LINES ON PROJECTIVE VARIETIES AND APPLICATIONS

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ABSTRACT. The first part of this note contains a review of basic properties of the variety of lines contained in an embedded projective variety and passing through a general point. In particular we provide a detailed proof that for varieties defined by quadratic equations the base locus of the projective second fundamental form at a general point coincides, as a scheme, with the variety of lines.

The second part concerns the problem of extending embedded projective manifolds, using the geometry of the variety of lines. Some applications to the case of homogeneous manifolds are included.

INTRODUCTION

The *principle* that the Hilbert scheme of lines contained in a (smooth) projective variety $X \subset \mathbb{P}^N$ and passing through a (general) point can inherit intrinsic and extrinsic geometrical properties of the variety, has emerged recently. This principle allowed to attack some problems in a *unified way*, provided non trivial connections between different theories and put some basic questions in a new light. A typical example is the Hartshorne Conjecture on complete intersections, see [H1, IR2] and also [Ru, IR1]. The technique of studying, or even reconstructing, X from the *variety of minimal rational tangents* introduced in the work of Hwang, Mok and others (a generalization of the Hilbert scheme of lines passing through a point) was applied to the theory of Fano manifolds (see e.g. [HM, HM2, HM3, Hw, HK]). On the other hand, Landsberg and others investigated some possible characterizations of special homogeneous manifolds via the projective second fundamental form (see e.g. [L2, L3, HY]).

The Hilbert schemes of lines through a general point of many homogeneous varieties with notable geometrical properties are also somehow *nested*, see Tables (2.4) and (2.5), or *part of a matrioska*. For this class of varieties, or more generally for classes where the principle holds, one starts an induction process which sometimes stops after only a few steps, see e.g. [Ru, Theorem 2.8, Corollary 3.1 and 3.2]. An example of this kind is the following: if $X \subset \mathbb{P}^N$ is a $LQEL$-manifold of type $\delta \geq 3$, then the Hilbert scheme of lines $\mathcal{L}_{x,X} \subset \mathbb{P}^{n-1}$, $n = \dim(X)$, passing through a general point $x \in X$ is a QEL-manifold of type $\delta - 2$, [Ru, Theorem 2.3]. Then starting the induction with $X \subset \mathbb{P}^N$ a $LQEL$-manifold of type $\frac{n}{2}$, one deduces immediately $n = 2, 4, 8$ or 16, yielding as a consequence a quick proof that Severi varieties appear only in these dimensions (see [Ru, Corollary 3.2], also for the definitions of $(L)QEL$-variety and of Severi variety, introduced by Zak, see e.g. [Za]).

The Hilbert scheme of lines through a point is closely related to the base locus of the (projective) second fundamental form, a classical tool used in projective differential geometry and reconsidered in modern algebraic geometry by Griffiths and Harris, [GH] and also [IL]. In this theory one tries to reconstruct a (homogeneous) variety from its second fundamental form (see e.g [L2, L3, HY]) by integrating local differential equations and obtaining global results. We note that the base locus of the second fundamental form at a general point of a smooth variety is typically not smooth, while this property is preserved by the Hilbert scheme of lines, see Proposition 1.1.

An important class where the two previous objects coincide is that of *quadratic varieties*, that is varieties $X \subset \mathbb{P}^N$ scheme theoretically defined by quadratic equations. All known *prime Fano manifolds of high index*, other than complete intersections (for example many homogeneous manifolds), are quadratic; moreover,

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they are embedded with *small codimension*. For quadratic varieties the Hilbert scheme of lines through a smooth point is also quadratic, see Proposition 1.2. Moreover, since it coincides with the base locus scheme of the second fundamental form, it may be scheme theoretically defined by at most $c = \mathrm{codim}(X)$ (quadratic) equations, see Corollary 1.6. If $X \subset \mathbb{P}^N$ is smooth and $x \in X$ is general, then $\mathcal{L}_{x,X} \subset \mathbb{P}^{n-1}$ is also smooth, see Proposition 1.1. Thus for quadratic manifolds, if $\mathcal{L}_{x,X}$ is also irreducible, a beautiful *matrioska* naturally appears. From this point of view, a quadratic manifold $X \subset \mathbb{P}^N$ with $3n > 2N$ is a complete intersection *because* $\mathcal{L}_{x,X} \subset \mathbb{P}^{n-1}$ is a smooth irreducible non-degenerate complete intersection, defined exactly by c quadratic equations, so that it has the *right dimension*, [IR2, Theorems 4.8 and 2.4] and Remark 1.7.

The aim of this note is twofold: In §1 we study in detail the intrinsic and extrinsic properties of the Hilbert scheme of lines passing through a smooth point of an equidimensional connected variety $X \subset \mathbb{P}^N$, providing an almost self contained treatment. In §2 we illustrate another incarnation of the principle presented above by studying the problem of extending smooth varieties uniruled by lines as hyperplane sections of irreducible varieties.

First we describe the possible singularities of $\mathcal{L}_{x,X}$, proving that a singular point of the Hilbert scheme of lines passing through a general point x of an irreducible variety produces a line joining x to a singular point of X, a stronger condition than the mere existence of a singular point on X, see Proposition 1.1. Then we relate the equations defining $X \subset \mathbb{P}^N$ with those of $\mathcal{L}_{x,X} \subset \mathbb{P}((t_{x,X})^*)$, see (1.7). This is applied to quadratic varieties showing that the Hilbert scheme of lines passing through a smooth point is a quadratic scheme, which coincides with the projectivized tangent cone at x to the scheme $T_xX \cap X$, see Proposition 1.2. After introducing the base locus of the second fundamental form of X at x, $B_{x,X} \subset \mathbb{P}((t_{x,X})^*)$, we show that in general $\mathcal{L}_{x,X} \subseteq B_{x,X}$ as schemes with equality holding, as schemes, if $X \subset \mathbb{P}^N$ is quadratic, see Corollary 1.6, [IR2, Theorem 2.4 and §4] and also Proposition 1.8 here. Then we recall some results about lines on prime Fano manifolds to illustrate further how geometric properties of $X \subset \mathbb{P}^N$ are transferred to $\mathcal{L}_{x,X} \subset \mathbb{P}((t_xX)^*)$, see Proposition 1.7 and Example 1.10.

In §2 we consider the classical problem of the existence of projective extensions $X \subset \mathbb{P}^{N+1}$ of a subvariety $Y \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$. It is well known that some special manifolds cannot be hyperplane sections of smooth varieties and that in some cases only the trivial extensions exist. These are given by cones over Y with vertex a point $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ (see e.g. [Se], [S1], [S2], [Te] and also §2 for precise definitions). Recently the interest in the above problem (and further generalizations of it) was renewed. Complete references, many results and a lot of interesting connections with other areas, such as deformation theory of isolated singularities, can be found in the monograph [Bă], especially relevant for this problem being Chapters 1 and 5.

Many sufficient conditions for the non-existence of non-trivial extensions of smooth varieties are known. These conditions are usually expressed, in the more general setting of extensions as ample divisors, by the vanishing of (infinitely many) cohomology groups of the twisted tangent bundle of Y (or of its normal bundle in $\mathbb{P}^N$). These results are general and concern a lot of applications, see *loc. cit.*, but even in the simplest cases the computation of these cohomology groups can be quite complicated. In any case their geometrical meaning is not so obvious to the non-expert in the field.

Here we prove a simple geometrical sufficient condition for non-extendability, Theorem 2.3, for smooth projective complex varieties uniruled by lines. The simplest version states that $Y \subset \mathbb{P}^N$ admits only trivial extensions $X \subset \mathbb{P}^{N+1}$ as soon as $\mathcal{L}_{y,X} \subset \mathbb{P}((t_yX)^*)$ admits no smooth extension (a weaker condition than the thesis!). Indeed, one easily shows in Proposition 2.2, via the results of §1, that also $\mathcal{L}_{y,X} \subset \mathbb{P}((t_yX)^*)$ is a projective extension of $\mathcal{L}_{y,Y} \subset \mathbb{P}((t_yY)^*)$ for $y \in Y$ general. Then under the hypothesis of Theorem 2.3 one deduces the existence of a line through y and a singular point $p_y \in X$. Then $p_y = p$ does not vary with $y \in Y$ general since X has at most a finite number of singular points so that $X \subset \mathbb{P}^{N+1}$ is a cone of vertex p. The range of applications of Theorem 2.3 is quite wide, see Corollary 2.4, 2.5, 2.7, allowing us to recover some results previously obtained by more sophisticated methods.

