RESONANCE EQUALS REDUCIBILITY FOR A-HYPERGEOMETRIC SYSTEMS

MATHIAS SCHULZE AND ULI WALTHER

Abstract. Classical Theorems of Gel'fand et al., and recent results of Beukers show essentially that non-confluent A-hypergeometric systems have reducible monodromy representation if and only if the continuous parameter is A-resonant.

We remove the confluence conditions.

1. Introduction

1.1. **GKZ-systems.** Let $\mathbb{Z}^n$ denote the free $\mathbb{Z}$-module with basis $e = e_1, \ldots, e_n$ and let $A = (a_{i,j})$ be an integer $d \times n$-matrix of rank d. The additive group $\mathbb{Z}A$ generated by the columns of A is a free group of rank d; let $\varepsilon = \varepsilon_1, \ldots, \varepsilon_d$ be a basis for $\mathbb{Z}A$. We may thus view A both as a map $\mathbb{Z}^n \to \mathbb{Z}A$ with respect to the bases above and as the finite subset $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of $\mathbb{Z}A$ consisting of the images of the e_i.

Throughout, we assume that the additive semigroup $\mathbb{N}A$ is *pointed* (i.e., 0 is the only unit in $\mathbb{N}A$). To this type of data, Gel'fand, Graev, Kapranov and Zelevinskiĭ [GGZ87, GZK89] associated in the 1980's a class of D-modules today called *GKZ-* or *A-hypergeometric systems* and defined as follows.

Let $x_A = x_1, \ldots, x_n$ be the coordinate system on $X := \operatorname{Spec}(\mathbb{C}[\mathbb{N}^n]) \cong \mathbb{C}^n$ induced by e, and let $\partial_A = \partial_1, \ldots, \partial_n$ be the corresponding partial derivative operators on $\mathbb{C}[x_A]$. Then the *Weyl algebra*

$$D_A = \mathbb{C}\langle x_A, \partial_A \mid [x_i, \partial_j] = \delta_{i,j},\ [x_i, x_j] = 0 = [\partial_i, \partial_j]\rangle$$

is the ring of algebraic differential operators on X. With $\mathbf{u}_+ = (\max(0, u_j))_j$ and $\mathbf{u}_- = \mathbf{u}_+ - \mathbf{u}$, write $\Box_{\mathbf{u}}$ for $\partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-}$ where here and elsewhere we freely use multi-index notation. The *toric relations of* A are then

$$\Box_A := \{\Box_{\mathbf{u}} \mid A\mathbf{u} = 0\} \subseteq R_A := \mathbb{C}[\partial_A],$$

while the *Euler vector fields* $E = E_1, \ldots, E_d$ to A are

$$E_i := \sum_{j=1}^{n} a_{i,j} x_i \partial_j. \tag{1.1}$$

Finally, for $\beta \in \mathbb{C}^d$, the *A-hypergeometric ideal* and *module* are the left D_A-ideal and -module

$$H_A(\beta) = D_A \cdot \langle E - \beta, \Box_A\rangle; \qquad M_A(\beta) = D_A / H_A(\beta).$$

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The structure of the solutions to the (always holonomic) modules $M_A(\beta)$ is tightly interwoven with the combinatorics of the pair $(A,\beta) \in (\mathbb{Z}A)^n \times \mathbb{C}A$, and A-hypergeometric structures are nearly ubiquitous. Indeed, research of the past two decades revealed that toric residues, generating functions for intersection numbers on moduli spaces, and special functions (Gauß, Bessel, Airy, etc.) may all be viewed as solutions to GKZ-systems. In other directions, varying Hodge structures on families of Calabi–Yau toric hypersurfaces as well as the space of roots of univariate polynomials with undetermined coefficients have A-hypergeometric structure. We refer to [SW08] for a detailed introduction as well as further references.

1.2. **Torus action.** Consider the algebraic d-torus $T = \operatorname{Spec}(\mathbb{C}[\mathbb{Z}^d])$ with coordinate functions $t = t_1, \ldots, t_d$ corresponding to $\varepsilon = \varepsilon_1, \ldots, \varepsilon_d$. One can view the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of A, as characters $\mathbf{a}_i(t) = t^{\mathbf{a}_i}$ on T, and the parameter vector $\beta \in \mathbb{C}^d$ as a character on its Lie algebra via $\beta(t_i \partial_{t_i}) = -\beta_i + 1$. A natural tool for investigating $M_A(\beta)$ is the torus action of T on the cotangent space $X^* = T_0^* X$ of X at 0 given by

$$t \cdot \partial_A = (t^{\mathbf{a}_1}\partial_1, \ldots, t^{\mathbf{a}_n}\partial_n).$$

