PARAHORIC BUNDLES ON A COMPACT RIEMANN SURFACE

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Contents

1. Introduction

Let Y be a smooth projective curve (defined over the ground field $\mathbb{C}$) with an action of a finite group Γ. Let X be the smooth projective curve Y/Γ and let $p : Y \to X$ be the quotient morphism. Let G be a reductive algebraic group over $\mathbb{C}$. We say that E is a (Γ, G)–bundle on Y if E is an algebraic principal G–bundle over Y and the action of Γ on Y lifts to an action on E.

If G is the full-linear group, the (Γ, G)–bundles on Y have an equivalent description as Γ–vector bundles on Y. Recall ([27], [18]) that if V is a Γ–vector bundle on Y, the vector bundle $W = p_*^{\Gamma}(V)$(invariant direct image by p) on X acquires a *parabolic structure* which consists of the data assigning a *flag to the fibre of* W *at every ramification point in* X *for the covering* p together with a tuple of *weights*; further,

Key words and phrases. Stable vector bundles, parahoric groups, parabolic bundles, group schemes, principal bundles, unitary representations.

The research of the first author was partially supported by the J.C. Bose Fellowship.

the "invariant direct image" functor $V \mapsto p_*^\Gamma(V)$ gives an equivalence of categories between the category of Γ–vector bundles on Y and the category of parabolic vector bundles on X (morphisms being taken as isomorphisms). This translates easily into an equivalent description of $(\Gamma, Gl(n))$–bundles on Y as principal $GL(n)$–bundles on X with *parabolic structures*. Now one can define the concepts of *stability* (resp. *semistability, polystability*) for Γ–vector bundles (or equivalently parabolic bundles on X) and construct the corresponding moduli spaces of isomorphism classes of polystable objects (fixing some invariants) as a normal projective variety. Further, for these moduli spaces the underlying topological spaces can be identified with isomorphism classes of certain unitary representations of Fuchsian groups (see [18], [27]), which generalize the results in [19] and [26].

The purpose of this paper is to generalize the above results when the group G is no longer the full-linear group. Since there are a number of papers which partially address this issue and some of them erroneously, we give a somewhat elaborate and leisurely introduction, which also views the problem from a proper historical perspective. Let us suppose hereafter that the group G is *semisimple and simply connected* (over $\mathbb{C}$) unless otherwise stated. One can again give an equivalent description of (Γ, G)–bundles on Y as certain intrinsically defined objects on X. However, this is somewhat subtler than the case when G is the full-linear group ; in particular, *it is not possible, in general, to associate in a natural manner a principal G–bundle on X to a (Γ, G)–bundle on Y*. We call these objects on X which give an equivalent description of (Γ, G)–bundles on Y, as *"parahoric bundles"*. These parahoric bundles are describable as pairs $(\mathcal{E}, w)$, where $\mathcal{E}$ is a point of a double coset space resembling the adèlic description of usual principal G–bundles, together with an additional structure w which we term *weights* (see Definition 5.1), in accordance with the classical terminology. Equivalently, these objects can be described as torsors on X under a Bruhat-Tits group scheme, together with weights (see Definition 5.10). We construct also moduli spaces of these objects; they are projective varieties and their points are isomorphism classes of polystable (Γ, G)–bundles. This construction is broadly along the lines of [2]. However, the connection with parahoric bundles plays a key role and leads to stronger and more precise results.

The parahoric bundles that we consider here have also been defined by Pappas and Rapoport, without however the notion of weights (see [21] and [22]). Heinloth has since settled many of their conjectures (see [11] which is over arbitrary ground fields). We were led independently to the description of these parahoric bundles in trying to interpret (Γ, G)–bundles on Y as objects on X (inspired by A. Weil's work [33], as was the case in [18] and [27]). The connection with (Γ, G)–bundles leads to some stronger results than [11], when we work over $\mathbb{C}$, e.g the construction of moduli spaces as projective varieties. In [11] Heinloth works with the moduli *stacks* and proves his results in that setting. The paper [12] considers some related issues from a symplectic perspective.

We shall give a detailed outline of the contents of this paper.

1.1. Let $q_1 : \mathbb{H} \to Y$ be a simply connected covering of Y. Then the fundamental group π_o of Y acts freely on $\mathbb{H}$ and $Y = \mathbb{H}/\pi_o$. We have the following commutative diagram:

(1.0.1)
$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\ q_1\ } & Y \\ & {\scriptstyle q_2}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

with $q_2 = p \circ q_1$.

