ON INTERSECTION LATTICES OF HYPERPLANE ARRANGEMENTS GENERATED BY GENERIC POINTS

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Abstract. We consider hyperplane arrangements generated by generic points and study their intersection lattices. These arrangements are known to be equivalent to discriminantal arrangements. We show a fundamental structure of the intersection lattices by decomposing the poset ideals as direct products of smaller lattices corresponding to smaller dimensions. Based on this decomposition we compute the Möbius functions of the lattices and the characteristic polynomials of the arrangements up to dimension six.

1. Introduction

Consider a set of n $(> d)$ generic points $P = \{ p_1, \dots, p_n \}$ in a d-dimensional vector space $V = K^d$ over a field K of characteristic zero. For $X \subset P$ let H_X denote the affine hull of X. Let

$$\mathcal{A} = \{ H_X \mid X \subset P, \#X = d \}$$

be the set of all hyperplanes defined by H_X for some $X \subset P$, $\#X = d$. Here we assume that points $p_1, \dots, p_n$ are generic in the sense of Athanasiadis [1999]. Then combinatorial properties of the arrangement $\mathcal{A}$ does not depend on the points. Since in this paper we are interested only in the combinatorial properties of $\mathcal{A}$, we denote the arrangement by $\mathcal{A}_{n,d}$. We decompose the poset ideals of the intersection lattice of $\mathcal{A}_{n,d}$ into direct products of smaller lattices corresponding to smaller dimensions. Based on this decomposition we give an explicit description of the Möbius functions and the characteristic polynomials of the intersection lattices for $d \le 6$ and for all $n > d$.

By Theorem 2.2 of Falk [1994], $\mathcal{A}_{n,d}$ is equivalent to the discriminantal arrangement $\mathcal{B}(n, n-d-1)$ of Manin and Schechtman [1989]. Relevant facts on the discriminantal arrangement are given in Section 5.6 of Orlik and Terao [1992], Bayer and Brandt [1997] and Athanasiadis [1999]. We prefer to work with $\mathcal{A}_{n,d}$ because we utilize the recursive structure of $\mathcal{A}_{n,d}$ with respect to d.

The organization of this paper is the following. In Section 2 we set up our definition and notation. In particular following Athanasiadis [1999] we interpret the intersection lattice of our arrangement in set theoretical terminology. We also give illustrations for $d \le 3$. In Section 3, we show the fundamental structure of the intersection lattice of $\mathcal{A}_{n,d}$, which is the main result of this paper. Based on the main result, in Section 4 we compute the Möbius function of the intersection lattice, the number of elements of a particular type of the intersection lattice, and the characteristic polynomials of the arrangements up to $d = 6$ and for all $n > d$.

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2. Definition and Notation

We denote the intersection lattice of $\mathcal{A}_{n,d}$ by

$$L(\mathcal{A}_{n,d}) = \{ H_1 \cap \cdots \cap H_k \mid H_1, \ldots, H_k \in \mathcal{A}_{n,d} \},$$

where the sets are ordered by reverse inclusion. Contrary to the usual convention, here we consider that $\emptyset = \bigcap_{X:\, \#X=d} H_X$ belongs to $L(\mathcal{A}_{n,d})$, so that $L(\mathcal{A}_{n,d})$ is not only a poset but also a lattice (cf. Proposition 2.3 of Stanley [2007]). In usual convention, this corresponds to the coning $c\mathcal{A}_{n,d}$ of $\mathcal{A}_{n,d}$, except that we do not add a coordinate hyperplane. The reason for this unconventional definition is that $\emptyset \in L(\mathcal{A}_{n,d})$ plays an essential role for recursive description of $L(\mathcal{A}_{n,d})$.

We now follow Athanasiadis [1999] to give an interpretation of $L(\mathcal{A}_{n,d})$ in set theoretical terminology.

Definition 2.1. For a finite set X, we define

$$\operatorname{codim}_d(X) = d + 1 - \#X.$$

For distinct finite sets $T_1, \ldots, T_l$, we define

$$\rho_d(\{ T_1, \ldots, T_l \}) = \operatorname{codim}_d T_1 + \cdots + \operatorname{codim}_d T_l,$$
$$D_d(\{ T_1, \ldots, T_l \}) = \operatorname{codim}_d(T_1 \cap \cdots \cap T_l) - \rho_d(\{ T_1, \ldots, T_l \}).$$

We also define $\rho_d(\emptyset) = \rho_d(\{\ \}) = 0$.

Remark 2.2. By definition, it follows that

$$D_d(\{ T_1, \ldots, T_l \}) = -(l-1)(d+1) + \#T_1 + \cdots + \#T_l - \#(T_1 \cap \cdots \cap T_l). \tag{1}$$

In particular for $T_1 \neq T_2$,

$$D_d(\{ T_1, T_2 \}) = \#(T_1 \cup T_2) - (d+1). \tag{2}$$

Remark 2.3. For $Y \subset X$, $\operatorname{codim}_{d-\#Y}(X \setminus Y) = \operatorname{codim}_d(X)$. This implies the following fact. Let $U \subset T_1 \cap \cdots \cap T_l$. Then

$$\rho_{d-\#U}(\{ T_1 \setminus U, \ldots, T_l \setminus U \}) = \rho_d(\{ T_1, \ldots, T_l \}),$$
$$D_{d-\#U}(\{ T_1 \setminus U, \ldots, T_l \setminus U \}) = D_d(\{ T_1, \ldots, T_l \}).$$

Definition 2.4. For $d > 0$ and $n > d$, we define $L(n,d)$ to be the set of $T \subset 2^{\{ 1,\ldots,n \}}$ satisfying the following two conditions:

1) $D_d(T') > 0$ for all $T' \subset T$ with $\#T' > 1$.
2) $0 \leq \#T_i \leq d$ for all $T_i \in T$.

Moreover we define the partial ordering $<$ on $L(n,d)$ by

$$T < T' \iff \begin{cases} \rho_d(T) < \rho_d(T') & \text{and} \\ \forall T_i \in T,\ \exists T'_j \in T' \text{ such that } T'_j \subset T_i. \end{cases} \tag{3}$$

Let $P = \{ p_1, \ldots, p_n \}$ be a collection of generic points in V in the sense of Section 1. For $X \subset \{ 1, \ldots, n \}$, $0 \leq \#X \leq d$, define H_X to be the affine hull $H_{\{ p_i \mid i \in X \}}$. Since $n > d$, there exists a subset $X' \subset \{ 1, \ldots, n \}$ such that $X' \cap X = \emptyset$ and $\#(X \cup X') = d + 1$. Hence

$$\begin{aligned} H_X &= H_{\{ p_{i_1}, \ldots, p_{i_l} \}} \\ &= \bigcap_{k \in X'} H_{\{ p_i \mid i \in X \cup X' \} \setminus \{ p_k \}} \in L(\mathcal{A}_{n,d}). \end{aligned}$$

This mapping induces a map from $L(n,d)$ to $L(\mathcal{A}_{n,d})$, or equivalently, $T \in L(n,d)$ corresponds to $H(T) = \bigcap_{T_i \in T} H_{T_i} \in L(\mathcal{A}_{n,d})$. By this correspondence, $L(n,d)$ is isomorphic to $L(\mathcal{A}_{n,d})$ as lattices (Athanasiadis [1999], Falk [1994]).

Remark 2.5. $L(n,d)$ is a graded poset with the rank function ρ_d. $\emptyset = \{\ \}$ is the minimum element of $L(n,d)$ with $\rho_d(\emptyset) = 0$ and $\{\emptyset\}$ ($\emptyset \subset \{1,\dots,n\}$) is the maximum element of $L(n,d)$ with $\rho_d(\{\emptyset\}) = d+1$. In the one-to-one correspondence between $L(n,d)$ and $L(\mathcal{A}_{n,d})$, $H(\emptyset) = V = K^d$ and $H(\{\emptyset\}) = \emptyset$ ($\subset K^d$). In the case $d = 0$, the condition 2) in Definition 2.4 implies $\#T_i = 0$ for $T_i \in T \in L(n,0)$. Hence $L(n,0)$ is the poset of two elements

$$L(n,0) = \{\emptyset, \{\emptyset\}\}$$

independent of n.

Let d be a nonnegative integer. We call a weakly-decreasing sequence $\delta = (\delta_1, \delta_2, \dots)$ of nonnegative integers such that $\sum_i \delta_i = d$ a *partition* of d. We write $\delta \vdash d$ to say that δ is a partition of d. We also regard a partition as a multiset of positive integers. For example, $\{\delta \vdash 3\} = \{(3),(2,1),(1,1,1)\}$, and $\{\delta \vdash 0\}$ is the set consisting of the unique partition of zero, which is denoted by (0).

Definition 2.6. Let $T = \{T_1,\dots,T_l\} \in L(n,d)$. Without loss of generality assume $\#T_1 \le \dots \le \#T_l$. We call

$$\gamma_d(T) = (\mathrm{codim}_d(T_1), \dots, \mathrm{codim}_d(T_l)) \vdash \rho_d(T)$$

the *type* of T.

