Ideal games, Ramsey sets and the Fréchet-Urysohn property

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Abstract

It is shown that Matet's characterization ([8]) of $\mathcal{H}$-Ramseyness relative to a selective coideal $\mathcal{H}$, in terms of games of Kastanas ([5]), still holds if we consider semiselectivity ([2]) instead of selectivity. Moreover, we prove that a coideal $\mathcal{H}$ is semiselective if and only if Matet's game-theoretic characterization of $\mathcal{H}$-Ramseyness holds. This gives a game-theoretic counterpart to a theorem of Farah [2], asserting that a coideal $\mathcal{H}$ is semiselective if and only if the family of $\mathcal{H}$-Ramsey subsets of $\mathbb{N}^{[\infty]}$ coincides with the family of those sets having the abstract $Exp(\mathcal{H})$-Baire property. Finally, we show that under suitable assumptions, semiselectivity is equivalent to the Fréchet-Urysohn property.

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MSC: 05D10, 03E02.

1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. Given an infinite set $A \subseteq \mathbb{N}$, the symbol $A^{[\infty]}$ (resp. $A^{[<\infty]}$) represents the collection of the infinite (resp. finite) subsets of A. Let $A^{[n]}$ denote the

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set of all the subsets of A with n elements. If $a \in \mathbb{N}^{[<\infty]}$ is an **initial segment** of $A \in \mathbb{N}^{[\infty]}$ then we write $a \sqsubset A$. Also, let $A/a := \{n \in A : max(a) < n\}$, and write A/n to mean $A/\{n\}$. For $a \in \mathbb{N}^{[<\infty]}$ and $A \in \mathbb{N}^{[\infty]}$ let

$$[a, A] := \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B \subseteq A\}.$$

The family $Exp(\mathbb{N}^{[\infty]}) := \{[a, A] : (a, A) \in \mathbb{N}^{[<\infty]} \times \mathbb{N}^{[\infty]}\}$ is a basis for **Ellentuck's topology**, also known as **exponential topology**. In [1], Ellentuck gave a characterization of Ramseyness in terms of the Baire property relative to this topology (see Theorem 2.1 below).

Let $(P, \leq)$ be a poset, a subset $D \subseteq P$ is **dense** in P if for every $p \in P$, there is $q \in D$ with $q \leq p$. $D \subseteq P$ is **open** if $p \in D$ and $q \leq p$ imply $q \in D$. P is σ-**distributive** if the intersection of countably many dense open subsets of P is dense. P is σ-**closed** if every decreasing sequence of elements of P has a lower bound.

Definition 1.1. *A family $\mathcal{H} \subset \wp(\mathbb{N})$ is a* ***coideal*** *if it satisfies:*

(i) $A \subseteq B$ and $A \in \mathcal{H}$ implies $B \in \mathcal{H}$, and

(ii) $A \cup B \in \mathcal{H}$ implies $A \in \mathcal{H}$ or $B \in \mathcal{H}$.

The complement $\mathcal{I} = \wp(\mathbb{N}) \setminus \mathcal{H}$ is the **dual ideal** of $\mathcal{H}$. In this case, as usual, we write $\mathcal{H} = \mathcal{I}^+$. We will suppose that coideals differ from $\wp(\mathbb{N})$. Also, we say that a nonempty family $\mathcal{F} \subseteq \mathcal{H}$ is $\mathcal{H}$-**disjoint** if for every $A, B \in \mathcal{F}$, $A \cap B \notin \mathcal{H}$. We say that $\mathcal{F}$ is a **maximal** $\mathcal{H}$-**disjoint family** if it is $\mathcal{H}$-disjoint and it is not properly contained in any other $\mathcal{H}$-disjoint family as a subset.

A subset $\mathcal{X}$ of $\mathbb{N}^{[\infty]}$ is **Ramsey** if for every $[a, A] \neq \emptyset$ with $A \in \mathbb{N}^{[\infty]}$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. Some authors have used the term "completely Ramsey" to express this property, reserving the term "Ramsey" for a weaker property. Galvin and Prikry [3] showed that all Borel subsets of $\mathbb{N}^{[\infty]}$ are Ramsey, and Silver [11] extended this to all analytic sets. Mathias in [9] showed that if the existence of an inaccessible cardinal is consistent with ZFC then it is consistent, with $ZF + DC$, that every subset of $\mathbb{N}^{[\infty]}$ is Ramsey. Mathias introduced the concept of a selective coideal (or a happy family), which has turned out to be of wide interest. Ellentuck [1] characterized the Ramsey sets as those having the Baire property with respect to the exponential topology of $\mathbb{N}^{[\infty]}$.

A game theoretical characterization of Ramseyness was given by Kastanas in [5], using games in the style of Banach-Mazur with respect to Ellentuck's topology.

In this work we will deal with a game-theoretic characterization of the following property:

Definition 1.2. *Let $\mathcal{H} \subset \mathbb{N}^{[\infty]}$ be a coideal. $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ is $\mathcal{H}$-**Ramsey** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X}$ is $\mathcal{H}$-**Ramsey null** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \cap \mathcal{X} = \emptyset$.*

$\mathcal{H}$-Ramseyness is also called **local Ramsey property**.

Mathias considered sets that are $\mathcal{H}$-Ramsey with respect to a selective coideal $\mathcal{H}$, and generalized Silver's result to this context. Matet [8] used games to characterize sets which are Ramsey with respect to a selective coideal $\mathcal{H}$. These games coincide with the games of Kastanas if $\mathcal{H}$ is $\mathbb{N}^{[\infty]}$ and with the games of Louveau [7] if $\mathcal{H}$ is a Ramsey ultrafilter.