We were led to the analysis of the problem of extending smooth varieties by the desire of understanding geometrically why in some well-known examples the geometry of $Y \subset \mathbb{P}^N$ forces that every extension is trivial and by the curiosity of explicitly constructing the cones extending Y. Moreover, this approach reveals that Scorza's result about the non-extendability of $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$ for $a+b \geq 3$, originally proved in [S2] and recovered later by many authors (see e.g. [Bă] and Corollary 2.4 here), implies the non-extendability of a lot of homogeneous varieties via the description of their Hilbert scheme of lines. From this perspective the Plücker embedding of $Y = \mathbb{G}(r,m)$, with $1 \leq r < m-1$ and for $r=1$ with $m \geq 4$, admits only trivial extensions because $\mathcal{L}_{y,Y} = \mathbb{P}^r \times \mathbb{P}^{m-r-1}$ admits only trivial extensions (see [DFF] for an ad-hoc proof following Scorza's approach). Besides the applications contained in Corollary 2.4 and 2.5, we also show that our analysis can be used to provide a direct proof that $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ admits only trivial extensions, see Proposition 2.6, a well-known classical fact originally proved by Scorza in [S1].

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1. Geometry of (the Hilbert scheme of) lines contained in a variety and passing through a (general) point

1.1. **Notation, definitions and preliminary results.** Let $X \subset \mathbb{P}^N$ be a (non-degenerate) connected equidimensional projective variety of dimension $n \geq 1$, defined over a fixed algebraically closed field of characteristic zero, which from now on will be simply called a *projective variety*. If X is smooth and irreducible, we shall call X a *manifold*. Let $X_{\mathrm{reg}} = X \setminus \mathrm{Sing}(X)$ be the smooth locus of X. Let $t_x X$ denote *the affine tangent space to X at x*, let $T_x X \subset \mathbb{P}^N$ denote *the projective tangent space to X at x of $X \subset \mathbb{P}^N$* and for an arbitrary scheme Z and for a closed point $z \in Z$ let $C_z Z$ denote *the affine tangent cone to Z at z*. Let $\mathcal{L}_{x,X}$ denote the Hilbert scheme of lines contained in X and passing through the point $x \in X$. For a line $L \subset X$ passing through x, we let $[L] \in \mathcal{L}_{x,X}$ be the corresponding point.

Let $\pi_x : \mathcal{H}_x \to \mathcal{L}_{x,X}$ denote the universal family and let $\phi_x : \mathcal{H}_x \to X$ be the tautological morphism. From now on we shall always suppose that $x \in X_{\mathrm{reg}}$. Note that π_x admits a section $s_x : \mathcal{L}_{x,X} \to \mathcal{E}_x \subset \mathcal{H}_x$, which is contracted by ϕ_x to the point x. Consider the blowing-up $\sigma_x : \mathrm{Bl}_x X \to X$ of X at x. For every $[L] \in \mathcal{L}_{x,X}$ the line $L = \phi_x(\pi_x^{-1}([L]))$ is smooth at x so that [IN, Lemma 4.3] and the universal property of the blowing-up ensure the existence of a morphism $\psi_x : \mathcal{H}_x \to \mathrm{Bl}_x X$ such that $\sigma_x \circ \psi_x = \phi_x$. So we have the following diagram

$$
\begin{tikzcd}
\mathcal{H}_x \arrow[r, "\psi_x"] \arrow[d, "\pi_x"'] \arrow[dr, "\phi_x"] & \mathrm{Bl}_x X \arrow[d, "\sigma_x"] \\
\mathcal{L}_{x,X} & X.
\end{tikzcd}
\tag{1.1}
$$

In particular, ψ_x maps the section $\mathcal{E}_x$ to E_x, the exceptional divisor of σ_x. Let $\widetilde{\psi}_x : \mathcal{E}_x \to E_x$ be the restriction of ψ_x to $\mathcal{E}_x$. We can define the morphism

$$
\tau_x = \tau_{x,X} = \widetilde{\psi}_x \circ s_x : \mathcal{L}_{x,X} \to \mathbb{P}((t_x X)^*) = E_x = \mathbb{P}^{n-1}, \tag{1.2}
$$

which associates to each line $[L] \in \mathcal{L}_{x,X}$ the corresponding tangent direction through x, i.e. $\tau_x([L]) = \mathbb{P}((t_x L)^*)$. The morphism τ_x is clearly injective and we claim that τ_x is a closed immersion. Indeed, by taking in the previous construction $X = \mathbb{P}^N$ the corresponding morphism $\tau_{x,\mathbb{P}^N} : \mathcal{L}_{x,\mathbb{P}^N} \to \mathbb{P}((t_x\mathbb{P}^N)^*) = \mathbb{P}^{N-1}$ is an isomorphism between $\mathcal{L}_{x,\mathbb{P}^N}$ and the exceptional divisor of $\mathrm{Bl}_x \mathbb{P}^N$. By definition the inclusion $X \subset \mathbb{P}^N$ induces a closed embedding $i_x : \mathcal{L}_{x,X} \to \mathcal{L}_{x,\mathbb{P}^N}$. If $j_x : \mathbb{P}((t_x X)^*) \to \mathbb{P}((t_x\mathbb{P}^N)^*)$ is the natural closed embedding, then we have the following commutative diagram

$$\begin{tikzcd} \mathcal{L}_{x,X} \arrow[r, "\tau_{x,X}"] \arrow[d, "i_x"'] & \mathbb{P}((t_xX)^*) \arrow[d, "j_x"] \\ \mathcal{L}_{x,\mathbb{P}^N} \arrow[r, "\tau_{x,\mathbb{P}^N}"] & \mathbb{P}((t_x\mathbb{P}^N)^*), \end{tikzcd} \tag{1.3}$$

proving the claim.

For $x \in X_{\mathrm{reg}}$ such that $\mathcal{L}_{x,X} \neq \emptyset$, we shall always identify $\mathcal{L}_{x,X}$ with $\tau_x(\mathcal{L}_{x,X})$ and we shall naturally consider $\mathcal{L}_{x,X}$ as a subscheme of $\mathbb{P}^{n-1} = \mathbb{P}((t_xX)^*)$. We denote by $\mathcal{C}_x$ the scheme theoretic image of $\mathcal{H}_x$, that is $\phi_x(\mathcal{H}_x) = \mathcal{C}_x \subset X$. Via (1.1) we deduce the following relation:

$$\mathbb{P}(C_x(\mathcal{C}_x)) = \mathcal{L}_{x,X}, \tag{1.4}$$

as subschemes of $\mathbb{P}((t_xX)^*)$, where $\mathbb{P}(C_x(\mathcal{C}_x))$ is the *projectivized tangent cone to* $\mathcal{C}_x$ *at* x, see [Mu, II,§3].

1.2. **Singularities of** $\mathcal{L}_{x,X}$**.** We begin by studying the intrinsic geometry of $\mathcal{L}_{x,X} \subset \mathbb{P}^{n-1}$. When it is clear from the context which variety $X \subset \mathbb{P}^N$ we are considering we shall write $\mathcal{L}_x$ instead of $\mathcal{L}_{x,X}$.

If $L \subset X_{\mathrm{reg}}$, then $N_{L/X}$ is locally free of rank $n-1$ and more precisely

$$N_{L/X} \simeq \oplus_{i=1}^{n-1}\mathcal{O}_{\mathbb{P}^1}(a_i), \tag{1.5}$$

with $a_i \leq 1$ because $N_{L/X}$ is a subsheaf of $N_{L/\mathbb{P}^N} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{N-1}$.

If $N_{L/X}$ is also generated by global sections, then

$$N_{L/X} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{s(L,X)} \oplus \mathcal{O}_{\mathbb{P}^1}^{n-1-s(L,X)}. \tag{1.6}$$

Therefore if $N_{L/X}$ is generated by global sections, then $\mathcal{L}_x$ is unobstructed at $[L]$, that is $h^1(N_{L/X}(-1)) = 0$, $\mathcal{L}_x$ is smooth at $[L]$ and $\dim_{[L]}(\mathcal{L}_x) = h^0(N_{L/X}(-1)) = s(L,X)$, where $s(L,X) \geq 0$ is the integer defined in (1.6).

For $x \in X_{\mathrm{reg}}$, let

$$S_x = S_{x,X} = \{[L] \in \mathcal{L}_x \text{ such that } L \cap \mathrm{Sing}(X) \neq \emptyset\ \} \subseteq \mathcal{L}_x.$$

Then $S_{x,X}$ has a natural scheme structure and the previous inclusion holds at the scheme theoretic level. If X is smooth, then $S_{x,X} = \emptyset$. Moreover, if $L \subset X$ is a line passing through $x \in X_{\mathrm{reg}}$, clearly $[L] \notin S_{x,X}$ if and only if $L \subset X_{\mathrm{reg}}$.

We now prove that a singular point of $\mathcal{L}_x$ produces a line passing through x and through a singular point of X, a stronger condition than the mere existence of a singular point on X. These results are well known to experts, at least for manifolds, see [Hw, Proposition 1.5] and also [Ru, Proposition 2.2]. In [DG], the singularities of the Hilbert scheme of lines contained in a projective variety are related to some geometrical properties of the variety.

