has a C^2-convergence as $r \to \infty$. The limiting function is then a Kähler potential of a Ricci-flat metric in TM. It is also interesting to observe that exhausting $\mathbb{C}^n$ through unbounded domains $T^r \mathbb{R}^n$ has achieved the same Ricci-flat as exhausting $\mathbb{C}^n$ through balls.

On the other hand, Stenzel [S] has worked on the same TM as well. By the transitivity of the rank-one symmetry, the defining equation for a Ricci-flat metric could be reduced to an ordinary differential equation. Working directly on the solvability and the completeness of this ordinary differential equation, Stenzel was able to show the existence a complete Ricci-flat metric in TM for compact rank-one M.

Although our resulting Ricci-flat metric through the rescaling process turns out to coincide with the one constructed by Stenzel in TM. We think the approach here is interesting and we are looking for some further investigation.

Cheng-Yau have proved the existence of a complete Kähler-Einstein metric of negative Ricci in bounded weakly pseudoconvex domains in $\mathbb{C}^n$ through the exhaustion by a family of bounded smooth strictly pseudoconvex subdomains. Later on Mok-Yau have generalized the existence to any bounded Stein domain and use the existence as a characterization of a bounded domain of holomorphy. The convergence of the exhaustion has strongly relied on the boundedness of the domain Ω. An essential point is $\Omega \Subset B(0, R)$ for some large R, so that the Poincaré metric of $B(0, R)$ could be used as a comparison to get hold a uniform lower bound of the exhaustion.

It is not clear whether such kind of metric exists in unbounded Stein domain or not. The difficulty is on the lower bound. In the last part of this article, we study the unbounded domain $T^\pi H^n$. Working on the ODEs, we show a uniform bound needed in the convergence argument could be obtained. We conclude there exists a complete Kähler-Einstein metric with negative Ricci curvature in $T^\pi H^n$.

We started from the ball example, a detailed convergence argument has been provided in the first section. In §2, fundamental properties on the Grauert tubes' setting and related ODEs are established. In §3, we take care of the rescaling process. By some suitable choice of the rescaling factors on the ODEs, the convergence could be achieved. §4 is on the existence of a complete Kähler-Einstein metric with negative Ricci curvature in $T^\pi H^n$. The key point is a lower bound estimate shown on Lemma 4.1. Some holomorphic sectional curvatures are also computed in Proposition 4.5.

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1 A Ricci-flat obtained from a rescaled exhaustion.

Let $B_r = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < r^2\}$ be the ball of radius r in $\mathbb{C}^n$ which has admitted a complete Kähler-Einstein metric of Ricci curvature $-(n+1)$:

$$\begin{aligned} g_r(z) &= -\partial\bar{\partial}\log(r^2 - |z|^2) \\ &= \sum_j \frac{1}{r^2 - |z|^2} dz_j d\bar{z}_j - \sum_{i,j} \frac{\bar{z}_i z_j}{(r^2 - |z|^2)^2} dz_i d\bar{z}_j. \end{aligned} \tag{1.1}$$

The metric $r^2 g_r$ is then a complete Kähler-Einstein metric of Ricci $\frac{-(n+1)}{r^2}$ in B_r and

$$\lim_{r\to\infty} r^2 g_r(z) = \sum_j dz_j \bar{z}_j$$

has created a complete Ricci-flat metric in the exhaustion space $\mathbb{C}^n$.

This is the example motivating this work of rescaling Kähler-Einstein metrics to achieve a Ricci-flat metric.

The exhaustion process could also be checked from the potential level. By [C-Y], there is a unique real-analytic function w^r in B_r such that

$$\begin{cases} \det(w^r_{i\bar{j}}) = e^{(n+1)w^r}, \\ (w^r_{i\bar{j}}) > 0 \ \text{ in } B_r, \quad w^r = \infty \text{ on } \partial B_r \end{cases} \tag{1.2}$$

and $\sum w^r_{i\bar{j}} dz_i dz_{\bar{j}}$ gives the unique complete Kähler-Einstein metric of Ricci curvature $-(n+1)$. For $h := -\log(-\varphi)$, the authors have derived the following formula in [C-Y]

$$\det(h_{i\bar{j}}) = \left(\frac{-1}{\varphi}\right)^{n+1} \det(\varphi_{i\bar{j}})(-\varphi + |d\varphi|^2).$$

Using this equation, the solution w^r of (1.2) can be computed explicitly

$$w^r(z) = -\log(r^2 - |z|^2) + \frac{2}{n+1}\log r. \tag{1.3}$$

It is clear that $w^r(z)$ is an increasing function of $|z|$ and its minimum has occurred at the origin:

$$a_r := \inf_{B_r} w^r = w^r(0) = \frac{-2n}{n+1}\log r \to -\infty \text{ as } r \to \infty. \tag{1.4}$$

For a fixed $z \in B_r$,

$$\frac{\partial}{\partial r} w^r(z) = \frac{-2(nr^2 + |z|^2)}{(n+1)r(r^2 - |z|^2)} < 0.$$

Hence $w^r(z) < w^s(z), \forall z \in B_s \Subset B_r$.

Due to (1.4), the family $\{w^r\}_{r>0}$ will diverge as r goes to infinity. Searching for some convergent Kähler potentials, we consider the following $\{\tilde{w}^r\}$ instead. Let $\tilde{w}^r$ be the unique solution of

$$\begin{cases} \det(\tilde{w}^r_{i\bar{j}}) = e^{\frac{(n+1)}{r^2}\tilde{w}^r}, \\ (\tilde{w}^r_{i\bar{j}}) > 0 \text{ in } B_r, \quad \tilde{w}^r = \infty \text{ on } \partial B_r. \end{cases} \tag{1.5}$$

For any $r > 0$, $\tilde{w}^r$ is then a Kähler potential of the complete Kähler-Einstein metric of Ricci $\frac{-(n+1)}{r^2}$ in B_r. A comparison between different $\tilde{w}^r$ could be derived.