Given a coideal $\mathcal{H} \subset \mathbb{N}^{[\infty]}$, let

$$Exp(\mathcal{H}) := \{[a, A] : (a, A) \in \mathbb{N}^{[<\infty]} \times \mathcal{H}\}.$$

In general, this is not a basis for a topology on $\mathbb{N}^{[<\infty]}$, but the following abstract version of the Baire property and related concepts will be useful:

Definition 1.3. *Let $\mathcal{H} \subset \mathbb{N}^{[\infty]}$ be a coideal. $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ has the abstract $Exp(\mathcal{H})$-**Baire property** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $[b, B] \subseteq [a, A]$ with $B \in \mathcal{H}$ such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X}$ is $Exp(\mathcal{H})$-**nowhere dense** if for every $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$ there exists $[b, B] \subseteq [a, A]$ with $B \in \mathcal{H}$ such that $[b, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X}$ is $Exp(\mathcal{H})$-**meager** if it is the union of countably many $Exp(\mathcal{H})$-nowhere dense sets.*

Given a decreasing sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ of infinite subsets of $\mathbb{N}$, a set B is a **diagonalization** of the sequence (or B **diagonalizes** the sequence) if and only if $B/n \subseteq A_n$ for each $n \in B$. A coideal $\mathcal{H}$ is **selective** if and only if every decreasing sequence in $\mathcal{H}$ has a diagonalization in $\mathcal{H}$.

A coideal $\mathcal{H}$ has the Q^+**-property**, if for every $A \in \mathcal{H}$ and every partition $(F_n)_n$ of A into finite sets, there is $S \in \mathcal{H}$ such that $S \subseteq A$ and $|S \cap F_n| \leq 1$ for every $n \in \mathbb{N}$.

Proposition 1.4. *[9] A coideal $\mathcal{H}$ is selective if and only if the poset $(\mathcal{H}, \subseteq^*)$ is σ-closed and $\mathcal{H}$ has the Q^+-property.*

Given a coideal $\mathcal{H}$ and a sequence $\{D_n\}_{n \in \mathbb{N}}$ of dense open sets in $(\mathcal{H}, \subseteq)$, a set B is a **diagonalization** of $\{D_n\}_{n \in \mathbb{N}}$ if and only if $B/n \in D_n$ for every $n \in B$. A coideal $\mathcal{H}$ is **semiselective** if for every sequence $\{D_n\}_{n \in \mathbb{N}}$ of dense open subsets of $\mathcal{H}$, the family of its diagonalizations is dense in $(\mathcal{H}, \subseteq)$.

Proposition 1.5. *[2] A coideal $\mathcal{H}$ is semiselective if and only if the poset $(\mathcal{H}, \subseteq^*)$ is σ-distributive and $\mathcal{H}$ has the Q^+-property.*

Since σ-closedness implies σ-distributivity, then semiselectivity follows from selectivity, but the converse does not hold (see [2] for an example).

In section 2 we list results of Ellentuck, Mathias and Farah that characterize topologically the Ramsey property and the local Ramsey property. In section 3 we define a family of games, and present the main result, which states that a coideal $\mathcal{H}$ is semiselective if and only if the $\mathcal{H}$-Ramsey sets are exactly those for which the associated games are determined. This generalizes results of Kastanas [5] and Matet [8]. The proof is given in section 4. In section 5 we relate semiselectivity of coideals with the Fréchet-Urysohn property, and show that in Solovay's model every semiselective coideal has the Fréchet-Urysohn property.

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2 Topological characterization of Ramseyness.

The following are the main results concerning the characterization of the Ramsey property and the local Ramsey property in topological terms.

Theorem 2.1. *[Ellentuck] Let $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ be given.*

(i) $\mathcal{X}$ is Ramsey if and only if $\mathcal{X}$ has the Baire property, with respect to Ellentuck's topology.

(ii) $\mathcal{X}$ is Ramsey null if and only if $\mathcal{X}$ is meager, with respect to Ellentuck's topology.

Theorem 2.2. *[Mathias] Let $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ and a selective coideal $\mathcal{H}$ be given.*

(i) $\mathcal{X}$ is $\mathcal{H}$-Ramsey if and only if $\mathcal{X}$ has the abstract Exp$(\mathcal{H})$-Baire property.

(ii) $\mathcal{X}$ is $\mathcal{H}$-Ramsey null if and only if $\mathcal{X}$ is Exp$(\mathcal{H})$-meager.

Theorem 2.3. *[Farah, Todorcevic] Let $\mathcal{H}$ be a coideal. The following are equivalent:*

(i) $\mathcal{H}$ is semiselective.

(ii) The $\mathcal{H}$-Ramsey subsets of $\mathbb{N}^{[\infty]}$ are exactly those sets having the abstract Exp$(\mathcal{H})$-Baire property, and the following three families of subsets of $\mathbb{N}^{[\infty]}$ coincide and are σ-ideals:

(a) $\mathcal{H}$-Ramsey null sets,

(b) Exp$(\mathcal{H})$-nowhere dense, and

(c) Exp$(\mathcal{H})$-meager sets.

In the next section we state results by Kastanas [5] and Matet [8] (Theorems 3.1 and 3.2 below) which are the game-theoretic counterparts of theorems 2.1 and 2.2, respectively; and we also present our main result (Theorem 3.3 below), which is the game-theoretic counterpart of Theorem 2.3.

3 Characterizing Ramseyness with games.

The following is a relativized version of a game due to Kastanas [5], employed to obtain a characterization of the family of completely Ramsey sets (i.e. $\mathcal{H}$-Ramsey for $\mathcal{H} = \mathbb{N}^{[\infty]}$). The same game was used by Matet in [8] to obtain the analog result when $\mathcal{H}$ is selective.