Proposition 1.1. *Let notation be as above and let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension $n \geq 2$. Then for $x \in X_{\mathrm{reg}}$ general:*

(1) *$\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth outside $S_{x,X}$, that is $\mathrm{Sing}(\mathcal{L}_x) \subseteq S_x$. In particular if $X \subset \mathbb{P}^N$ is smooth and if $x \in X$ is general, then $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a smooth variety.*

(2) *If $\mathcal{L}_x^j$, $j = 1, \ldots, m$, are the irreducible components of $\mathcal{L}_x$ and if*

$$\dim(\mathcal{L}_x^l) + \dim(\mathcal{L}_x^p) \geq n-1 \quad \textit{for some } l \neq p,$$

then $\mathcal{L}_x$ is singular, X is singular and there exists a line $[L] \in \mathcal{L}_x$ such that $L \cap \mathrm{Sing}(X) \neq \emptyset$.

Proof. There exists an open dense subset $U \subseteq X_{\rm reg}$ such that for every line $L \subset X_{\rm reg}$ such that $L \cap U \neq \emptyset$ the normal bundle $N_{L/X}$ is generated by global sections, see for example [De, Proposition 4.14]. Combining this result with the above discussion, we deduce that for every $x \in U$ the variety $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth outside S_x, proving the first assertion.

The condition on the dimensions of two irreducible components of $\mathcal{L}_x$ in (2) ensures that these components have to intersect in $\mathbb{P}^{n-1}$. A point of intersection is a singular point of $\mathcal{L}_x \subset \mathbb{P}^{n-1}$. This forces X to be singular by the first part and also the existence of a line $[L] \in S_x$, which by definition cuts $\operatorname{Sing}(X)$. □

1.3. **Equations for $\mathcal{L}_{x,X} \subset \mathbb{P}((t_xX)^*)$.** We now follow and expand the treatment outlined in [IR2, Theorem 2.4] by looking at the equations defining $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ for $x \in X_{\rm reg}$.

Let $X = V(f_1, \ldots, f_m) \subset \mathbb{P}^N$ be a projective equidimensional connected variety, not necessarily irreducible, let $x \in X_{\rm reg}$, let $n = \dim(X)$ and let $c = \operatorname{codim}(X) = N - n$. Thus we are assuming that $X \subset \mathbb{P}^N$ is scheme theoretically the intersection of $m \geq 1$ hypersurfaces of degrees $d_1 \geq d_2 \geq \ldots \geq d_m \geq 2$. Moreover it is implicitly assumed that m is minimal, i.e. none of the hypersurfaces contains the intersection of the others. Define, following [IR2], the integer

$$d := \sum_{i=1}^{c} (d_i - 1) \geq c.$$

With these definitions $X \subset \mathbb{P}^N$ (or more generally a scheme $Z \subset \mathbb{P}^N$) is called *quadratic* if it is scheme theoretically an intersection of quadrics, which means that we can assume $d_1 = 2$. In particular $X \subset \mathbb{P}^N$ is quadratic if and only if $d = c$.

We can choose homogeneous coordinates $(x_0 : \ldots : x_N)$ on $\mathbb{P}^N$ such that $x = (1 : 0 : \ldots : 0)$, $T_xX = V(x_{n+1}, \ldots, x_N)$. Let $\mathbb{A}^N = \mathbb{P}^N \setminus V(x_0)$ with affine coordinates $(y_1, \ldots, y_N)$, that is $y_l = \frac{x_l}{x_0}$ for every $l = 1, \ldots, N$. Let $\widetilde{\mathbb{P}}^N = \operatorname{Bl}_x \mathbb{P}^N$ with exceptional divisor $E' \simeq \mathbb{P}((t_x\mathbb{P}^N)^*) = \mathbb{P}^{N-1}$ and let $\widetilde{X} = \operatorname{Bl}_x X$ with exceptional divisor $E = \mathbb{P}((t_xX)^*) = \mathbb{P}^{n-1}$. Looking at the graph of the projection from x onto $V(x_0)$ we can naturally identify the projectivization of $\mathbb{A}^N \setminus \mathbf{0} = \mathbb{A}^N \setminus x$ with E' and with the projective hyperplane $V(x_0) = \mathbb{P}^{N-1}$.

Let $f_i = f_i^1 + f_i^2 + \cdots + f_i^{d_i}$, with f_i^j homogeneous of degree j in the variables $(y_1, \ldots, y_N)$. So $f_1^1 = \ldots = f_m^1 = 0$ are the equations of $t_xX = T_xX \cap \mathbb{A}^N \subset \mathbb{A}^N$, which reduce to $y_{n+1} = \ldots = y_N = 0$ by the previous choice of coordinates, yielding

$$V(f_1^1, \cdots, f_m^1) = \mathbb{P}((t_xX)^*) \subset \mathbb{P}((t_x\mathbb{P}^N)^*) = \mathbb{P}^{N-1}.$$

With the previous identifications $\mathcal{L}_{x,\mathbb{P}^N} = E' = \mathbb{P}^{N-1} = \mathbb{P}((t_x\mathbb{P}^N)^*)$. We now write a set of equations defining $\mathcal{L}_x \subset E \subset E'$ as a subscheme of E' and of E. By definition $\mathbf{y} = (y_1 : \ldots : y_n)$ are homogeneous coordinates on $E \subset E'$. For every $j = 2, \ldots, m$ and for every $i = 1, \ldots, m$, let

$$\widetilde{f}_i^j(\mathbf{y}) = f_i^j(y_1, \ldots, y_n, 0, 0, \ldots, 0, 0).$$

Then we have that $\mathcal{L}_x \subset E'$ is the scheme

$$V(f_1^1, f_1^2, \cdots, f_1^{d_1}, \cdots, f_m^1, f_m^2, \cdots, f_m^{d_m}) \subset E',$$

while $\mathcal{L}_x \subset E$ is the scheme

$$V(\widetilde{f}_1^2, \cdots, \widetilde{f}_1^{d_1}, \cdots, \widetilde{f}_m^2, \cdots, \widetilde{f}_m^{d_m}), \tag{1.7}$$

so that it is scheme theoretically defined by at most $\sum_{i=1}^m (d_i - 1)$ equations.

The equations of $T_xX \cap X \cap \mathbb{A}^N = t_xX \cap X \cap \mathbb{A}^N$, as a subscheme of $\mathbb{A}^N$, are

$$V(f_1^1, \ldots, f_m^1, f_1^1 + f_1^2 + \cdots + f_1^{d_1}, \ldots, f_m^1 + f_m^2 + \cdots + f_m^{d_m}) =$$

$$V(f_1^1, \ldots, f_m^1, f_1^2 + \cdots + f_1^{d_1}, \ldots, f_m^2 + \cdots + f_m^{d_m}) \subset \mathbb{A}^N. \tag{1.8}$$

Thus the equations of $T_xX \cap X \cap \mathbb{A}^N = t_xX \cap X \cap \mathbb{A}^N$ as a subscheme of $t_x(X \cap \mathbb{A}^N) = t_xX$ are

$$V(\widetilde{f}_1^2 + \cdots + \widetilde{f}_1^{d_1}, \ldots, \widetilde{f}_m^2 + \cdots + \widetilde{f}_m^{d_m}) \subset t_xX = \mathbb{A}^n. \tag{1.9}$$

Let $I = \langle \widetilde{f}_1^2 + \cdots + \widetilde{f}_1^{d_1}, \ldots, \widetilde{f}_m^2 + \cdots + \widetilde{f}_m^{d_m} \rangle \subset \mathbb{C}[y_1, \ldots, y_n] = S$ and let I^* be the ideal generated by the *initial forms* of elements of I. Remark that if I is homogeneous and generated by forms of the same degree, then clearly $I = I^*$. Then the affine tangent cone to $T_xX \cap X$ at x is $C_x(T_xX \cap X) = \operatorname{Spec}(\frac{S}{I^*})$ so that

$$\mathbb{P}(C_x(T_xX \cap X)) = \operatorname{Proj}(\frac{S}{I^*}), \tag{1.10}$$

see [Mu, III, § 3].

Let $J \subset S$ be the homogeneous ideal generated by the polynomials in (1.7) defining $\mathcal{L}_{x,X}$ scheme theoretically, that is $\mathcal{L}_{x,X} = \operatorname{Proj}(\frac{S}{J}) \subset \mathbb{P}((t_xX)^*)$. Clearly $I^* \subseteq J$, yielding the closed embedding of schemes

$$\mathcal{L}_{x,X} \subseteq \mathbb{P}(C_x(T_xX \cap X)). \tag{1.11}$$

If $X \subset \mathbb{P}^N$ is quadratic, then $I = I^* = J$. In conclusion we have proved the following results.