Let $\mathcal{R}_p \subset X$ be the points of X over which the map p is ramified and let n_x be the *ramification index* at $x \in \mathcal{R}_p$. Let $m = |\mathcal{R}_p|$. Then $q_2 : \mathbb{H} \to X$ is characterized by the property that $\mathbb{H}$ is simply connected, and q_2 has ramification index n_x over $x \in \mathcal{R}_p$ and is unramified elsewhere. Let us suppose that $\mathbb{H}$ is the *upper half-plane* (this is the case when the genus of Y is ≥ 2). Then $Aut(\mathbb{H})$ is isomorphic to $PSL(2, \mathbb{R})$ and furthermore, $X = \mathbb{H}/\pi$, π being a discrete subgroup of $PSL(2, \mathbb{R})$ acting freely on $q_2^{-1}(X - \mathcal{R}_p)$. Let $z \in \mathbb{H}$ lie over $x \in \mathcal{R}_p$. Then it is well-known that the isotropy subgroup π_z at z is a cyclic group of order n_x. Note that π is a Fuchsian group in $PSL(2, \mathbb{R})$. We see that π_o identifies with a normal subgroup of π and $\Gamma = \pi/\pi_o$. On the other hand if we start with a covering $q_2 : \mathbb{H} \to X$ with the above properties, then there is a normal subgroup π'_o of π of finite index such that π'_o acts freely on $\mathbb{H}$ so that if we set $\Gamma' = \pi/\pi'_o$, $Y' = \mathbb{H}/\pi'_o$ and $q'_1 : \mathbb{H} \to Y'$ the canonical quotient map, we have $X = Y'/\Gamma'$ and a commutative diagram as (1.0.1) above. Note that since the action of π_o on $\mathbb{H}$ is free, the set of isomorphism classes of (π, G)–bundle on $\mathbb{H}$ gets identified with the set of isomorphism classes of (Γ, G)–bundles on Y.

The map $q_1 : \mathbb{H} \to Y$ is a local isomorphism; in fact, if $z \in \mathbb{H}$ maps to $y \in Y$, then q_1 induces an isomorphism $\pi_z \xrightarrow{\sim} \Gamma_y$ of isotropy subgroups of π and Γ respectively, as well as an isomorphism of a sufficiently small (analytic) neighbourhood of z onto that of y, respecting the actions of the isotropy groups. Now a (Γ, G)–bundle E on Y is *locally* a (Γ_y, G)–bundle at y. Recall that this (Γ_y, G)–bundle is *defined by a representation* (see for example [27], [33], [9, Proposition 1, page 06] and also more recently [30, Lemma 2.5]); i.e, if N_y is a sufficiently small Γ_y–stable neighbourhood of y, then this bundle is isomorphic to the (Γ_y, G)–bundle $N_y \times G$, for the *twisted action* of Γ_y-action on $E \times G$ given by a representation $\rho_y : \Gamma_y \longrightarrow G$, defined as follows:

(1.0.2)
$$\gamma \cdot (u, g) = (\gamma u, \rho_y(\gamma) g), \ u \in N_y, \ \gamma \in \Gamma_y.$$

It is easily seen that these (Γ_y, G)–bundles given by representations are isomorphic as (Γ_y, G)–bundles if and only if the defining representations are equivalent. We call the representations ρ_y the *local representations* associated to a (Γ, G)–bundle. *We observe that for a (Γ, G)–bundle E, the equivalence class of the local representation ρ_y is an invariant of its isomorphism class.*

1.2. Definition. *The* local type *of E at y is defined as the equivalence class of the local representation ρ_y and is denoted by $\boldsymbol{\tau}_y$.*