Let $\mathcal{H} \subseteq \mathbb{N}^{[\infty]}$ be a fixed coideal. For each $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ we define a two-player game $G_{\mathcal{H}}(a, A, \mathcal{X})$ as follows: player I chooses an element $A_0 \in \mathcal{H} \restriction A$; II answers by playing $n_0 \in A_0$ such that $a \subseteq n_0$, and $B_0 \in \mathcal{H} \cap (A_0/n_0)^{[\infty]}$; then I chooses $A_1 \in \mathcal{H} \cap B_0^{[\infty]}$; II answers by playing $n_1 \in A_1$ and $B_1 \in \mathcal{H} \cap (A_1/n_1)^{[\infty]}$; and so on. Player I wins if and only if $a \cup \{n_j : j \in \mathbb{N}\} \in \mathcal{X}$.

I	A_0		A_1		$\cdots$	A_k		$\cdots$
II		n_0, B_0		n_1, B_1	$\cdots$		n_k, B_k	$\cdots$

A **strategy** for a player is a rule that tells him (or her) what to play based on the previous moves. A strategy is a **winning strategy for player I** if player I wins the game whenever she (or he) follows the strategy, no matter what player II plays. Analogously, it can be defined what is a winning strategy for player II. The precise definitions of strategy for two players games can be found in [6, 10].

Let $s = \{s_0, \ldots, s_k\}$ be a nonempty finite subset of $\mathbb{N}$, written in its increasing order, and $\overrightarrow{B} = \{B_0, \ldots, B_k\}$ be a sequence of elements of $\mathcal{H}$. We say that the pair $(s, \overrightarrow{B})$ is a **legal position for player II** if $(s_0, B_0), \ldots, (s_k, B_k)$ is a sequence of possible consecutive moves of II in the game $G_{\mathcal{H}}(a, A, \mathcal{X})$, respecting the rules. In this case, if σ is a winning strategy for player I in the game, we say that $\sigma(s, \overrightarrow{B})$ is a **realizable move of player I** according to σ. Notice that if $r \in B_k/s_k$ and $C \in \mathcal{H} \upharpoonright B_k/s_k$ then $(s_0, B_0), \ldots, (s_k, B_k), (r, C)$ is also a sequence of possible consecutive moves of II in the game. We will sometimes use the notation $(s, \overrightarrow{B}, r, C)$, and say that $(s, \overrightarrow{B}, r, C)$ is a legal position for player II and $\sigma(s, \overrightarrow{B}, r, C)$ is a realizable move of player I according to σ.

We say that the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is **determined** if one of the players has a winning strategy.

Theorem 3.1. *[Kastanas] $\mathcal{X}$ is Ramsey if and only if for every $A \in \mathbb{N}^{[\infty]}$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathbb{N}^{[\infty]}}(a, A, \mathcal{X})$ is determined.*

Theorem 3.2. *[Matet] Let $\mathcal{H}$ be a selective coideal. $\mathcal{X}$ is $\mathcal{H}$-Ramsey if and only if for every $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is determined.*

Now we state our main result:

Theorem 3.3. *Let $\mathcal{H}$ be a coideal. The following are equivalent:*

1. *$\mathcal{H}$ is semiselective.*
2. *$\forall \mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $\mathcal{X}$ is $\mathcal{H}$-Ramsey if and only if for every $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is determined.*

So Theorem 3.3 is a game-theoretic counterpart to Theorem 2.3 in the previous section, in the sense that it gives us a game-theoretic characterization of semiselectivity. Obviously, it also gives us a characterization of $\mathcal{H}$-Ramseyness, for semiselective $\mathcal{H}$, which generalizes the main results of Kastanas in [5] and Matet in [8] (Theorems 3.1 and 3.2 above).

It is known that every analytic set is $\mathcal{H}$-Ramsey for $\mathcal{H}$ semiselective (see Theorem 2.2 in [2] or Lemma 7.18 in [15]). We extend this result to the projective hierarchy. Please see [6] or [10] for the definitions of *projective set* and of *projective determinacy.*

Corollary 3.4. *Assume projective determinacy for games over the reals. Let $\mathcal{H}$ be a semiselective projective coideal. Then, every projective set is $\mathcal{H}$-Ramsey.*

Proof. Let $\mathcal{X}$ be a projective subset of $\mathbb{N}^{[\infty]}$. Fix $A \in \mathcal{H}$, $a \in \mathbb{N}^{[<\infty]}$. By the projective determinacy over the reals, the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is determined. Then, Theorem 3.3 implies that $\mathcal{X}$ is $\mathcal{H}$-Ramsey. □

4 Proof of the main result

Throughout the rest of this section, fix a semiselective coideal $\mathcal{H}$. Before proving Theorem 3.3, in Propositions 4.1 and 4.7 below we will deal with winning strategies of players in a game $G_{\mathcal{H}}(a, A, \mathcal{X})$.

Proposition 4.1. *For every $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$, I has a winning strategy in $G_{\mathcal{H}}(a, A, \mathcal{X})$ if and only if there exists $E \in \mathcal{H} \upharpoonright A$ such that $[a, E] \subseteq \mathcal{X}$.*

Proof. Suppose σ is a winning strategy for I. We will suppose that $a = \emptyset$ and $A = \mathbb{N}$ without loss of generality.