Lemma 1.1.

$$\tilde{w}^r(z) = r^2 \log\frac{r^2}{r^2 - |z|^2} \geq 0;$$
$$\tilde{w}^r(z) < \tilde{w}^s(z), \forall z \in B_s \Subset B_r.$$

Proof. Let $u^r = w^r - a_r$, then

$$\det(u^r_{i\bar{j}}) = e^{(n+1)u^r} e^{(n+1)a_r}.$$

$$\begin{aligned} \det\left((e^{\frac{n+1}{-n}a_r} u^r)_{i\bar{j}}\right) &= e^{(n+1)u^r} \\ &= \exp\{\frac{(n+1)}{r^2}(e^{\frac{n+1}{-n}a_r} u^r)\}, \end{aligned}$$

$\left(e^{\frac{n+1}{-n}a_r}u^r_{i\bar{j}}\right) > 0$ in B_r and $e^{\frac{n+1}{-n}a_r}u^r = \infty$ on ∂B_r. By the uniqueness of the solution in (1.5),

$$\begin{aligned}\tilde{w}^r(z) &= e^{\frac{n+1}{-n}a_r}u^r(z)\\ &= r^2(w^r(z) - a_r)\\ &= r^2(-\log(r^2 - |z|^2) + \log r^2) \geq 0,\ z \in B_r,\ \forall r > 0.\end{aligned} \tag{1.6}$$

Taking derivative of $\tilde{w}^r$ with respect to r,

$$H^r(z) := \frac{\partial}{\partial r}\tilde{w}^r(z) = 2r\log\frac{r^2}{r^2 - |z|^2} - \frac{2r|z|^2}{r^2 - |z|^2}. \tag{1.7}$$

Notice that H^r is actually a real-valued function of $|z|^2$ and $H^r(0) = 0$. Viewing $|z|^2 = a \in [0, r^2)$ and taking H^r as a function of a, we now take derivative with respect to a,

$$\frac{\partial}{\partial a}H^r(a) = \frac{-2ra}{(r^2 - a)^2} = \frac{-2r|z|^2}{(r^2 - |z|^2)^2} \leq 0. \tag{1.8}$$

Thus for any $z \in B_r$, we have

$$H^r(z) \leq H^r(0) = 0,\ \forall z \in B_r,\ \forall r > 0.$$

This together with (1.7) has shown that for z fixed $\tilde{w}^r(z)$ is decreasing with respect to r. The comparison $\tilde{w}^r(z) < \tilde{w}^s(z), \forall z \in B_s \Subset B_r$ is achieved. $\square$

Theorem 1.2. *The family $\{\tilde{w}^r\}$ converges C^2-smoothly to a C^2 function in $\mathbb{C}^n$.*

$$\lim_{r\to\infty}\tilde{w}^r(z) = |z|^2$$

Proof. By Lemma 1.1, the limit has existed because $\{\tilde{w}^r\}$ are uniformly bounded on any compact subset. Viewing $r^2 = \mu$ and using the L'Hôpital's rule repeatedly, the limit can be computed explicitly

$$\begin{aligned}\lim_{r\to\infty}\tilde{w}^r(z) &= \lim_{\mu\to\infty}\frac{\log\frac{\mu - |z|^2}{\mu}}{-\frac{1}{\mu}}\\ &= \lim_{\mu\to\infty}\frac{\mu|z|^2}{\mu - |z|^2}\\ &= |z|^2.\end{aligned}$$

The first and 2nd order convergence can also be computed directly.

$$\lim_{r\to\infty}\tilde{w}^r_i(z) = \lim_{r\to\infty}\frac{r^2\bar{z}_i}{r^2 - |z|^2} = \bar{z}_i;$$

$$\tilde{w}^r_{i\bar{j}}(z) = \frac{\delta_{ij}r^2(r^2 - |z|^2) + r^2\bar{z}_i z_j}{(r^2 - |z|^2)^2}.$$

The resulting metric $\lim_{r\to\infty}\tilde{w}^r_{i\bar{j}}(z) = \delta_{ij}$ is exactly the Euclidean metric. $\square$

2 Properties of Kähler-Einstein potentials in tangent bundles of rank-one symmetric spaces

Terminologies used in this section come from [K2] and references listed there, we will explain very briefly what a Grauert tube is.

For any real-analytic Riemannian manifold M, there exists a neighborhood U of M in TM where the adapted complex structure can be endowed with. The adapted complex structure is the unique complex structure turning every leaf of the Riemannian foliation into a holomorphic curve. With respect to this complex structure, the length square function is real-analytic and strictly plurisubharmonic.

A *Grauert tube of radius r over center M* is the disk bundle

$$T^rM \ = \{(x,v) \in TM : x \in M, |v| < \frac{r}{2}\}$$

equipped with the adapted complex structure. Each M has associated with a maximal possible radius, denoted by $r_{max}(M)$, such that the adapted complex structure could be defined in $T^{r_{max}(M)}M$. In this article, $\rho(x,v) = 4|v|^2$.

The maximal radius $r_{max}(M) = \infty$ when M is a compact symmetric space of rank-one, and for any $0 < r < \infty$ the Grauert tube T^rM is a Stein manifold with bounded strictly pseudoconvex boundary. The existence of a complete Kähler-Einstein metric with negative Ricci curvature is guaranteed by [C-Y]. Furthermore, as shown in [S], [A] and [K2], this Kähler-Einstein metric has a Kähler potential solely depending on the length square.

Let $\mathcal{S}_M$ denote the density function of a Riemannian symmetric space M of rank-one. It was shown in §5 [K2] that for any $\lambda > 0$ and any $r < r_{max}(M)$, there exists a Kähler potential $h_r(\sqrt{\rho})$ for the complete Kähler-Einstein metric of Ricci curvature $-\lambda < 0$ in the Grauert tube T^rM. Let $u = \sqrt{\rho}$, the potential function is unique and satisfies the following, *c.f.* (5.1) [K2],

$$h_r''(u)(h_r'(u))^{n-1}\exp(-\lambda h_r(u)) = u^{n-1}\hat{\mathcal{S}}_M(u); \ \lim_{u\to r} h_r(u) = \infty. \tag{2.1}$$

The derivatives are taken with respect to u where $\hat{\mathcal{S}}_M(u) := \mathcal{S}_M(-u^2)$.

$$\hat{\mathcal{S}}(u) = 2^{n-1}u^{1-n}\left(cosh\ \frac{u}{2}\right)^k (sinh\ \frac{u}{2})^{n-1},$$

where $k = n-1$ for the round sphere and the real projective space and $k = 1, 3, 7$ for the complex projective space, the quaternionic projective space and the Caley plane, respectively.