Example 2.7. For any d, $\gamma_d(\emptyset) = (0)$ and $\gamma_d(\{\emptyset\}) = (d+1)$.

Definition 2.8. For $T \in L(n,d)$, we define $\mathcal{I}_{n,d}(T)$ to be the poset ideal generated by T, i.e., $\mathcal{I}_{n,d}(T) = \{S \in L(n,d) \mid S \le T\}$.

Finally we define the Möbius function $\mu_{n,d}$ of the poset $L(n,d)$, which will be studied in Section 4. Define $\mu_{n,d}$ by

$$\mu_{n,d}(T,T) = 1, \qquad \sum_{S:\ T \le S \le T'} \mu_{n,d}(T,S) = 0,\ T < T'.$$

We write $\mu_{n,d}(T) = \mu_{n,d}(\emptyset, T)$. The characteristic polynomial $\chi_{n,d}(t)$ of the poset $L(n,d)$ (cf. Section 3.10 of Stanley [1997]) is defined by

$$\chi_{n,d}(t) = \sum_{T \in L(n,d)} \mu_{n,d}(T) t^{d+1-\rho_d(T)}. \tag{4}$$

Note that the usual characteristic polynomial $\chi(\mathcal{A}_{n,d}, t)$ of the non-central arrangement $\mathcal{A}_{n,d}$ is given as

$$\chi(\mathcal{A}_{n,d}, t) = \sum_{T \in L(n,d),\, T \ne \{\emptyset\}} \mu_{n,d}(T) t^{d-\rho_d(T)} = \frac{\chi_{n,d}(t) - \mu_{n,d}(\{\emptyset\})}{t}.$$

Conversely from $\chi(\mathcal{A}_{n,d}, t)$ we can evaluate $\mu_{n,d}(\{\emptyset\}) = -\chi(\mathcal{A}_{n,d}, 1)$ since $\chi_{n,d}(1) = 0$. Equivalently

$$\mu_{n,d}(\{\emptyset\}) = -\sum_{T \in L(n,d),\, T \ne \{\emptyset\}} \mu_{n,d}(T). \tag{5}$$

2.1. **Illustration of the posets up to dimension three.** We illustrate the above definitions with $d = 0,\dots,3$. For $d = 0$ we already saw $L(n,0) = \{\emptyset, \{\emptyset\}\}$. In particular $\mu_{n,0}(\{\emptyset\}) = -1$.

Let $d = 1$. In $L(n,1)$, in addition to the minimum $\emptyset$ and the maximum $\{\emptyset\}$, there are n rank one elements $\{\{i\}\}$, $i = 1,\dots,n$, with $\mu_{n,1}(\{\{i\}\}) = -1$. Hence $\chi(\mathcal{A}_{n,1}, t) = t - n$. The value $\mu_{n,1}(\{\emptyset\}) = n - 1$ is relevant for $d > 1$.

The case $d = 2$ is already discussed in Section 7 of Manin and Schechtman [1989] and Section 5.6 of Orlik and Terao [1992]. However we present it here from our viewpoint. As shown in Figure 1, each line (rank one element) is labeled

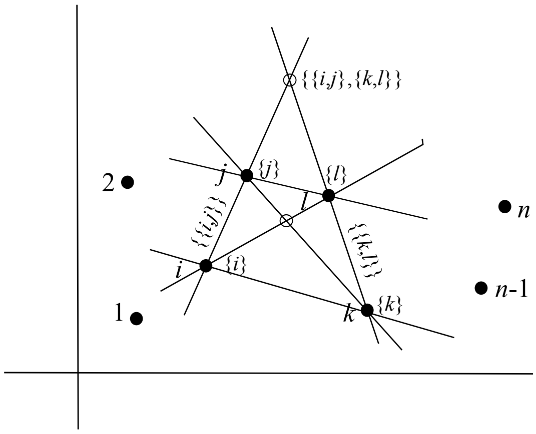

FIGURE 1. Arrangement for dimension two

by a pair of points, such as $T = \{\{i,j\}\}$, which is a line connecting points p_i and p_j. There are two types of points (rank two elements). The first type is an element of type $(2) \vdash 2$. Each element $\{\{i\}\}$ of type $(2) \vdash 2$ corresponds to an original point in P. The second type is an element of type $(1,1) \vdash 2$. Each element $T = \{\{i,j\},\{k,l\}\}$ of type $(1,1)$ corresponds the intersection of two lines, depicted by a white circle in Figure 1. The Möbius function is evaluated as $\mu_{n,2}(\{\{i\}\}) = n-2$ and $\mu_{n,2}(\{\{i,j\},\{k,l\}\}) = 1$.

Remark 2.9. In this paper we are assuming that $n > d$ so that $\mathcal{A}_{n,d}$ is a non-central arrangement. We usually think of n as "sufficiently large" compared to d. Relevant quantities are polynomials in n and these polynomials are determined by sufficiently large n. However our polynomials hold for all $n > d$ with appropriate qualifications. For example, the second type $\{\{i,j\},\{k,l\}\}$ of $L(n,2)$ exists if and only if $n \geq 4$. As long as $n \geq 4$, $\mu_{n,2}(\{\{i,j\},\{k,l\}\}) = 1$. In general, when we write $T \in L(n,d)$, this T has to exist in $L(n,d)$. Actually we are interested in the existence of some T' with the same type as T, i.e. $\gamma_d(T') = \gamma_d(T)$. The existence implies that n has to be larger than or equal to some specific value, say $n_{\gamma_d(T)}$, depending on the type of T. As shown in Section 4.2, $n_{\gamma_d(T)}$ is the minimum n such that the number of elements of $L(n,d)$ of the type $\gamma_d(T)$ is positive.

We now count the number of elements of $L(n,2)$. This is also needed to evaluate $\mu_{n,2}(\{\emptyset\})$. There are $\binom{n}{2}$ lines. There are n points of the first type and

$$\frac{1}{2}\binom{n}{2}\binom{n-2}{2} = 3\binom{n}{4}$$

points of the second type. As discussed in Remark 2.9, this $3\binom{n}{4}$ is positive if and only if $n \geq 4$.

Therefore for $n \geq 3$

$$\chi(\mathcal{A}_{n,2},t) = t^2 - \binom{n}{2}t + 3\binom{n}{4} + n(n-2)$$
$$= t^2 - \binom{n}{2}t + 3\binom{n}{4} + 2\binom{n}{2} - n,$$

(6) $$\mu_{n,2}(\{\emptyset\}) = -3\binom{n}{4} - \binom{n}{2} + n - 1.$$

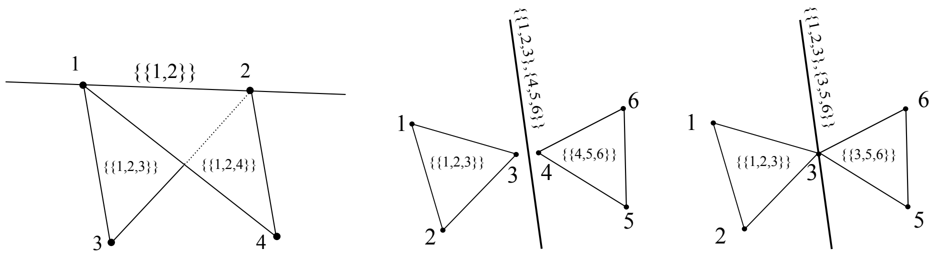

FIGURE 2. Rank two elements for dimension three

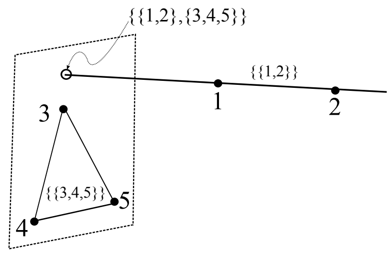

FIGURE 3. Rank three element for dimension three of type $(2,1) \vdash 3$

These quantities are polynomials in n and we prefer to write these polynomials as integer combinations of binomial coefficients $\binom{n}{k}$. Note that, in view of Remark 2.9, $\binom{n}{k} = 0$ for integer $k > n$.

We now discuss the case of $d = 3$.

We first look at rank two elements (lines) of $\mathcal{A}_{n,3}$. There are two types of elements. The first type is an element of type $(2) \vdash 2$. Each element of type (2), such as $T = \{\{1,2\}\}$, corresponds to the line connecting two points as in the leftmost picture of Figure 2. $\{\{1,2\}\}$ is understood as the intersection of all hyperplanes $\{\{1,2,i\}\}$, $i = 3,\ldots,n$. The second type is an element of type $(1,1) \vdash 2$. Each elements of type $(1,1)$ corresponds to an intersection of two hyperplanes, such as $H(\{\{1,2,3\}\}) \cap H(\{\{4,5,6\}\})$. As shown in the rightmost picture in Figure 2, two points (p_3 and p_4 in the picture) may overlap in this case without violating 1) of Definition 2.4. This type of element exists for $n \geq 5$ (cf. Remark 2.9).