Let $A_0 = \sigma(\emptyset)$ be the first move of I using σ. We will define a tree T of finite subsets of A_0; and for each $s \in T$ we will also define a family $M_s \subseteq A_0^{[\infty]}$ and a family $N_s \subseteq (A_0^{[\infty]})^{|s|}$, where $|s|$ is the length of s. Put $\{p\} \in T$ for each $p \in A_0$ and let

$$M_{\{p\}} \subseteq \{\sigma(p, B) : \; B \in \mathcal{H} \upharpoonright A_0\}$$

be a maximal $\mathcal{H}$-disjoint family, and set

$$N_{\{p\}} = \{\{B\} : \; \sigma(p, B) \in M_{\{p\}}\}.$$

Suppose we have defined $T \cap A_0^{[n]}$ and we have chosen a maximal $\mathcal{H}$-disjoint family M_s of realizable moves of player I of the form $\sigma(s, \overrightarrow{B})$ for every $s \in T \cap A_0^{[n]}$. Let

$$N_s = \{\overrightarrow{B} : \; \sigma(s, \overrightarrow{B}) \in M_s\}.$$

Given $s \in T \cap A_0^{[n]}$, $\overrightarrow{B} \in N_s$ and $r \in \sigma(s, \overrightarrow{B})/s$, we put $s \cup \{r\} \in T$. Then choose a maximal $\mathcal{H}$-disjoint family

$$M_{s\cup\{r\}} \subseteq \{\sigma(s, \overrightarrow{B}, r, C) : \; \overrightarrow{B} \in N_s, \; C \in \mathcal{H} \upharpoonright \sigma(s, \overrightarrow{B})/r\}.$$

Put

$$N_{s\cup\{r\}} = \{(\overrightarrow{B}, C) : \; \sigma(s, \overrightarrow{B}, r, C) \in M_{s\cup\{r\}}\}.$$

Now, for every $s \in T$, let

$$\mathcal{U}_s = \{E \in \mathcal{H} : (\exists F \in M_s) \; E \subseteq F\} \;\text{ and}$$

$$\mathcal{V}_s = \{E \in \mathcal{H} : (\forall F \in M_{s\setminus\{max(s)\}}) \; max(s) \in F \;\rightarrow F \cap E \notin \mathcal{H}\}.$$

Claim 4.2. *For every $s \in T$, $\mathcal{U}_s \cup \mathcal{V}_s$ is dense open in $(\mathcal{H} \upharpoonright A_0, \subseteq)$.*

Proof. Fix $s \in T$ and $A \in \mathcal{H} \upharpoonright A_0$. If $(\forall F \in M_{s\setminus\{max(s)\}})\ max(s) \in F\ \rightarrow F \cap A \notin \mathcal{H}$ holds, then $A \in \mathcal{V}_s$. Otherwise, fix $F \in M_{s\setminus\{max(s)\}}$ such that $max(s) \in F$ and $F \cap A \in \mathcal{H}$. Let $\overrightarrow{B} \in N_{s\setminus\{max(s)\}}$ be such that $\sigma(s \setminus \{max(s)\}, \overrightarrow{B}) = F$. Notice that since $max(s) \in F$ then

$$(s \setminus \{max(s)\}, \overrightarrow{B}, max(s), F \cap A/max(s))$$

is a legal position for player II. Then, using the maximality of M_s, choose $\hat{F} \in M_s$ such that

$$E := \sigma(s \setminus \{max(s), \overrightarrow{B}, max(s), F \cap A/max(s)) \cap \hat{F}$$

is in $\mathcal{H}$. So $E \in \mathcal{U}_s$ and $E \subseteq A$. This completes the proof of claim 4.2. □

Claim 4.3. *There exists $E \in \mathcal{H} \upharpoonright A_0$ such that for every $s \in T$ with $s \subset E$, $E/s \in \mathcal{U}_s$.*

Proof. For each $n \in \mathbb{N}$, let

$$\mathcal{D}_n = \bigcap_{max(s)=n} \mathcal{U}_s \cup \mathcal{V}_s.$$

$$\mathcal{U}_n = \bigcap_{max(s)=n} \mathcal{U}_s\ ,$$

(if there is no $s \in T$ with $max(s) = n$, then we put $\mathcal{D}_n = \mathcal{U}_n = \mathcal{H} \upharpoonright A_0$). By Claim 4.2, every $\mathcal{D}_n$ is dense open in $(\mathcal{H} \upharpoonright A_0, \subseteq)$. Using semiselectivity, choose a diagonalization $\hat{E} \in \mathcal{H} \upharpoonright A_0$ of the sequence $(\mathcal{D}_n)_n$. Let

$$E_0 := \{n \in \hat{E} : \hat{E}/n \in \mathcal{U}_n\} \text{ and } E_1 := \hat{E} \setminus E_0.$$

Let us prove that $E_1 \notin \mathcal{H}$:

Suppose $E_1 \in \mathcal{H}$. By the definitions, $(\forall n \in E_1)\ \hat{E}/n \notin \mathcal{U}_n$. Let $n_0 = min(E_1)$ and fix $s_0 \subset \hat{E}$ such that $max(s_0) = n_0$ and satisfying, in particular, the following:

$$(\forall F \in M_{s_0\setminus\{n_0\}})\ n_0 \in F\ \rightarrow\ F \cap E_1/n_0 \notin \mathcal{H}.$$

Notice that $|s_0| > 1$, by the construction of the M_s's.

Now, let $m = max(s_0 \setminus \{n_0\})$. Then $m \in E_0$ and therefore $\hat{E}/m \in \mathcal{U}_m \subseteq \mathcal{U}_{s_0\setminus\{n_0\}}$. So there exists $F \in M_{s_0\setminus\{n_0\}}$ such that $\hat{E}/m \subseteq F$. Since $m < n_0$ then $n_0 \in F$. But $F \cap E_1/n_0 = E_1/n_0 \in \mathcal{H}$. A contradiction.

