A thin-thick Decomposition for Hardy Martingales

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Abstract

We prove thin-thick decompositions, for the class of Hardy martingales and thereby strenghten its square function characterization. We apply the underlying method to several classical martinale inequalities, for which we give new proofs .

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Contents

1 Introduction

Let $\mathbb{T}^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty}\}$ denote the countable product of the torus $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$,equipped with its normalized Haar measure $\mathbb{P}$. A natural filtration of $\sigma-$ algebras on $\mathbb{T}^{\mathbb{N}}$ is given by the coordinate projections

$$P_k : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^k, \quad (x_i)_{i=1}^{\infty} \to (x_i)_{i=1}^{k}.$$

Define $\mathcal{F}_k$ to be the $\sigma-$ algebra on $\mathbb{T}^{\mathbb{N}}$ generated by P_k.

Let $F = (F_k)$ be an $L^1(\mathbb{T}^{\mathbb{N}})-$bounded martingale on the filtered probability space $(\mathbb{T}^{\mathbb{N}}, (\mathcal{F}_k), \mathbb{P})$. Conditioned on $\mathcal{F}_{k-1}$ the martingale differnce $\Delta F_k = F_k - F_{k-1}$ defines

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an element in the Lebesque space of integrable, function of vanishing mean $L^1_0(\mathbb{T})$. By definition the martingale $F = (F_k)$ belongs to the class of Hardy martingales, if, conditioned on $\mathcal{F}_{k-1}$,

$$\Delta F_k = F_k - F_{k-1} \quad \text{defines an element in the Hardy space } H^1_0(\mathbb{T}).$$

Hardy martingales, introduced by Garling [9], arise throughout Complex and Functional Analysis. For instance in renorming problems for Banach spaces [5, 22], vector valued Littlewood Paley Theory [23], embedding problems [2], isomorphic classification problems [3, 17], factorization problems [19], similarity problems [20], boundary convergence of vector valued analytic functions [8, 10, 13, 12], Jensen measures [4, 1].

As pointed out by Garling [9], two robustness properties of Hardy martingales are particularly important for their use in Analysis.

1. The class of Hardy martingales is closed under martingale transforms.
2. For Hardy martingales, their L^1 norm is determined by square functions. There exist $c, C > 0$ so that for any Hardy martingale $F = (F_k)$,

$$c\mathbb{E}|F_n| \le \mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2} \le C\mathbb{E}|F_n|. \tag{1}$$

In the present paper we strengthen the square function characterization (1) for Hardy martingales. We prove that every Hardy martingale $F = (F_k)_{k=1}^n$ can be written as

$$F = G + B$$

where $G = (G_k)_{k=1}^n$ and $B = (B_k)_{k=1}^n$ are again Hardy martingales so that

$$\mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^{n} |\Delta B_k|) \le C\mathbb{E}|F_n|. \tag{2}$$

and

$$|\Delta G_k| \le A_0|F_{k-1}|, \qquad k \le n. \tag{3}$$

The estimate (2) implies of course the right hand side of the square function estimate (1) since the triangle inequality and the Burkholder-Gundy martingale inequality [11] give

$$\begin{aligned}\mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2} &\le \mathbb{E}(\sum_{k=1}^{n} |\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^{n} |\Delta B_k|) \\ &\le 2\mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^{n} |\Delta B_k|).\end{aligned}$$

The uniform previsible estimate (3) should be compared with uniform previsible estimates appearing in the classical Davis and Garsia inequality [11, Chapters III and IV]. (See also Section 4.3.) For general martingales the Davis decomposition [11, Chapter III] guarantees only uniform estimates by previsible *and increasing* functionals such as $\max_{m \le k-1} |F_m|$. Hence a routine application of the Davis decomposition could yield only

$$|\Delta G_k| \le A_0 \max_{m \le k-1} |F_m|.$$

The present paper exploits a basic and general iteration principle extracted from the work of J. Bourgain [2]. In its simplest form it yields a comparison theorem between square functions as follows: Assume that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are non- negative, integrable functions so that the following set of estimates hold true,

$$\mathbb{E}(\sum_{m=1}^{k-1} u_m^2 + v_k^2)^{1/2} \leq \mathbb{E}(\sum_{m=1}^{k} u_m^2)^{1/2}, \qquad k \leq n.$$

Then we have

$$\mathbb{E}(\sum_{m=1}^{n} v_m^2)^{1/2} \leq 2\mathbb{E}(\sum_{m=1}^{n} u_m^2)^{1/2}.$$

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2 Basic Iteration

In this section we review J. Bourgain's iteration method introduced in [2]. It provides upper estimates for the norm in $L^1(\ell^2)$. By its generality the iteration method can easily be adapted to a variety of different situations. In this paper we apply it to obtain proofs of four different martingale inequalities.

Consider first the elementary Lemma.

Lemma 2.1 *Let $0 \leq s \leq 1$, and $A, B \geq 0$. Then*

$$Bs \leq s^2 A + (A^2 + B^2)^{1/2} - A. \tag{4}$$

Proof. Since $0 \leq s \leq 1$, we have $1 - s^2 < (1 - s^2)^{1/2}$. Multiply by $A > 0$, add Bs and use the Cauchy-Schwartz inequality. This gives

$$A(1 - s^2) + Bs \leq A(1 - s^2)^{1/2} + Bs \leq (A^2 + B^2)^{1/2}.$$

Subtracting $A(1 - s^2)$ gives (4).

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Let $(\Omega, \mathbb{P})$ be a probability space and write $\mathbb{E}$ to denote expectation in $(\Omega, \mathbb{P})$.