Finally we look at rank three elements (points) of $\mathcal{A}_{n,3}$. We will not repeat remarks on existence of these elements of $L(n,3)$. There are three types of rank three elements, corresponding to three partitions of 3. Each element $\{\{i\}\}$ of the first type $(3) \vdash 3$, corresponds to an original point in P. Each element the second type $(2,1) \vdash 3$ corresponds to an intersection of a line of type $(2) \vdash 2$ and a hyperplane, e.g. $H(\{\{1,2\}\}) \cap H(\{\{3,4,5\}\})$ as shown in Figure 3. The third type is $(1,1,1) \vdash 3$, corresponding to an intersection of three hyperplanes as depicted by a white circle in Figure 4. Without violating 1) of Definition 2.4, there are four patterns of overlaps of points.

As will be proved in Section 4, the Möbius function depends only on the above types (i.e. the overlaps of points do not affect the Möbius function) and it is given as follows.

$$
\begin{aligned}
\mu_{n,3}(\{\{1,2\}\}) &= -\mu_{n,3}(\{\{1,2\},\{3,4,5\}\}) = \mu_{n-2,1}(\{\emptyset\}) \\
&= n-3, \\
\mu_{n,3}(\{\{1\}\}) &= \mu_{n-1,2}(\{\emptyset\})
\end{aligned}
\tag{7}
$$

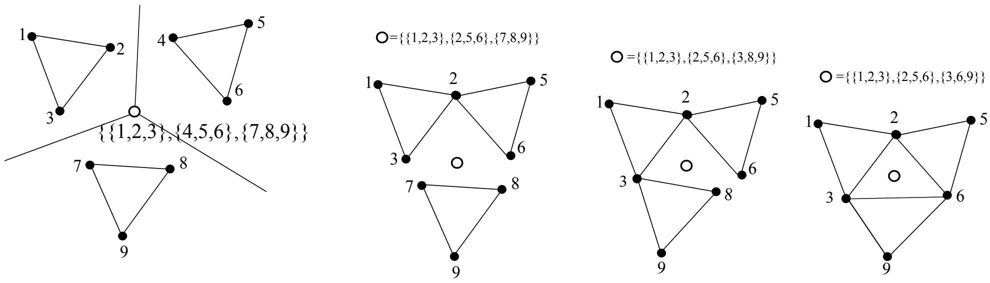

FIGURE 4. Rank three elements for dimension three of type $(1,1,1) \vdash 3$

TABLE 1. Number of elements for $d = 3$

(1)	(2)	(1,1)	(3)	(2,1)	(1,1,1)
$\binom{n}{3}$	$\binom{n}{2}$	$10\binom{n}{6} + 15\binom{n}{5}$	n	$10\binom{n}{5}$	$280\binom{n}{9} + 840\binom{n}{8} + 630\binom{n}{7} + 120\binom{n}{6}$

$$= -3\binom{n-1}{4} - \binom{n-1}{2} + n - 2,$$

and $\mu_{n,3}(T) = (-1)^{\rho_3(T)}$ for all other T, $T \neq \{\emptyset\}$.

We need the numbers of elements of $L(n, 3)$ to evaluate $\mu_{n,3}(\{\emptyset\})$. These are tabulated in Table 1. An element of a particular type exists if and only if the number of elements is positive in Table 1. For example, T of type $(1,1) \vdash 2$ exists if and only if $10\binom{n}{6} + 15\binom{n}{5} > 0$, i.e. $n \geq 5$. From Table 1 and (7) we obtain (for $n \geq 4$)

$$\begin{aligned}\chi(\mathcal{A}_{n,3}, t) = t^3 - \binom{n}{3}t^2 + \big[-\binom{n}{2} + 3\binom{n}{3} + 15\binom{n}{5} + 10\binom{n}{6}\big]t \\ - \big[n - 2\binom{n}{2} + 3\binom{n}{3} + 35\binom{n}{5} + 180\binom{n}{6} + 630\binom{n}{7} \\ + 840\binom{n}{8} + 280\binom{n}{9}\big], \\ \mu_{n,3}(\{\emptyset\}) = -1 + n - \binom{n}{2} + \binom{n}{3} + 20\binom{n}{5} + 170\binom{n}{6} + 630\binom{n}{7} \\ + 840\binom{n}{8} + 280\binom{n}{9}.\end{aligned}$$

3. MAIN RESULT

In this section we show the following main theorem.

Theorem 3.1. *Let $T \in L(n,d)$, $T \neq \emptyset$. Then the ideal $\mathcal{I}_{n,d}(T)$ is isomorphic to the direct product $\prod_{T_i \in T} \mathcal{I}_{n,d}(\{T_i\})$ as posets. They are also isomorphic to $\prod_{T_i \in T} L(n-\#T_i, d-\#T_i)$.*

The second part of this theorem is a consequence of the following lemma.

Lemma 3.2. *For $\{T_1\} \in L(n,d)$, $\mathcal{I}_{n,d}(\{T_1\})$ is isomorphic to $L(n-\#T_1, d-\#T_1)$ as posets.*

Proof. Suppose that $S = \{S_1, \ldots, S_l\} \in \mathcal{I}_{n,d}(\{T_1\})$. Then $\{S_1, \ldots, S_l\} \leq \{T_1\}$, so $S_i \supset T_1$, $\forall i$, by (3). Hence $\{S_1 \setminus T_1, \ldots, S_l \setminus T_1\} \in L(n - \#T_1, d - \#T_1)$ by

Remark 2.3. Therefore we have a map

$$\mathcal{I}_{n,d}(\{ T_1 \}) \ni \{ S_1, \dots, S_l \} \mapsto \{ S_1 \setminus T_1, \dots, S_l \setminus T_1 \} \in L(n - \#T_1, d - \#T_1),$$

which is seen to be one-to-one and onto, and preserves the partial order. □

To prove the first part of Theorem 3.1, we show one proposition and three lemmas.

Proposition 3.3. *Let $T < T' \in L(n,d)$. For each $T_i \in T$, there uniquely exists $T'_j \in T'$ such that $T'_j \subset T_i$.*

Proof. It suffices to show the uniqueness. Let $T_i \in T$ and $T'_j, T'_k \in T'$, $j \neq k$, satisfy $T'_j, T'_k \subset T_i$. This means $T'_j \cup T'_k \subset T_i$. Since $D_d(\{ T'_j, T'_k \}) > 0$, by (2), $\#T_i \geq \#(T'_j \cup T'_k) > d+1$. This conflicts with $\#T_i \leq d$. □

Lemma 3.4. *Let $T = \{ T_1, \dots, T_l \}$ and $S = \{ S_1, \dots, S_l \}$. If $T_i \subset S_i$ for all i, then $D_d(S) \geq D_d(T)$.*

Proof. Let $S'_i = S_i \setminus T_i$. Then by (1)

$$\begin{aligned} D_d(S) - D_d(T) &= \sum_{i=1}^{l} (\#S_i - \#T_i) + \# \bigcap_{i=1}^{l} T_i - \# \bigcap_{i=1}^{l} S_i \\ &= \sum_{i=1}^{l} \#S'_i + \# \bigcap_{i=1}^{l} T_i - \# \bigcap_{i=1}^{l} S_i. \end{aligned}$$

Since $\bigcap_i S_i = \bigcap_i (S'_i \cup T_i) = (\bigcap_i T_i) \cup \big(S'_1 \cap \bigcap_j S_j\big) \cup \big(S'_2 \cap \bigcap_j S_j\big) \cup \cdots \cup \big(S'_l \cap \bigcap_j S_j\big)$,

$$\# \bigcap_{i=1}^{l} S_i \leq \# \bigcap_{i=1}^{l} T_i + \#(S'_1 \cap \bigcap_{j=1}^{l} S_j) + \cdots + \#(S'_l \cap \bigcap_{j=1}^{l} S_j).$$

This implies

$$\begin{aligned} D_d(S) - D_d(T) &\geq \sum_{i=1}^{l} \#S'_i + \# \bigcap_{i=1}^{l} T_i - \# \bigcap_{i=1}^{l} T_i - \sum_{i=1}^{l} \#(S'_i \cap \bigcap_{j=1}^{l} S_j) \\ &= \sum_{i=1}^{l} \#S'_i - \sum_{i=1}^{l} \#(S'_i \cap \bigcap_{j=1}^{l} S_j) \\ &= \sum_{i=1}^{l} \#(S'_i \setminus \bigcap_{j=1}^{l} S_j). \end{aligned}$$

Hence $D_d(S) - D_d(T) \geq 0$. □

Lemma 3.5. *Let $T = \{ T_1, \dots, T_l \}$, $S^{(1)} = \left\{ S^{(1)}_1, \dots, S^{(1)}_{m_1} \right\}$, $S^{(2)} = \left\{ S^{(2)}_1, \dots, S^{(2)}_{m_2} \right\}$, $\dots$, $S^{(l)} = \left\{ S^{(l)}_1, \dots, S^{(l)}_{m_l} \right\}$, and $S = S^{(1)} \cup \cdots \cup S^{(l)}$. Assume $T_i \subset S^{(i)}_j$ for all i, j. If $D_d(T) > 0$ and $D_d(S^{(i)}) > 0$ for all i, then $D_d(S) > 0$.*