If E is represented by a (π, G)–bundle E' on $\mathbb{H}$, we see that its local type at a point $z \in \mathbb{H}$ is the same as that of E at the image $y \in Y$ of z. Now if $y, y' \in Y$ lie over $x \in \mathcal{R}_p$, inner conjugation by a suitable element $\gamma \in \Gamma$ induces an isomorphism $\gamma^* : \Gamma_y \to \Gamma_{y'}$ and one sees that

$$\gamma^* : \Gamma_y \to \Gamma_{y'} \tag{1.0.3}$$

is a well-determined isomorphism i.e independent of the choice of γ. This is a consequence of the fact, which is easily seen, that the normalizer of Γ_y in Γ coincides with Γ_y itself and also the fact that Γ_y is *abelian.* Thus if we choose a subset $\mathcal{R}_p^* \subset Y$, which maps bijectively onto $\mathcal{R}_p$, we see that the local type of E at any ramification point is determined, once we know the local type of E at every $y \in \mathcal{R}_p^*$. We denote by $\boldsymbol{\tau}(\mathcal{R}_p^*)$ the set $\{\boldsymbol{\tau}_y \mid y \in \mathcal{R}_p^*\}$. Let us denote by

$$Bun_Y^{\boldsymbol{\tau}(\mathcal{R}_p^*)}(\pi, G) = \left\{ \begin{array}{l} \text{isomorphism classes of } (\pi, G) \text{ bundles} \\ \text{with fixed local type } \boldsymbol{\tau}(\mathcal{R}_p^*) \end{array} \right\} \tag{1.0.4}$$

1.3**.** Let g be the genus of X. Recall that π can be identified with the group on the letters $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m$, modulo the relations

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_1 \cdots C_m = I. \tag{1.0.5}$$

$$C_1^{n_1} = C_2^{n_2} = \cdots = C_m^{n_m} = I. \tag{1.0.6}$$

Let π_i be the cyclic subgroup of π of order n_i generated by C_i. Then one can identify π_i with the isotropy at some $z_i \in \mathbb{H}$ such that the set $\{z_i \mid 1 \leq i \leq m\}$, maps bijectively onto $\mathcal{R}_p$. Let y_i be the image of z_i in Y and let $\mathcal{R}_p^* = \{y_i \mid 1 \leq i \leq m\}$. Let $\rho : \pi \to G$ be a homomorphism. Let $E(\rho)$ denote the (π, G)–bundle on $\mathbb{H}$ defined by the *twisted action* given by (1.0.2).

1.4. Definition. *The* type of a representation ρ *is defined to be the set of conjugacy classes in G of the images $\rho(C_i)$ and is denoted by $\{\boldsymbol{\tau}_i\}$.*

We observe that the *local type $\boldsymbol{\tau}_i$ of the bundle $E(\rho)$ at y_i* in the sense of Definition 1.2 is equivalently given by the *conjugacy class of* $\rho(C_i)$ in G. Thus if $\boldsymbol{\tau}(\mathcal{R}_p^*) = \{\boldsymbol{\tau}_i\}$, then we have

$$\rho \text{ is of type } \{\boldsymbol{\tau}_i\} \iff E(\rho) \text{ is of local type } \boldsymbol{\tau}(\mathcal{R}_p^*) \tag{1.0.7}$$

Fix a maximal compact subgroup K_G of G. If the representation ρ factors through K_G, one says that $E(\rho)$ is a *unitary* (π, G)–bundle. Let $R^{\boldsymbol{\tau}}(\pi, K_G)$ be the set of unitary representations of type $\boldsymbol{\tau}(\mathcal{R}^*_p)$. Then we get a canonical map

$$\psi : R^{\boldsymbol{\tau}(\mathcal{R}^*_p)}(\pi, K_G) \to Bun_Y^{\boldsymbol{\tau}(\mathcal{R}^*_p)}(\pi, G) \tag{1.0.8}$$

1.5. Let $D_x = Spec(A)$, where A is the complete discrete valuation ring obtained by taking the completion of the local ring $\mathcal{O}_{X,x}$ and let $K = K_x$ be its quotient field. Similarly, for $y \in \mathcal{R}^*_p$, let $N_y = Spec(B)$, where B is the integral closure of A in $L = K(\omega)$, where ω is a primitive d^{th}–root of z, z being the uniformizer of A. Let $p : N_y \to D_x$ be the totally ramified covering projection. Let E be the (Γ, G)–bundle on Y and $y \in \mathcal{R}^*_p$. Consider the restriction of E to N_y. Then as we have seen above in (1.0.2), as a (Γ_y, G) bundle we can identify $E|_{N_y}$ with the trivial bundle $N_y \times G$ together with the twisted Γ_y–action.