The coordinate ring R_A of X^* is contains the *toric ideal* $I_A = R_A \cdot \Box_A$. For $\mathbf{1}_A = (1, \ldots, 1) \in X$, I_A is the ideal of the closure of the orbit $T \cdot \mathbf{1}_A$ of $\mathbf{1}_A$ with coordinate ring

$$S_A := R_A/I_A \cong \mathbb{C}[t^{\mathbf{a}_1}, \ldots, t^{\mathbf{a}_n}] \cong \mathbb{C}[\mathbb{N}A].$$

If $\mathbf{1}$ is in the row span of A then A is *homogeneous*.

The contragredient action of T on R_A given by

$$(t \cdot P)(\partial_A) = P(t^{-\mathbf{a}_1}\partial_1, \ldots, t^{-\mathbf{a}_n}\partial_n),$$

for $P \in R_A$, defines a $\mathbb{Z}A$-grading on R_A and on the coordinate ring $\mathbb{C}[x_A, \partial_A]$ of T^*X by

$$-\deg(\partial_j) = \mathbf{a}_j = \deg(x_j). \tag{1.2}$$

Note that for $\mathbb{Z}A$-homogeneous $P \in R_A$ the commutator $[E_i, P]$ equals $\deg_i(P)P$ where $\deg_i(-)$ is the i-th component of $\deg(-)$. As $\partial_i x_i - x_i \partial_i = 1$, (1.2) also defines a $\mathbb{Z}A$-grading on the sheaves of differential operators $\mathscr{D}_X$ and $\mathscr{D}_{X^*}$ under which E and $\Box_A$ are homogeneous.

Note that A and β naturally define an algebraic $\mathscr{D}_T$-module

$$\mathscr{M}(\beta) := \mathscr{D}_T / \mathscr{D}_T \langle \partial_{t_i} t_i + \beta_i \mid i = 1, \ldots, d \rangle, \tag{1.3}$$

$\mathscr{O}_T$-isomorphic to $\mathscr{O}_T$ but equipped with a twisted $\mathscr{D}_T$-module structure expressed symbolically as

$$\mathscr{M}(\beta) = \mathscr{O}_T \cdot t^{-\beta-1}$$

on which $\mathscr{D}_T$ acts via the product rule.

The orbit inclusion

$$\phi \colon T \to T \cdot \mathbf{1} \hookrightarrow \mathbb{C}^n,$$

when combined with the Fourier transform F, gives rise to a direct image functor $F \circ \phi_+ \colon \mathscr{D}_T\text{-mods} \to \mathscr{D}_{X^*}\text{-mods}$.

1.3. **Questions, results, techniques.** A powerful way of studying $M_A(\beta)$ is to consider it as a 0-th homology of the *Euler–Koszul complex* $K_\bullet(S_A, \beta)$ of $E - \beta$ on $D_A/D_A \cdot \Box_A \cong \mathbb{C}[x_A] \otimes_{\mathbb{C}} S_A$. This idea can be traced back to [GZK89] and was developed into a functor in [MMW05]. Results from [MMW05] show that $K_\bullet(S_A, \beta)$ is a resolution of $M_A(\beta)$ if and only if β is not in the *A-exceptional locus* $\mathscr{E}_A$, a well-understood (finite) subspace arrangement of $\mathbb{C}^n$ describing the parameters β that exhibit unusual(ly large) solution space for $H_A(\beta)$.

A parameter is *non-resonant* if it is not contained in the (analytically) locally finite subspace arrangement of *resonant* parameters

$$\operatorname{Res}(A) := \bigcup_\tau (\mathbb{Z}A + \mathbb{C}\tau)\,,$$

the union being taken over all linear subspaces $\tau \subseteq \mathbb{Q}^n$ that form a boundary component of the rational polyhedral cone $\mathbb{Q}_+A$.

A fundamental theorem of [GKZ90, Thm. 2.11] is that, in the homogeneous saturated case (i.e., I_A defines a projectively normal projective variety), the generic monodromy representation on the solution space of $H_A(\beta)$ is irreducible for non-resonant β. The crucial tool for this proof is the Riemann–Hilbert correspondence of Kashiwara and Mebkhout, relating regular holonomic D-modules to perverse sheaves.