Proof. Let $m = \sum_{i=1}^{l} m_i$. Then

$$-(m-1)(d+1) = -(l-1)(d+1) - \sum_{i=1}^{l} (m_i - 1)(d+1).$$

Hence

$$
\begin{aligned}
&D_d(S)\\
&= -(m-1)(d+1) + \sum_{i=1}^{l}\sum_{j=1}^{m_i} \#S_j^{(i)} - \#\bigcap_{i=1}^{l}\bigcap_{j=1}^{m_i} S_j^{(i)},\\
&= -(l-1)(d+1) - \sum_{i=1}^{l}(m_i-1)(d+1) + \sum_{i=1}^{l}\sum_{j=1}^{m_i} \#S_j^{(i)} - \#\bigcap_{i=1}^{l}\bigcap_{j=1}^{m_i} S_j^{(i)}\\
&= \sum_{i=1}^{l}(-(m_i-1)(d+1) + \sum_{j=1}^{m_i} \#S_j^{(i)}) - (l-1)(d+1) - \#\bigcap_{i=1}^{l}\bigcap_{j=1}^{m_i} S_j^{(i)}.
\end{aligned}
$$

Since $D_d(S^{(i)}) + \#\bigcap_{j=1}^{m_i} S_j^{(i)} = -(m_i-1)(d+1) + \sum_{j=1}^{m_i} \#S_j^{(i)}$,

$$
\begin{aligned}
D_d(S) &= \sum_{i=1}^{l} D_d(S^{(i)}) - (l-1)(d+1) + \sum_{i=1}^{l} \#\bigcap_{j=1}^{m_i} S_j^{(i)} - \#\bigcap_{i=1}^{l}\bigcap_{j=1}^{m_i} S_j^{(i)}\\
&= \sum_{i=1}^{l} D_d(S^{(i)}) + D_d\Big(\Big\{ \bigcap_{j=1}^{m_1} S_j^{(1)}, \dots, \bigcap_{j=1}^{m_l} S_j^{(l)} \Big\}\Big).
\end{aligned}
$$

Since $T_i \subset \bigcap_{j=1}^{m_i} S_j^{(i)}$, it follows from Lemma 3.4 that

$$
D_d(S) \geq \sum_{i=1}^{l} D_d(S^{(i)}) + D_d(T) > 0.
$$

□

Lemma 3.6. *Let $T \in L(n,d)$, $T_1 \in T$ and $T' = T \setminus \{ T_1 \}$. Then $\mathcal{I}_{n,d}(\{ T_1 \}) \times \mathcal{I}_{n,d}(T')$ and $\mathcal{I}_{n,d}(T)$ are isomorphic as posets.*

Proof. Let $T \in L(n,d)$, $T_1 \in T$ and $T' = T \setminus \{ T_1 \}$. For $(S,S') \in \mathcal{I}_{n,d}(\{ T_1 \}) \times \mathcal{I}_{n,d}(T')$, let us define $\varphi(S,S') = S \cup S'$. Then $\varphi(S,S') \in L(n,d)$ by Lemma 3.5. By definition $\varphi(S,S') \leq T$. Hence φ is a map from $\mathcal{I}_{n,d}(\{ T_1 \}) \times \mathcal{I}_{n,d}(T')$ to $\mathcal{I}_{n,d}(T)$. Moreover, if (S,S') and (S'',S''') satisfy $S \leq S''$ and $S' \leq S'''$, then $\varphi(S,S') \leq \varphi(S'',S''')$. On the other hand, we can define the following map ψ from $\mathcal{I}_{n,d}(T)$ to $\mathcal{I}_{n,d}(\{ T_1 \}) \times \mathcal{I}_{n,d}(T')$:

$$
\psi(S) = (\{ S_i \mid T_1 \subset S_i \}, \{ S_i \mid T_1 \not\subset S_i \}),
$$

which is the inverse map of φ. Hence $\mathcal{I}_{n,d}(\{ T_1 \}) \times \mathcal{I}_{n,d}(T')$ and $\mathcal{I}_{n,d}(T)$ are isomorphic as posets. □

Applying Lemma 3.6 recursively, we have Theorem 3.1.

4. Computation of Möbius function and the characteristic polynomial

In this section we apply Theorem 3.1 to compute the Möbius function and the characteristic polynomial of the intersection lattice $L(n,d)$ for $d \leq 6$. This section is divided into four subsections.

In Section 4.1 we derive an explicit formula for the value of the Möbius function of $L(n,d)$ and show that it only depends on the type of $T \in L(n,d)$. Next in Section 4.2 we derive a formula for the number of elements of the same type as $T \in L(n,d)$. Then in Section 4.3 we derive some identities for these numbers, which are useful for checking the results of computations by computer. Finally in Section 4.4 we present lists of the numbers of elements and the characteristic polynomials for $d \leq 6$.

4.1. **Möbius function of the intersection lattice.** We first obtain the value of Möbius function of $L(n,d)$.

Proposition 4.1. *For $T \in L(n,d)$, $T \neq \emptyset$,*

$$\mu_{n,d}(T) = \prod_{T_i \in T} \mu_{n-\#T_i, d-\#T_i}(\{\,\emptyset\,\}).$$

Note that for $T = \emptyset$ we have $\mu_{n,d}(\emptyset) = 1$. Also, as discussed at the beginning of Section 2.1, $\mu_{n-\#T_i,d-\#T_i}(\{\,\emptyset\,\}) = -1$ if $d = \#T_i$.

Proposition 4.1 is an immediate consequence of Theorem 3.1 and the following well-known lemma.

Lemma 4.2 (Proposition 3.8.2 of Stanley [1997]). *Let P and P' be posets, and $P \times P'$ the direct product of posets P and P'. Then $\mu_P(S,T) \cdot \mu_{P'}(S',T') = \mu_{P\times P'}((S,S'),(T,T'))$ for $S,T \in P$ and $S',T' \in P'$, where μ denotes the Möbius function for each poset.*

Proposition 4.1 shows that the Möbius function of $L(n,d)$ is completely determined by the values of $\mu_{n+k-d,k}(\{\,\emptyset\,\})$, $0 \le k \le d$. In particular for $T \neq \{\,\emptyset\,\}$, $\mu_{n,d}(T)$ is a product of $\mu_{n+k-d,k}(\{\,\emptyset\,\})$ for k smaller than d. As seen in the examples of Section 2.1, $\mu_{n',d'}(\{\,\emptyset\,\})$ is a polynomial in n'. Hence $\mu_{n,d}(T)$, $T \neq \{\,\emptyset\,\}$, can be immediately obtained from $\mu_{n',d'}(\{\,\emptyset\,\})$ for $d' < d$. Therefore for the recursion on d, the essential step is to compute $\mu_{n,d}(\{\,\emptyset\,\})$ by (5), which will be discussed in the next subsection.

As a corollary to Proposition 4.1 we have the following result.

Corollary 4.3. *Let $T = \{\,T_1,\dots,T_l\,\} \in L(n,d)$ and $T' = \{\,T'_1,\dots,T'_l\,\} \in L(n,d')$ satisfy $\operatorname{codim}_d(T_i) = \operatorname{codim}_{d'}(T'_i)$ for each i. Define $\bar\mu_{u,d}(T) = \mu_{d+u,d}(T)$, $u \ge 1$. Then*

$$\bar\mu_{u,d}(T) = \bar\mu_{u,d'}(T').$$

In this sense the value of the Möbius function depends only on the multiset of codimensions, i.e., the type $\gamma_d(T)$ of T. Therefore from now on we denote $\mu_{n,d}(T) = \mu_{n,d}(\gamma)$ if $\gamma_d(T) = \gamma$.

4.2. **Number of elements of the intersection lattice.** The results of the previous subsection implies that the terms of the summations in (4) and (5) can be grouped into different types. Then the question is how to obtain the number of elements of the same type in $L(n,d)$, denoted by $\lambda_{n,d}(\gamma)$ below. In this subsection we give an explicit expression for $\lambda_{n,d}(\gamma)$ in Proposition 4.7.