Hence, $E_1 \notin \mathcal{H}$ and therefore $E_0 \in \mathcal{H}$. Then $E := E_0$ is as required. □

Claim 4.4. *Let E be as in Claim 4.3 and $s \cup \{r\} \in T$ with $s \subset E$ and $r \in E/s$. If $E/s \subseteq \sigma(s, \overrightarrow{B})$ for some $\overrightarrow{B} \in N_s$, then there exists $C \in \mathcal{H} \upharpoonright \sigma(s, \overrightarrow{B})/r$ such that $E/r \subseteq \sigma(s, \overrightarrow{B}, r, C)$ and $(\overrightarrow{B}, C) \in N_{s\cup\{r\}}$.*

Proof. Fix s and r as in the hypothesis. Suppose $E/s \subseteq \sigma(s, \overrightarrow{B})$ for some $\overrightarrow{B} \in N_s$. Since $E/r \in \mathcal{U}_{s\cup\{r\}}$, there exists $(\overrightarrow{D}, C) \in N_{s\cup\{r\}}$ such that $E/r \subseteq \sigma(s, \overrightarrow{D}, r, C)$. Notice that $E/r \subseteq \sigma(s, \overrightarrow{B}) \cap \sigma(s, \overrightarrow{D})$. Since M_s is $\mathcal{H}$-disjoint, then $\sigma(s, \overrightarrow{D})$ is neccesarily equal to $\sigma(s, \overrightarrow{B})$ and therefore $\sigma(s, \overrightarrow{B}, r, C) = \sigma(s, \overrightarrow{D}, r, C)$. Hence $(\overrightarrow{B}, C) \in N_{s\cup\{r\}}$ and $E/r \subseteq \sigma(s, \overrightarrow{B}, r, C)$. □

Claim 4.5. *Let E be as in Claim 4.3. Then $[\emptyset, E] \subseteq \mathcal{X}$.*

Proof. Let $\{k_i\}_{i\geq 0} \subseteq E$ be given. Since $E/k_0 \in \mathcal{U}_{\{k_0\}}$, there exists $B_0 \in N_{\{k_0\}}$ such that $E/k_0 \subseteq \sigma(k_0, B_0)$. Thus, by the choice of E and applying Claim 4.4 iteratively, we prove that $\{k_i\}_{i\geq 0}$ is generated in a run of the game in which player I has used his winning strategy σ. Therefore $\{k_i\}_{i\geq 0} \in \mathcal{X}$. □

The converse is trivial. This completes the proof of Proposition 4.1. □

Now we turn to the case when player II has a winning strategy. The proof of the following is similar to the proof of Proposition 4.3 in [8]. First we show a result we will need in the sequel, it should be compared with lemma 4.2 in [8].

Lemma 4.6. *Let $B \in \mathcal{H}$, $f : \mathcal{H} \upharpoonright B \to \mathbb{N}$, and $g : \mathcal{H} \upharpoonright B \to \mathcal{H} \upharpoonright B$ be given such that $f(A) \in A$ and $g(A) \subseteq A/f(A)$. Then there is $E_{f,g} \in \mathcal{H} \upharpoonright B$ with the property that for each $p \in E_{f,g}$ there exists $A \in \mathcal{H} \upharpoonright B$ such that $f(A) = p$ and $E_{f,g}/p \subseteq g(A)$.*

Proof. For each $n \in \{f(A) : A \in \mathcal{H} \upharpoonright B\}$, let

$$U_n = \{E \in \mathcal{H} \upharpoonright B : \ (\exists A \in \mathcal{H} \upharpoonright B) \ (f(A) = n \wedge E \subseteq g(A))\}$$

and

$$V_n = \{E \in \mathcal{H} \upharpoonright B : \ (\forall A \in \mathcal{H} \upharpoonright B) \ (f(A) = n \ \rightarrow | \ g(A) \setminus E \ |= \infty)\}.$$

The set $D_n = U_n \cup V_n$ is dense open in $\mathcal{H} \upharpoonright B$. Choose $E \in \mathcal{H} \upharpoonright B$ such that for each $n \in E$, $E/n \in D_n$. Let

$$E_0 = \{n \in E : \ E/n \in U_n\} \text{ and } E_1 = \{n \in E : \ E/n \in V_n\}.$$

Now, suppose $E_1 \in \mathcal{H}$. Then, for each $n \in E_1$, $E_1/n \in V_n$. Let $n_1 = f(E_1)$. So $n_1 \in E_1$ by the definition of f. But, by the definition of g, $g(E_1) \subseteq E_1/n_1$ and so $E_1/n_1 \notin V_{n_1}$; a contradiction. Therefore, $E_1 \notin \mathcal{H}$. Hence $E_0 \in \mathcal{H}$, since $\mathcal{H}$ is a coideal. The set $E_{f,g} := E_0$ is as required. □

Proposition 4.7. *For every $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$, II has a winning strategy in $G_{\mathcal{H}}(a, A, \mathcal{X})$ if and only if $\forall A' \in \mathcal{H} \upharpoonright A$ there exists $E \in \mathcal{H} \upharpoonright A'$ such that $[a, E] \cap \mathcal{X} = \emptyset$.*