In Theorems 3.1 and 3.2 we prove that reducibility of the monodromy in a generic point of a GKZ-system is (essentially) equivalent to resonance of the parameter defining the GKZ-system. We do not assume I_A to be homogeneous, so this generalizes to the confluent case both the result mentioned above in [GKZ90] (which assume homogeneity and saturatedness of I_A) as well as a recent converse theorem of Beukers [Beu10] (in which homogeneity is still assumed, but non-saturated semigroups are admitted). Confluence rules out the use of Riemann–Hilbert, but surprisingly Euler–Koszul arguments provide an approach that is simultaneously significantly simpler and more widely applicable.

2. Pyramids

Definition 2.1. For any subset F of the columns of A we write $\overline{F}$ for the complement $A \smallsetminus F$.

A *face of* A is a subset $F \subset A$ subject to the condition that there a linear functional $\phi_F : \mathbb{Z}A \to \mathbb{Z}$ that vanishes on F but not on any element of $\overline{F}$.

For a given face F we set

$$I_A^F := I_A + R_A \cdot \partial_{\overline{F}}$$

and note that $R_A/I_A^F = S_F$ as R_A-module.

Definition 2.2. Let F be a face of A. The parameter $\beta \in \mathbb{C}^d$ is *F-resonant* if $\beta \in \mathbb{Z}A + \mathbb{C}G$ for a proper subface G of F.

If β is G-resonant for all faces properly containing F, but not for F itself, we call F a *resonance center for* β.

A resonance center is a minimal face F for which $\beta \in \mathbb{Z}A + \mathbb{C}F$. Every parameter β has a resonance center; A is the only center of resonance for β if and only if β is non-resonant in the usual sense (i.e., $\beta \notin \operatorname{Res}(A)$).

It is easy to have several resonance centers for β. For example, consider $\beta = (1/2, 1)$ on the quadric cone $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$; β has both extremal rays as resonance centers.

Remark 2.3. By [MMW05], $H_A(\beta)$ has rank (dimension of the analytic solution space) equal to $\mathrm{vol}_A(A)$ for $\beta \notin \mathscr{E}_A$, and the rank is larger for $\beta \in \mathscr{E}_A$. Here $\mathrm{vol}_A(F)$ denotes, for $F \in \mathbb{Z}A$, the simplicial volume of F taken in the lattice $\mathbb{Z}A$.

Definition 2.4. We say that A is a(n iterated) *pyramid over the face* F if $d = \dim_{\mathbb{Z}}(\mathbb{Z}A)$ equals $|\overline{F}| + \dim_{\mathbb{Z}}(\mathbb{Z}F)$.

Remark 2.5. The following are equivalent, cf. [Wal07, Lem. 3.13]:

(1) A is a pyramid over F;
(2) $\mathbf{a}_j \notin \mathbb{C}(A \smallsetminus \{\mathbf{a}_j\})$ for any $j \notin F$;
(3) $\mathbb{Z}A = \mathbb{Z}\mathbf{a}_j \oplus \mathbb{Z}(A \smallsetminus \{\mathbf{a}_j\})$;
(4) $\mathrm{vol}_F(F) = \mathrm{vol}_A(A)$;
(5) for every $\beta \in \mathbb{C}A$, the coefficients c_j for $j \notin F$ in the sum $\beta = \sum_A c_j \mathbf{a}_j$, are uniquely determined by β;
(6) the generators $\square_A$ of I_A do not involve ∂_j for any $j \in \overline{F}$
(7) $S_F \otimes_{\mathbb{C}} \mathbb{C}[\partial_{\overline{F}}] = S_A$ as R_A-modules.

Notation 2.6. Suppose F is any face of A and $\beta \in \mathbb{Z}F$. Viewing F as a subset of $\mathbb{Z}F$ in its own right one then has a GKZ-system $M_F(\beta)$. Note that $I_F \subseteq I_A$, and the Euler operators in $H_F(\beta)$ are the restrictions $\{\sum_{j\in F} a_{i,j} x_j \partial_j - \beta_i\}$ of those appearing in $H_A(\beta)$. We routinely abuse notation and view $H_F(\beta)$ as system of differential equations on $\mathbb{C}F$ as well as on $\mathbb{C}A$, as it suits us.

Suppose now that A is a pyramid over F, and let $\beta \in \mathbb{Z}A$. The splitting 2.5.(3) induces splittings $\mathbb{C}A \to \mathbb{C}\mathbf{a}_j$ for $j \notin F$; write $\beta_j^{\overline{F}}$ for the image of β. Note that if $\beta = \mathbf{a}_j$, $j \notin F$, then $\beta_j^{\overline{F}} = 1$.