Theorem 2.2 *Let $n \in \mathbb{N}$. Let $u_1, \ldots, u_n \in L^1(\Omega)$, and form the partial sums*

$$Z_k = \sum_{m=1}^{k} u_m, \qquad 1 \leq k \leq n.$$

Assume that $v_1, \ldots, v_n$, and $w_1, \ldots, w_n$ be non-negative in $L^1(\Omega)$, so that the following estimates hold

$$\mathbb{E}(|Z_{k-1}|^2 + v_k^2)^{1/2} + \mathbb{E}w_k \leq \mathbb{E}|Z_k| \quad \text{for } 1 \leq k \leq n. \tag{5}$$

Then

$$\mathbb{E}(\sum_{k=1}^{n} v_k^2)^{1/2} + \mathbb{E}\sum_{k=1}^{n} w_k \leq 2(\mathbb{E}|Z_n|)^{1/2}(\mathbb{E}\max_{k \leq n}|Z_k|)^{1/2}. \tag{6}$$

PROOF. Let $0 \le \epsilon \le 1$ be defined by

$$\epsilon^2 = (\mathbb{E}|Z_n|)(\mathbb{E}\max_{k\le n}|Z_k|)^{-1}. \quad (7)$$

Choose next non negative $s_k \in L^\infty$ so that $\sum_{k=1}^n s_k^2 \le \epsilon^2$. Apply Lemma 2.1 with

$$A = |Z_{k-1}|,\ B = v_k \ \text{ and } \ s = s_k.$$

This yields the point-wise estimates

$$v_k s_k \le s_k^2|Z_{k-1}| + (|Z_{k-1}|^2 + v_k^2)^{1/2} - |Z_{k-1}| \quad (8)$$

Integrating the point-wise estimates (8) gives

$$\mathbb{E}(v_k s_k) \le \mathbb{E}(s_k^2|Z_{k-1}|) + \mathbb{E}(|Z_{k-1}|^2 + v_k^2)^{1/2} - \mathbb{E}|Z_{k-1}|^2. \quad (9)$$

Next apply the hypothesis (5) to the central term $\mathbb{E}(|Z_{k-1}|^2 + v_k^2)^{1/2}$ appearing in the integrated estimates (9). This gives

$$\mathbb{E}(v_k s_k) \le \mathbb{E}(s_k^2|Z_{k-1}|) + \mathbb{E}|Z_k| - \mathbb{E}|Z_{k-1}| - \mathbb{E}w_k. \quad (10)$$

Taking the sum over $k \le n$ and exploiting the telescoping nature of the right hand side of (10) yields,

$$\begin{aligned}\mathbb{E}(\sum_{k=1}^n v_k s_k) + \sum_{k=1}^n \mathbb{E}w_k &\le \mathbb{E}|Z_n| + \mathbb{E}(\sum_{k=1}^n s_k^2|Z_{k-1}|) \\ &\le \mathbb{E}|Z_n| + \epsilon^2\mathbb{E}\max_{k\le n}|Z_{k-1}|\end{aligned} \quad (11)$$

Since (11) holds for every choice of $s_k \in L^\infty$ such that $\sum_{k=1}^n s_k^2 \le \epsilon^2$, we may take the supremum and obtain, by duality, the square function estimate

$$\epsilon\mathbb{E}(\sum_{k=1}^n v_k^2)^{1/2} + \sum_{k=1}^n \mathbb{E}w_k \le \mathbb{E}|Z_n| + \epsilon^2\mathbb{E}\max_{k\le n}|Z_{k-1}|.$$

It remains to divide by $0 \le \epsilon \le 1$ and take into account (7). This gives

$$\begin{aligned}\mathbb{E}(\sum_{k=1}^n v_k^2)^{1/2} + \sum_{k=1}^n \mathbb{E}w_k &\le \epsilon^{-1}\mathbb{E}|Z_n| + \epsilon\mathbb{E}\max_{k\le n}|Z_{k-1}| \\ &= 2(\mathbb{E}|Z_n|)^{1/2}(\mathbb{E}\max_{k\le n}|Z_k|)^{1/2}.\end{aligned} \quad (12)$$

■

Theorem 2.2 gives estimates between plain integrals; in particular martingale structures are neither part of its hypothesis nor of its conclusion. Nevertheless in Section 3 we employ Theorem 2.2 to prove an inequality for Hardy martingales. We use it to estimate the L^1 norm of perturbed square functions by the L^1 norm of the martingale itself. In Section 4 we discuss classical martingale inequalties involving different forms of square functions. There we will use a version of Theorem 2.2 that is suitably adapted to estimating quadratic expressions.

3 Decomposing Hardy Martingales

In this section we state and prove the main theorems of this paper. In the first paragraphs we collect probabilistic results used later in the proof. We record here a stochastic proof of Bourgain's complex convexity inequality. This underlines the probabilistic nature of Theorem 3.3.

Hardy Spaces, Brownian Motion and Complex Convexity. Let $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$, equipped with its normalized Haar measure dm. For $h \in L^p(\mathbb{T})$ we say that h belongs to the Hardy space $H^p(\mathbb{T})$ if the harmonic extension of h to the unit disk is analytic. If moreover $\int_{\mathbb{T}} h dm = 0$ we write $h \in H^p_0(\mathbb{T})$. Recall J. Bourgain's complex convexity inequality [2]: There exists $\alpha_0 > 0$ so that

$$\int_{\mathbb{T}} (|z|^2 + \alpha_0^2 |h|^2)^{1/2} dm \le \int_{\mathbb{T}} |z + h| dm, \qquad z \in \mathbb{C}, \quad h \in H^1_0(\mathbb{T}). \tag{13}$$

M. Schmuckenschläger informed me that the Bourgain's proof [2] of (13) gives $\alpha_0^2 = 1/27$.

Let (B_t) denote complex 2D-Brownian motion on Wiener space, and $((\mathcal{F}_t), \mathbb{P})$, the associated filtered probability space. Put

$$\tau = \inf\{t > 0 : |B_t| > 1\}.$$

With the following proposition we verify Bourgain's complex convexity inequality. The proof uses Ito's formula and the Cauchy-Riemann equations.