Proposition 1.2. *Let $X \subset \mathbb{P}^N$ be a (non-degenerate) projective variety, let $x \in X_{\mathrm{reg}}$ be a point and let notation be as above. If $X \subset \mathbb{P}^N$ is quadratic, then*

$$T_xX \cap X \cap \mathbb{A}^N = C_x(T_xX \cap X) \subset t_xX \tag{1.12}$$

and

$$\mathcal{L}_{x,X} = \mathbb{P}(C_x(T_xX \cap X)) \subset \mathbb{P}((t_xX)^*). \tag{1.13}$$

In particular if $X \subset \mathbb{P}^N$ is quadratic, then the scheme $\mathcal{L}_{x,X} \subset \mathbb{P}((t_xX)^*)$ is quadratic.

1.4. **$\mathcal{C}_x$ versus $T_xX \cap X$ for a quadratic variety.** The closed embedding (1.11) holds at the scheme theoretic level. If $\mathcal{L}_{x,X}$ were reduced, or better smooth, it would be enough to prove that there exists an inclusion as sets. Since $x \in X_{\mathrm{reg}}$ was arbitrary we cannot control a priori the structure of $\mathcal{L}_{x,X}$ even if $X \subset \mathbb{P}^N$ is a manifold. Recall that by Proposition 1.1 $\mathcal{L}_{x,X}$ is smooth as soon as X is a manifold and $x \in X$ is a general point.

From now on we shall suppose $X \subset \mathbb{P}^N$ quadratic. Then

(1) $(\mathcal{C}_x)_{\mathrm{red}} = (T_xX \cap X)_{\mathrm{red}}$;
(2) if $X \subset \mathbb{P}^N$ is a manifold and if $x \in X$ is a general point, then $\mathcal{C}_x = (T_xX \cap X)_{\mathrm{red}}$;
(3) the strict transforms of $\mathcal{C}_x$ and of $T_xX \cap X$ on $\operatorname{Bl}_x X$ cut the exceptional divisor $E = \mathbb{P}((t_xX)^*)$ of $\operatorname{Bl}_x X$ in the same scheme $\mathcal{L}_{x,X}$ (see (1.4) and (1.13));
(4) if $x \in X$ is a general point on a quadratic manifold $X \subset \mathbb{P}^N$ and if I^* is saturated, then $T_xX \cap X$ is reduced in a neighborhood of x so that it coincides with $\mathcal{C}_x$ in a neighborhood of x. Indeed since $T_xX \cap X \cap \mathbb{A}^n = \operatorname{Spec}(\frac{S}{I})$, with $I = I^* = J$ homogeneous and saturated, it follows that $T_xX \cap X$ is reduced at x; therefore it is reduced also in a neighborhood of x, usually smaller than $X \cap \mathbb{A}^N$, as shown in Example 1.4 below.

Already for quadratic manifolds there exist many important differences between $\mathbb{P}(C_x(T_xX \cap X)) \subset \mathbb{P}((t_xX)^*)$ and $C_x(T_xX \cap X) = T_xX \cap X \cap \mathbb{A}^N \subset t_xX$ and also between $T_xX \cap X$ and the cone $\mathcal{C}_x \subseteq T_xX \cap X$. We shall discuss some examples in order to analyze closer these important schemes containing a lot of geometrical information.

Example 1.3. ($T_xX \cap X$ *non-reduced only at* x) Remark that $t_x(T_xX \cap X) = t_xX$ so that $\langle C_x(T_xX \cap X) \rangle = t_xX$, while in some cases $\mathbb{P}(C_x(T_xX \cap X))$ is degenerate in $\mathbb{P}((t_xX)^*)$. Consider a rational normal scroll $X \subset \mathbb{P}^N$, different from the Segre varieties $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, $n \geq 2$, and a general point $x \in X$. It is well known that $X \subset \mathbb{P}^N$ is quadratic so that $\mathcal{L}_{x,X} = \mathbb{P}(C_x(T_xX \cap X)) \subset \mathbb{P}((t_xX)^*)$ by (1.11). On the other hand, if

$\mathbb{P}^{n-1}_x$ is the unique $\mathbb{P}^{n-1}$ of the ruling passing through $x \in X$, it is easy to see, letting notation as above, that in this case

$$\mathcal{L}_{x,X} = \mathbb{P}(\mathbb{P}^{n-1}_x \cap \mathbb{A}^n) = \mathbb{P}^{n-2} \subset \mathbb{P}((t_x X)^*) = \mathbb{P}^{n-1}.$$

This is possible because in this example $T_xX \cap X$ and $C_x(T_xX \cap X)$ are not reduced at x. Indeed, the point $x \in C_x(T_xX \cap X)$ corresponds to the irrelevant ideal of S. I^* is not saturated, because the equation defining the hyperplane $\mathcal{L}_{x,X}$ belongs to the saturation of I^*, but is not in I^* (I^* is generated by quadratic polynomials!).

Also for quadratic manifolds for which I^* is saturated, the scheme $T_xX \cap X$ can be quite mysterious outside x, and hence different, as a scheme, from the reduced cone $\mathcal{C}_x \subseteq T_xX \cap X$. In any case, they essentially coincide outside x in a neighborhood of x inside t_xX, and their projective tangent cones are the same at x by (1.13) and (1.1) (see also (1.2)). Thus, when I^* is saturated, the cones of vertex x $T_xX \cap X$ and $\mathcal{C}_x$ coincide modulo some embedded components, possibly appearing in $T_xX \cap X$, outside x.

Example 1.4. (*Embedded components in $T_xX \cap X$ outside x but not at x*). Let $X \subset \mathbb{P}^N$ be a quadratic manifold. Keeping notation as above let $Q_i = V(f_i) \subset \mathbb{P}^N$. Then $T_xX = \cap_{i=1}^m T_xQ_i$ and $X = \cap_{i=1}^m Q_i$ as schemes. Thus $T_xX \cap X = T_xX \cap (\cap_{i=1}^m (T_xQ_i \cap Q_i))$. Each *true* quadric $T_xQ_i \cap Q_i \cap T_xX \subset T_xX$ (that is we are considering only the quadrics Q_i not containing T_xX) is singular at x so that it can be considered also as a quadric in $\mathbb{P}((t_xX)^*)$, yielding the homogeneous ideal $I = I^*$.

Since $X \subset \mathbb{P}^N$ is quadratic, there exist c quadrics, let us say $Q_1, \ldots, Q_c$ such that $T_xX = \cap_{j=1}^c T_xQ_j$. Thus we can write the scheme $T_xX \cap X$ as:

$$T_xX \cap X = T_xX \cap (\cap_{j=1}^c (T_xQ_j \cap Q_j)) \cap (Q_{c+1} \cap \ldots \cap Q_m)$$

Now the $r \leq c$ effective quadrics in T_xX given by $T_xQ_l \cap Q_l \cap T_xX = T_xX \cap Q_l$, $l = 1, \ldots, c$ are cones with vertex at x and naturally projectivize. What could be somehow surprising is that these r quadrics are sufficient to define $\mathcal{L}_{x,X}$ scheme theoretically (or also $\mathcal{C}_x$, at least locally around x), while $T_xX \cap (Q_{c+1} \cap \ldots \cap Q_m)$ could define some embedded component, not necessarily supported at x. Let us consider an explicit example, communicated to me by Paltin Ionescu, revealing the subtle nature of the above picture and of what we shall prove in the next section.

Let $X = \mathbb{G}(1,4) = Q_1 \cap \ldots \cap Q_5 \subset \mathbb{P}^9$. Since $X \subset \mathbb{P}^9$ is scheme theoretically defined by these quadrics, fixing (a general) $x \in X$, every hyperplane $H \subset \mathbb{P}^N$ tangent at x, that is $H \supset T_xX$, is of the form T_xQ, with Q a quadric vanishing on X and belonging to the linear span of $Q_1, \ldots, Q_5$. It is well known that $\mathcal{L}_{x,X} = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ so that $\mathcal{C}_x = \phi_x(H_x) \subset X$ is a degree 3 reduced subscheme contained in $T_xX \cap X$ such that $\mathcal{C}_x = (T_xX \cap X)_{\mathrm{red}}$. In particular $\dim(T_xX \cap X) = 4$ and since I^* is saturated $T_xX \cap X$ and $\mathcal{C}_x$ coincide in a neighborhood of x.

We can suppose that $H_i = T_xQ_i$, $i = 1, 2, 3$, are three general hyperplanes containing T_xX so that $H_1 \cap H_2 \cap H_3 = T_xX$ and $H_1 \cap H_2 \cap H_3 \cap X = T_xX \cap X$ as schemes. Since $\mathbb{G}(1,4)$ is self-dual, $H_1 \cap X$ is singular along a $\mathbb{P}^2_x \subset T_xX \cap X$; the scheme $H_1 \cap H_2 \cap X$ will be singular also along a line $L_{1,2}$ passing through x so that, at least in a neighborhood U of x, $H_1 \cap H_2 \cap H_3 \cap X$ is singular only at x and it will coincide with $\mathcal{C}_x$ in a neighborhood $V \subseteq U$ of x. In particular we have a complete description of the singularity of $T_xX \cap X$ at x. In general it is very difficult to control the singularities of $T_xX \cap X$ outside x and also in this case there are more singular points of a very particular nature.