Writing β^F for the image of β under the splitting $\mathbb{C}A \to \mathbb{C}F$, we have

$$\beta = \beta^F + \sum_{j \in \overline{F}} \beta_j^{\overline{F}}.$$

Remark 2.7. If A is a pyramid over F then the following conditions hold:

(8) the ideal $H_A(\beta)$ contains $x_j\partial_j - \beta_j^{\overline{F}}$ for $j \notin F$;
(9) $M_F(\beta)(x_A) = M_A(\beta)(x_A)$ for $\beta \in \mathbb{C}F$;
(10) the solutions of $H_A(\beta)$ are the solutions of $H_F(\beta_F)$, multiplied with the unique solution to the system

$$\{x_j\partial_j \bullet f = \beta_j^{\overline{F}} \cdot f\}_{i=1}^d.$$

Remark 2.8. Let A be a pyramid over F and choose $\beta \in \mathbb{C}F$. Then $\beta \in \mathscr{E}_A$ precisely if it is rank-jumping for $\beta \in \mathscr{E}_F$. Indeed, $\mathscr{E}_A$ is the Zariski closure of the degrees of the homogeneous elements of all $H^i_{\mathfrak{m}_A}(S_A)$, $i < d$. But $S_F[\partial_{\overline{F}}] = S_A$ by the pyramid condition. So $H^i_{\mathfrak{m}_A}(S_A) = H^i_{\mathfrak{m}_F}(S_F) \otimes_{\mathbb{C}} \mathbb{C}[\partial_{\overline{F}}]$. The claim follows for $\beta \in \mathbb{C}F$.

Lemma 2.9. *Let $\beta \in \mathbb{Z}A$. If there is a resonance center F for β over which A is a pyramid then F is the only resonance center for β.*

Proof. Let F be a resonance center for β over which A is a pyramid. Let G be a second resonance center and suppose G meets the complement of F: $\mathbf{a}_k \in G \cap (\overline{F})$. Since $\mathbb{Z}\mathbf{a}_k$ is a direct summand of $\mathbb{Z}A$, it is also a direct summand of $\mathbb{Z}G$. It follows that $G \smallsetminus \{\mathbf{a}_k\}$ is a face G' of A.

As F is a resonance center, $\beta = z_k a_k + \sum_{j\neq k} z_j \mathbf{a}_j + \sum_F c_j \mathbf{a}_j$ where $z_j \in \mathbb{Z}$ and $c_j \in \mathbb{C}$. Since G is a resonance center, $\beta = c'_k \mathbf{a}_k + \sum_{j\neq k} z'_j \mathbf{a}_j + \sum_{G'} c'_j \mathbf{a}_j$, where $z'_j \in \mathbb{Z}$, $c'_j \in \mathbb{C}$. By Remark 2.5, the coefficients for $\mathbf{a}_k$ in these sums are identical, $c'_k = z_k \in \mathbb{Z}$. It follows that $\beta = \left(z_k \mathbf{a}_k + \sum_{j\neq k} z'_j \mathbf{a}_j\right) + \sum_{G'} c'_j \mathbf{a}_j \in \mathbb{Z}A + \mathbb{C}G'$. This contradicts G being a center of resonance, and thus G must contain F. But then G can only be a center of resonance if $F = G$. □

3. Resonance vs. reducibility

In this section we generalize Theorem 2.11 in [GKZ90], Theorem 3.4 in [Wal07], and Theorem 1.3 in [Beu10].

Theorem 3.1. *Let F be a resonance center for $\beta \in \mathbb{Z}A$. If A is not a pyramid over F then $M_A(\beta)$ has reducible monodromy.*

Proof. We have $\beta \in \mathbb{Z}A + \mathbb{C}F$. Let $\gamma \in \mathbb{C}F$ differ from β by an element of $\mathbb{Z}A$. By [Wal07, Thm. 3.15], we need to show the reducibility of $M_A(\gamma)$.

Consider the surjection $M_A(\gamma) \twoheadrightarrow H_0(R_A/I_A^F, \gamma) = D_A/D_A \cdot (E-\beta, I_A^F)$ induced by the surjection $S_A \twoheadrightarrow S_F$. It suffices to show that the rank of the target is positive but less than $\mathrm{vol}(A)$.