Proposition 3.1 *Let $z \in \mathbb{C}$, $h \in H^1_0(\mathbb{T})$ and let $\rho \le \tau$ be a stopping time. Then for $\alpha^2 \le 1/6$*

$$\mathbb{E}(|z|^2 + \alpha^2 |h(B_\rho)|^2)^{1/2} \le \mathbb{E}|z + h(B_\rho)|. \tag{14}$$

Proof. We may put $|z| = 1$. By Ito's formula [7] we have the identities

$$\mathbb{E}(1 + \alpha^2 |h(B_\rho)|^2)^{1/2} = 1 + \frac{1}{2}\mathbb{E}\int_0^\rho \Delta((1 + \alpha^2|h(B_s)|^2)^{1/2}) ds,$$

and

$$\mathbb{E}(|1 + h(B_\rho)|) = 1 + \frac{1}{2}\mathbb{E}\int_0^\rho \Delta(|1 + h(B_s)|) ds.$$

We calculate the Laplacians and evaluate the integrands on the right hand side as follows

$$\Delta(|1 + h(z)|) = \frac{|h'(z)|^2}{|1 + h(z)|},$$

and

$$\Delta((1 + \alpha^2|h(z)|^2)^{1/2}) = \alpha^2 |h'(z)|^2 (2 + \alpha^2 |h(z)|^2)(1 + \alpha^2|h(z)|^2)^{-3/2}$$

An elementary calculation shows that for $\alpha^2 \le 1/6$,

$$\alpha^2 |1 + w|(2 + \alpha^2|w|^2) \le (1 + \alpha^2|w|^2)^{3/2}, \qquad w \in \mathbb{C}.$$

Hence for $\alpha^2 \le 1/6$,

$$\frac{\alpha^2(2 + \alpha^2|h(B_s)|^2)}{(1 + \alpha^2|h(B_s)|^2)^{3/2}} \le \frac{1}{|1 + h(B_s)|}. \tag{15}$$

Multiplying both sides of (15) by $|h'(B_s)|^2$ and integrating gives

$$\mathbb{E}\int_0^\rho \Delta((1 + \alpha^2|h(B_s)|^2)^{1/2}) ds \le \mathbb{E}\int_0^\rho \Delta(|1 + h(B_s)|) ds.$$

■

Remarks.

1. The above proof applies, mutatis mutandis, to Conformal Martingales on Wiener Space. Let X, Y be real-valued and integrable on Wiener space, $((\mathcal{F}_t), \mathbb{P})$. Assume X, Y have identical quadratic variation,and vanishing co-variance process,
$$\langle X\rangle_t - \langle Y\rangle_t = \langle X, Y\rangle_t = 0, \qquad t \geq 0.$$
The proof of Proposition 3.1 shows that for $Z = X + iY$, $w \in \mathbb{Z}$ and $\alpha^2 < 1/6$,
$$\mathbb{E}(|w|^2 + \alpha^2|Z|^2)^{1/2} \leq \mathbb{E}|w + Z|. \tag{16}$$

2. A short analytic proof of (13) was obtained by M. Schmuckenschläger who based his agrument on Green's identity in the following form:
$$\partial_r \int_0^{2\pi} \varphi(re^{it})dt = r^{-1} \iint_{D_r} \Delta\varphi(z)dA(z),$$
where $D_r = \{z \in \mathcal{C} : |z| \leq r\}$, Δ denotes the Laplacian and $dA(z)$ the area measure. Thus Green's formula replaces the use of Brownian Motion Ito's lemma.

Hardy Martingales. Let $\mathbb{T}^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty}\}$ denote the countable product of the torus $\mathbb{T}$ equipped with its product Haar measure.

Let $n \in \mathbb{N}$, and denote by $\mathcal{F}_n$ the the $\sigma-$algebra on $\mathbb{T}^{\mathbb{N}}$ generated by the cylinder sets
$$\{(A_1, \ldots, A_n, \mathbb{T}, \ldots, \mathbb{T}, \ldots,)\},$$
where A_i, $i \leq n$ are measurable subsets of $\mathbb{T}$. Thus $(\mathcal{F}_n)$ is an increasing sequence of $\sigma-$algebras canonically linked to the product structure of $\mathbb{T}^{\mathbb{N}}$. Subsequently we let $\mathbb{E}_n$ denote the conditional expectation with respect to the $\sigma-$algebra $\mathcal{F}_n$. Let $F = (F_n)$ be an $(\mathcal{F}_n)$ martingale in $L^1(\mathbb{T}^{\mathbb{N}})$. Denote its difference sequence by
$$\Delta F_n = F_n - F_{n-1}.$$
By definition $F = (F_n)$ is a Hardy martingale if for almost all $(x_1, \ldots, x_{n-1}) \in \mathbb{T}^{n-1}$ fixed, the function
$$y \to \Delta F_n(x_1, \ldots, x_{n-1}, y),$$
defines an element in $H_0^1(\mathbb{T})$.

As shown by J. Bourgain [2], the complex convexity inequality (13) combined with Theorem 2.2 yields the following square function estimate for Hardy martingales
$$\mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2} \leq 2\alpha_0^{-1}(\mathbb{E}|F_n|)^{1/2}(\mathbb{E}\max_{k\leq n} |F_k|)^{1/2}. \tag{17}$$
The B. Davis martingale inequality [11, p.37]
$$\mathbb{E}\max_{k\leq n} |F_k| \leq \sqrt{10}\mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2}, \tag{18}$$

and (17) imply that

$$\mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2} \le C_0 \mathbb{E}|F_n|, \tag{19}$$

with $C_0 = 4\times\alpha_0^{-2}\times\sqrt{10}$. The converse is a consequence of (18). With a different constant and by a different method, the estimate (19) was abtained by B. Garling [9].

We next state the main theorem of the present paper. It provides a thin-thick decomposition for Hardy martingales and strengthens the square function characterization (19).