1.6. Definition. *Define* U_y *to be the group:*

$$\mathsf{U}_y = Aut_{(\Gamma_y, G)}(E|_{N_y}) \tag{1.0.9}$$

of (Γ_y, G) *automorphisms of* E *over* N_y. *We call* U_y the unit group *(or more precisely* the local unit group *at* $y \in Y$*) associated to* E.

In Theorem 2.3 we prove the basic fact that the unit group U_y determines a *parahoric subgroup* (in the sense of Bruhat-Tits ([6]) of $G(K)$, K being the quotient field of A. Conversely, we show that any parahoric subgroup of $G(K)$ can be obtained in this manner and the conjugacy class of U_y is in fact independent of the choice of $y \in Y$ above $x \in X$. In Section 3, we place this result in the more general setting of Bruhat-Tits theory.

For every $x \in \mathcal{R}_p$, let us choose a $\mathfrak{U}_x$ in the conjugacy class of U_y for $y \in Y$ above $x \in X$. Call $\mathfrak{U}_x$ *"a unit group at x"* associated to E. We remind the reader that only the conjugacy class of $\mathfrak{U}_x$ is well-determined associated to a point $x \in X$.

Then we have the following identification of $Bun_Y^{\boldsymbol{\tau}(\mathcal{R}^*_p)}(\pi, G)$ with the adèlic type double coset space (see Section 4 below)

$$Bun_Y^{\boldsymbol{\tau}(\mathcal{R}^*_p)}(\pi, G) \simeq \Big[\prod_{x \in \mathcal{R}_p} \mathfrak{U}_x \backslash^{\prod_{x \in \mathcal{R}_p} G(K_x)} / G(K(X)) \Big] \tag{1.0.10}$$

K_x being the quotient field of the local rings at $x \in \mathcal{R}_p$ and $K(X)$ being the quotient field of X.

For simplicity of notation, we fix a single point $x \in \mathcal{R}_p$, noting however that everything of what we state works for an arbitrary set $\mathcal{R}_p$ of ramifications. Let $K = K_x$. Fix an abstract parahoric subgroup $\mathfrak{P}_\Omega(K) \subset G(K)$ (see (3.0.6) for the definitions). Consider the double coset space

$$\mathfrak{M}_X(\mathfrak{P}_\Omega(K)) = \left[\mathfrak{P}_\Omega(K)\backslash^{G(K)}/G(X-x)\right]$$

We call an element of $\mathcal{E} \in \mathfrak{M}_X(\mathfrak{P}_\Omega(K))$ a *quasi-parahoric bundle* on X (Definition 4.2). The notion of a *weight* provides an identification $w_\theta : \mathfrak{P}_\Omega(K)) \simeq \mathsf{U}_y$ (see (5.0.1)). Equivalently, a weight is given by fixing $\theta \in Y(T) \otimes \mathbb{Q}$ i.e a *rational* 1-PS. The pair $(\mathcal{E}, w_\theta)$ is called a *parahoric bundle* (see Definition 5.1). In other words, a quasi-parahoric structure identifies the double coset space and the weight isolates a specific unit group U_y in the conjugacy class. We remark that in the classical setting of parabolic vector bundles, the notion of weight, apart from providing the correct stability properties, plays the key role in providing the *polarization* on a suitable total family for carrying out the geometric invariant theoretic construction of the moduli space of parabolic bundles.

Now we can translate the notion of local type $\boldsymbol{\tau}$ (or more generally $\boldsymbol{\tau}(\mathcal{R}_p^*)$) of a (Γ, G)–bundle E into that of *weights* for the quasi-parahoric bundle $\mathcal{E}$ on X associated to E by (1.0.10). An isomorphism between parahoric bundles therefore means that they have the same weights and are isomorphic as quasi-parahoric bundles. Thus parahoric bundles on X are indeed the objects which give an equivalent description of (Γ, G)–bundles on Y.

1.7. Recall that there is an affine Bruhat-Tits group scheme $\mathcal{G}_{\Omega,X}$ of finite type over X (see Definition 4.5). Let $Bun_X(\mathcal{G}_{\Omega,X})$ denote the set (or more precisely the stack) of isomorphism classes of $\mathcal{G}_{\Omega,X}$–torsors on X (see [11, Proposition 1, page 502]). Then one has the following identification (4.0.22)

$$\mathfrak{M}_X(\mathfrak{P}_\Omega(K)) \simeq Bun_X(\mathcal{G}_{\Omega,X})$$

The notion of weight again gives us the defintion of parahoric $\mathcal{G}_{\Omega,X}$–torsors (Definition 5.10) and therefore, the set of isomorphism classes of parahoric $\mathcal{G}_{\Omega,X}$–torsors (E, w_θ) is an equivalent description for the double coset space $Bun_Y^{\boldsymbol{\tau}}(\pi, G)$.