We may also consider two non-compact symmetric spaces: the real hyperbolic space H^n and the Euclidean space $\mathbb{R}^n$. For H^n: $\hat{\mathcal{S}}(u) = \left(\frac{\sin u}{u}\right)^{n-1}, r_{max}(H^n) = \pi$. For $\mathbb{R}^n$: $\hat{\mathcal{S}}(u) = 1$, $r_{max}(\mathbb{R}^n) = \infty$.

In $T^r M$, (2.1) can be expressed as

$$\begin{cases} h_r''(u)(h_r'(u))^{n-1} = e^{\lambda h_r(u)} \mathcal{D}_M(u), \ u \in [0, r) \\ \lim_{u \to r} h_r(u) = \infty \end{cases} \tag{2.2}$$

where $\mathcal{D}(u) = u^{n-1}$ for $\mathbb{R}^n$; $\mathcal{D}(u) = (\sin(u))^{n-1}$ for H^n; $\mathcal{D}(u) = (\sinh(u))^{n-1}$ for the round sphere; $\mathcal{D}(u) = 2^{n-1}\left(cosh\ \frac{u}{2}\right)^k (sinh\ \frac{u}{2})^{n-1}, k = 1, 3, 7$ for the complex projective space, the quaternionic projective space and the Caley plane, respectively. For all the above cases $\mathcal{D}_M(u) \geq 0$ and $\mathcal{D}_M(u) = 0$ if and only if $u = 0$.

Some well-known properties of h_r have been discussed in [K2], we summarize some of them here for future application. $h_r(\rho)$ is a real-analytic function of ρ, near the center, it has the asymptotic expression $h_r(u) = a + bu^2 + cu^4 + O(6),\ b > 0$.

Proposition 2.1.

(1) $h_r'(u) > 0, h_r''(u) > 0,\ \forall u \in (0, r);\ \inf_{u \in [0,r)} h_r(u) = h_r(0)$;

(2) $h_r'(0) = 0$;

(3) $h_r'(t) = \left(\int_0^t n e^{\lambda h_r(u)} \mathcal{D}(u)\right)^{\frac{1}{n}},\ \forall t \in (0, r)$.

(4) $\lim_{u \to r} h_r'(u) = \infty,\ \lim_{u \to r} h_r''(u) = \infty$

Proof. Since the right hand side of (2.2) is positive for all $u \in (0, r)$, neither $h_r''(u)$ nor $h_r'(u)$ has any zero point in $u \in (0, r)$. That is, $h_r'(u)$ is either positive or negative for all $u \in (0, r)$. Since $\lim_{u \to r} h_r(u) = \infty$, the only possibility is $h_r'(u) > 0, \forall u \in (0, r)$. The positivity of $h_r''(u)$ follows.

It was shown in Prop.5.1 of [K2] that h_r is a real-analytic function of $\rho = u^2$, hence $h_r'(0) = \frac{dh_r}{du}(0) = 0$. By (2.2)

$$\begin{aligned} \frac{d}{du}(h_r'(u))^n &= n e^{\lambda h_r(u)} \mathcal{D}(u), \\ (h_r'(t))^n &= \int_0^t n e^{\lambda h_r(u)} \mathcal{D}(u). \end{aligned} \tag{2.3}$$

Through the relation $h_r(u) = \int_0^u h_r'(t) dt + h_r(0)$, the condition $\lim_{u \to r} h_r(u) = \infty$ has implied that

$$\lim_{u \to r} h_r'(u) = \infty \text{ and similarly, } \lim_{u \to r} h_r''(u) = \infty. \tag{2.4}$$

□

We also need some estimate for $h_r''(0)$.

Lemma 2.2. $h_r''(0) = e^{\frac{\lambda}{n} h_r(0)} > 0$.

Proof. Let

$$F_r(u) = h_r'(u)\mathcal{D}_M^{\frac{-1}{n-1}}(u). \tag{2.5}$$

The equation (2.2) could be written as

$$h_r''(u)(F_r(u))^{n-1} = e^{\lambda h_r(u)},\ u \in [0, r). \tag{2.6}$$

From (2.5)

$$h_r''(u) = F_r'(u)\mathcal{D}_M^{\frac{1}{n-1}}(u) + F_r(u)\left(\mathcal{D}_M^{\frac{1}{n-1}}\right)'(u) \tag{2.7}$$

For $\mathbb{R}^n$, the real hyperbolic space H^n and all the compact rank-one symmetric spaces, $\mathcal{D}^{\frac{1}{n-1}}(0) = 0$ and $\left(\mathcal{D}^{\frac{1}{n-1}}\right)'(0) = 1$. Hence

$$h_r''(0) = F_r(0) \tag{2.8}$$

Plugging (2.8) into (2.6) with $u = 0$, the lemma is concluded. □

Let K be a compact subset of $T^sM \Subset T^rM$. We would like to develop a comparison between $h_r(u)$ and $h_s(u)$ for $u \in K$.

Lemma 2.3. $s < r$. *Let h_r, h_s be solutions of (2.2) in T^rM and in T^sM, respectively. Then $h_r(u) \le h_s(u), \forall u \in [0, s)$.*

Proof. $F(u) := h_s(u) - h_r(u)$ is a continuous function defined in $[0, s)$, thus $F^{-1}(-\infty, 0]$ is a closed subset of $[0, s)$ which could be written as union of closed connected intervals in $[0, s)$. There are two cases.

Case 1. If $0 \notin F^{-1}(-\infty, 0]$, then $h_r(u) < h_s(u), \forall u \in [0, s)$.