Indeed, $H_1 \cap H_2 = \mathbb{P}^7 \supset T_xX = \mathbb{P}^6$. We claim that $W_x = H_1 \cap H_2 \cap X = \mathcal{C}_x \cup Q_x$, where Q_x is a four dimensional quadric hypersurface cutting $\mathcal{C}_x$ along a quadric $\overline{Q}_x$ not passing through x. Then the hyperplane $T_xX \subset H_1 \cap H_2$ cuts W_x in the scheme $T_xX \cap X = \mathcal{C}_x \cup \overline{Q}_x$, with $\overline{Q}_x \subset \mathcal{C}_x$, showing that $\overline{Q}_x$ is an embedded component of $T_xX \cap X$.

In the case of rational normal scrolls discussed in Example 1.3 we saw that $T_xX \cap X \setminus x = \mathcal{C}_x \setminus x$ as schemes, the affine tangent cones are different affine schemes, but the projectivized tangent cones coincide.

By choosing suitable quadrics $Q_1, \ldots, Q_c$ we shall see in subsection 1.6 that the complete intersection $Y = Q_1 \cap \ldots \cap Q_c$ coincides locally with X around x. Thus $T_xY \cap Y$ and $T_xX \cap X$ coincide locally around x. In particular the intersection of their strict transform on $\mathrm{Bl}_x X$ with the exceptional divisor is the same, so that $\mathcal{L}_{x,X} = \mathcal{L}_{x,Y}$ and the last scheme can be defined scheme theoretically by $r \leq c$ linearly independent quadrics by (1.7).

In any case the double nature of $T_xX \cap X$ as a subscheme of T_xX and X plays a central role for its infinitesimal properties at x, measured exactly by $\mathbb{P}(C_x(T_xX \cap X)) \subset \mathbb{P}((t_xX)^*)$.

It is useful to think of $\mathbb{P}(C_x(T_xX \cap X)) \subset \mathbb{P}((t_xX)^*)$ as being the base locus scheme of the projection of X from T_xX, as we shall do in the next section. We shall provide in this way another reason why $\mathcal{L}_{x,X}$ can be defined scheme theoretically by at most c quadratic equations for an arbitrary point $x \in X_{\mathrm{reg}}$.

1.5. **Tangential projection and second fundamental form.** There are several possible equivalent definitions of the projective second fundamental form $|II_{x,X}| \subseteq \mathbb{P}(S^2(t_xX))$ of a connected equidimensional projective variety $X \subset \mathbb{P}^N$ at $x \in X_{\mathrm{reg}}$, see for example [IL, 3.2 and end of Section 3.5]. We use the one related to tangential projections, as in [IL, Remark 3.2.11].

Suppose $X \subset \mathbb{P}^N$ is non-degenerate, as always, let $x \in X_{\mathrm{reg}}$ and consider the projection from T_xX onto a disjoint $\mathbb{P}^{c-1}$

$$\pi_x : X \dashrightarrow W_x \subseteq \mathbb{P}^{c-1}. \tag{1.14}$$

The map π_x is not defined along the scheme $T_xX \cap X$, which contains x, and it is associated to the linear system of hyperplane sections cut out by hyperplanes containing T_xX, or equivalently by the hyperplane sections singular at x.

Let $\phi : \mathrm{Bl}_x X \to X$ be the blow-up of X at x, let

$$E = \mathbb{P}((t_xX)^*) = \mathbb{P}^{n-1} \subset \mathrm{Bl}_x X$$

be the exceptional divisor and let H be a hyperplane section of $X \subset \mathbb{P}^N$. The induced rational map $\widetilde{\pi}_x : \mathrm{Bl}_x X \dashrightarrow \mathbb{P}^{c-1}$ is defined as a rational map along E since $X \subset \mathbb{P}^N$ is not a linear space, see also the discussion below.

The restriction of $\widetilde{\pi}_x$ to E is given by a linear system in $|\phi^*(H) - 2E|_{|E} \subseteq |-2E_{|E}| = |\mathcal{O}_{\mathbb{P}((t_xX)^*)}(2)| = \mathbb{P}(S^2(t_xX))$, whose base locus scheme will be denoted by $B_{x,X}$.

Consider the strict transform scheme of $T_xX \cap X$ on $\mathrm{Bl}_x X$, denoted from now on by $\widetilde{T} = \mathrm{Bl}_x(T_xX \cap X)$. Then $\widetilde{T}$ is the base locus scheme of $\widetilde{\pi}_x$ and the restriction of $\widetilde{\pi}_x$ to E has base locus scheme equal to

$$\widetilde{T} \cap E = \mathbb{P}(C_x(T_xX \cap X)) = B_{x,X} \subset \mathbb{P}((t_xX)^*). \tag{1.15}$$

Definition 1.5. The *second fundamental form* $|II_{x,X}| \subseteq \mathbb{P}(S^2(t_xX))$ of a connected equidimensional non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension $n \geq 2$ at a point $x \in X_{\mathrm{reg}}$ is the non-empty linear system of quadric hypersurfaces in $\mathbb{P}((t_xX)^*)$ defining the restriction of $\widetilde{\pi}_x$ to E and $B_{x,X} \subset \mathbb{P}((t_xX)^*)$ is the so called *base locus scheme of the second fundamental form of X at x.*

Clearly $\dim(|II_{x,X}|) \leq c-1$ and $\widetilde{\pi}_x(E) \subseteq W_x \subseteq \mathbb{P}^{c-1}$. Let $\widetilde{I} \subset S$ be the homogeneous ideal generated by the $r \leq c$ linearly independent quadratic forms in the second fundamental form of X at x. Then via (1.15) we obtain

$$\mathrm{Proj}(\frac{\mathrm{S}}{\widetilde{\mathrm{I}}}) = B_{x,X} = \mathbb{P}(C_x(T_xX \cap X)) = \mathrm{Proj}(\frac{\mathrm{S}}{\mathrm{I}^*}) \subset \mathbb{P}((t_xX)^*). \tag{1.16}$$

In conclusion we have proved the following results by combining (1.15) with (1.13) and (1.16).

Corollary 1.6. *Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety, let $x \in X_{\mathrm{reg}}$ be a point and let notation be as above. Then:*

(1) $\mathcal{L}_{x,X} \subseteq B_{x,X}$;

(2) *if $X \subset \mathbb{P}^N$ is quadratic, then equality holds and $\mathcal{L}_{x,X} \subset \mathbb{P}((t_xX)^*)$ can be defined scheme theoretically by the $r \leq c$ quadratic equations defining the second fundamental form of X at x.*

Remark 1.7. The previous result has many important applications. We recall that, as proved in [IR2], if $X \subset \mathbb{P}^N$ is a quadratic manifold and if $c \leq \frac{n-1}{2}$, then, for $x \in X$ general, $\mathcal{L}_{x,X} \subset \mathbb{P}((t_xX)^*)$ is the complete intersection of the c linearly independent quadratic polynomials defining $|II_{x,X}|$. Then $\mathcal{L}_{x,X}$ has dimension $n-1-c$ from which it follows that $X \subset \mathbb{P}^N$ is a complete intersection. This proves the Hartshorne Conjecture on complete intersections in the quadratic case and also leads to the classification of quadratic Hartshorne manifolds, see [IR2, Theorem 2.4 and Section 4] for details.

The paper [PR1] considers also irreducible projective varieties $X \subset \mathbb{P}^{2n+1}$ which are 3–covered by twisted cubics, i.e. such that through three general points of $X \subset \mathbb{P}^{2n+1}$ there passes a twisted cubic contained in X. A key remark for the classification of these varieties is [PR1, Theorem 5.2], which among other things shows that for such an X the equality $\mathcal{L}_{x,X} = B_{x,X}$ holds for $x \in X$ general. A posteriori all the known examples of varieties 3–covered by twisted cubics are projectively equivalent to the so called *twisted cubics over Jordan algebras*, which are quadratic, see *loc. cit* for definitions and details and also [PR2] for a proof. This fact has also many important consequences for the theory of Jordan algebras and for the classification of *quadro-quadric* Cremona transformations, as shown in the forthcoming paper [PR2].

1.6. **Approach to $B_{x,X} = \mathcal{L}_{x,X}$ via** [BEL]**.** For manifolds $X \subset \mathbb{P}^N$ there is another approach based on a construction of [BEL] elaborating and generalizing an idea due to Severi, see *loc. cit.* It can be used to give a proof of a weaker form of Corollary 1.6 (in the sense that we prove it only for $x \in X$ general); this approach illustrates the local nature of the second fundamental form and the phenomena described in Example 1.4. Let us remark that the treatment in the general setting developed in the previous sections is unavoidable because the point $x \in X$ is not necessarily general on the complete intersection $Y \supseteq X$ we now construct.

