

# ON THE PARAMETER DOMAIN OF WISHART DISTRIBUTIONS AND THEIR INFINITE DIVISIBILITY

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ABSTRACT. A complete characterization of Wishart distributions on the cones of positive semi-definite matrices is provided in terms of a description of their maximal parameter domain. This result is new in that also degenerate scale parameters are included. For such cases, the standard constraints on the range of the shape parameter may be relaxed. Furthermore, the infinitely divisible Wishart distributions are revealed as suitable transformations and embeddings of one dimensional gamma distributions. This note completes the findings of Lévy (1937) concerning infinite divisibility and Gindikin (1975) regarding the existence issue.

## 1. INTRODUCTION

The Wishart distribution  $\Gamma(p; \sigma)$  on the cone  $S_d^+$  of symmetric positive semi-definite  $d \times d$  matrices is defined (whenever it exists) by its Laplace transform

$$\mathcal{L}(\gamma(p; \sigma))(u) = (\det(I + \sigma u))^{-p}, \quad u \in -S_d^+, \quad (1.1)$$

where  $p > 0$  denotes its shape parameter and  $\sigma \in S_d^+$  is the scale parameter. In the non-degenerate case where  $\sigma$  is invertible, and for a discrete set of shape parameters, these distributions have been introduced in 1928 by Wishart [7] as sums of "squares" of centered multivariate normal distributions (quite similarly to the chi-squared distributions in dimension one).

In 1937, Lévy [4] showed that  $\gamma(p; \sigma)$  on  $S_2^+$  is not infinitely divisible for invertible  $\sigma$ , which means that for some sequence of shape parameters  $p_k \downarrow 0$ ,  $\gamma(p_k; \sigma)$  cannot exist. Gindikin [2], Shanbhag [6] and Peddada & Richards [5]<sup>1</sup> subsequently showed:

**Theorem G.** *For non-degenerate  $\sigma$  the following are equivalent:*

- (i) *The right side of (1.1) is the Laplace transform of a probability measure.*
- (ii)  *$p$  belongs to the Gindikin ensemble*

$$\Lambda_d = \left\{ \frac{j}{2}, \quad j = 1, 2, \dots, d-2 \right\} \cup \left[ \frac{d-1}{2}, \infty \right).$$

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<sup>1</sup>Contrary to [5] we exclude the point mass at zero, i.e. the Gindikin ensemble does not contain 0. That's why we have chosen  $p > 0$ .

Aim of the present note is to extend this characterization by also allowing for degenerate  $\sigma \in S_d^+$ , and, to determine all Wishart distributions which are infinitely divisible.

It is worth noting that – in view of the right side of (1.1) and Lévy’s continuity theorem – the maximal parameter domain

$$\Theta_d := \{(p, \sigma) \in \mathbb{R}_+ \times S_d^+ \mid \Gamma(p; \sigma) \text{ is a probability measure}\}$$

is closed, and that only for invertible scale parameters the issues of existence and infinite divisibility have been settled in the literature (where one easily follows from the other). Hence the following two statements answer very natural question on the Wishart family.

**Theorem 1.1.** *Let  $d \in \mathbb{N}$ ,  $p \geq 0$  and  $\sigma \in S_d^+$  with  $\text{rank}(\sigma) = r$ . The following are equivalent:*

- (i) *The right side of (1.1) is the Laplace transform of a non-trivial probability measure  $\Gamma(p; \sigma)$  on  $S_d^+$ .*
- (ii)  *$p \in \Lambda_r$ .*

**Theorem 1.2.** *Let  $d \in \mathbb{N}$ ,  $p \geq 0$  and  $\sigma \in S_d^+$ . The following are equivalent:*

- (i)  *$\Gamma(p; \sigma)$  is infinitely divisible.*
- (ii)  *$\text{rank}(\sigma) = 1$ .*

**Remark 1.3.** • It should be pointed out that Theorem 1.1 does not follow from the classification of positive Riesz distributions, when  $\text{rank}(\sigma) < d$ , see, e.g., [1, Theorem VII.3.2]. In fact let us consider the special case, where  $\sigma = \text{diag}(1, 0, \dots, 0) \in S_d^+$ , and denote by  $\pi_{d \rightarrow r}$  the projection onto the  $r$ -th subminors of  $S_d^+$ , that is

$$\pi_{d \rightarrow r} : S_d^+ \rightarrow S_r^+, \quad \pi_{d \rightarrow r}(a) = (a_{ij})_{1 \leq i, j \leq r}. \quad (1.2)$$

Then the right side of (1.1) takes the form

$$(\det(\pi_{d \rightarrow r}(1 + u)))^{-p} = (1 + u_{11})^{-p}, \quad (1.3)$$

which the Laplace transform of an infinitely divisible probability measure on  $S_d^+$  due to the theorems 1.1–1.2. However, the ”corresponding” Riesz distribution<sup>2</sup> (which is also infinitely divisible) has Laplace transform

$$(\det(\pi_{d \rightarrow r}((I + u)^{-1})))^p = \left( \frac{1 + u_{22}}{\det(I + u)} \right)^p,$$

which obviously differs from eq. (1.3). Nevertheless, both functions are characteristic functions of probability measures on the sub-cones of positive matrices of rank 1.

- It is possible to generalize the Theorems of this paper to symmetric cones. In that more general setting the set  $\Lambda$  must be replaced by the so-called Wallach set, see, e.g., [1, Theorem VII.3.1]. We avoid this setting to make the presentation short and accessible to a larger group of readers.