Since $h_r(0) < h_s(0)$ and $\lim_{u\to s} h_s(u) = \infty$, either $h_r < h_s$ in the whole interval $[0, s)$ or there exists an $\alpha \in (0, s)$ such that $h_r(\alpha) = h_s(\alpha)$ and $h_r(u) < h_s(u)$ for any $u \in [0, \alpha)$. If such α exists,

$$\begin{aligned}(h_r'(\alpha))^n &= \int_0^\alpha ne^{\lambda h_r(u)}\mathcal{D}(u) \\ &< \int_0^\alpha ne^{\lambda h_s(u)}\mathcal{D}(u) \\ &= (h_s'(\alpha))^n\end{aligned} \tag{2.9}$$

which shows $h_r'(\alpha) < h_s'(\alpha)$. Thus there exists some $\epsilon > 0$ such that $h_r(u) < h_s(u)$ for any $u \in (\alpha, \alpha + \epsilon)$. The point α is then a local minimum of the function F, then $0 = F'(\alpha) = h_s'(\alpha) - h_r'(\alpha)$, a contradiction. Therefore, $h_r < h_s$ in the whole interval $[0, s)$.

Case 2. If $0 \in F^{-1}(-\infty, 0]$, then 0 is an isolated point and $h_r(u) \le h_s(u), \forall u \in [0, s)$.

Let I be the interval containing 0, we claim $I = \{0\}$. Since I is a closed subset of $[0, s)$ containing 0, it is either $[0, s)$ or there exists a $\delta \geq 0$ such that $I = [0, \delta]$ is a maximal connected interval in $F^{-1}(-\infty, 0]$.

The first case has been ruled out because $\lim_{u \to s} h_s(u) = \infty$. The second case means $h_s(u) \leq h_r(u)$ in $[0, \delta]$, $h_s(\delta) = h_r(\delta)$ and $h_s(\delta + \epsilon) > h_r(\delta + \epsilon)$ for $0 < \epsilon \ll 1$. Similar calculation as (2.9) has led to $h_r'(\delta) \geq h_s'(\delta)$ if $\delta > 0$. This shows for $0 < \epsilon \ll 1$, $h_r(\delta + \epsilon) \geq h_s(\delta + \epsilon)$, a contradiction. Thus $\delta = 0$ and $I = \{0\}$.

Let $(0, \beta)$ be a maximal open interval such that $h_r < h_s$ in $(0, \beta)$. If $\beta < s$, then $h_r(\beta) = h_s(\beta)$ and $h_r > h_s$ in $(\beta, \beta + \epsilon')$ for some $\epsilon' \ll 1$. Following (2.9), $h_r'(\beta) < h_s'(\beta)$. Since $h_r(\beta) = h_s(\beta)$, it is not possible to have $h_r > h_s$ in $(\beta, \beta + \epsilon')$. Therefore, $\beta = s$ and $h_r(u) \leq h_s(u), \forall u \in [0, s)$. □

Remark 2.4. *Lemma 2.3 immediately implies that there is a unique solution for the equation (2.2).*

We would like to explain briefly the condition $\lim_{u \to r} h_r(u) = \infty$ has implied the corresponding metric is complete on $T^r M$.

The distance from center to the boundary is (c.f. p.157. [S])

$$L = \frac{1}{\sqrt{2}} \int_0^r \sqrt{h_r''(u)}\, du. \tag{2.10}$$

For L to be infinity, it is sufficient to show

$$\sqrt{h_r''(u)} > \frac{1}{r - u} \tag{2.11}$$

when $u \to r$, i.e., we need a comparison of $\sqrt{h_r''(u)} = \left(e^{\lambda h_r(u)} \frac{\mathcal{D}_M(u)}{(h_r'(u))^{n-1}} \right)^{\frac{1}{2}}$ with $\frac{1}{r-u}$. If $\frac{\mathcal{D}_M(u)}{(h_r'(u))^{n-1}} > 0$ as $u \to r$, the distance $L = \infty$ since $\sqrt{h_r''(u)}$ then grows exponentially.

The worst case is $\frac{\mathcal{D}_M(u)}{(h_r'(u))^{n-1}} \to 0$ as $u \to r$. After a translation, we may set the origin at $\{u = r\}$ and call the new coordinate by x. Let the order of vanishing at $x = 0$ of $\left(\frac{\mathcal{D}_M(x)}{(h_r'(x))^{n-1}} \right)^{\frac{1}{2}} \simeq x^k, k \in \mathbb{N}$; the order of going infinity of $h_r(x)$ is $x^{-\alpha}, \alpha > 0$. Repeatedly applying the L'Hôpital's rule shows, for $\alpha > 0$,

$$\lim_{x \to 0} x^k e^{x^{-\alpha}} = \infty.$$

Thus (2.11) holds and $L = \infty$.

3 Complete Ricci-flat metric in TM through a rescaling process

In this section, we fix M to be $\mathbb{R}^n$ or a compact symmetric space of rank-one, *i.e.*, it is either $\mathbb{R}^n$ or one of the round sphere, the real projective space, the complex projective space, the quaternionic projective space or the Caley plane.

The adapted complex structure is defined on the whole tangent bundle TM. For any $r > 0$ and any $\lambda_r > 0$, there exists a unique real-analytic function $f_r(u), u = \sqrt{\rho}$, satisfying the ODE

$$\begin{cases} f_r''(u)(f_r'(u))^{n-1} = e^{\lambda_r f_r(u)}\mathcal{D}_M(u),\ u \in [0, r); \\ \lim_{u\to r} f_r(u) = \infty. \end{cases} \tag{3.1}$$

Indeed, f_r is a Kähler potential of the complete Kähler-Einstein metric of Ricci curvature $-\lambda_r$ in T^rM. Since the whole tangent bundle TM is exhausted by Grauert tubes $\{T^rM\}_{0<r<\infty}$, a C^2-convergence of the family $\{f_r\}_{r>0}$ will lead to the existence of a Kähler metric with Ricci curvature $-\lim_{r\to\infty}\lambda_r$.

Let $\{\lambda_r\}_{r>0}$ be a decreasing sequence of positive numbers, $\lambda_r < \lambda_s$ whenever $r > s$, such that $\lim_{r\to\infty}\lambda_r = 0$.