It was proved in [BEL] that given a manifold $X = V(f_1, \ldots, f_m) \subset \mathbb{P}^N$ as above, we can choose $g_i \in H^0(\mathcal{I}_X(d_i))$, $i = 1, \ldots, c$ such that

$$Y = V(g_1, \ldots, g_c) = X \cup X', \tag{1.17}$$

where X' (if nonempty) meets X in a divisor D. Moreover from (1.17) it follows

$$\mathcal{O}_X(D) \simeq \det(\frac{\mathcal{I}_X}{\mathcal{I}_X^2}) \otimes \mathcal{O}_X(\sum_{i=1}^{c} d_i) \simeq \mathcal{O}_X(d-n-1) \otimes \omega_X^*, \tag{1.18}$$

see also [BEL, pg. 597]. We now illustrate the usefulness of this construction by proving some facts and results contained in [IR2, Theorem 2.4].

Suppose that $X \subset \mathbb{P}^N$ is a quadratic manifold and consider a point $x \in U = X \setminus \mathrm{Supp}(D)$. By definition $Y \setminus \mathrm{Supp}(D) = U \amalg V$, where $V = X' \setminus \mathrm{Supp}(D)$. Consider the two schemes $T_xX \cap X \cap U$ and $T_xY \cap Y \cap U$. Since $t_xX = t_xY$ and since $Y \cap U = X \cap U$ by the above construction, we obtain the equality as schemes

$$C_x(T_xX \cap X) = C_x(T_xX \cap X \cap U) = C_x(T_xY \cap Y \cap U) = C_x(T_xY \cap Y).$$

Via (1.13) we deduce the following equality as subschemes of $\mathbb{P}((t_xX)^*)$:

$$\mathcal{L}_{x,Y} = \mathbb{P}(C_x(T_xY \cap Y)) = \mathbb{P}(C_x(T_xX \cap X)) = \mathcal{L}_{x,X}. \tag{1.19}$$

Since $\mathcal{L}_{x,Y}$ can be scheme-theoretically defined by $r \leq c$ linearly independent quadratic equations, the same is true for $\mathcal{L}_{x,X}$. Now, without assuming anymore that X is quadratic, since $x \in X$ is general, $\mathcal{L}_{x,X}$ is smooth and hence reduced. Clearly a line L passing through x is contained in X if and only if it is contained in Y, yielding $\mathcal{L}_{x,X} = (\mathcal{L}_{x,Y})_{\mathrm{red}}$, see [IR2, Theorem 2.4]. We proved:

Proposition 1.8. *Let $X \subset \mathbb{P}^N$ be a manifold, let notation be as above and let $x \in U$ be a general point. Then:*

(1) $\mathcal{L}_{x,X} = (\mathcal{L}_{x,Y})_{\mathrm{red}}$ *so that* $\mathcal{L}_{x,X}$ *can be defined set theoretically by the* $r \leq d$ *equations defining* $\mathcal{L}_{x,Y}$ *scheme theoretically. In particular, if* $d \leq n-1$, *then* $\mathcal{L}_{x,X} \neq \emptyset$.
(2) *If* $X \subset \mathbb{P}^N$ *is quadratic, then* $\mathcal{L}_{x,X} = \mathcal{L}_{x,Y}$ *so that* $\mathcal{L}_{x,X} \subset \mathbb{P}((t_xX)^*)$ *is a quadratic manifold defined scheme theoretically by at most* c *quadratic equations.*

1.7. **Lines on prime Fano manifolds.** Let $X \subset \mathbb{P}^N$ be a (non-degenerate) manifold of dimension $n \geq 2$. For a general point $x \in X$ we know that $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth, Proposition 1.1. There are well-known examples when $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is not irreducible, such as $X = \mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$ Segre embedded, and also examples where $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is degenerate, see Example 1.3 and also table (2.5) below. A relevant class of manifolds where the properties of smoothness, irreducibility and non-degeneracy of $X \subset \mathbb{P}^N$ are transfered to $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ consists of prime Fano manifolds of high index, which we now define.

A manifold $X \subset \mathbb{P}^N$ is called a *prime Fano manifold* if $-K_X$ is ample and if $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle\mathcal{O}(1)\rangle$. The *index of* X is the positive integer defined by $-K_X = i(X)H$, with H a hyperplane section of $X \subset \mathbb{P}^N$.

Let us recall some fundamental facts. Part (1) below is well known and follows from the previous discussion except for a fundamental Theorem of Mori which implies that for prime Fano manifolds of index greater than $\frac{n+1}{2}$, necessarily $\mathcal{L}_x \neq \emptyset$, see [Mo] and [Ko, Theorem V.1.6].

Proposition 1.9. *Let* $X \subset \mathbb{P}^N$ *be a projective manifold and let* $x \in X$ *be a general point. Then*
(1) *If* $\mathcal{L}_x \neq \emptyset$, *then for every* $[L] \in \mathcal{L}_x$ *we have* $\dim_{[L]}(\mathcal{L}_x) = -K_X \cdot L - 2$. *In particular for prime Fano manifolds of index* $i(X) \geq \frac{n+3}{2}$ *the variety* $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ *is irreducible (and in particular non-empty!).*
(2) ([Hw]) *If* $X \subset \mathbb{P}^N$ *is a prime Fano manifold of index* $i(X) \geq \frac{n+3}{2}$, *then* $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ *is a non-degenerate manifold of dimension* $i(X)-2$.

Let us finish this section by looking at another significant example in which meaningful geometrical properties of $X \subset \mathbb{P}^N$ are reflected in similar properties of $\mathcal{L}_x \subset \mathbb{P}^{n-1}$, when this is non-empty.

Example 1.10. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of type $(d_1, d_2, \ldots, d_c)$ with $d_c \geq 2$. Then:
- if $n+1-d > 0$, then X is a Fano manifold and $i(X) = n+1-d$;
- if $n \geq 3$, then $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle\mathcal{O}(1)\rangle$;
- if $i(X) \geq 2$, then $\mathcal{L}_x \neq \emptyset$ and for every $[L] \in \mathcal{L}_x$ we have
$$\dim_{[L]}(\mathcal{L}_x) = (-K_X \cdot L) - 2 = i(X) - 2 = n-1-d \geq 0,$$
so that $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a smooth complete intersection of type
$$(2, \ldots, d_1; 2, \ldots, d_2; \ldots; 2, \ldots d_{c-1}; 2, \ldots, d_c)$$
since it is scheme theoretically defined by the d equations in (1.7).

2. A condition for non-extendability

Definition 2.1. Let us consider $H = \mathbb{P}^N$ as a hyperplane in $\mathbb{P}^{N+1}$. Let $Y \subset \mathbb{P}^N = H$ be a smooth (non-degenerate) irreducible variety of dimension $n \geq 1$. An irreducible variety $X \subset \mathbb{P}^{N+1}$ will be called *an extension of* Y if

(1) $\dim(X) = \dim(Y) + 1$;

(2) $Y = X \cap H$ as a scheme.

For every $p \in \mathbb{P}^{N+1} \setminus H$, the irreducible cone
$$X = S(p, Y) = \bigcup_{y \in Y} < p, y > \subset \mathbb{P}^{N+1}$$

is an extension of $Y \subset \mathbb{P}^N = H$, which will be called *trivial*. Let us observe that for any extension $X \subset \mathbb{P}^{N+1}$ of $Y \subset \mathbb{P}^N$ we necessarily have $\#(\mathrm{Sing}(X)) < \infty$ since X is smooth along the very ample divisor $Y = X \cap H$. We also remark that in our definition Y is a fixed hyperplane section. In the classical approach usually it was required that H was a general hyperplane section of X, see for example [S1]. Under these more restrictive hypotheses one can always suppose that a general point on Y is also a general point on X.

2.1. **Extensions of $\mathcal{L}_{x,Y} \subset \mathbb{P}^{n-1}$ via $\mathcal{L}_{x,X} \subset \mathbb{P}^n$.** Let $y \in Y$ be a general point and let us consider an extension $X \subset \mathbb{P}^{N+1}$ of Y and an irreducible component $\mathcal{L}^j_{y,Y}$ of $\mathcal{L}_{y,Y} \subset \mathbb{P}^{n-1}$, which is a smooth irreducible variety by Proposition 1.1. The results of §1 yield that this property is immediately translated in terms of Hilbert schemes of lines. Indeed we deduce the following result, where part (4) requires an ad hoc proof since in our hypotheses the point $y \in Y$ is general on Y, but not necessarily on X, so that we cannot apply Proposition 1.1.