Note that if for every $x \in \mathcal{R}_p$ the unit group $\mathfrak{U}_x$ gets identified with the canonical "hyperspecial" parahoric subgroup $G(\mathcal{O}_{X,x})$ (upto conjugacy by elements of $G(K_x)$) (see 3.0.5), then the double coset space (1.0.10) identifies with the set of isomorphism classes of principal G–bundles on X, which is indeed the usual adèlic way of defining G–bundles. If on the other hand, $\mathfrak{U}_x \subsetneq G(\mathcal{O}_{X,x})$ for every $x \in \mathcal{R}_p$, then under the evaluation map $ev : G(\mathcal{O}_{X,x}) \to G(\mathbb{C})$, the subgroup $\mathfrak{U}_x$ maps to a standard parabolic subgroup of G, so that in this case a quasi-parahoric bundle could indeed be called a *quasi-parabolic G–bundle* in the familiar sense of the term when G is the full-linear group, i.e the data consists of a principal G–bundle on X together with a parabolic subgroup of G ("a flag") for every $x \in \mathcal{R}_p$.

In general there is a third case, namely there are parahoric subgroups of $G(K_x)$ which cannot be conjugated to subgroups of $G(\mathcal{O}_{X,x})$ and indeed, most parahoric subgroups of $G(K_x)$ fall under this third category (see [6]). It is this case which highlights the precise reason why we need to give a subtler description of (Γ, G)–bundles on Y as parahoric bundles on X which do not support a principal G–bundle on X. Evidence to this effect was shown using Tannakian considerations in [2], leading to the definition of a ramified bundle in [3]. More concrete examples were shown in [28] indicating what to expect in general. In this context note that [30, Theorem 2.3] is *incorrect*.

We remark that in the classical case when G is the full-linear group, every parahoric subgroup of $G(K_x)$ is in fact conjugate to a subgroup of $G(\mathcal{O}_{X,x})$, which explains why in this case we get only the usual parabolic structures. The striking cases which arise out of the present study are the "non-hyperspecial" maximal parahoric subgroups where a number of new phenomena seem to show up. These correspond, on the side of the representations of the Fuchsian group (see 1.0.5), to those maps $\rho : \pi \to K_G$ such that centralizers of the images of the elements $\rho(C_i)$ are *semisimple* and seem linked to the classical questions of non-existence of complex structures to homogeneous spaces under K_G. For these and other connections we refer the reader to the last section.

1.8. One defines the notion of *stable (resp. semistable, polystable)* (Γ, G)–bundles on Y following A. Ramanathan (see [3]). This can be translated into intrinsic definitions of such notions for parahoric bundles or equivalently for parahoric $\mathcal{G}$–torsors and the notion of weights is indispensable for these notions as in [18].

Let $M_Y^{\boldsymbol{\tau}(\mathcal{R}_p^*)}(\Gamma, G)$ (resp. $M_X(\mathcal{G})$) denote the set of isomorphism classes of polystable (Γ, G)–bundles on Y of local type $\boldsymbol{\tau}(\mathcal{R}_p^*)$ (resp. parahoric $\mathcal{G}$–bundles with fixed weights w_θ). We summarize the main results on moduli spaces in the following theorem:

1.9. Theorem.

(1) *(Corollary 7.19) Let $\mathcal{G}_{\Omega,X}$ be an affine Bruhat-Tits group scheme of finite type over X (see Definition 4.5). Then, there is an equivalence of categories between* stable *parahoric $\mathcal{G}_{\Omega,X}$–torsors, (E, w_θ) and* stable (Γ, G)*–bundles of local type $\boldsymbol{\tau}$ as well as with* irreducible *unitary representations of type $\boldsymbol{\tau}$ of the group π.*
(2) *(Theorem 7.15) There is a canonical structure of an irreducible normal projective variety on $M_Y^{\boldsymbol{\tau}(\mathcal{R}_p^*)}(\Gamma, G)$ (equivalently on $M_X(\mathcal{G})$). The set of points of this variety is non-empty and there is a stable object if the genus of X is larger than 1.*