A comparison analogous to Lemma 2.3 has played an essential role in the convergent argument. In this rescaling setting, we are looking for a comparison between f_r and f_s where f_r satisfies (3.1) and f_s is the unique solution of the following:

$$\begin{cases} f_s''(u)(f_s'(u))^{n-1} = e^{\lambda_s f_s(u)}\mathcal{D}_M(u),\ u \in [0, s); \\ \lim_{u\to s} f_s(u) = \infty. \end{cases} \tag{3.2}$$

In each T^rM, we fix the Ricci curvature to be -1 and let h_r be the unique solution of (2.2) in T^rM with $\lambda = 1$:

$$\begin{cases} h_r''(u)(h_r'(u))^{n-1} = e^{h_r(u)}\mathcal{D}_M(u),\ u \in [0, r); \\ \lim_{u\to r} h_r(u) = \infty. \end{cases} \tag{3.3}$$

Each $h_r(u)$ is an increasing function in u. We denote it's minimum as a_r,

$$a_r := h_r(0) = \inf_{u\in[0,r)} h_r(u). \tag{3.4}$$

Lemma 3.1. *$\{a_r\}_{r>0}$ is a decreasing sequence and* $\lim_{r\to\infty} a_r = -\infty$.

Proof. By Lemma 2.3, the sequence $\{a_r\}_{r>0}$ is decreasing.

Suppose $\lim_{r\to\infty} a_r = c$ for some real number c. Given $\epsilon > 0$ there exists N such that $|a_r - c| < \epsilon$ whenever $r \geq N$. In other words,

$$|a_l - a_N| < 2\epsilon \text{ for any } l > N. \tag{3.5}$$

Observing from the equations

$$\begin{aligned} h_l'(t) &= \left(\int_0^t n e^{h_l(u)} \mathcal{D}(u)\right)^{\frac{1}{n}},\ t \in (0,l); \\ h_N'(t) &= \left(\int_0^t n e^{h_N(u)} \mathcal{D}(u)\right)^{\frac{1}{n}},\ t \in (0,N), \end{aligned} \tag{3.6}$$

the difference of h_l and h_N is decided by their initial values a_l and a_N. Since $|a_l - a_N| < 2\epsilon$ for any $l > N$ and $\lim_{u\to N} h_N(u) = \infty$, there exists a positive number α such that $h_l(N+\alpha) = \infty$. This is not possible if l is taken to be sufficiently large. $\lim_{r\to\infty} a_r = -\infty$ is concluded. □

Define the function

$$H_r(u) = h_r(u) - a_r. \tag{3.7}$$

For any given $r > 0$, $H_r(0) = 0$ and $H_r(u) \geq 0, \forall u \in [0,r)$ and

$$\left(e^{\frac{-a_r}{n}} H_r\right)'' \left(e^{\frac{-a_r}{n}} H_r'\right)^{n-1} = \exp\left(e^{\frac{a_r}{n}} e^{\frac{-a_r}{n}} H_r\right) \mathcal{D}_M. \tag{3.8}$$

The function

$$K_r := e^{\frac{-a_r}{n}} H_r \tag{3.9}$$

satisfies

$$K_r''(u)(K_r'(u))^{n-1} = \exp(e^{\frac{a_r}{n}} K_r(u)) \mathcal{D}_M(u),\ \forall u \in [0,r) \tag{3.10}$$

and $\lim_{u\to r} K_r(u) = \infty$. By the uniqueness of the solution, K_r is a Kähler potential of the complete Kähler-Einstein metric of Ricci curvature $-e^{\frac{a_r}{n}}$ in $T^r M$. By (3.10),

$$\begin{aligned} K_r''(u)(K_r'(u))^{n-1} &= \exp(e^{\frac{a_r}{n}} K_r(u)) \mathcal{D}_M(u) \\ &= \exp(h_r - a_r) \mathcal{D}_M(u). \end{aligned} \tag{3.11}$$

Then

$$(K_r'(u))^n = n \int \exp(h_r - a_r) \mathcal{D}_M(u). \tag{3.12}$$

If we can show $h_r - a_r < h_s - a_s$, then we have

$$K_r'(u) < K_r'(u). \tag{3.13}$$

We also have

$$(h_r'(u))^n = n \int \exp h_r \mathcal{D}_M(u). \tag{3.14}$$

Since $h_r < h_s$, then $h_r' < h_s'$. Now

$$\begin{aligned} h_r - a_r &= \int_0^1 h_r'(tu) u dt \\ &< \int_0^1 h_s'(tu) u dt \\ &= h_s - a_s. \end{aligned} \tag{3.15}$$

$K_r(0) = K_s(0)$, (3.13) will imply that

$$K_r(u) \le K_s(u),\ \forall u \in [0, s) \tag{3.16}$$

which has provided a uniform upper bound for the family $\{K_r\}_{r>0}$ in compact subsets. A uniform lower bound is easily obtained since for any $r > 0$ and any $u \in [0, r)$,

$$K_r(u) = e^{\frac{-a_r}{n}} H_r(u) \ge 0.$$

The following proposition has been concluded.

Proposition 3.2. $\{K_r\}_{r>0}$ *has converged uniformly on any compact subset of* $[0, \infty)$ *to a continuous function* K*.*

$$\lim_{r\to\infty} K_r(u) := K(u),\ u \in [0, \infty).$$

The goal is to show this function K is a Kähler potential of a Ricci-flat metric in TM. For this to work, it is sufficient to find some uniform bounds for the first derivatives and the second derivatives of the family $\{K_r\}_{r>0}$ in compact subsets. For a fixed compact set $A \subset [0, \infty)$, the restriction of $\mathcal{D}_M$ in A is bounded. By (3.16), for any $r > s$,

$$\begin{aligned} e^{\frac{a_r}{n}} K_r(u) &\le e^{\frac{a_r}{n}} K_s(u),\ \forall u \in [0, s); \\ &\le K_s(u),\ \forall u \in [0, s),\ r \gg 1, \end{aligned} \tag{3.17}$$

where the last inequality comes from the fact that $\lim_{r\to\infty} a_r = -\infty$. On the other hand, $e^{\frac{a_r}{n}} K_r(u) = H_r(u) \ge 0$. Therefore, there exists $c > 0$ such that

$$0 \le \mathcal{D}_M(u) \le \exp(e^{\frac{a_r}{n}} K_r(u))\mathcal{D}_M(u) \le c,\ u \in A,\ r \gg 1. \tag{3.18}$$