Let

$$\lambda_{n,d}(\gamma) = \#\{\,T \in L(n,d) \mid \gamma_d(T) = \gamma\,\}$$

denote the number of $T \in L(n,d)$ of type γ. Then (4) and (5) are written as follows.

$$\chi_{n,d}(t) = \mu_{n,d}(\{\,\emptyset\,\}) + \sum_{i=0}^{d}\sum_{\gamma\vdash i} \lambda_{n,d}(\gamma)\mu_{n,d}(\gamma)t^{d+1-i}, \tag{8}$$

$$\mu_{n,d}(\{\,\emptyset\,\}) = -\sum_{i=0}^{d}\sum_{\gamma\vdash i} \lambda_{n,d}(\gamma)\mu_{n,d}(\gamma). \tag{9}$$

For stating Proposition 4.7 we need some more definitions. For a partition $\gamma = (\gamma_1,\dots,\gamma_l)$ of a nonnegative integer let

$$\operatorname{Stab}_{S_l}(\gamma) = \left\{\,\sigma \in S_l \;\middle|\; \gamma_i = \gamma_{\sigma(i)}, i = 1,\dots,l\,\right\}$$

denote the stabilizer of the symmetric group S_l fixing γ. Then we have

$$\#\,\mathrm{Stab}_{S_l}(\gamma) = \prod_k m_k(\gamma)!,$$

where $m_k(\gamma)$ denotes the multiplicity $\#\{\, i \mid \gamma_i = k \,\}$ of $k \in \{\, 1, \dots, d \,\}$ in γ.

Denote the elements of $2^{\{\,1,\dots,l\,\}}$ as

$$2^{\{\,1,\dots,l\,\}} = \{\, \emptyset, \{\,1\,\}, \dots, \{\,l\,\}, \{\,1,2\,\}, \dots, \{\,1,\dots,l\,\}\,\} = \{\, I_1, \dots, I_{2^l} \,\},$$

where $I_1 = \emptyset$. Let $N(n,d;\gamma)$, $\gamma = (\gamma_1, \dots, \gamma_l)$, $l \geq 1$, denote the set of maps ν from the power set $2^{\{\,1,\dots,l\,\}}$ to the set $\mathbb{N}$ of nonnegative integers satisfying the following:

1) $\sum_{I\,:\,i \in I} \nu(I) = d + 1 - \gamma_i$ for all $i = 1, \dots, l$.
2) $\sum_{I'\,:\,I \subset I'} \nu(I') < d + 1 - \sum_{i \in I} \gamma_i$ for all I such that $\#I \geq 2$.
3) $\sum_{I \in 2^{\{\,1,\dots,l\,\}}} \nu(I) = n$.

Example 4.4. In the case when $\gamma = (d+1)$, ν is a map from $2^{\{\,1\,\}} = \{\, \emptyset, \{\,1\,\}\,\}$ to $\mathbb{N}$. By the condition 1), $\sum_{I\,:\,1 \in I} \nu(I) = \nu(\{\,1\,\}) = 0$. Hence, by 3), $\nu(\emptyset) = n$. $N(n,d;(d+1))$ consists of this ν only.

Remark 4.5. Consider the elements of $\{\,1,\dots,n\,\}$ as "symbols". Each T_i in $T = \{\, T_1, \dots, T_l \,\}$ is a subset of $\{\,1,\dots,n\,\}$ and therefore contains $\#T_i$ symbols. Also call T_i a "block". We can think of $T = \{\, T_1, \dots, T_l \,\}$ as putting symbols $1, \dots, n$ in the blocks $T_1, \dots, T_l$. Some symbol appears in several blocks. For $I \subset \{\,1,\dots,l\,\}$, $\nu(I)$ denotes the number of symbols commonly contained in T_i, $i \in I$, but not contained in any other T_i, $i \notin I$. The condition 3) on ν means that n symbols $1, \dots, n$ are classified by the blocks containing them. The condition 1) on ν corresponds to the size of each block $\#T_i = d + 1 - \gamma_i$. The condition 2) on ν is essential and corresponds to 1) of Definition 2.4.

Example 4.6. First let us consider the case when $\gamma = (\gamma_1)$. In this case

$$\lambda_{n,d}((\gamma_1)) = \binom{n}{d+1-\gamma_1}. \tag{10}$$

Next let us consider the case when $\gamma = (\gamma_1, \gamma_2)$. Let $t_i = d + 1 - \gamma_i$, $i = 1, 2$. For an element $T = \{\, T_1, T_2 \,\}$ of type γ, by (2),

$$\begin{aligned} 0 &\leq \#(T_1 \cap T_2) = \#T_1 + \#T_2 - \#(T_1 \cup T_2) \\ &\leq \#T_1 + \#T_2 - d - 2 = t_1 + t_2 - d - 2 = d - \gamma_1 - \gamma_2. \end{aligned}$$

It also follows by definition that $\#(T_1 \cap T_2) \leq \min(t_1, t_2)$. Write $\nu = \#(T_1 \cap T_2)$. If $\gamma_1 > \gamma_2$, then $(\mathrm{codim}(T_2), \mathrm{codim}(T_1)) \neq (t_1, t_2)$ as ordered pairs. Hence, for the case $\gamma_1 > \gamma_2$,

$$\lambda_{n,d}((\gamma_1,\gamma_2)) = \sum_{\nu=0}^{\min(t_1,t_2,d-\gamma_1-\gamma_2)} \binom{n}{\nu}\binom{n-\nu}{t_1-\nu}\binom{n-t_1}{t_2-\nu}. \tag{11}$$

On the other hand, if $\gamma_1 = \gamma_2$, then $(\mathrm{codim}(T_2), \mathrm{codim}(T_1)) = (\gamma_1, \gamma_2)$ as ordered pairs for all elements $T = \{\, T_1, T_2 \,\}$ of type γ. Since $T_1 \neq T_2$, for the case $\gamma_1 = \gamma_2$,

$$\lambda_{n,d}((\gamma_1,\gamma_2)) = \frac{1}{2} \sum_{\nu=0}^{\min(t_1,t_2,d-\gamma_1-\gamma_2)} \binom{n}{\nu}\binom{n-\nu}{t_1-\nu}\binom{n-t_1}{t_2-\nu}. \tag{12}$$

Now we present the following proposition.

Proposition 4.7. *For $\gamma \neq (0)$, $\lambda_{n,d}(\gamma)$ is given as follows.*

(13)
$$\lambda_{n,d}(\gamma) = \frac{1}{\prod_{k=1}^{d} m_k(\gamma)!} \sum_{\nu \in N(n,d;\gamma)} \frac{n!}{\nu(I_1)!\nu(I_2)!\cdots\nu(I_{2^l})!}$$
$$= \frac{1}{\prod_{k=1}^{d} m_k(\gamma)!} \sum_{\nu \in N(n,d;\gamma)} \frac{(\nu(I_2)+\cdots+\nu(I_{2^l}))!}{\nu(I_2)!\cdots\nu(I_{2^l})!} \binom{n}{\nu(I_2)+\cdots+\nu(I_{2^l})}.$$

Before giving a proof of this proposition, we give some explanation on the range of summation in (13). We can consider $(\nu(I_1),\dots,\nu(I_{2^l}))$ as a 2^l-dimensional vector of non-negative integers. The equalities and the inequalities in 1),2),3) for ν specify a polytope. Hence $N(n,d;\gamma)$ can be identified with the set of integer points in a polytope in $\mathbb{R}^{2^l}$. Since the dimension 2^l of the vector increases exponentially with l, the number of terms in (13) increases doubly exponentially in l. In our computation for $d=6$ and $\gamma=(1,1,1,1,1,1)$, $\#N(n,6;(1,1,1,1,1,1)) = 109719496370$. Computing a sum of this many polynomials is quite heavy.

The equalities and inequalities in 1),2) for ν concern only $\nu(I)$, $I \neq \emptyset$, and the bounds for these nonnegative integers are given in terms of γ and d only. Therefore the range for $\nu(I), I \neq \emptyset$, in $N(n,d;\gamma)$ does not depend on n. n only appears through 3):
$$\nu(\emptyset) = \nu(I_1) = n - (\nu(I_2)+\cdots+\nu(I_{2^l})).$$
Therefore in the right-hand side of (13) the sum is a finite sum not depending on n and n only appears in the binomial coefficient $\binom{n}{\nu(I_2)+\cdots+\nu(I_{2^l})}$.

Now we give a proof of Proposition 4.7.

Proof of Proposition 4.7. Let us consider l-tuples $(T_1,\dots,T_l)$ of subsets of $\{1,\dots,n\}$. We define $\tilde{L}(n,d;\gamma)$ to be the set of l-tuples $(T_1,\dots,T_l)$ of subsets in $\{1,\dots,n\}$ satisfying the following:

1) $\{T_1,\dots,T_l\} \in L(n,d)$.
2) $\operatorname{codim}_d T_i = \gamma_i$ for each i.