Proof. Let τ be a winning strategy for II in $G_{\mathcal{H}}(a, A, \mathcal{X})$ and let $A' \in \mathcal{H} \upharpoonright A$ be given. We are going to define a winning strategy σ for I, in $G_{\mathcal{H}}(a, A', \mathbb{N}^{[\infty]} \setminus \mathcal{X})$, in such a way that we will get the required result by means of Proposition 4.1. So, in a play of the game $G_{\mathcal{H}}(a, A', \mathbb{N}^{[\infty]} \setminus \mathcal{X})$, with II's successive moves being (n_j, B_j), $j \in \mathbb{N}$, define $A_j \in \mathcal{H}$ and E_{f_j, g_j} as in Lemma 4.6, for f_j and g_j such that

(1) For all $\hat{A} \in \mathcal{H} \upharpoonright A'$,
$$(f_0(\hat{A}), g_0(\hat{A})) = \tau(\hat{A});$$

(2) For all $\hat{A} \in \mathcal{H} \upharpoonright B_j \cap g_j(A_j)$,
$$(f_{j+1}(\hat{A}), g_{j+1}(\hat{A})) = \tau(A_0, \cdots, A_j, \hat{A});$$

(3) $A_0 \subseteq A'$, and $A_{j+1} \subseteq B_j \cap g_j(A_j)$;

(4) $n_j = f_j(A_j)$ and $E_{f_j, g_j}/n_j \subseteq g_j(A_j)$.

Now, let $\sigma(\emptyset)=E_{f_0,g_0}$ and $\sigma((n_0, B_0), \cdots, (n_j, B_j))=E_{f_{j+1},g_{j+1}}$.

Conversely, let A_0 be the first move of I in the game. Then there exists $E \in \mathcal{H} \upharpoonright A_0$ such that $[a, E] \cap \mathcal{X} = \emptyset$. We define a winning strategy for player II by letting her (or him) play $(\min E, E \setminus \{\min E\})$ at the first turn, and arbitrarily from there on. □

We are ready now for the following:

Proof of Theorem 3.3. If $\mathcal{H}$ is semiselective, then part 2 of Theorem 3.3 follows from Propositions 4.1 and 4.7.

Conversely, suppose part 2 holds and let $(\mathcal{D}_n)_n$ be a sequence of dense open sets in $(\mathcal{H}, \subseteq)$. For every $a \in \mathbb{N}^{[<\infty]}$, let

$$\mathcal{X}_a = \{B \in [a, \mathbb{N}] : B/a \text{ diagonalizes some decreasing } (A_n)_n \text{ such that } (\forall n)\ A_n \in \mathcal{D}_n\}$$

and define
$$\mathcal{X} = \bigcup_{a \in \mathbb{N}^{[<\infty]}} \mathcal{X}_a.$$

Fix $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ with $[a, A] \neq \emptyset$, and define a winning strategy σ for player I in $G_{\mathcal{H}}(a, A, \mathcal{X})$, as follows: let $\sigma(\emptyset)$ be any element of $\mathcal{D}_0$ such that $\sigma(\emptyset) \subseteq A$. At stage k, if II's successive moves in the game are (n_j, B_j), $j \leq k$, let $\sigma((n_0, B_0), \ldots, (n_k, B_k))$ be any element of $\mathcal{D}_{k+1}$ such that $\sigma((n_0, B_0), \ldots, (n_k, B_k)) \subseteq B_k$. Notice that $a \cup \{n_0, n_1, n_2, \ldots\} \in \mathcal{X}_a$.

So the game $G_{\mathcal{H}}(a, A, \mathcal{X})$ is determined for every $A \in \mathcal{H}$ and $a \in \mathbb{N}^{[<\infty]}$ with $[a, A] \neq \emptyset$. Then, by our assumptions, $\mathcal{X}$ is $\mathcal{H}$-Ramsey. So given $A \in \mathcal{H}$, there exists $B \in \mathcal{H} \upharpoonright A$ such that $B^{[\infty]} \subseteq \mathcal{X}$ or $B^{[\infty]} \cap \mathcal{X} = \emptyset$. The second alternative does not hold, so $\mathcal{X} \cap \mathcal{H}$ is dense in $(\mathcal{H}, \subseteq)$. Hence, $\mathcal{H}$ is semiselective. □

5 The Fréchet-Urysohn property and semiselectivity.

We say that an coideal $\mathcal{H} \subseteq \mathbb{N}^{[\infty]}$ has the *Fréchet-Urysohn property* if

$$(\forall A \in \mathcal{H})\ (\exists B \in A^{[\infty]})\ (B^{[\infty]} \subseteq \mathcal{H}).$$

The following characterization of the Fréchet-Urysohn property is taken from [12, 14]. It provides a method to construct ideals with that property. Given $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$, define the *orthogonal of* $\mathcal{A}$ as $\mathcal{A}^{\perp} := \{A \in \mathbb{N}^{[\infty]} : (\forall B \in \mathcal{A})\ (|A \cap B| < \infty)\}$. Notice that $\mathcal{A}^{\perp}$ is an ideal.

Proposition 5.1. *A coideal $\mathcal{H} \subseteq \mathbb{N}^{[\infty]}$ has the Fréchet-Urysohn property if and only if $\mathcal{H} = (\mathcal{A}^{\perp})^{+}$ for some $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$.*

Proposition 5.2. *Let $\mathcal{H}$ be a coideal. The following are equivalent:*

(i) $\mathcal{H}$ is $\mathcal{H}$-Ramsey.

(ii) $\mathcal{H}$ has the Fréchet-Urysohn property.

Proof. Suppose $\mathcal{H}$ has the Fréchet-Urysohn property. Let a be a finite set and $A \in \mathcal{H}$. Let $B \subseteq A$ be such that $B^{[\infty]} \subseteq \mathcal{H}$. Then $[a, B] \subseteq \mathcal{H}$ and thus $\mathcal{H}$ is $\mathcal{H}$-Ramsey.