Proposition 2.2. *Let $X \subset \mathbb{P}^{N+1}$ be an irreducible projective variety which is an extension of the non-degenerate manifold $Y \subset \mathbb{P}^N$. Let $n = \dim(Y) \geq 1$ and let $y \in Y$ be an arbitrary point such that $\mathcal{L}_{y,Y} \neq \emptyset$. Then:*

(1) $\mathcal{L}_{y,X} \cap \mathbb{P}((t_yY)^*) = \mathcal{L}_{y,Y}$ *as schemes.*
(2) *if $y \in Y$ is general, then $\dim_{[L]}(\mathcal{L}_{y,X}) = \dim_{[L]}(\mathcal{L}_{y,Y}) + 1$ and $[L]$ is a smooth point of $\mathcal{L}_{y,X}$ for every $[L] \in \mathcal{L}_{y,Y}$.*
(3) *if $y \in Y$ is general and if $\mathcal{L}^j_{y,Y}$ is an irreducible component of positive dimension, then there exists an irreducible component $\mathcal{L}^j_{y,X}$ such that $\mathcal{L}^j_{y,Y} = \mathcal{L}^j_{y,X} \cap \mathbb{P}((t_yY)^*)$ as schemes.*
(4) *If $y \in Y$ is general, then $\mathrm{Sing}(\mathcal{L}_{y,X}) \subseteq S_{y,X}$.*

Proof. Let $Y = X \cap H$, with $H = \mathbb{P}^N \subset \mathbb{P}^{N+1}$ a hyperplane and let notation be as in subsection 1.3. The conclusion in (1) immediately follows from (1.7).

Let us pass to (2) and consider an arbitrary line $[L] \in \mathcal{L}^j_{y,Y}$, an irreducible component of the smooth not necessarily irreducible variety $\mathcal{L}_{y,Y}$. We have an exact sequence of normal bundles

$$0 \to N_{L/Y} \to N_{L/X} \to N_{Y/X|L} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \to 0. \tag{2.1}$$

Since $y \in Y$ is general, $N_{L/Y}$ is generated by global sections, see the proof of Proposition 1.1, so that (1.6) yields

$$N_{L/X} \simeq N_{L/Y} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{s(L,Y)+1} \oplus \mathcal{O}_{\mathbb{P}^1}^{n-s(L,Y)-1}. \tag{2.2}$$

Thus also $N_{L/X}$ is generated by global sections, $\mathcal{L}_{y,X}$ is smooth at $[L]$ and $\dim_{[L]}(\mathcal{L}_{y,X}) = \dim_{[L]}(\mathcal{L}_{y,Y}) + 1$, proving (2).

Therefore if $y \in Y$ is general, there exists a unique irreducible component of $\mathcal{L}_{y,X} \subset \mathbb{P}((t_yX)^*)$, let us say $\mathcal{L}^j_{y,X}$, containing $[L]$ and by the previous calculation $\dim(\mathcal{L}^j_{y,X}) = s(L,Y) + 1 = \dim(\mathcal{L}^j_{y,Y}) + 1$. Recall that by part (1) we have $t_{[L]}\mathcal{L}_{y,Y} = t_{[L]}\mathcal{L}_{y,X} \cap \mathbb{P}((t_yY)^*)$ so that

$$\mathcal{L}^j_{y,Y} \subseteq \mathcal{L}^j_{y,X} \cap \mathbb{P}((t_yY)^*) \subseteq \mathcal{L}_{y,Y} \subset \mathbb{P}^{n-1} = \mathbb{P}((t_yY)^*), \tag{2.3}$$

yielding that $\mathcal{L}^j_{y,Y}$ is an irreducible component of $\mathcal{L}^j_{y,X} \cap \mathbb{P}((t_yY)^*)$ as well as an irreducible component of the smooth variety $\mathcal{L}_{y,Y}$. Hence, if $\dim(\mathcal{L}^j_{y,Y}) \geq 1$, we have the equality $\mathcal{L}^j_{y,Y} = \mathcal{L}^j_{y,X} \cap \mathbb{P}((t_yY)^*)$ as schemes, i.e. under this hypothesis $\mathcal{L}^j_{y,X} \subset \mathbb{P}((t_yX)^*)$ (or better $(\mathcal{L}^j_{y,X})_{\mathrm{red}}$) is a projective extension of the smooth positive dimensional irreducible variety $\mathcal{L}^j_{y,Y} \subset \mathbb{P}((t_xY)^*)$. Indeed, $\dim(\mathcal{L}^j_{y,Y}) \geq 1$ forces $\dim(\mathcal{L}^j_{y,X}) \geq 2$ so that it is sufficient to recall that $\mathcal{L}_{y,X}$ is smooth along $\mathcal{L}_{y,Y}$ by the previous discussion and also that an arbitrary hyperplane section of the irreducible variety $(\mathcal{L}^j_{y,X})_{\mathrm{red}}$ is connected by the Fulton-Hansen Theorem, [FH]. More precisely, if $\dim(\mathcal{L}^j_{y,Y}) \geq 1$, then equality as schemes holds in (2.3), proving part (3).

By [De, Proposition 4.9] there exists a non-empty open subset $U \subseteq X$ such that $N_{\widetilde{L}/X}$ is generated by global sections for every line $\widetilde{L} \subset X_{\rm reg}$ intersecting U. If $U \cap Y \neq \emptyset$, then (4) clearly holds. Suppose $Y \cap U = \emptyset$. Let $[\widetilde{L}] \in \mathcal{L}_{y,X} \setminus S_{y,X}$. If $\widetilde{L} \cap U \neq \emptyset$, then $[\widetilde{L}]$ is a smooth point of $\mathcal{L}_{y,X}$ by the previous analysis. If $\widetilde{L} \cap U = \emptyset$, then $\widetilde{L} \subset Y$ by the generality of $y \in Y$ and $N_{\widetilde{L}/X}$ is generated by global sections by (2.2), concluding the proof of (4). $\square$

Now we are in position to prove the main result of this section and to deduce some applications.

Theorem 2.3. *Let notation be as above and let $y \in Y$ be a general point. Then:*

(1) *Suppose there exist two distinct irreducible components $\mathcal{L}^1_{y,X}$ and $\mathcal{L}^2_{y,X}$ of $\mathcal{L}_{y,X} \subset \mathbb{P}((t_yX)^*)$, extending two irreducible components $\mathcal{L}^1_{y,Y}$, respectively $\mathcal{L}^2_{y,Y}$, of $\mathcal{L}_{y,Y}$ in the sense specified above. If $\mathcal{L}^1_{y,X} \cap \mathcal{L}^2_{y,X} \neq \emptyset$, then $X \subset \mathbb{P}^{N+1}$ is a cone over $Y \subset \mathbb{P}^N$ of vertex a point $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$.*
(2) *If $\mathcal{L}_{y,Y} \subset \mathbb{P}((t_yY)^*)$ is a manifold whose extensions are singular, then every extension of $Y \subset \mathbb{P}^N$ is trivial.*

Proof. By the above discussion, we get that in both cases, for $y \in Y$ general, the variety $S_{y,X} \subseteq \mathcal{L}_{y,X}$ is not empty so that for $y \in Y$ general there exists a line $L_y \subseteq X$ passing through y and through a singular point $p_y \in L_y \cap \mathrm{Sing}(X)$. Since Y is irreducible and since $\mathrm{Sing}(X)$ consists of a finite number of points, there exists $p \in \mathrm{Sing}(X)$ such that $p \in L_y$ for $y \in Y$ general. This implies that $X = S(p, Y)$ is a cone over Y with vertex p. $\square$

The first easy consequence is a result due to Scorza (see [S2] and also [Bă]), proved by him under the stronger assumption that $Y = X \cap H$ is a general hyperplane section of X. Under these more restrictive hypotheses, the analysis before the proof of Theorem 2.3 could be simplified via Proposition 1.1, since we may assume that the general point $y \in Y$ is also general on X.

Corollary 2.4. *Let $1 \leq a \leq b$ be integers, let $n = a + b \geq 3$ and let $Y \subset \mathbb{P}^{ab+a+b}$ be a smooth irreducible variety projectively equivalent to the Segre embedding $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$. Then every extension of Y in $\mathbb{P}^{ab+a+b+1}$ is trivial.*

Proof. For $y \in Y$ general, it is well known that $\mathcal{L}_{y,Y} = \mathcal{L}^1_{y,Y} \amalg \mathcal{L}^2_{y,Y} \subset \mathbb{P}^{a+b-1} = \mathbb{P}^{n-1}$ with $\mathcal{L}^1_{y,Y} = \mathbb{P}^{a-1}$ and $\mathcal{L}^2_{y,Y} = \mathbb{P}^{b-1}$, both linearly embedded. Observe that $b - 1 \geq 1$. By (2.3) and the discussion following it, there exist two irreducible components $\mathcal{L}^j_{y,X}$, $j = 1, 2$, of $\mathcal{L}_{y,X} \subset \mathbb{P}^n = \mathbb{P}^{a+b}$ with $\dim(\mathcal{L}^1_{y,X}) = a$ and $\dim(\mathcal{L}^2_{y,X}) = b$. If $a \neq b$ then clearly $\mathcal{L}^1_{y,X} \neq \mathcal{L}^2_{y,X}$. If $a = b \geq 2$, then $\mathcal{L}^1_{y,X} \neq \mathcal{L}^2_{y,X}$ because an arbitrary hyperplane section of a variety of dimension at least 2 is connected, see [FH]. Since $a + b = n$, $\mathcal{L}^1_{y,X} \cap \mathcal{L}^2_{y,X} \neq \emptyset$ and the conclusion follows from the first part of Theorem 2.3. $\square$

The previous result has some interesting consequences via iterated applications of the second part of Theorem 2.3. Indeed, let us consider the following homogeneous varieties (also known as irreducible hermitian symmetric spaces), in their homogeneous embedding, and the description of the Hilbert scheme of lines passing through a general point, see [Hw, §1.4.5]. Smooth extensions of homogeneous manifolds were also considered in [Wa], see also [Kn].