Let $(T_1,\dots,T_l) \in \tilde{L}(n,d;\gamma)$. Then, since $\{T_1,\dots,T_l\} \in L(n,d)$, by (2), $\#(T_i \cup T_j) > d$ for $i \neq j$. Since $\#T_i$ and $\#T_j$ are less than or equal to $d+1$, we obtain $T_i \neq T_j$. Therefore, for $(T_1,\dots,T_l) \in \tilde{L}(n,d;\gamma)$ and $\sigma \in \operatorname{Stab}_{S_l}(\gamma)$, we have
$$(T_1,\dots,T_l) \neq (T_{\sigma(1)},\dots,T_{\sigma(l)}) \in \tilde{L}(n,d;\gamma)$$
if σ is not the identity. This implies

(14)
$$\#\tilde{L}(n,d;\gamma) = \lambda_{n,d}(\gamma)\cdot\#\operatorname{Stab}_{S_l}(\gamma) = \lambda_{n,d}(\gamma)\cdot\prod_k m_k(\gamma)!.$$

For $T=(T_1,\dots,T_l) \in \tilde{L}(n,d;\gamma)$ and a subset $I \subset \{1,\dots,l\}$, define $\tau(T,I)$ by
$$\tau(T,I) = \{\, t \mid t \in T_i \iff i \in I \,\}$$
$$= \bigcap_{i\in I} T_i \setminus \bigcup_{i \notin I} T_i.$$

Moreover, for $\nu \in N(n,d;\gamma)$ let us define $\tilde{L}(n,d;\gamma,\nu)$ by
$$\tilde{L}(n,d;\gamma,\nu) = \left\{ T \in \tilde{L}(n,d;\gamma) \;\middle|\; \forall I \subset \{1,\dots,l\}, \nu(I) = \#\tau(T,I) \right\}.$$

Then, by definition, we have the following decomposition of $\tilde{L}(n,d;\gamma)$:
$$\tilde{L}(n,d;\gamma) = \coprod_{\nu\in N(n,d;\gamma)} \tilde{L}(n,d;\gamma,\nu).$$

Now note that

$$\#\tilde{L}(n,d;\gamma,\nu) = \frac{n!}{\nu(I_1)!\nu(I_2)!\cdots\nu(I_{2^l})!}.$$

Therefore

$$\#\tilde{L}(n,d;\gamma) = \sum_{\nu\in N(n,d;\gamma)} \frac{n!}{\nu(I_1)!\nu(I_2)!\cdots\nu(I_{2^l})!}.$$

This together with (14) proves the proposition. □

4.3. **Identities for the number of elements.** We have coded the finite sum in (13) in a computer program and evaluated $\lambda_{n,d}(\gamma)$ up to $d = 6$. In the next subsection we present our computational results. However the range of summation in (13) is somewhat complicated and our code was error-prone. Therefore it is desirable to have some way of checking our results. Here we present some identities among $\lambda_{n,d}(\gamma)$'s, which can be used for checking purposes.

Again we need some more definitions for stating the identities. Let T_i be a subset of $\{1,\dots,n\}$ of size $d = \#T_i$. Then $H(\{T_i\})$ is a hyperplane of $\mathcal{A}_{n,d}$. By abuse of terminology, we also call $\{T_i\}$ itself a hyperplane. Let $\{T_1\},\dots,\{T_m\}$ be m distinct hyperplanes. Consider the intersection $H(\{T_1\})\cap\cdots\cap H(\{T_m\})$ of corresponding hyperplanes of $\mathcal{A}_{n,d}$, or equivalently the join of these hyperplanes $\{T_1\},\dots,\{T_m\}$ in $L(n,d)$:

$$\{T_1\}\vee\cdots\vee\{T_m\}\in L(n,d).$$

It seems hard to explicitly describe $S_1,\dots,S_{l'}$ such that $S = \{S_1,\dots,S_{l'}\} = \{T_1\}\vee\cdots\vee\{T_m\}$. However we can count the number of $\{T_1,\dots,T_m\}$, ($\#T_i = d$, $\forall i$), such that $\{T_1\}\vee\cdots\vee\{T_m\}$ is an element of a particular type of $L(n,d)$. This will give us the desired identities.

For a particular $T\in L(n,d)$ define

$$\kappa_{n,d}(m,T) = \#\left\{ \{T_1,\dots,T_m\} \;\middle|\; \begin{array}{c}\{T_1\},\dots,\{T_m\} : \text{distinct hyperplanes},\\ T = \{T_1\}\vee\cdots\vee\{T_m\}\end{array} \right\},$$

which is the number of ways of choosing m distinct hyperplanes such that their join is T. By Theorem 3.1, $\kappa_{n,d}(m,T)$ only depends on the type $\gamma_d(T)$ of T. Hence we can write

$$\kappa_{n,d}(m,T) = \kappa_{n,d}(m,\gamma) \quad \text{if} \quad \gamma_d(T) = \gamma.$$

Note that there are $\binom{\binom{n}{d}}{m}$ ways to choose m distinct hyperplanes from $\{1,\dots,n\}$. Therefore we have the following identity:

$$\binom{\binom{n}{d}}{m} = \kappa_{n,d}(m,\{\emptyset\}) + \sum_{i=0}^{d}\sum_{\gamma\vdash i}\lambda_{n,d}(\gamma)\cdot\kappa_{n,d}(m,\gamma). \tag{15}$$

If we can compute $\kappa_{n,d}(m,\gamma)$, these identities for various m can be used to check computations of $\lambda_{n,d}(\gamma)$. Hence it remains to show how to evaluate $\kappa_{n,d}(m,\gamma)$, which is again based on recursion on d.

First we consider $\kappa_{n,d}(m,\gamma)$ for some special γ. Write $(1^h) = (\underbrace{1,1,\dots,1}_{h})$. Then

$$\kappa_{n,d}(m,(1^h)) = \delta_{mh},$$

where δ_{mh} is Kronecker's delta. Also note that

$$\kappa_{n,d}(m,T) = 0 \quad \text{if} \quad \rho_d(T) > m.$$

In particular

$$\kappa_{n,d}(m,\{\emptyset\}) = 0 \;\; \text{for} \;\; 1\le m\le d.$$

Based on these observations, there are two uses of (15). For $m = 1, \dots, d$, we can use (15) to check $\lambda_{n,d}(\gamma)$ for $\gamma \vdash i \le d$. With $m > d$, (15) gives the values of $\kappa_{n,d}(m, \{\emptyset\})$.

Now we show how $\kappa_{n,d}(m, \gamma)$ is evaluated from $\kappa_{n,d'}(m, \{\emptyset\})$ with $d' < d$. We list $\kappa_{n,d}(m, \{\emptyset\})$ for $d = 0, 1, 2$. For $d = 0$ we define $\kappa_{n,0}(m, \{\emptyset\}) = 1$. For $d = 1$, since the intersection of more than one point is empty, $\kappa_{n,1}(m, \{\emptyset\}) = \binom{n}{m}$ for $m > 1$. For $d = 2$, the intersection of $m > 2$ lines is non-empty if and only if they contain a common point p_i. Therefore for $m > 2$,

$$\kappa_{n,2}(m, \{\emptyset\}) = \binom{\binom{n}{2}}{m} - n\binom{n-1}{m}.$$

Finally as another consequence of the main theorem we have the following proposition. It allows us to evaluate $\kappa_{n,d}(m, \gamma)$ recursively from $\kappa_{n,d'}(m, \{\emptyset\})$, $d' < d$.

Proposition 4.8. *For $T = \{T_1, \dots, T_l\} \in L(n, d)$, $T \ne \{\emptyset\}$, and a positive integer m, define*

$$M(m, T) = \left\{ (m_1, \dots, m_l) \in \mathbb{Z}^l_{>0} \;\middle|\; \begin{array}{c} m_i \ge \operatorname{codim}_d(T_i), \forall i. \\ m_i = 1 \textit{ for } \operatorname{codim}_d(T_i) = 1. \\ m = m_1 + \cdots + m_l. \end{array} \right\}.$$

Then

$$\kappa_{n,d}(m, T) = \sum_{(m_1,\dots,m_l) \in M(m,T)} \prod_{i=1}^{l} \kappa_{n-\#T_1, d-\#T_1}(m_i, \{\emptyset\}).$$

Note that in the product a term with $d = \#T_i$ does not contribute to the product since $\kappa_{n,0}(m, \{\emptyset\}) \equiv 1$. We omit a detailed proof of the proposition.