Conversely, suppose $\mathcal{H}$ is $\mathcal{H}$-Ramsey. Let $A \in \mathcal{H}$. Since $\mathcal{H}$ is $\mathcal{H}$-Ramsey, there is $B \subseteq A$ in $\mathcal{H}$ such that $[\emptyset, B] \subseteq \mathcal{H}$ or $[\emptyset, B] \cap \mathcal{H} = \emptyset$. Since the second alternative does not hold, then $B^{[\infty]} \subseteq \mathcal{H}$ and thus $\mathcal{H}$ is Fréchet-Urysohn.

□

The following result is probably known but we include its proof for the sake of completeness.

Proposition 5.3. *Every coideal $\mathcal{H}$ with the Fréchet-Urysohn property is semiselective.*

Proof. Let $\mathcal{H}$ be a Fréchet-Urysohn coideal. Suppose $\{D_n\}_{n \in \mathbb{N}}$ is a sequence of dense open sets in $(\mathcal{H}, \subseteq)$ and $B \in \mathcal{H}$. Since $B \in \mathcal{H}$ and $\mathcal{H}$ has the Fréchet-Urysohn property, then there is $A \subseteq B$ in $\mathcal{H}$ such that $A^{[\infty]} \subseteq \mathcal{H}$. Let $D \subseteq A$ be any diagonalization of $\{D_n\}_{n \in \mathbb{N}}$. Then $D \in \mathcal{H}$. This shows that the collection of diagonalizations is dense in $(\mathcal{H}, \subseteq)$. Thus $\mathcal{H}$ is semiselective. □

The converse of the previous result is not true in general, since a non principal ultrafilter cannot have the Fréchet-Urysohn property. However, as every analytic set is $\mathcal{H}$-Ramsey [2], from 5.2 we immediately get the following.

Proposition 5.4. *Every analytic semiselective coideal is Fréchet-Urysohn.*

The previous result can be extended from suitable axioms.

Theorem 5.5. *Assume projective determinacy over the reals. Then, every projective semiselective coideal is Fréchet-Urysohn.*

Proof. It follows from corollary 3.4 and proposition 5.2. □

Farah [2] shows that if there is a supercompact cardinal, then every semiselective coideal in $L(\mathbb{R})$ has the Fréchet-Urysohn property.

As we show below, it is also an easy consequence of results of [2] and [9] that in Solovay's model every semiselective coideal has the Fréchet-Urysohn property.

Recall that the Mathias forcing notion $\mathbb{M}$ is the collection of all the sets of the form

$$[a, A] := \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B \subseteq A\},$$

ordered by $[a, A] \leq [b, B]$ if and only if $[a, A] \subseteq [b, B]$.

If $\mathcal{H}$ is a coideal, then $\mathbb{M}_{\mathcal{H}}$, the Mathias partial order with respect to $\mathcal{H}$ is the collection of all the $[a, A]$ as above but with $A \in \mathcal{H}$, ordered in the same way.

A coideal $\mathcal{H}$ has the *Mathias property* if it satisfies that if x is $\mathbb{M}_{\mathcal{H}}$-generic over a model M, then every $y \in x^{[\infty]}$ is $\mathbb{M}_{\mathcal{H}}$-generic over M. And $\mathcal{H}$ has the *Prikry property* if for every $[a, A] \in \mathbb{M}_{\mathcal{H}}$ and every formula φ of the forcing language of $\mathbb{M}_{\mathcal{H}}$, there is $B \in \mathcal{H} \restriction A$ such that $[a, B]$ decides φ.

Theorem 5.6. *([2], Theorem 4.1) For a coideal $\mathcal{H}$ the following are equivalent.*

1. *$\mathcal{H}$ is semiselective.*
2. *$\mathbb{M}_{\mathcal{H}}$ has the Prikry property.*
3. *$\mathbb{M}_{\mathcal{H}}$ has the Mathias property.*

Suppose M is a model of ZFC and there is a inaccessible cardinal λ in M. The Levy partial order $Col(\omega, < \lambda)$ produces a generic extension $M[G]$ of M where λ becomes $\aleph_1$. Solovay's model is obtained by taking the submodel of $M[G]$ formed by all the sets hereditarily definable in $M[G]$ from a sequence of ordinals (see [9], or [4]).

In [9], Mathias shows that if $V = L$, λ is a Mahlo cardinal, and $V[G]$ is a generic extension obtained by forcing with $Col(\omega, < \lambda)$, then every set of real numbers defined in the generic extension from a sequence of ordinals is $\mathcal{H} - Ramsey$ for $\mathcal{H}$ a selective coideal. This result can be extended to semiselective coideals under suitable large cardinal hypothesis.

Theorem 5.7. *Suppose λ is a weakly compact cardinal. Let $V[G]$ be a generic extension by $Col(\omega, < \lambda)$. Then, if $\mathcal{H}$ is a semiselective coideal in $V[G]$, every set of real numbers in $L(\mathbb{R})$ of $V[G]$ is $\mathcal{H}$-Ramsey.*

Proof. Let $\mathcal{H}$ be a semiselective coideal in $V[G]$. Let $\mathcal{A}$ be a set of reals in $L(\mathbb{R})^{V[G]}$; in particular, $\mathcal{A}$ is defined in $V[G]$ by a formula φ from a sequence of ordinals. Let $[a, A]$ be a condition of the Mathias forcing $\mathbb{M}_{\mathcal{H}}$ with respect to the semiselective coideal $\mathcal{H}$. Let finally $\dot{\mathcal{H}}$ be a name for $\mathcal{H}$. Notice that $\dot{\mathcal{H}} \subseteq V_\lambda$.