Theorem 3.2 *Let* $A_0 = 4\alpha_0^{-1}$ *and* $C_1 = 4\alpha_0^{-1}\sqrt{10}\times A_0$. *Every Hardy martingale* $F = (F_k)_{k=1}^n$ *can be decomposed as*

$$F = G + B \tag{20}$$

where $G = (G_k)_{k=1}^n$ *and* $B = (B_k)_{k=1}^n$ *are again Hardy martingales so that the following holds:*

1. *Integral bounds:*

$$\mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^{n} |\Delta B_k|) \le C_1 \mathbb{E}|F_n|. \tag{21}$$

2. *Previsible uniform estimates:*

$$|\Delta G_k| \le A_0 |F_{k-1}|, \qquad k \le n. \tag{22}$$

Comments. We emphasize several points in which the decomposition for Hardy martingales given above is distinct from the classical Davis and Garsia inequality [11, Theorems III.3.5 and IV.4.3], holding for general martingales. We refer also to Section 4 where we give an alternative proof of the classical Davis and Garsia inequality based on the iteration method.

1. The right hand side of (21) involves just the L^1 norm $\mathbb{E}|F_n|$ and not the square function $\mathbb{E}(\sum_{k=1}^n |\Delta F_k|^2)^{1/2}$.

2. The decomposition (20) of F_k as

$$F_k = G_k + B_k,$$

yields *analytic* martingale differences ΔG_k and ΔB_k in the following sense. For $(x_1, \ldots, x_{k-1}) \in \mathbb{T}^{k-1}$ fixed the martingale difference

$$y \to \Delta G_k(x_1, \ldots, x_{k-1}, y),$$

defines an element in $H_0^\infty(\mathbb{T})$. Hence the decomposing martingales $G = (G_k)_{k=1}^n$ and $B = (B_k)_{k=1}^n$ are in fact Hardy martingales.

3. The right hand side of the previsible estimate (22) involves just the value of the martingale at time $k-1$, and *not* the entire history of the martingale *up to time* $k-1$. This fact reflects J.Bourgain's complex convexity inequality (13) and, apparently, does not follow from (19).

We obtain Theorem 3.2 from the basic iteration theorem using as input the following thin-thick decomposition in $H^1_0(\mathbb{T})$.

Theorem 3.3 *Let $A_0 = 4\alpha_0^{-1}$. To $h \in H^1_0(\mathbb{T})$ and $z \in \mathbb{C}$, put*

$$\rho = \inf\{t < \tau : |h(B_t)| > 2\alpha_0^{-1}|z|\}, \quad \text{and} \quad g(e^{i\theta}) = \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta}).$$

Then $g \in H^\infty_0(\mathbb{T})$ satisfies the integral bounds

$$(|z|^2 + A_0^{-2}\int_{\mathbb{T}} |g|^2 dm)^{1/2} + A_0^{-1}\int_{\mathbb{T}} |h-g|dm \le \int_{\mathbb{T}} |z+h|dm,$$

and the uniform estimate

$$|g| \le A_0|z|.$$

Comments.

1. In the course of proving the decomposition Theorem 3.3 we use stopping time arguments on the holomorphic martingale $h(B_t), t \le \tau$ together with J. Bourgain's complex convexity inequality (13).
2. We gave a proof of the complex convexity inequality (14) using Ito's formula and Cauchy Riemann equations. Hence the thin-thick decomposition of Theorem 3.3 has natural counterparts in pure stochastic settings.

In Lemma 3.4 we separate the stopping time argument from the rest of the proof. We use below the following observation of Varopoulos. Let $\rho \le \tau$ be a stopping time and let $h \in H^1_0(\mathbb{T})$. Then taking the expectation of $h(B_\rho)$ conditioned to $\{B_\tau = e^{i\theta}\}$ gives again an element in $H^1_0(\mathbb{T})$. Thus

$$\theta \to \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta})$$

is in $H^1_0(\mathbb{T})$. See [21].

Lemma 3.4 *Let $h \in H^1_0(\mathbb{T})$ and $M > 0$. Put*

$$\rho = \inf\{t < \tau : |h(B_t)| > 2M\}, \quad \text{and} \quad g(e^{i\theta}) = \mathbb{E}(h(B_\rho)|B_\tau = e^{i\theta}).$$

Then $g \in H^\infty_0(\mathbb{T})$ satisfies the integral estimate

$$(M^2 + \frac{1}{12}\int_{\mathbb{T}} |g|^2 dm)^{1/2} + \frac{1}{4}\int_{\mathbb{T}} |h-g|dm \le \int_{\mathbb{T}} (M^2 + |h|^2)^{1/2} dm, \tag{23}$$

and the uniform bound

$$|g| \le 2M. \tag{24}$$

PROOF. By homogeneity assume $M = 1$. Put

$$X = h(B_\tau), \quad X_0 = h(B_\rho), \quad X_1 = X - X_0,$$

and

$$A = \{\rho < \tau\}, \quad B = \Omega \setminus A.$$

Note first that $\mathbb{E}X_0 = \mathbb{E}X_1 = 0$, and

$$|X_0| \le 2, \qquad \mathrm{supp}(X_1) \subseteq A, \qquad \mathbb{E}|X_1| \le 2\mathbb{E}(1_A|X|). \tag{25}$$

Moreover by inspection,

$$\mathbb{E}(1_A|X_0|) \ge 2\mathbb{P}(A).$$

We will prove next that

$$\mathbb{E}(1 + |X|^2)^{1/2} \ge (1 + \frac{1}{12}\mathbb{E}|X_0|^2)^{1/2} + \frac{1}{4}\mathbb{E}|X_1|.$$

To this end we consider separately the contribution of the sets A and B to $\mathbb{E}(1+|X|^2)^{1/2}$ and verify the following two estimates

$$\mathbb{E}(1_A(1 + |X|^2)^{1/2}) \ge \mathbb{P}(A) + \frac{1}{4}\mathbb{E}|X_1|$$

and

$$\mathbb{E}(1_B(1 + |X|^2)^{1/2}) \ge \mathbb{P}(B) + \frac{1}{12}\mathbb{E}|X_0|^2.$$