Consider the GKZ-system $M_F(\gamma)$ given by

$$M_F(\gamma) = D_F/D_F \cdot (I_F, \{E_i^F - \gamma_i\})$$

where $E_i^F = \sum_{j\in F} \mathbf{a}_{i,j} x_j \partial_j$ is the part of E_i supported in F. Since F is a resonance center for β (and hence for γ as well), $M_F(\gamma)$ is non-resonant. By [Ado94, Thm. 5.15], $\mathrm{rk}(M_F(\gamma)) = \mathrm{vol}_F(F)$. By Remark 2.5,

$$\mathrm{rk}(M_F(\gamma)) = \mathrm{vol}_F(F) < \mathrm{vol}_A(A) \leq \mathrm{rk}(M_A(\gamma)).$$

Finally, note that by [MMW05, Lem. 4.8] we have

$$\begin{aligned} \mathbb{C}(x_A) \otimes_{\mathbb{C}[x_A]} H_0(R_A/I_F^A, \gamma) &= \mathbb{C}(x_A) \otimes_{\mathbb{C}[x_A]} \mathbb{C}[x_A] \otimes_{\mathbb{C}[x_F]} M_F(\gamma) \\ &= \mathbb{C}(x_A) \otimes_{\mathbb{C}[x_F]} M_F(\gamma) \\ &= \mathbb{C}(x_A) \otimes_{\mathbb{C}(x_F)} (\mathbb{C}(x_F) \otimes_{\mathbb{C}[x_F]} M_F(\gamma)) \end{aligned}$$

so that $M_F(\gamma)$ and $H_0(R_A/I_F^A, \gamma)$ have the same rank. By [MMW05, Lem. 4.9] this rank is at least the volume of F and hence positive. □

Theorem 3.2. *Let F be a resonance center for β. If A is a pyramid over F then $M_A(\beta)$ has irreducible monodromy.*

Proof. First consider the case $F = A$. By [SW09, Thm. 3.5] and [SW09, Cor. 3.7], $M_A(\beta)$ is the unique homology group of $K_\bullet(S_A, \beta)$ and $M_A(\beta)$ agrees with the Fourier transform of the direct image $\phi_+(\mathscr{M}_\beta)$.

Now ϕ can be factored into 1) the identification of the abstract d-torus with the orbit of $\mathbf{1} \in \mathbb{C}^n$, followed by the inclusion of this orbit into the n-torus $(\mathbb{C}^*)^n$, completed by 3) the inclusion $(\mathbb{C}^*)^n \hookrightarrow \mathbb{C}^n$.

Step 1 clearly preserves irreducibility, and by Kashiwara equivalence so does Step 2. Step 3 preserves irreducibility as well because D-affinity of both the target and the source of the inclusion map allow to detect submodules on global sections, which agree because we are looking at an open embedding. As Fourier transforms preserve composition chains, $M_A(\beta)$ is irreducible provided that $\mathscr{M}(\beta)$ is an irreducible $\mathscr{D}_T$-module. But the ideal generated by $t\partial_t - \beta$ is the same as the (clearly) maximal ideal generated by $\partial_t - \beta/t$.

Suppose now that F is a proper face. Choose $\gamma \in \mathbb{C}F$ with $\beta - \gamma \in \mathbb{Z}A$. Then $H_F(\gamma)$ is irreducible by the first part of the proof. But the pyramid condition assures $M_F(\gamma)(x_A) = M_A(\gamma)(x_A)$, so we are done by [Wal07, Thm. 3.13]. $\square$

References

[Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), no. 2, 269–290. MR 96c:33020

[Beu10] Frits Beukers, *Irreducibility of a-hypergeometric systems*, arXiv:1007.4644, 2010.

[GGZ87] I. M. Gel′fand, M. I. Graev, and A. V. Zelevinskiĭ, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19. MR MR902936 (88j:58118)

[GKZ90] I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), no. 2, 255–271. MR MR1080980 (92e:33015)

[GZK89] I. M. Gel′fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26. MR 90m:22025

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), no. 4, 919–941 (electronic). MR MR2163866

[SW08] Mathias Schulze and Uli Walther, *Irregularity of hypergeometric systems via slopes along coordinate subspaces*, Duke Math. J. **142** (2008), no. 3, 465–509. MR 2412045 (2009b:13067)

[SW09] ———, *Hypergeometric D-modules and twisted Gauß-Manin systems*, J. Algebra **322** (2009), no. 9, 3392–3409. MR 2567427

[Wal07] Uli Walther, *Duality and monodromy reducibility of A-hypergeometric systems*, Math. Ann. **338** (2007), no. 1, 55–74. MR 2295504 (2008e:32043)

M. Schulze, Oklahoma State University, Dept. of Mathematics, 401 MSCS, Stillwater, OK 74078, USA

E-mail address: mschulze@math.okstate.edu

U. Walther, Purdue University, Dept. of Mathematics, 150 N. University St., West Lafayette, IN 47907, USA

E-mail address: walther@math.purdue.edu