Since $V[G]$ satisfies that $\mathcal{H}$ is semiselective, the following statement holds in $V[G]$: For every sequence $D = (D_n : n \in \omega)$ of open dense subsets of $\mathcal{H}$ and for every $x \in \mathcal{H}$ there is $y \in \mathcal{H}$, $y \subseteq x$, such that y diagonalizes the sequence D.

Therefore, there is $p \in G$ such that, in V, the following statement holds:

$$\forall \dot{D} \forall \tau (p \Vdash_{Col(\omega,<\lambda)} (\dot{D} \text{ is a name for a sequence of dense open subsets of } \dot{\mathcal{H}} \text{ and } \tau \in \dot{\mathcal{H}}) \longrightarrow (\exists x (x \in \dot{\mathcal{H}}, x \subseteq \tau, x \text{ diagonalizes } \dot{D}))).$$

Notice that every real in $V[G]$ has a name in V_λ, and names for subsets of $\mathcal{H}$ or countable sequences of subsets of $\mathcal{H}$ are contained in V_λ. Also, the forcing $Col(\omega, < \lambda)$ is a subset of V_λ. Therefore the same statement is valid in the structure $(V_\lambda, \in, \dot{\mathcal{H}}, Col(\omega, < \lambda))$. This statement is Π^1_1 over this structure, and since λ is Π^1_1-indescribable, there is $\kappa < \lambda$ such that in $(V_\kappa, \in, \dot{\mathcal{H}} \cap V_\kappa, Col(\omega, < \lambda) \cap V_\kappa)$ it holds

$$\forall \dot{D} \forall \tau (p \Vdash_{Col(\omega,<\kappa)} (\dot{D} \text{ is a name for a sequence of dense open subsets of } \dot{\mathcal{H}} \cap V_\kappa \text{ and } \tau \in \dot{\mathcal{H}} \cap V_\kappa) \longrightarrow (\exists x (x \in \dot{\mathcal{H}} \cap V_\kappa, x \subseteq \tau, x \text{ diagonalizes } \dot{D}))).$$

We can get κ inaccessible, since there is a Π^1_1 formula expressing that λ is inaccessible. Also, κ is such that p and the names for the real parameters in the definition of $\mathcal{A}$ and for A belong to V_κ.

If we let $G_\kappa = G \cap Col(\omega, < \kappa)$, then $G_\kappa \subseteq Col(\omega, < \kappa)$, and is generic over V. Also, $p \in G_\kappa$. $\dot{\mathcal{H}} \cap V_\kappa$ is a $Col(\omega, < \kappa)$-name in V which is interpreted by G_κ as $\mathcal{H} \cap V[G_\kappa]$, thus $\mathcal{H} \cap V[G_\kappa] \in V[G_\kappa]$. And since every subset (or sequence of subsets) of $\mathcal{H} \cap V[G_\kappa]$ which belongs to $V[G_\kappa]$ has a name contained in V_κ, we have that, in $V[G_\kappa]$, $\mathcal{H} \cap V_\kappa$ is semiselective, and in consequence it has both the Prikry and the Mathias properties.

Now the proof can be finished as in [9]. Let $\dot{r}$ be the canonical name of a $\mathbb{M}_{\mathcal{H} \cap V[G_\kappa]}$ generic real, and consider the formula $\varphi(\dot{r})$ in the forcing language of $V[G_\kappa]$. By the Prikry property of $\mathcal{H} \cap V[G_\kappa]$, there is $A' \subseteq A$, $A' \in \mathcal{H} \cap V[G_\kappa]$, such that $[a, A']$ decides $\varphi(\dot{r})$. Since 2^{2^ω} computed in $V[G_\kappa]$ is countable in $V[G]$, there is (in $V[G]$) a $\mathbb{M}_{\mathcal{H} \cap V[G_\kappa]}$-generic real x over $V[G_\kappa]$ such that $x \in [a, A']$. To see that there is such a generic real in $\mathcal{H}$ we argue as in 5.5 of [9] using the semiselectivity of $\mathcal{H}$ and the fact that $\mathcal{H} \cap V[G_\kappa]$ is countable in $V[G]$ to obtain an element of $\mathcal{H}$ which is generic. By the Mathias property of $\mathcal{H} \cap V[G_\kappa]$, every $y \in [a, x \setminus a]$ is also $\mathbb{M}_{\mathcal{H} \cap V[G_\kappa]}$-generic over $V[G_\kappa]$, and also $y \in [a, A']$. Thus $\varphi(x)$ if and only if $[a, A'] \Vdash \varphi(\dot{r})$, if and only if $\varphi(y)$. Therefore, $[a, x \setminus a]$ is contained in $\mathcal{A}$ or is disjoint from $\mathcal{A}$.

□

As in [9], we obtain the following.

Corollary 5.8. *If ZFC is consistent with the existence of a weakly compact cardinal, then so is $ZF + DC$ and "every set of reals is $\mathcal{H}$-Ramsey for every semiselective coideal $\mathcal{H}$".*

Corollary 5.9. *Suppose there is a weakly compact cardinal. Then, there is a model of $ZF + DC$ in which every semiselective coideal $\mathcal{H}$ has the Fréchet-Urysohn property.*

Proof. By theorem 5.7, in $L(\mathbb{R})$ of the Levy collapse of a weakly compact cardinal, every set of reals is $\mathcal{H}$-Ramsey for every semiselective coideal . Thus, by proposition 5.2, every semiselective coideal $\mathcal{H}$ in this model has the Fréchet-Urysohn property. □

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