Let $\mathcal{F}_\rho$ denote the stopping time $\sigma-$algebra generated by ρ. We may then rewrite

$$X_0 = \mathbb{E}(X|\mathcal{F}_\rho).$$

For $\omega \in A$, we have $|X_0(\omega)| \ge 2 1_A(\omega)$. Recall that $A = \{\rho < \tau\}$, hence A is $\mathcal{F}_\rho$ measurable, and

$$\mathbb{E}(1_A|X|) \ge \mathbb{E}(|1_A \mathbb{E}(X|\mathcal{F}_\rho)|) \ge 2\mathbb{P}(A). \tag{26}$$

Using (26) and (25) we get

$$\begin{aligned} \mathbb{E}(1_A(1 + |X|^2)^{1/2}) &\ge \mathbb{E}(1_A|X|) \\ &\ge \frac{1}{2}\mathbb{E}(1_A|X|) + \mathbb{P}(A) \ge \frac{1}{4}\mathbb{E}|X_1| + \mathbb{P}(A). \end{aligned} \tag{27}$$

Next for $\omega \in B$, $|X(\omega)| \le 2$. Recall next the elementary estimate $(1+x)^{1/2} \ge 1+x/3$ for $0 \le x \le 1$. Hence

$$(1 + |X(\omega)|^2)^{1/2} \ge (1 + \frac{1}{4}|X(\omega)|^2)^{1/2} \ge 1 + \frac{1}{12}|X(\omega)|^2, \qquad \omega \in B.$$

Next take expectations and use $\mathbb{E}|X^2| \ge \mathbb{E}|X_0^2|$ to obtain

$$\mathbb{E}(1_B(1 + |X|^2)^{1/2}) \ge \mathbb{E}(1_B(1 + \frac{1}{12}|X|^2)) \ge \mathbb{P}(B) + \frac{1}{12}\mathbb{E}|X_0|^2. \tag{28}$$

Add the estimates (27)and (28). This gives,

$$\begin{aligned}\mathbb{E}(1+|X|^2)^{1/2} &\geq \mathbb{P}(B) + \frac{1}{12}\mathbb{E}|X_0|^2 + \mathbb{P}(A) + \frac{1}{4}\mathbb{E}|X_1| \\ &\geq (1+\frac{1}{12}\mathbb{E}|X_0|^2)^{1/2} + \frac{1}{4}\mathbb{E}|X_1|.\end{aligned} \tag{29}$$

Finally we use the above bounds for X, X_1, X_0 on Wiener space to get estimates for

$$g(e^{i\theta}) = \mathbb{E}(X_0 | B_\tau = e^{i\theta}).$$

By a well known observation of Varopoulos [21] $g \in H^1_0(\mathbb{T})$. Moreover, since g is obtained by conditional expectation from X_0 we get

$$|g| \leq 2, \quad \int_{\mathbb{T}} |g|^2 dm \leq \mathbb{E}|X_0|^2, \quad \text{and} \quad \int_{\mathbb{T}} |h-g| dm \leq \mathbb{E}|X_1|$$

Since $X = h(B_\tau)$,

$$\int_{\mathbb{T}} (1+|h|^2)^{1/2} dm = \mathbb{E}(1+|X|^2)^{1/2}.$$

Combining with (29) gives

$$\begin{aligned}\int_{\mathbb{T}} (1+|h|^2)^{1/2} dm &\geq (1+\frac{1}{12}\mathbb{E}|X_0|^2)^{1/2} + \frac{1}{4}\mathbb{E}|X_1| \\ &\geq (1+\frac{1}{12}\int_{\mathbb{T}} |g|^2 dm)^{1/2} + \frac{1}{4}\int_{\mathbb{T}} |h-g| dm\end{aligned} \tag{30}$$

■

We next merge the conclusion of Lemma 3.4 with the complex convexity estimate (13).

Proof of Theorem 3.3. Apply Lemma 3.4 to $\alpha_0 h$ and $M = |z|$. This gives $g \in H^\infty(\mathbb{T})$ so that

$$|g| \leq 2\alpha_0^{-1}|z|,$$

and

$$(|z|^2 + 12\alpha_0^2 \int_{\mathbb{T}} |g|^2 dm)^{1/2} + 4\alpha_0 \int_{\mathbb{T}} |h-g| dm \leq \int_{\mathbb{T}} (|z|^2 + \alpha_0^2 |h|^2)^{1/2} dm. \tag{31}$$

It remains to invoke (13), asserting that the right hand side of (31) is bounded by

$$\int_{\mathbb{T}} |z+h| dm.$$

■

Finally we give the details of the proof of Theorem 3.2. We show how to apply Theorem 3.3 to obtain the thin-thick decomposition for Hardy martingales.

Proof of Theorem 3.2. Fix $k \le n$ and $(x_1, \dots, x_{k-1}) \in \mathbb{T}^{k-1}$. By assumtion the martingale difference

$$h(y) = \Delta F_k(x_1, \dots, x_{k-1}, y)$$

defines an element in $H_0^1(\mathbb{T})$. Put

$$z = F_{k-1}(x_1, \dots, x_{k-1}),$$

and apply Theorem 3.3 to h and z. This gives a decomposition

$$h = g + b,$$

with $g \in H_0^\infty(\mathbb{T})$, and $b \in H_0^1(\mathbb{T})$, so that

$$|g| \le A_0|z|,$$

and

$$(|z|^2 + A_0^{-2} \int_{\mathbb{T}} |g|^2 dm)^{1/2} + A_0^{-1} \int_{\mathbb{T}} |b| dm \le \int_{\mathbb{T}} |z + h| dm,$$

Define next with $(x_1, \dots, x_{k-1}) \in \mathbb{T}^{k-1}$ fixed above

$$\Delta G_k(x_1, \dots, x_{k-1}, y) = g(y), \qquad \text{and} \quad \Delta B_k(x_1, \dots, x_{k-1}, y) = b(y).$$