Lemma 3.3. *In any compact set* $A \subset [0, \delta] \subset [0, \infty)$*, the family* $\{K_r'\}$ *is uniformly bounded:*

$$0 \le K_r'(t) \le \sqrt[n]{nc}\ \sqrt[n]{\delta},\ t \in [0, \delta],\ \forall r \gg 1.$$

Proof. The fact that $K_r'(0) = 0$ along with (3.10) and (3.18) then implies for $t \in [0, \delta],\ r \gg 1$,

$$\begin{aligned} (K_r'(t))^n &= \int_0^t n \exp(e^{\frac{a_r}{n}} K_r(u))\mathcal{D}_M(u)du \le ntc; \\ (K_r'(t))^n &= \int_0^t n \exp(e^{\frac{a_r}{n}} K_r(u))\mathcal{D}_M(u)du \ge \int_0^t n\mathcal{D}_M(u)du \end{aligned} \tag{3.19}$$

for some $c > 0$. Thus,

$$0 \le \left(\int_0^t n\mathcal{D}_M(u)du\right)^{\frac{1}{n}} \le K_r'(t) \le \sqrt[n]{nc}\ \sqrt[n]{\delta},\ t \in [0, \delta],\ \forall r \gg 1. \tag{3.20}$$

This uniform estimate in $[0, \delta]$ of course has implied a uniform estimate in the compact set A. ☐

The next step is to find some uniform bound on the second derivatives.

Lemma 3.4. *Given a compact set $A \subset [0,\delta] \subset [0,\infty)$, there exists a constant $C > 0$ such that*

$$-C \leq K_r''(u) \leq C,\ u \in A,\ \forall r \gg 1.$$

Proof. A uniform lower bound for $K_r''(u)$ is already available since

$$K_r''(u) = e^{\frac{-ar}{n}} H_r''(u) = e^{\frac{-ar}{n}} h_r''(u) > 0, \forall u \in [0,r), r \in (0,\infty). \tag{3.21}$$

For a uniform upper bound, we consider two kinds of compact sets: $A = [\epsilon,\delta], \epsilon > 0$, and $A = [0,\delta]$. For the first case, (3.20) shows there exists a constant $d > 0$ such that

$$d \leq K_r'(\epsilon),\ \forall r \gg 1.$$

By (3.21), K_r' is an increasing function in u, so

$$K_r'(u) \geq K_r'(\epsilon) \geq d \text{ for any } u \geq \epsilon.$$

(3.10) (3.18) and (3.21) have concluded a uniform upper bound:

$$0 \leq K_r''(u) \leq \frac{c}{(K_r'(u))^{n-1}} \leq \frac{c}{d^{n-1}},\ u \in [\epsilon,\delta],\ \forall r \gg 1. \tag{3.22}$$

For the second case, it amounts to show $K_r''(0)$ have a uniform upper bound for $r \gg 1$. By (3.21),

$$K_r''(0) = e^{\frac{-ar}{n}} h_r''(0) = e^{\frac{-ar}{n}} e^{\frac{h_r(0)}{n}} = 1, \forall\, r \in (0,\infty), \tag{3.23}$$

where the second equality is obtained from Lemma 2.2. The lemma is therefore concluded. ☐

Theorem 3.5. *Let M be a compact rank-one symmetric space or $M = \mathbb{R}^n$. There exists a complete Ricci-flat metric in TM obtained from an exhaustion by Kähler-Einstein metrics in $\{T^rM\}_{0<r<\infty}$.*

Proof. Let the function K in TM be the one defined in Proposition 3.2. Then lemmas 3.3 and 3.4 have asserted the uniform convergence of first derivatives and second derivatives in any compact subset of $[0,\infty)$. Therefore,

$$K'(u) = \lim_{r\to\infty} K_r'(u);\ K''(u) = \lim_{r\to\infty} K_r''(u),\ u \in [0,\infty).$$

Since K_r is a Kähler potential of a Kähler-Einstein metric of Ricci curvature $-e^{\frac{ar}{n}}$ in T^rM, the function K is a Kähler potential of some Kähler-Einstein metric in TM

with Ricci curvature $-\lim_{r\to\infty} e^{\frac{a_r}{n}}$. By Lemma 3.1, $\lim_{r\to\infty} a_r \to -\infty$, therefore K is a Kähler potential of a Ricci-flat metric in TM. Furthermore,

$$K''(u)(K'(u))^{n-1} = \mathcal{D}_M(u),\ \forall u \in [0,\infty), \tag{3.24}$$

which implies

$$(K'(t))^n = \int_0^t n\mathcal{D}_M(u)du,\ \forall t \in (0,\infty) \tag{3.25}$$

Examining the list after (2.2), it is clear that $\mathcal{D}_M$ grows exponentially when M is a compact rank-one symmetric space. Hence $K'(t)$ grows exponentially as well. On the other hand,

$$K'(t) = \int_0^t K''(t)du,\ \forall t \in (0,\infty) \tag{3.26}$$

so $K''(t)$ goes to ∞ in an exponential way as $t \to \infty$.

When $M = \mathbb{R}^n$, $(K'(t))^n = \int_0^t nu^{n-1}du = t^n$ and $K''(t) = 1,\ \forall t \in (0,\infty)$.

Since the distance from the center to the boundary is given by $\frac{1}{\sqrt{2}}\int_0^\infty \sqrt{K''(u)}\,du$ which equals to ∞ for all the above mentioned cases. The metrics are complete. $\square$

The resulting Kähler potential in $T\mathbb{R}^n$ is a solution of $h''(u)(h'(u))^{n-1} = u^{n-1}$ which has taken the form $h(u) = \frac{u^2}{2}+c$ for any constant c. It is known in the Euclidean case that $u(z) = |y|$ and the Kähler potential $h(z) = \frac{|y|^2}{2} + c = \frac{-1}{8}\sum_j(z_{\bar{j}}^2 + z_j^2 - 2z_j z_{\bar{j}})$. The corresponding Ricci-flat metric is simply the standard Euclidean metric in $\mathbb{C}^n$.

It is interesting to observe that from two different exhaustions, one through the balls discussed in §2, and the other one through $T^r\mathbb{R}^n$ studied in this section, the resulting Ricci-flat metrics are the same.