(3) *(Theorem 6.4 and Theorem 7.5) For simplicity, let* $|\mathcal{R}_p| = 1$ *and let* $M_Y^{\boldsymbol{\tau}(\alpha)}(\Gamma, G)$ *be the moduli space of* (Γ, G)*–bundles of local type* $\boldsymbol{\tau}(\alpha)$ *(see Definition 6.3). Then the dimension of the moduli space is given by* $dim(G)(g-1) + dim(\frac{G}{P_\alpha}) - \mu(\alpha)$ *where* $\mu(\alpha) = \#\{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta \neq \alpha} x_\beta.\beta\}$*, and where* $P_\alpha \subset G$ *is the standard parabolic subgroup associated to the simple root* α *(see* (3.0.3) *for the definition of* c_α*).*

(4) *(Corollary 7.17) Let* $\overline{K}_G = K_G/centre$*. If* $\eta : \pi \to G$ *is a unitary representation of type* $\boldsymbol{\tau}(\mathcal{R}_p^*)$*, then the bundle* $E(\eta)$ *lies in* $M_Y^{\boldsymbol{\tau}(\mathcal{R}_p^*)}(\Gamma, G)$*; moreover, the map* ψ *defined in* (1.0.8) *induces a map* ψ^**:*

$$\psi^* : R^{\boldsymbol{\tau}(\mathcal{R}_p^*)}(\pi, K_G)/\overline{K}_G \to M_Y^{\boldsymbol{\tau}(\mathcal{R}_p^*)}(\Gamma, G) \tag{1.0.11}$$

which is a homeomorphism *of the underlying topological spaces.*

We have assumed above that the group G is semisimple and simply connected. In fact, the construction of the moduli spaces when G is reductive (not just semisimple and simply connected) can be carried out as an easy consequence of the semisimple and simply connected case. However, if G is not simply connected, the moduli stack need not be connected (see 7.22).

In the context of the identification (1.0.11), we came across a very interesting unpublished note by V. Drinfeld on the conjugacy classes of elements of G whose centralizers are *semisimple*. There are some striking parallels with the present paper and some interesting connections. We have summarized these in the final remarks.

The parahoric moduli spaces are linked to each other by Hecke correspondences (see 4.0.24). These relations are easily expressed in the language of stacks. It would be interesting to express these relations as morphisms between moduli spaces which have been constructed above as projective varieties.

Acknowledgements: The first author thanks Gopal Prasad for some helpful discussions on Bruhat-Tits theory.

2. Fonctions non-abéliennes et bornologie

2.0.1. **Some preliminaries.** Let G be a semisimple, simply connected algebraic group defined over $\mathbb{C}$ and we fix a maximal torus T of G. Let $X(T) := Hom(T, \mathbb{G}_m)$ be the character group and $Y(T) := Hom(\mathbb{G}_m, T)$ the group of 1-parameter subgroups of T. Let S be a system of simple roots and let $R = R(T, G) \subset X(T)$ be the root system associated to the adjoint representation of G.

Denote by $(\ , \ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form. The set S determines a system of positive roots $R^+ \subset R$ and a Borel subgroup $B \subset G$ with unipotent radical U. We now order the set $R^+ = \{r_i\}, i = 1, \ldots, q$. We then have a family $\{u_r : \mathbb{G}_a \to G \mid r \in R\}$ of *root homomorphisms* of groups such that one gets an isomorphism:

$$\prod_{i=1,\ldots,q} u_{r_i} : \prod_{i=1,\ldots,q} \mathbb{G}_a \to U \tag{2.0.1}$$

For every root $r \in R$, we denote by $T_r = Ker(r)^0$, and $Z_r = Z_G(T_r)$. The derived group $[Z_r, Z_r]$ is of rank 1 and there exists a unique 1PS, $r^\vee : \mathbb{G}_m \to T \cap [Z_r, Z_r]$ such that $T = Im(r^\vee).T_r$ and $(r^\vee, r) = 2$. The element $r^\vee$ is the coroot (or 1–PS) associated to r. The $\{r^\vee \mid r \in R\}$ form a system of roots $R^\vee$.