Then we get the identity $\Delta F_k = \Delta G_k + \Delta B_k$ and the estimates

$$|\Delta G_k| \le A_0 |F_{k-1}|$$

together with

$$(|F_{k-1}|^2 + A_0^{-2} \mathbb{E}_{k-1} |\Delta G_k|^2)^{1/2} + A_0^{-1} \mathbb{E}_{k-1} |\Delta B_k| \le \mathbb{E}_{k-1} |F_k|.$$

Taking expectations gives

$$\mathbb{E}(|F_{k-1}|^2 + A_0^{-2} \mathbb{E}_{k-1} |\Delta G_k|^2)^{1/2} + A_0^{-1} \mathbb{E} |\Delta B_k| \le \mathbb{E} |F_k|.$$

Now apply Theorem 2.2 with

$$u_k = \Delta F_k, \quad v_k = (\mathbb{E}_{k-1} |\Delta G_k|^2)^{1/2} / A_0 \qquad \text{and} \qquad w_k = |\Delta B_k| / A_0, \quad k \le n.$$

This gives the estimate

$$\mathbb{E}(\sum_{k=1}^n \mathbb{E}_{k-1} |\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^n |\Delta B_k|) \le 2A_0 (\mathbb{E}|F_n|)^{1/2} (\mathbb{E} \sup_{k \le n} |F_k|)^{1/2}. \tag{32}$$

Next use the inequalities of B. Davis (18) and Bourgain/Garling (19)

$$\frac{1}{\sqrt{10}} \mathbb{E} \sup_{k \le n} |F_k| \le \mathbb{E}(\sum_{k=1}^n |\Delta F_k|^2)^{1/2} \le C \mathbb{E} |F_n|, \tag{33}$$

where $C = 4\alpha_0^{-2} \times \sqrt{10}$. Inserting (33) into (32) gives (22).

■

4 Further Applications

We continue with applications of the iteration principle to classical martingale inequalities. We deduce the previsible projection theorem, the comparison theorem between square functions and conditional square functions, and prove the Davis and Garsia inequality.

We start with a variant of Theorem 2.2 that is adapted to bounding quadratic expressions. Let $(\Omega, \mathbb{P})$ be probability space and denote by $\mathbb{E}$ the expectation in $(\Omega, \mathbb{P})$.

Theorem 4.1 *Let $n \in \mathbb{N}$. Let $u_1, \ldots, u_n$ be non-negative in $L^1(\Omega)$, and*

$$M_k = (\sum_{m=1}^{k} u_m^2)^{1/2} \quad \textit{for } 1 \leq k \leq n.$$

Assume that $v_1, \ldots, v_n$, and $w_1, \ldots, w_n$ be non-negative in $L^1(\Omega)$, so that the following estimates hold

$$\mathbb{E}(M_{k-1}^2 + v_k^2)^{1/2} + \mathbb{E}w_k \leq \mathbb{E}M_k \quad \textit{for } 1 \leq k \leq n. \tag{34}$$

Then

$$\mathbb{E}(\sum_{k=1}^{n} v_k^2)^{1/2} + \mathbb{E}\sum_{k=1}^{n} w_k \leq 2\mathbb{E}(\sum_{k=1}^{n} u_k^2)^{1/2}. \tag{35}$$

Proof. Choose non negative $s_k \in L^\infty$ so that $\sum_{k=1}^{n} s_k^2 \leq 1$. Apply Lemma 2.1 with

$$A = M_{k-1}, \; B = v_k \; \text{ and } \; s = s_k.$$

This yields the pointwise estimates

$$v_k s_k \leq s_k^2 M_{k-1} + (M_{k-1}^2 + v_k^2)^{1/2} - M_{k-1} \tag{36}$$

Integrating the point-wise estimates (36) gives

$$\mathbb{E}(v_k s_k) \leq \mathbb{E}(s_k^2 M_{k-1}) + \mathbb{E}(M_{k-1}^2 + v_k^2)^{1/2} - \mathbb{E}M_{k-1}. \tag{37}$$

Next apply the hypothesis (34) to the central term $\mathbb{E}(M_{k-1}^2 + v_k^2)^{1/2}$ appearing in the integrated estimates (37). This gives

$$\mathbb{E}(v_k s_k) \leq \mathbb{E}(s_k^2 M_{k-1}) + \mathbb{E}M_k - \mathbb{E}M_{k-1} - \mathbb{E}w_k. \tag{38}$$

Taking the sum over $k \leq n$ and exploiting the telescoping nature of the right hand side of (38) yields,

$$\begin{aligned} \mathbb{E}(\sum_{k=1}^{n} v_k s_k) + \sum_{k=1}^{n} \mathbb{E}w_k &\leq \mathbb{E}M_n + \mathbb{E}(\sum_{k=1}^{n} s_k^2 M_{k-1}) \\ &\leq \mathbb{E}M_n + \mathbb{E}\max_{k\leq n} M_{k-1} \end{aligned} \tag{39}$$

Since (39) holds for every choice of $s_k \in L^\infty$ such that $\sum_{k=1}^{n} s_k^2 \leq 1$, we may take the supremum and obtain, by duality, the square function estimate

$$\mathbb{E}(\sum_{k=1}^{n} v_k^2)^{1/2} + \sum_{k=1}^{n} \mathbb{E}w_k \leq \mathbb{E}M_n + \mathbb{E}\max_{k\leq n} M_{k-1}.$$

It remains to use that clearly $M_n = \max_{k\leq n} M_k$.

■

4.1 Previsible Projections

Let $(\Omega, (\mathcal{F}_k), \mathbb{P})$, be filterd probability space. Integration in $(\Omega, \mathbb{P})$ is written as $\mathbb{E}$. Conditional expectation with respect to $\mathcal{F}_k$ is denoted by $\mathbb{E}_k$. Let (F_k) be a martingale in $(\Omega, (\mathcal{F}_k), \mathbb{P})$, and $\Delta F_k = F_k - F_{k-1}$.