Stenzel [S] has also worked on the tangent bundle of compact rank-one symmetric spaces. By the transitivity of the rank-one symmetry, the defining Monge-Ampère equation for a Ricci-flat metric could be reduced to an ordinary differential equation. Working directly on the solvability and the completeness of this ordinary differential equation, Stenzel was able to show the existence a complete Ricci-flat metric in TM for compact rank-one M.

Although our resulting Ricci-flat metric through the rescaling process turns out to be the same with the one constructed by Stenzel in TM. We think the approach here is interesting and we are looking for some further investigation.

4 Kähler-Einstein metric of negative Ricci in $T^\pi H$

Using an exhaustion by smooth bounded strictly pseudoconvex domains, Cheng-Yau were able to confirm the existence of a Kähler-Einstein metric with negative Ricci curvature in any bounded pseudoconvex domain. With very mild regularity condition, the metric could be proved to be complete. Later on, Mok-Yau have extended the existence and completeness to any bounded Stein domain by showing the exhaustion has a uniform convergent limit. The convergence of the exhaustion has strongly relied on the boundedness of the domain D. An essential point is $D \Subset B(0,R)$ for some large R, so that the Kähler-Einstein metrics in the exhaustion family could be made comparison with the Poincaré metric of $B(0,R)$ to get hold of a uniform lower bound.

It is not clear whether such kind of metric exists in unbounded pseudoconvex domain in $\mathbb{C}^n$ or not. In this case, the comparison theorem of [C-Y] has provided a uniform upper bound for the sequence. However, it is no longer clear whether there exists any uniform lower bound or not.

In this section, we consider Grauert tubes over the real-hyperbolic space H^n, a non-compact rank-one symmetric space. Since H^n is co-compact, there exists a discrete subgroup $G \subset Isom(H^n)$ so that $\hat{H} = H/G$ is a compact real-analytic Riemannian manifold. It was shown in [K1][K2] that the maximal radius for $\hat{H}^n$ is π.

For any $r < \pi$, $T^r\hat{H}$ is a bounded Stein domain with smooth strictly pseudoconvex boundary in $T^\pi\hat{H}$ and the existence of a complete Kähler-Einstein metric with negative Ricci is guaranteed. Exhaustion by such an increasing family of Stein domains, the manifold $T^\pi\hat{H}$ itself is then a Stein manifold.

By the nature of the adapted complex structure, the universal covering T^rH of $T^r\hat{H}$ has shared the same maximal radius, i.e., $r_{max}(H) = \pi$ and the existence of a complete Kähler-Einstein metric with negative Ricci in T^rH, $r < \pi$, is also guaranteed. Furthermore, $T^\pi H$ is a Stein manifold since it is the universal covering of the Stein manifold $T^\pi\hat{H}$.

It is not clear at all whether this $T^\pi H$ could sit inside any Kähler manifold as a bounded domain or not. Thus, we can't apply the main theorem in [M-Y] to conclude the existence of a Kähler-Einstein metric with negative Ricci in it. The goal of this section is to show the existence of such a metric in $T^\pi H$.

$\mathcal{D}_{H^n}(u) = (\sin(u))^{n-1}$ and $r_{max}(H) = \pi$. Therefore, Grauert tube T^rH^n has existed for any $r \in (0,\pi)$, and a Kähler potential for the complete Kähler-Einstein metric of Ricci -1 in T^rM is given by

$$\begin{cases} h_r''(u)(h_r'(u))^{n-1} = e^{h_r(u)}(\sin(u))^{n-1}, \ u \in [0,r); \\ \lim\limits_{u\to r} h_r(u) = \infty. \end{cases} \tag{4.1}$$

Furthermore, h_r is real-valued and real-analytic.

The Kähler manifold $T^\pi H$ is exhausted by the family $\{T^rH : 0 < r < \pi\}$ in the sense that T^rH is an increasing family of Grauert tubes and $T^\pi H = \cup_{0<r<\pi} T^rH$. T^rH is not relatively compact in $T^\pi H$ since H is not compact.

Let $a_r \in [0,r)$ be the largest number such that $h_r(a_r) = 0$. If there is no such a_r, then $h_r(u) > 0$ for all $u \in [0,r)$.

Lemma 4.1. $h_r(0) \geq -\pi \sqrt[n]{n}\sqrt[n]{\pi},\ \forall r \in (0,\pi)$.

Proof. Given $r \in (0,\pi)$, if $h_r(u) > 0$ for all $u \in [0,r)$ the statement automatically holds. Without loss of generality, we may assume $a_r \geq 0$.

Since h_r is a monotonically increasing function of u, $h_r(u) \leq 0,\ \forall u \in [0,a_r]$. For any given $t \in [0,a_r]$, the equation (4.1) implies

$$(h_r'(t))^n = \int_0^t n e^{h_r(u)}(\sin(u))^{n-1} \leq nt. \tag{4.2}$$

Since $h_r'(u)$ is increasing in u,

$$\begin{aligned} -h_r(0) = h_r(a_r) - h_r(0) &= \int_0^{a_r} h_r'(t)dt \\ &\leq \int_0^{a_r} h_r'(a_r)dt \\ &\leq a_r \sqrt[n]{n}\sqrt[n]{a_r} \\ &\leq \pi \sqrt[n]{n}\sqrt[n]{\pi}. \end{aligned} \tag{4.3}$$

The lemma is concluded. □

As an increasing function of u, Lemma 4.1 has provided a uniform lower bound for $h_r(u)$. That is,

$$inf_{u\in[0,r)} h_r(u) \geq h_r(0) \geq -\pi \sqrt[n]{n}\sqrt[n]{\pi},\ \forall r \in (0,\pi). \tag{4.4}$$

Summing up Lemma 2.3 and Lemma 4.1, it is clear that for given compact set $K \subset [0,\delta] \subset [0,\pi)$, there exists a $c > 0$ such that

$$-c \leq h_r(u) \leq c,\ \ \forall u \in K,\ \forall r > \delta \tag{4.5}$$

and the limit exists

$$h(u) := \lim_{r\to\pi} h_r(u),\ u \in [0,\pi). \tag{4.6}$$

The goal is to show this function h is a Kähler potential of a Kähler-Einstein metric in $T^\pi H$. For this to work, it is sufficient to find some uniform bounds for the first derivatives and the second derivatives of the family $\{h_r\}$ in compact subsets.