(2.4)

	Y	$\mathcal{L}_{y,Y}$	$\tau_y : \mathcal{L}_{y,Y} \to \mathbb{P}((t_yY)^*)$
1	$\mathbb{G}(r, m)$	$\mathbb{P}^r \times \mathbb{P}^{m-r-1}$	Segre embedding
2	$SO(2r)/U(r)$	$\mathbb{G}(1, r-1)$	Plücker embedding
3	E_6	$SO(10)/U(5)$	miminal embedding
4	$E_7/E_6 \times U(1)$	E_6	Severi embedding
5	$Sp(r)/U(r)$	$\mathbb{P}^{r-1}$	quadratic Veronese embedding

There are also the following homogeneous contact manifolds with Picard number one associated to a complex simple Lie algebra $\mathbf{g}$, whose Hilbert scheme of lines passing through a general point is known. Let us observe that in these examples the variety $\mathcal{L}_{y,Y} \subset \mathbb{P}^{n-1} = \mathbb{P}((t_yY)^*)$ is degenerate and its linear span is exactly $\mathbb{P}((D_y)^*) = \mathbb{P}^{n-2}$, there D_y is the tangent space at y of the distribution associated to the contact structure on Y, i.e. there is the following factorization $\tau_y : \mathcal{L}_{y,Y} \to \mathbb{P}((D_y)^*) \subset \mathbb{P}((t_yY)^*)$. For more details one can consult [Hw, §1.4.6].

(2.5)

	$\mathbf{g}$	$\mathcal{L}_{y,Y}$	$\tau_x : \mathcal{L}_{x,Y} \to \mathbb{P}((D_y)^*$
6	F_4	$Sp(3)/U(3)$	Segre embedding
7	E_6	$\mathbb{G}(2,5)$	Plücker embedding
8	E_7	$SO(12)/U(6)$	minimal embedding
9	E_8	$E_7/E_6 \times U(1)$	minimal embedding
10	$\mathbf{s}o_{m+4}$	$\mathbb{P}^1 \times Q^{m-2}$	Segre embedding

By case 1') we shall denote a variety as in 1) of (2.4) satisfying the following numerical conditions: $r < m-1$; if $r = 1$, then $m \geq 4$. By 2') we shall denote a variety as in 2) with $r \geq 5$.

Corollary 2.5. *Let $Y \subset \mathbb{P}^N$ be a manifold as in Examples 1'), 2'), 3), 4), 7), 8), 9) above. Then every extension of Y is trivial.*

Proof. In cases 2'), 3), 4) and 9) in the statement the variety $\mathcal{L}_{y,Y} \subset \mathbb{P}^{n-1}$ of one example is the variety $Y \subset \mathbb{P}^N$ occurring in the next one. Thus for these cases, by the second part of Theorem 2.3, it is sufficient to prove the result for case 1'). For this variety the conclusion follows from Corollary 2.4. For the remaining cases, the variety $\mathcal{L}_{y,Y} \subset \mathbb{P}^{n-1}$ is either as in case 1') with $(r,m) = (2,5)$ or as in case 2) with $r = 6$ and the conclusion follows once again by the second part of Theorem 2.3. □

The next result is also classical and well-known but we provide a direct geometric proof. Under the assumption that the hyperplane section $H \cap X = Y$ is general, it was proved by C. Segre for $n = 2$ in [Se] and by Scorza in [S1], see also [Te], for arbitrary $n \geq 2$ (and also for arbitrary Veronese embeddings $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{N(d)}$, with $n \geq 2$ and $d \geq 2$).

Proposition 2.6. *Let $n \geq 2$ and let $Y \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ be a manifold projectively equivalent to the quadratic Veronese embedding $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$. Then every extension of Y is trivial.*

Proof. Let $y \in Y$ be a general point and let $N = \frac{n(n+3)}{2}$. Since $\mathcal{L}_{y,Y} = \emptyset$, then $\mathcal{L}_{y,X} \subset \mathbb{P}^n$, if not empty, consists of at most a finite number of points and through $y \in X$ there passes at most a finite number of lines contained in X. Consider a conic $C \subset Y$ passing through y. Then $N_{C/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{n-1}$. The exact sequence of normal bundles

$$0 \to N_{C/Y} \to N_{C/X} \to N_{Y/X|C} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \to 0,$$

yields

$$N_{C/X} \simeq N_{C/Y} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(2).$$

Thus there exists a unique irreducible component $\mathcal{C}_{y,X}$ of the Hilbert scheme of conics contained in $X \subset \mathbb{P}^{N+1}$ passing through $y \in X$ to which $[C]$ belongs. Moreover $\dim(\mathcal{C}_{y,X}) = n+1$ and the conics parametrized by $\mathcal{C}_{y,X}$ cover X. Hence there exists a one dimensional family of conics through y and a general point $x \in X$. By Bend and Break, see for example [De, Proposition 3.2], there is at least a singular conic through y and x. Since $X \subset \mathbb{P}^{N+1}$ is not a linear space, there exists no line joining y and a general x, i. e. the singular conics through x and y are reduced. Thus given a general point x in X, there exists a line $L_x \subset X$ through x, not passing through y, and a line $L_y \subset X$ through y such that $L_y \cap L_x \neq \emptyset$. Since there are a finite number of lines contained in X and passing through y, we can conclude that given a general point $x \in X$, there exists a fixed line passing through y, $\widetilde{L}_y$, and a line L_x through x such that $L_x \cap \widetilde{L}_y \neq \emptyset$.

Moreover, a general conic $[C_{x,y}] \in \mathcal{C}_{y,X}$ and passing through a general point x is irreducible, does not pass through the finite set $\mathrm{Sing}(X)$ and has ample normal bundle verifying $h^0(N_{C_{x,y}/X}(-1)) = h^0(N_{C/X}(-1)) = n+1$. This means that the deformations of $C_{x,y}$ keeping x fixed cover an open subset of X and also that through general points $x_1, x_2 \in X$ there passes a one dimensional family of irreducible conics. The plane spanned by one of these conics contains x_1 and x_2 so that it has to vary with the conic. Otherwise the fixed plane would be contained in X and $X \subset \mathbb{P}^{N+1}$ would be a linearly embedded $\mathbb{P}^{N+1}$, which is contrary to our assumptions. In conclusion through a general point $z \in < x_1, x_2 >$ there passes at least a one dimensional family of secant lines to X so that

$$\dim(SX) \leq 2(n+1) - 1 = 2n+1 < N+1 = \frac{n(n+3)}{2} + 1, \tag{2.6}$$

yielding $SX \subsetneq \mathbb{P}^{N+1}$.

Suppose the point $p_x = \widetilde{L}_y \cap L_x$, for $y \in Y$ general, varies on $\widetilde{L}_y$. Then the linear span of two general tangent spaces $T_{x_1}X$ and $T_{x_2}X$ would contain the line $\widetilde{L}_y$. Since $T_z SX =< T_{x_1}X, T_{x_2}X >$ by the Terracini Lemma, we deduce that a general tangent space to SX contains $\widetilde{L}_y$ and a fortiori y. Since $SX \subsetneq \mathbb{P}^{N+1}$, the variety $SX \subset \mathbb{P}^{N+1}$ would be a cone whose vertex, which is a linear space, contains $\widetilde{L}_y$ and a fortiori $y \in Y$. By the generality of $y \in Y$ we would deduce that $Y \subset \mathbb{P}^N$ is degenerate.

Thus $p_x = \widetilde{L}_y \cap L_x$ does not vary with $x \in X$ general. Let us denote this point by p. Then clearly $X \subset \mathbb{P}^{N+1}$ is a cone with vertex p over Y. □

Corollary 2.7. *Let $Y \subset \mathbb{P}^N$ be a manifold either as in 5) above with $r \geq 3$ or as in 6) above. Then every extension of Y is trivial.*

Proof. By (2.4) we know that in case 5) with $r \geq 3$ we have $n - 1 = \frac{(r-1)(r+2)}{2}$ and the variety $\mathcal{L}_{y,Y} \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\nu_2(\mathbb{P}^{r-1}) \subset \mathbb{P}^{\frac{(r-1)(r+2)}{2}}$. To conclude we apply Proposition 2.6 and the second part of Theorem 2.3. Case 6) follows from case 5) with $r = 3$ by the second part of Theorem 2.3. □

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