4.4. **Number of elements and the characteristic polynomial up to dimension six.** In this section we present our computational results for $4 \le d \le 6$, since the cases $d \le 3$ were already discussed in Section 2.1. We just recall

$$\mu_{n,0}(\{\emptyset\}) = -1, \qquad \mu_{n,1}(\{\emptyset\}) = n - 1,$$
$$\mu_{n,2}(\{\emptyset\}) = -3\binom{n}{4} - \binom{n}{2} + n - 1.$$

From now on, to save space, we use the following abbreviated notation.

$$n_k = \binom{n}{k}.$$

Then, for example, $\mu_{n,3}(\{\emptyset\})$ is displayed as

$$\mu_{n,3}(\{\emptyset\}) = -1 + n - n_2 + n_3 + 20n_5 + 170n_6 + 630n_7 + 840n_8 + 280n_9.$$

We now present the computational results for $d = 4$. Because of (10), (11), (12), we only show $\lambda_{n,d}((\gamma_1, \dots, \gamma_l))$ where $l \ge 3$. Also, for further notational simplification, we omit the subscripts and write e.g. $\lambda(1, 1)$ instead of $\lambda_{n,4}((1, 1))$.

$$\lambda(1,1,1) = 15n_6 + 1470n_7 + 11340n_8 + 30240n_9 + 37450n_{10} + 23100n_{11} + 5775n_{12},$$
$$\lambda(2,1,1) = 1260n_7 + 10080n_8 + 23940n_9 + 21000n_{10} + 5775n_{11},$$
$$\lambda(1,1,1,1) = 2100n_7 + 120855n_8 + 1640520n_9 + 9585450n_{10} + 29799000n_{11} + 54365850n_{12} + 60660600n_{13} + 41166125n_{14} + 15765750n_{15} + 2627625n_{16}.$$

$$\chi(\mathcal{A}_{n,4},t) = t^4 - n_4 t^3 + \Big[- n_3 + 4n_4 + 45n_6 + 70n_7 + 35n_8\Big] t^2$$
$$+ \Big[- n_2 + 3n_3 - 6n_4 - 180n_6 - 1995n_7 - 11620n_8 - 30240n_9 - 37450n_{10} - 23100n_{11} - 5775n_{12}\Big] t$$
$$+ \Big[- n + 2n_2 - 3n_3 + 4n_4 + 250n_6 + 8995n_7 + 184835n_8 + 1873620n_9 + 9963100n_{10} + 30070425n_{11} + 54435150n_{12} + 60660600n_{13} + 41166125n_{14} + 15765750n_{15} + 2627625n_{16}\Big].$$

$$\mu_{n,4}(\{\emptyset\}) = -1 + n - n_2 + n_3 - n_4 - 115n_6 - 7070n_7 - 173250n_8 - 1843380n_9 - 9925650n_{10} - 30047325n_{11} - 54429375n_{12} - 60660600n_{13} - 41166125n_{14} - 15765750n_{15} - 2627625n_{16}.$$

The results for $d = 5$ are as follows.

$$\lambda(1,1,1) = 105n_7 + 9240n_8 + 102060n_9 + 453600n_{10} + 1089550n_{11} + 1561560n_{12} + 1336335n_{13} + 630630n_{14} + 126126n_{15},$$
$$\lambda(2,1,1) = 105n_7 + 15960n_8 + 170100n_9 + 642600n_{10} + 1166550n_{11} + 1136520n_{12} + 585585n_{13} + 126126n_{14},$$
$$\lambda(1,1,1,1) = 42000n_8 + 2796255n_9 + 52475850n_{10} + 464829750n_{11} + 2391764760n_{12} + 7945667730n_{13} + 18019621620n_{14} + 28608004425n_{15} + 31876244400n_{16} + 24459299865n_{17} + 12318095790n_{18} + 3666482820n_{19} + 488864376n_{20},$$
$$\lambda(3,1,1) = 3360n_8 + 37800n_9 + 138600n_{10} + 219450n_{11} + 152460n_{12} + 36036n_{13},$$
$$\lambda(2,2,1) = 5040n_8 + 56700n_9 + 201600n_{10} + 300300n_{11} + 194040n_{12} + 45045n_{13},$$
$$\lambda(2,1,1,1) = 47040n_8 + 3859380n_9 + 77275800n_{10} + 682882200n_{11} + 3311930160n_{12} + 9818128320n_{13} + 18834816000n_{14} + 23991267300n_{15} + 20272652400n_{16} + 10985154180n_{17} + 3473510040n_{18} + 488864376n_{19},$$
$$\lambda(1,1,1,1,1) = 70560n_8 + 28259280n_9 + 1892400300n_{10} + 49372299900n_{11} + 678800152800n_{12} + 5726202381900n_{13} + 32397151296510n_{14} + 129991147035750n_{15} + 383340007050000n_{16} + 849257881311840n_{17} + 1429769976354720n_{18} + 1833899747359680n_{19} + 1780941069507600n_{20} + 1287845979720300n_{21} + 672060801181770n_{22} + 239171396233770n_{23} + 51946728593760n_{24} + 5194672859376n_{25}.$$

$$\chi(\mathcal{A}_{n,5},t) = t^5 - n_5 t^4 + \Big[- n_4 + 5n_5 + 105n_7 + 280n_8 + 315n_9 + 126n_{10}\Big] t^3$$
$$+ \Big[- n_3 + 4n_4 - 10n_5 - 630n_7 - 11760n_8 - 105084n_9 - 454860n_{10} - 1089550n_{11} - 1561560n_{12} - 1336335n_{13} - 630630n_{14} - 126126n_{15}\Big] t^2$$
$$+ \Big[- n_2 + 3n_3 - 6n_4 + 10n_5 + 1540n_7 + 112371n_8 + 3739176n_9 + 57660120n_{10} + 479155600n_{11} + 2413802160n_{12} + 7965127170n_{13} + 18028954944n_{14} + 28609896315n_{15} + 31876244400n_{16} + 24459299865n_{17} + 12318095790n_{18} + 3666482820n_{19} + 488864376n_{20}\Big] t$$
$$+ \Big[- n + 2n_2 - 3n_3 + 4n_4 - 5n_5 - 1729n_7 - 444808n_8 - 51417954n_9 - 2407629420n_{10} - 54882065700n_{11} - 712167312780n_{12} - 5852028673491n_{13} - 32709595374456n_{14} - 130517797815405n_{15} - 383948623858800n_{16} - 849734640219465n_{17} - 1430012864760480n_{18} - 1833972588151704n_{19}$$

$$
\begin{aligned}
&- 1780950846795120n_{20} - 1287845979720300n_{21} - 672060801181770n_{22}\\
&- 239171396233770n_{23} - 51946728593760n_{24} - 5194672859376n_{25}\Big].
\end{aligned}
$$

$$
\begin{aligned}
\mu_{n,5}(\{\emptyset\}) = {}& -1 + n - n_2 + n_3 - n_4 + n_5 + 714n_7 + 343917n_8 + 47783547n_9\\
&+ 2350424034n_{10} + 54403999650n_{11} + 709755072180n_{12}\\
&+ 5844064882656n_{13} + 32691567050142n_{14} + 130489188045216n_{15}\\
&+ 383916747614400n_{16} + 849710180919600n_{17} + 1430000546664690n_{18}\\
&+ 1833968921668884n_{19} + 1780950357930744n_{20} + 1287845979720300n_{21}\\
&+ 672060801181770n_{22} + 239171396233770n_{23} + 51946728593760n_{24}\\
&+ 5194672859376n_{25}.
\end{aligned}
$$

Finally the results for $d = 6$ are as follows.

$$
\begin{aligned}
\lambda(1,1,1) &= 420n_8 + 40600n_9 + 620550n_{10} + 4158000n_{11} + 16046800n_{12} + 39399360n_{13}\\
&\quad + 63588525n_{14} + 67267200n_{15} + 44900856n_{16} + 17153136n_{17} + 2858856n_{18},\\
\lambda(2,1,1) &= 840n_8 + 105210n_9 + 1486800n_{10} + 8339100n_{11} + 25225200n_{12} + 46576530n_{13}\\
&\quad + 54444390n_{14} + 39414375n_{15} + 16144128n_{16} + 2858856n_{17},\\
\lambda(1,1,1,1) &= 105n_8 + 388080n_9 + 32389875n_{10} + 847573650n_{11} + 11095663425n_{12}\\
&\quad + 88232164020n_{13} + 470574214110n_{14} + 1778211935500n_{15} + 4911176169900n_{16}\\
&\quad + 10078325056800n_{17} + 15457185789045n_{18} + 17651874149910n_{19}\\
&\quad + 14793239711250n_{20} + 8833453364736n_{21} + 3557221631964n_{22}\\
&\quad + 865778809896n_{23} + 96197645544n_{24},\\
\lambda(3,1,1) &= 210n_8 + 45360n_9 + 642600n_{10} + 3326400n_{11} + 8523900n_{12} + 12132120n_{13}\\
&\quad + 9900891n_{14} + 4414410n_{15} + 840840n_{16},\\
\lambda(2,2,1) &= 280n_8 + 65520n_9 + 919800n_{10} + 4596900n_{11} + 11226600n_{12} + 15315300n_{13}\\
&\quad + 12192180n_{14} + 5360355n_{15} + 1009008n_{16},\\
\lambda(2,1,1,1) &= 892080n_9 + 83349000n_{10} + 2170822500n_{11} + 26591796000n_{12}\\
&\quad + 189359450280n_{13} + 876055780600n_{14} + 2806801697700n_{15}\\
&\quad + 6458643391200n_{16} + 10866964308200n_{17} + 13416110908200n_{18}\\
&\quad + 12029730132420n_{19} + 7626284265600n_{20} + 3240681948504n_{21}\\
&\quad + 828136252944n_{22} + 96197645544n_{23},\\
\lambda(1,1,1,1,1) &= 1829520n_9 + 817016760n_{10} + 72235270800n_{11} + 2647690791900n_{12}\\
&\quad + 53345363951880n_{13} + 682682216596380n_{14} + 6039039035429400n_{15}\\
&\quad + 38946366176117400n_{16} + 189638773413289200n_{17} + 713826716560797840n_{18}\\
&\quad + 2110340393930648880n_{19} + 4950304696313776800n_{20}\\
&\quad + 9265441477593100800n_{21} + 13857900072549583050n_{22}\\
&\quad + 16518003442667606880n_{23} + 15574944975706176060n_{24}\\
&\quad + 11462658924203487000n_{25} + 6442333859931445476n_{26}\\
&\quad + 2669265333214159680n_{27} + 768162249080226000n_{28}\\
&\quad + 137087416758932640n_{29} + 11423951396577720n_{30},\\
\lambda(4,1,1) &= 7560n_9 + 113400n_{10} + 589050n_{11} + 1432200n_{12} + 1747746n_{13}\\
&\quad + 1009008n_{14} + 210210n_{15},\\
\lambda(3,2,1) &= 30240n_9 + 453600n_{10} + 2263800n_{11} + 5128200n_{12} + 5765760n_{13}\\
&\quad + 3111108n_{14} + 630630n_{15},\\
\lambda(3,1,1,1) &= 181440n_9 + 20594700n_{10} + 585169200n_{11} + 7466867100n_{12} + 53238345160n_{13}\\
&\quad + 236878922280n_{14} + 700100200800n_{15} + 1424183961200n_{16}\\
&\quad + 2028644217600n_{17} + 2026021217220n_{18} + 1394893019520n_{19}\\
&\quad + 633079366920n_{20} + 171102531600n_{21} + 20912531640n_{22},\\
\lambda(2,2,2) &= 7560n_9 + 113400n_{10} + 554400n_{11} + 1201200n_{12} + 1261260n_{13}\\
&\quad + 630630n_{14} + 126126n_{15},\\
\lambda(2,2,1,1) &= 430920n_9 + 48365100n_{10} + 1342768350n_{11} + 16595271000n_{12} + 114090786810n_{13}\\
&\quad + 489169180500n_{14} + 1396656261000n_{15} + 2757820665600n_{16}\\
&\quad + 3836191659300n_{17} + 3764834613540n_{18} + 2561038249770n_{19}\\
&\quad + 1152905153400n_{20} + 309695582196n_{21} + 37642556952n_{22},
\end{aligned}
$$