Lemma 4.2. *In any compact set $K \subset [0,\delta] \subset [0,r)$, the family $\{h_r'\}_{r>\delta}$ is uniformly bounded: there exists a $c > 0$ such that*

$$0 \leq h_r'(t) \leq \sqrt[n]{n}\ \sqrt[n]{\delta}\ e^{\frac{c}{n}},\ \forall\, t \in K,\ r > \delta.$$

Proof. A constant c could be chosen from (4.5), then for $r > \delta$, the following inequality holds in K:

$$ne^{-c}(\sin(u))^{n-1} \leq ne^{h_r(u)}(\sin(u))^{n-1} \leq ne^c. \tag{4.7}$$

Since $h'_r(0) = 0$, (4.2) then implies

$$n\, e^{-c}\int_0^t (\sin u)^{n-1} \leq (h'_r(t))^n \leq n\; t\; e^c,\ \forall t \in [0,\delta],\ r > \delta. \tag{4.8}$$

Thus

$$0 \leq h'_r(t) \leq \sqrt[n]{n}\ \sqrt[n]{\delta}\; e^{\frac{c}{n}},\ \forall t \in [0,\delta],\ r > \delta. \tag{4.9}$$

This uniform estimate in $[0,\delta]$ of course has implied a uniform estimate in K. ☐

The next step is to derive a uniform bound on the second derivatives.

Lemma 4.3. *Given compact set $K \subset [0,\delta] \subset [0,r)$, there exist a constant $C > 0$ such that*

$$-C \leq h''_r(u) \leq C,\ \forall u \in K,\ r > \delta.$$

Proof. Lemma 2.2 has provided a uniform lower bound $h''_r(u) > 0, \forall u \in [0,r), r \in (0,\pi)$. For a uniform upper bound, we consider two kinds of compact sets: $K = [\epsilon,\delta], \epsilon > 0$ and $K = [0,\delta]$.

For the first case, (4.8) shows there exists a constant $d > 0$ such that

$$d \leq h'_r(\epsilon),\ \forall r > \delta.$$

As $h'' \geq 0$, $h'_r(u) \geq h'_r(\epsilon) \geq d$ for any $u > \epsilon$. By (4.1) and (4.9),

$$0 \leq h''_r(u) \leq \frac{e^c}{(h'_r(u))^{n-1}} \leq \frac{e^c}{d^{n-1}},\ u \in K, \forall r > \delta. \tag{4.10}$$

For the second case, it amounts to show $h''_r(0)$ have a uniform upper bound for $r > \delta$. By Lemma 2.2 and (4.5) there exists a constant L such that

$$h''_r(0) = e^{\frac{h_r(0)}{n}} \leq L,\ \forall r > \delta. \tag{4.11}$$

(4.10) along with (4.11) has proved the lemma. ☐

Theorem 4.4. *There exists a complete Kähler-Einstein metric of Ricci curvature -1 in $T^\pi H$.*

Proof. Lemmas 4.2 and 4.3 have concluded that the two families $\{h'_r\}$ and $\{h''_r\}$ have converged uniformly in any compact subset of $[0,\pi)$. This shows the convergence $h_r(u) \to h(u)$ is good up to second orders. The function h is then a Kähler potential of a Kähler-Einstein metric of Ricci curvature -1 in $T^\pi H$ and $\lim_{u\to\pi} h(u) = \infty$. Furthermore,

$$h''(u)(h'(u))^{n-1} = e^{h(u)}(\sin(u))^{n-1},\ \forall u \in [0,\pi). \tag{4.12}$$

Although $\lim_{u\to\pi}\sin(u) = 0$, the rapid exponential growth of $e^{h(u)}$ still guarantees that $\lim_{u\to\pi} e^{h(u)}(\sin(u))^{n-1} = \infty$ in the exponential way. The equation

$$(h'(t))^n = \int_0^t ne^{h(u)}(\sin(u))^{n-1}du,\ \forall t \in (0,\pi) \tag{4.13}$$

has shown that the h' is increasing to ∞ in an exponential way near π. The fact that the second derivative h'' has increased to ∞ exponentially near π is obtained through the following equation

$$h'(t) = \int_0^t h''(u)du.$$

We thus conclude the metric is complete. □

After this work has been done, we found the authors in [B-H-H] have discovered that maximal Grauert tubes over any rank-one space is Hermitian symmetric. And it is well-known, c.f. [H], that any Hermitian symmetric space is biholomorphic to a bounded domain in $\mathbb{C}^n$. $T^\pi H^n$ is then a bounded domain of holomorphy in $\mathbb{C}^n$. The existence of a complete Kähler-Einstein metric with negative Ricci curvature in $T^\pi H^n$ could be concluded from [M-Y] directly.

At the end of [K2], we have shown near the center, the holomorphic sectional curvatures along the Monge-Ampère are negative when the center of the Grauert tubes is of compact rank-one or is the Euclidean space. We were not able to reach any definite result for the real-hyperbolic space. With the machineries developed in this section, we conclude:

Proposition 4.5. *Let k_r denote the complete Kähler-Einstein metric of Ricci curvature $-(n+1)$ in the Grauert tube T^rH. There exists an $\epsilon > 0$ such that for any $r \in (0, \frac{\pi}{2}+\epsilon)$ holomorphic sectional curvatures of k_r along the Monge-Ampère leaves of T^rH are negative near the center H.*

Proof. It amounts to show $b = \frac{1}{2}\exp\frac{n+1}{n}a > \frac{n-1}{6(n+1)}$ where $a = h_r(0)$. The Grauert tube $T^{\frac{\pi}{2}}H$ is biholomorphic to the ball and an explicit solution $h_{\frac{\pi}{2}}(u) = -\log\cos u$ to the defining ODE. Since $h_{\frac{\pi}{2}}(0) = 0$, we conclude $h_r(0) \geq 0$ for any $r \in (0, \frac{\pi}{2})$. By the continuity, there exists an $\epsilon > 0$ such that $h_{\frac{\pi}{2}+\epsilon}(0)$ is quite close to 0. In this cases, $b \simeq \frac{1}{2} > \frac{n-1}{6(n+1)}$. □

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