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<sup>2</sup>This is a member of the natural exponential family generated by the standard Riesz distribution with Laplace transform  $(\det(\pi_{d \rightarrow r}(u^{-1})))^p$

Notation:  $I_d$  is the  $d \times d$  unit matrix, and if there arises no confusion, we simply write  $I$ .  $\det$  and  $\text{tr}$  are determinant and trace operator, and  $\text{rank}$  denotes the matrix rank. For matrices  $A, B$  of  $r \times r$  and  $t \times t$  dimension,  $\text{diag}(A, B)$  denotes the corresponding block-diagonal  $(r+t) \times (r+t)$  matrix.

*Proof of Theorem 1.1.* Let  $U$  be an orthogonal matrix such that  $U\sigma U^\top = \text{diag}(D, 0, \dots, 0)$  where  $D = \text{diag}(\sigma_1, \dots, \sigma_r)$ . We introduce the linear automorphism  $g_U$  as the map

$$g_U : S_d^+ \rightarrow S_d^+, \quad g_U(\xi) := U\xi U^\top.$$

Proof of (i) $\Rightarrow$ (ii): We denote by  $\Gamma_*$  the push-forward measure of  $\Gamma(p; \sigma)$  under  $g_U$ , which means that for Borel sets  $A \in S_d^+$  we have  $\Gamma_*(A) = \Gamma(p; \sigma)(g_U^{-1}(A)) = \Gamma(p; \sigma)(U^\top A U)$ . By [3, eq. (1.3.4)] the functional determinant of  $g_U$  equals 1, hence for all  $s \in -S_d^+$  we have

$$\begin{aligned} \mathcal{L}(\Gamma_*)(s) &:= \int_{S_d^+} e^{\text{tr}(s\xi)} \Gamma_*(d\xi) = \int_{S_d^+} e^{\text{tr}(s\xi)} \Gamma(p; \sigma)(d(U^\top \xi U)) \\ &= \int_{S_d^+} e^{\text{tr}(sU\eta U^\top)} \Gamma(p; \sigma)(d\eta) = \int_{S_d^+} e^{\text{tr}(U^\top s U \eta)} \Gamma(p; \sigma)(d\eta) \\ &= \det(I + \sigma U^\top s U)^{-p} = \det(I + U\sigma U^\top s)^{-p} \\ &= \det(I_d + \text{diag}(D, 0)s)^{-p} = \det(I_r + Ds)^{-p}. \end{aligned} \tag{1.4}$$

By Theorem G, we have that there exists a probability measure  $\Gamma(p; D)$  on  $S_d^+$  with Laplace transform

$$\det(I_r + Du)^{-p}, \quad u \in -S_r^+.$$

We may therefore conclude that  $\Gamma_*$  equals  $\Gamma(p; D)^{*, \pi_{d \rightarrow r}}$ , the pullback of  $\Gamma(p; D)$  under  $\pi_{d \rightarrow r}$  (see the defining equation (1.2)). In fact, for all  $s \in -S_d^+$  we obtain

$$\mathcal{L}(\Gamma(p; D)^{*, \pi_{d \rightarrow r}})(s) = \int_{S_d^+} e^{\text{tr}(\pi_{d \rightarrow r}(s)\xi)} \Gamma(p; D)(d\xi),$$

which equals eq. (1.4). But since  $\text{diag}(\sigma_1, \dots, \sigma_r)$  is of full rank  $r$ , we have by Theorem G that  $p \in \Lambda_r$ , and we have shown that (ii) holds.

Proof of (ii) $\Rightarrow$ (i): We reverse the preceding arguments: By Theorem G, there exists a probability measure  $\Gamma(p, D)$  on  $S_r^+$  for  $p \in \Lambda_r$ . Let  $\pi^{r \rightarrow d}$  be the embedding

$$\pi^{r \rightarrow d} : S_r^+ \rightarrow S_d^+, \quad \pi^{r \rightarrow d}(A) := \text{diag}(A, 0).$$

Then the Laplace transform of the push-forward  $\Gamma(p, D)_*^{\pi^{r \rightarrow d}}$  of  $\Gamma(p, D)$  under  $\pi^{r \rightarrow d}$  equals

$$\mathcal{L}(\Gamma(p, D)_*^{\pi^{r \rightarrow d}}) = \det(I_d + \text{diag}(D, 0, \dots, 0)\pi_{d \rightarrow r}(s))^{-p},$$

Hence by eq. (1.4) the push-forward of  $\Gamma(p, D)_*^{\pi^{r \rightarrow d}}$  under  $g_{U^{-1}} = g_U^{-1} = g_{U^\top}$  (hence given by  $g_U^{-1}(\xi) = U^\top \xi U$ ) equals  $\Gamma(p, \sigma)$ , which proves (i).  $\square$

*Proof of Theorem 1.2.* (i) $\Rightarrow$ (ii): By infinite divisibility and the right side of eq. (1.1) we see that  $\Gamma(p/n; \sigma)$  is a probability measure for each  $n \geq 1$ . But by definition,  $\Gamma_d \subseteq [1/2, \infty)$  if and only if  $d > 1$ . Hence by Theorem 1.1 we have that  $\text{rank}(\sigma) = 1$ .

The converse implication (ii) $\Rightarrow$ (i) may be proved in a similar way: By Theorem 1.1 and in view of the fact that  $\Gamma_1 = (0, \infty)$ , we have that  $\Gamma(p/n; \sigma)$  is a probability measure for all  $n \geq 1$ . Therefore by the right side of eq. (1.1) we see that  $\Gamma(p; \sigma)$  is infinitely divisible.  $\square$

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