We prove next square function estimates for the sequence of previsible projections $\mathbb{E}_{k-1}(|\Delta F_k|)$. With different methods the following result was obtained in [16, 2, 6] and [15, Section 5.6].

Proposition 4.2

$$\mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1}^2(|\Delta F_k|))^{1/2} \leq 2\mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2}.$$

Proof. Let $u : [0,1] \to \mathbb{C}$ be integrable and $M > 0$. We verify next the following elementary inequality,

$$(M^2 + (\int_0^1 |u(t)|dt)^2)^{1/2} \leq \int_0^1 (M^2 + |u(t)|^2)^{1/2} dt. \tag{40}$$

By normalization we may choose $M = 1$. Fix $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ so that

$$(1 + (\int_0^1 |u(t)|dt)^2)^{1/2} = a + b\int_0^1 |u(t)|dt$$

Now estimate simply as follows

$$\begin{aligned} a + b\int_0^1 |u(t)|dt &= \int_0^1 a + b|u(t)|dt \\ &\leq (a^2 + b^2)^{1/2} \int_0^1 (1 + |u(t)|^2)^{1/2} dt. \end{aligned}$$

Since $a^2 + b^2 = 1$ the estimate (40) is verified.

By the Theorem 4.1 the proof of Proposition 4.2 is now immediate. Fix $k \leq n$, and form the square function

$$M_k = (\sum_{m=1}^{k} |\Delta F_m|^2)^{1/2}.$$

An immediate application of (40) is,

$$(M_{k-1}^2 + \mathbb{E}_{k-1}(|\Delta F_k|^2))^{1/2} \leq \mathbb{E}_{k-1} M_k.$$

Taking expectations gives

$$\mathbb{E}(M_{k-1}^2 + \mathbb{E}_{k-1}(|\Delta F_k|^2))^{1/2} \leq \mathbb{E} M_k.$$

Now apply Theorem 4.1 with

$$u_k = |\Delta F_k|, \quad v_k = (\mathbb{E}_{k-1}(|\Delta F_k|^2))^{1/2} \quad \text{and} \quad w_k = 0,$$

to get the conclusion.

∎

4.2 Burkholder-Gundy Inequality

We prove the Burkholder-Gundy estimate, see [11, Theorem III.4.3] or [15, Section 5.6], comparing the martingale square function to the conditioned square functions.

Let (F_k) be an integrable martingale in a filtered probability space $(\Omega, (\mathcal{F}_k), \mathbb{P})$, with differences $\Delta F_k = F_k - F_{k-1}$.

Proposition 4.3

$$\mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2} \leq 2\mathbb{E}\left(\sum_{k=1}^{n} \mathbb{E}_{k-1}(|\Delta F_k|^2)\right)^{1/2}$$

Proof. Let $u : \mathbb{T} \to \mathbb{C}$ integrable and fix $M > 0$. By Minkowski's inequality,

$$\int_0^1 (M^2 + |u(t)|^2)^{1/2} dt \leq (M^2 + \int_0^1 |u(t)|^2 dt)^{1/2}. \tag{41}$$

By Theorem 4.1 we reduced Proposition 4.3 to (41). Indeed, fix $k \leq n$ and form the conditioned square function

$$M_k = (\sum_{m=1}^{k} \mathbb{E}_{m-1}(|\Delta F_m|^2))^{1/2}.$$

By (41),

$$\mathbb{E}_{k-1}(M_{k-1}^2 + |\Delta F_k|^2)^{1/2} \leq \mathbb{E}_{k-1} M_k.$$

Taking expectations gives

$$\mathbb{E}(M_{k-1}^2 + |\Delta F_k|^2)^{1/2} \leq \mathbb{E} M_k.$$

Use Theorem 4.1 with

$$u_k = (\mathbb{E}_{k-1}(|\Delta F_m|^2))^{1/2}, \quad v_k = |\Delta F_k|, \quad \text{and} \quad w_k = 0.$$

■

4.3 Davis and Garsia Inequality

Let $(\Omega, (\mathcal{F}_k), \mathbb{P})$ be a filtered probability space.

The martingale transform techniques of Garsia [11, Theorem IV.1.2], applied to the original Davis decomposition [11, Theorem III.3.5] of a martingale $F = (F_k)_{k=1}^n$ into a previsible part $G = (G_k)_{k=1}^n$ and $B = (B_k)_{k=1}^n$ gives the inequality

$$\mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1} |\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^{n} |\Delta B_k|) \leq C\mathbb{E}(\sum_{k=1}^{n} |\Delta F_k|^2)^{1/2}. \tag{42}$$

Thus the inequality (42) is a consequence of separate theorems due to Davis *and* Garsia respectively.

We proceed by giving a new proof of (42) based on Theorem 4.1 and a martingale thin-thick decomposition.

Theorem 4.4 *Every martingale* $F = (F_k)_{k=1}^n$ *in* L^1 *admits a decomposition as*

$$F = G + B$$

where $G = (G_k)_{k=1}^n$ *and* $B = (B_k)_{k=1}^n$ *are martingales so that the following holds:*

1. *Integral bounds:*

$$\mathbb{E}(\sum_{k=1}^n \mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^n |\Delta B_k|) \leq 8\mathbb{E}(\sum_{k=1}^n |\Delta F_k|^2)^{1/2}. \quad (43)$$

2. *Previsible uniform estimates:*

$$|\Delta G_k|^2 \leq 2\sum_{m=1}^{k-1} |\Delta F_m|^2, \qquad k \leq n. \quad (44)$$

Comments. Following are two remarks relating to the inequality (43) and to the nature of the uniform previsible estimates(44).