For each $r \in R$ the *root homomorphism*

$$u_r : \mathbb{G}_a \to G \tag{2.0.2}$$

is such that

$$t.u_r(a).t^{-1} = u_r(r(t).a) \tag{2.0.3}$$

for any $\mathbb{C}$–algebra A and for any $t \in T(A)$, $a \in A$ such that the tangent map du_r induces an isomorphism

$$du_r : Lie(\mathbb{G}_a) \to (LieG)_r$$

The functor $A \mapsto u_r(\mathbb{G}_a) = u_r(A)$ gives $U_r(A) \subset G(A)$. This determines a closed subgroup U_r of G and is called the *root group* corresponding to r.

Denote by $\{\alpha^* \mid \alpha \in S\}$ to be the basis dual to $\{\alpha \in S\}$, i.e $(\alpha^*, r) = \delta_{\alpha,r}$. Define

$$\mathbb{E} := Y(T) \otimes_{\mathbb{Z}} \mathbb{R} \tag{2.0.4}$$

$$\mathbb{E}' := X(T) \otimes_{\mathbb{Z}} \mathbb{R} \tag{2.0.5}$$

Most often, we in fact work with $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.0.2. **Parahoric subgroups.** Let K be a field equipped with a discrete valuation $v : K^\times \to \mathbb{Z}$ and we shall also assume that K is *complete*. Let A be the ring of integers, with residue field $\mathbb{C}$.

A subset $M \subset G(K)$ is said to be *bounded* if for any regular function $f \in K[G]$, the values $v(f(m))$ is bounded below, when m runs over all elements of M. In particular, we may talk of *bounded subgroups*. A subgroup $M \subset G(K)$ is therefore bounded if the "order of poles" of elements of M is bounded. This can be made precise by taking a faithful representation of $G \hookrightarrow GL(n)$ so that elements of M are represented by matrices with entries in K.

Let $\theta \in \mathbb{E}$ be an element of $\mathbb{E}$. Denote by $\mathfrak{P}_\theta(K) \subset G(K)$ the subgroup generated by $T(A)$ and the root groups $U_r(z^{m_r}A)$ for all the roots $r \in R$, where

$$m_r = m_r(\theta) = -[(\theta, r)] \tag{2.0.6}$$

the notation $[h]$ stands for the biggest integer smaller than h. In other words, we have:

$$\mathfrak{P}_{\theta}(K) = \langle T(A),\ U_r(z^{m_r(\theta)}A),\ r \in R \rangle \tag{2.0.7}$$

The group $\mathfrak{P}_{\theta}(K)$ is a bounded subgroup, more precisely it is a *parahoric subgroup* of $G(K)$ in the sense of Bruhat-Tits and conversely, any parahoric subgroup is bounded in the above sense (cf. [6]) . In fact, since we work with a semisimple and simply connected group G, all *parahoric groups* are, upto conjugacy by elements of $G(K)$, precisely the collection of groups $\{\mathfrak{P}_{\theta}(K)\}_{\theta \in \mathbb{E}}$, and as such we will work with these groups (see [32, Page 662]). In particular, associated to the "origin" $0 \in \mathbb{E}$ we have the group $\mathfrak{P}_0(K)$, which is nothing but the maximal bounded subgroup $G(A) \subset G(K)$. We will return to this in greater generality in §3.

Note that if $\theta \in Y(T)$ itself, then there exists $t \in T(K)$ such that

$$\mathfrak{P}_{\theta}(K) = t.\mathfrak{P}_0(K).t^{-1} \tag{2.0.8}$$

2.0.3. **Non-abelian functions and parahoric subgroups.** Let X and Y be as in the introduction and let $p : Y \to X$ be the covering projection. Let $\Gamma = Gal(Y/X)$. Let E be a (Γ, G)–bundle on Y. We stick to the notations in 1.5 in Section 1. We can identify D_x and N_y with analytic discs with centres (0) which correspond to analytic neighbourhoods of points $x \in X$ and $y \in Y$, for Riemann surfaces X and Y. We denote the stabilizer of the Γ–action at $y \in Y$ by Γ_y.

Observe that $D_x = N_y/\Gamma_y$. In other words, ω is the coordinate function on N_y so that $z = \omega^d$ is a coordinate function on D_x, where d is the order of the cyclic group Γ_y.

Consider the restriction of E to N_y. Then, by (1.0.2), as a (Γ_y, G) bundle we can identify E with the trivial bundle $N_y \times G$ (G action given by multiplication on the right and twisted action of Γ_y). Let U_y be the unit group at $y \in Y$ (see Definition 1.6).