$$\begin{aligned}
\lambda(2,1,1,1,1) = {} & 2872800n_9 + 1676581200n_{10} + 164904790050n_{11} + 6298213521600n_{12} \\
& + 126458717144760n_{13} + 1557569654921280n_{14} + 12897279885364875n_{15} \\
& + 76164200116804800n_{16} + 333794628241774700n_{17} \\
& + 1115520582743887320n_{18} + 2894974312598468100n_{19} \\
& + 5900420897320950000n_{20} + 9496944246098058750n_{21} \\
& + 12073014477589665600n_{22} + 12056810514853269165n_{23} \\
& + 9346203461860705440n_{24} + 5507792588210012100n_{25} \\
& + 2383822869473725680n_{26} + 714306478210645320n_{27} \\
& + 132360264456900480n_{28} + 11423951396577720n_{29}, \\
\lambda(1,1,1,1,1,1) = {} & 4011840n_9 + 11413776150n_{10} + 3444031510920n_{11} + 341035483477150n_{12} \\
& + 16334107213023600n_{13} + 458689729433265330n_{14} \\
& + 8450977741650944500n_{15} + 109792467460902806580n_{16} \\
& + 1056347419381332078000n_{17} + 7792389750829016643310n_{18} \\
& + 45197004798213378970860n_{19} + 209996223288982641611100n_{20} \\
& + 792475775069757320141600n_{21} + 2453913578583257706865950n_{22} \\
& + 6280395970196377852122300n_{23} + 13348940867374682005436000n_{24} \\
& + 23623379361553534532970000n_{25} + 34820458479536167782093750n_{26} \\
& + 42668658867461724953856000n_{27} + 43277385167426660997596850n_{28} \\
& + 36064655494545316433394600n_{29} + 24416711436628708549852500n_{30} \\
& + 13209334741333731036156120n_{31} + 5572094138384063443144992n_{32} \\
& + 1765284170332337557943040n_{33} + 394963790210911659497760n_{34} \\
& + 55628702846607275985600n_{35} + 3708580189773818399040n_{36}.
\end{aligned}$$

$$\begin{aligned}
\chi(\mathcal{A}_{n,6},t) = {} & t^6 - n_6t^5 + \Big[- n_5 + 6n_6 + 210n_8 + 840n_9 + 1575n_{10} + 1386n_{11} + 462n_{12}\Big]t^4 \\
& + \Big[- n_4 + 5n_5 - 15n_6 - 1750n_8 - 49420n_9 - 638190n_{10} - 4174170n_{11} \\
& - 16052344n_{12} - 39399360n_{13} - 63588525n_{14} - 67267200n_{15} - 44900856n_{16} \\
& - 17153136n_{17} - 2858856n_{18}\Big]t^3 \\
& + \Big[- n_3 + 4n_4 - 10n_5 + 20n_6 + 6545n_8 + 808269n_9 + 40056051n_{10} + 907650744n_{11} \\
& + 11350086517n_{12} + 88888289490n_{13} + 471662471280n_{14} + 1779383330725n_{15} \\
& + 4911968241180n_{16} + 10078630954392n_{17} + 15457237248453n_{18} \\
& + 17651874149910n_{19} + 14793239711250n_{20} + 8833453364736n_{21} \\
& + 3557221631964n_{22} + 865778809896n_{23} + 96197645544n_{24}\Big]t^2 \\
& + \Big[- n_2 + 3n_3 - 6n_4 + 10n_5 - 15n_6 - 12992n_8 - 6922440n_9 - 1253314020n_{10} \\
& - 85462425510n_{11} - 2844981329190n_{12} - 55072066969920n_{13} \\
& - 692509356770231n_{14} - 6077776702187190n_{15} - 39056313885629040n_{16} \\
& - 189868515062882656n_{17} - 714183552466036653n_{18} - 2110751767192248180n_{19} \\
& - 4950652071117753000n_{20} - 9265650239791905960n_{21} - 13857984617732497242n_{22} \\
& - 16518024125161398840n_{23} - 15574947284449669116n_{24} - 11462658924203487000n_{25} \\
& - 6442333859931445476n_{26} - 2669265333214159680n_{27} - 768162249080226000n_{28} \\
& - 137087416758932640n_{29} - 11423951396577720n_{30}\Big]t \\
& + \Big[- n + 2n_2 - 3n_3 + 4n_4 - 5n_5 + 6n_6 + 13020n_8 + 30306276n_9 + 21482580105n_{10} \\
& + 4476460758924n_{11} + 385724720114965n_{12} + 17372731634141884n_{13} \\
& + 473573588684378182n_{14} + 8594435149519438219n_{15} + 110777945174652868112n_{16} \\
& + 1061369389494699244960n_{17} + 7811915116061435525146n_{18} \\
& + 45256066666226567391240n_{19} + 210137040306764298507780n_{20} \\
& + 792742452459639413305574n_{21} + 2454315914651904858849294n_{22} \\
& + 6280878741810399702094119n_{23} + 13349398508879560267264124n_{24} \\
& + 23623717675088602759587900n_{25} + 34820649359673839255319726n_{26} \\
& + 42668738231115243168001080n_{27} + 43277408079933868947476370n_{28} \\
& + 36064659595743867804796080n_{29} + 24416711779347250447184100n_{30} \\
& + 13209334741333731036156120n_{31} + 5572094138384063443144992n_{32}
\end{aligned}$$

$$
\begin{aligned}
&+ 176528417033233755794304 0n_{33} + 39496379021091165949776 0n_{34} \\
&+ 5562870284660727598560 0n_{35} + 3708580189773818399040n_{36}\Big].
\end{aligned}
$$

$$
\begin{aligned}
\mu_{n,6}(\{\,\emptyset\,\}) = -1 &+ n - n_2 + n_3 - n_4 + n_5 - n_6 - 5033n_8 - 24143525n_9 - 20268685521n_{10} \\
&- 4391901811374n_{11} - 382891072820410n_{12} - 17317748416062094n_{13} \\
&- 472881550926490706n_{14} - 8588359152133314554n_{15} \\
&- 110738893772690579396n_{16} - 1061179531058250163560n_{17} \\
&- 7811200947966203878090n_{18} - 45253955932111249292970n_{19} \\
&- 210132089669486420466030n_{20} - 792733186818233074764350n_{21} \\
&- 2454302056670844347984016n_{22} - 6280862223787140319505175n_{23} \\
&- 13349382933932372015240552n_{24} - 23623706212429678556100900n_{25} \\
&- 34820642917339979323874250n_{26} - 42668735561849909953841400n_{27} \\
&- 43277407311771619867250370n_{28} - 36064659458656451045863440n_{29} \\
&- 24416711767923299050606380n_{30} - 13209334741333731036156120n_{31} \\
&- 5572094138384063443144992n_{32} - 1765284170332337557943040n_{33} \\
&- 394963790210911659497760n_{34} - 55628702846607275985600n_{35} \\
&- 3708580189773818399040n_{36}.
\end{aligned}
$$

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