1. The lower estimate (43) is sharp in the following sense. *Any* martingale decomposition of $F = (F_k)_{k=1}^n$ as
$$F_k = G'_k + B'_k$$
gives rise to a reciprocal upper estimate. By the triangle inequality and the conditional square function estimate Theorem 4.3,
$$\begin{aligned}\mathbb{E}(\sum_{k=1}^n |\Delta F_k|^2)^{1/2} &\leq \mathbb{E}(\sum_{k=1}^n |\Delta G'_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^n |\Delta B'_k|) \\ &\leq 2\mathbb{E}(\sum_{k=1}^n \mathbb{E}_{k-1}|\Delta G'_k|^2)^{1/2} + \mathbb{E}(\sum_{k=1}^n |\Delta B'_k|).\end{aligned}$$

2. The right hand side of the previsible uniform estimates (44) depends not only on the value of the martingale of F at time $k-1$ but also on its history up to time $k-1$. To wit (44) involves
$$|\Delta F_1|, \ldots, |\Delta F_{k-1}|.$$
This aspect contrasts the uniform previsible estimates in the thin-thick decomposition for Hardy martingales (22).

We prove Theorem 4.4 by feeding Theorem 4.1 with a thin-thick decomposition for $L^1(\Omega)$. It uses just truncation and is a simplified version of Lemma 3.4.

Lemma 4.5 *To each* $h \in L^1(\Omega)$ *satisfying* $\mathbb{E}h = 0$ *and* $M > 0$ *put*

$$g = 1_D h - \mathbb{E}(1_D h) \quad \text{where} \quad D = \{|h| \leq 2M\}. \quad (45)$$

Then

$$|g| \leq 2M,$$

and

$$(M^2 + \frac{1}{12}\mathbb{E}|g|^2)^{1/2} + \frac{1}{4}\mathbb{E}|h - g| \leq \mathbb{E}(M^2 + |h|^2)^{1/2}.$$

PROOF. By rescaling we may put $M = 1$. Let A denote the complement of D, thus $A = \{|h| > 2\}$. To $x \in A$, we have $|h(x)| \geq 2 1_A(x)$. Hence

$$\mathbb{E}(1_A|h|) \geq 2\mathbb{P}(A),$$

and

$$\mathbb{E}(1_A(1+|h|^2)^{1/2}) \geq \mathbb{E}(1_A|h|) \geq \frac{1}{2}\mathbb{E}(1_A|h|) + \mathbb{P}(A). \tag{46}$$

Next for $x \in D$, $|h(x)| \leq 2$. Hence

$$(1+|h(x)|^2)^{1/2} \geq (1+\frac{1}{4}|h(x)|^2)^{1/2} \geq 1 + \frac{1}{12}|h(x)|^2, \qquad x \in D.$$

Taking expectations gives

$$\mathbb{E}(1_D(1+|h|^2)^{1/2}) \geq \mathbb{E}(1_D(1+\frac{1}{12}|h|^2)).$$

Next recall that in (45) we defined $g = 1_D h - \mathbb{E}(1_D h)$. Hence $\mathbb{E}(1_D|h|^2) \geq \mathbb{E}|g|^2$, and

$$\mathbb{E}(1_D(1+|h|^2)^{1/2}) \geq \mathbb{P}(D) + \frac{1}{12}\mathbb{E}|g|^2. \tag{47}$$

Adding (46) and (47) gives

$$\mathbb{E}(1+|h|^2)^{1/2} \geq \mathbb{P}(D) + \frac{1}{12}\mathbb{E}|g^2| + \mathbb{P}(A) + \frac{1}{2}\mathbb{E}(1_A|h|). \tag{48}$$

Note that $\mathbb{E}h = 0$ implies $\mathbb{E}(1_A h) = -\mathbb{E}(1_D h)$ and

$$h - g = h - 1_D h + \mathbb{E}(1_D h) = 1_A h - \mathbb{E}(1_A h).$$

Moreover

$$2\mathbb{E}(1_A|h|) \geq \mathbb{E}|1_A h - \mathbb{E}(1_A h)|. \tag{49}$$

Inserting (49) into (48) gives the result

$$\begin{aligned} \mathbb{E}(1+|h|^2)^{1/2} &\geq \mathbb{P}(A) + \mathbb{P}(D) + \frac{1}{12}\mathbb{E}|g^2| + \frac{1}{4}\mathbb{E}|h-g| \\ &\geq (1+\frac{1}{12}\mathbb{E}|g|^2)^{1/2} + \frac{1}{4}\mathbb{E}|h-g|. \end{aligned} \tag{50}$$

■

Proof of Theorem 4.4. Let $k \leq n$ and put

$$M_{k-1} = (\sum_{m=1}^{k-1} |\Delta F_m|^2)^{1/2}.$$

Lemma 4.5 gives a decomposition of ΔF_k as follows. Put

$$D_k = \{|\Delta F_k| \leq 2M_{k-1}\}, \quad \text{and} \quad \Delta G_k = 1_{D_k}\Delta F_k - \mathbb{E}_{k-1}(1_{D_k}\Delta F_k).$$

Define ΔB_k by the decomposition

$$\Delta F_k = \Delta G_k + \Delta B_k.$$

Then $\Delta G_k, \Delta B_k$ are $\mathcal{F}_k$ measurable, and

$$\mathbb{E}_{k-1}(\Delta G_k) = \mathbb{E}_{k-1}(\Delta B_k) = 0.$$

By construction

$$|\Delta G_k| \leq 4M_{k-1}.$$

By Lemma 4.5

$$(M_{k-1}^2 + \frac{1}{12}\mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \frac{1}{4}\mathbb{E}_{k-1}(|\Delta B_k|) \leq \mathbb{E}_{k-1}M_k. \tag{51}$$

Take expectations of (51) to obtain

$$\mathbb{E}(M_{k-1}^2 + \frac{1}{12}\mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2} + \frac{1}{4}\mathbb{E}(|\Delta B_k|) \leq \mathbb{E}M_k. \tag{52}$$

Now apply Theorem 4.1 with

$$u_k = |\Delta F_k|, \quad v_k = (\mathbb{E}_{k-1}|\Delta G_k|^2)^{1/2}/4, \quad \text{and} \quad w_k = |\Delta B_k|/4, \qquad k \leq n.$$

■

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