We work with notations fixed above. Let $\rho : \Gamma_y \to G$ be a representation, where Γ_y is a cyclic group of order d. Let $\ell = rank(G)$ and we represent the maximal torus $T \subset G$ in the diagonal form as follows:

$$T = \begin{bmatrix} t_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & t_\ell \end{bmatrix} \tag{2.0.9}$$

Since Γ_y is cyclic, we can suppose that the representation ρ of Γ_y in G factors through T (by a suitable conjugation).

2.1. LEMMA. *Let Γ_y be a cyclic group of order d acting on N_y as above. Then we have a canonical identification*

$$Hom(\Gamma_y, T) \simeq \big(\frac{1}{d}Y(T)\big)(mod\, Y(T)) \tag{2.0.10}$$

Proof: The action of Γ_y on N_y canonically determines a character as follows. Since N_y is "1–dimensional", the action determines an action of Γ_y on the tangent space T_y to N_y at y. We denote this character by χ_o. Fix a generator γ in Γ_y. We can choose the coordinate function ω of N_y. Then the character χ_o is given by:

$$\chi_o(\gamma).\omega = \zeta.\omega$$

where ζ is a primitive d^{th}–root of unity.

Given a representation $\rho \in Hom(\Gamma_y, T)$, the image $\rho(\gamma)$ takes the form

$$\rho(\gamma) = \begin{bmatrix} \chi_o(\gamma)^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \chi_o(\gamma)^{a_\ell} \end{bmatrix} \tag{2.0.11}$$

i.e $\rho(\gamma)$ takes the form

$$\rho(\gamma) = \begin{bmatrix} \zeta^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \zeta^{a_\ell} \end{bmatrix} \text{ with } a_i \in \mathbb{Z}. \tag{2.0.12}$$

We can suppose that $|a_i| < d$ for all i (or even $0 \le a_i < d$) and take

$$\eta_i = a_i/d, \text{ so that } |\eta_i| < 1 \tag{2.0.13}$$

Note that the numbers $\{a_1, a_2, \ldots, a_\ell\}$ are determined uniquely modulo d.

In terms of the local coordinates ω and z, we may identify the function ω^{a_i} with z^{η_i} where $z = \omega^d$. Define the "meromorphic" (or "rational") map $\Delta : N_y \longrightarrow T$, or equivalently a morphism on the punctured disc $N_y - (0)$ as follows:

$$\Delta = \Delta(\omega) = \begin{bmatrix} \omega^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \omega^{a_\ell} \end{bmatrix} = \begin{bmatrix} z^{\eta_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & z^{\eta_\ell} \end{bmatrix} \tag{2.0.14}$$

Then we have

$$\Delta(\gamma u) = \rho(\gamma)\Delta(u), \quad u \in N_y \tag{2.0.15}$$

where Δ can be taken as a function $\Delta : N_y \longrightarrow G$ (through $T \hookrightarrow G$).

Consider the restriction of Δ to the *punctured disc* and view it as a 1PS, i.e $\Delta|_{Spec(L)} : \mathbb{G}_{m,L} \to G$. More precisely, the data of giving the function Δ together with its Γ_y–equivariance automatically gives a rational 1–PS of G, i.e an element $\theta_\Delta \in Y(T) \otimes \mathbb{Q}$ and the key point to note is that

$$d.\theta_\Delta = \Delta \tag{2.0.16}$$

i.e $\theta_\Delta \in \left(\frac{1}{d}Y(T)\right)(mod\ Y(T))$. Thus, the association $\rho \mapsto \theta_\Delta$ gives the required identification.

Q.E.D

2.2. *Remark.* We note that the tuple of numbers $\{a_1, a_2, \ldots, a_\ell\}$ are determined uniquely modulo d through the above identification.

Recall the definition of the unit group U_y (Definition 1.6). The aim of this section is to prove the following:

2.3. Theorem. *The unit group U_y is isomorphic to a parahoric subgroup $\mathfrak{P}_{\theta_\Delta}(K)$ of $G(K)$ associated to the element $\theta_\Delta \in Y(T) \otimes \mathbb{Q}$. Conversely, if $\mathfrak{P}_\theta(K)$ is any parahoric subgroup of $G(K)$ then there exists a positive integer d, a field extension $L = K(\omega)$ of degree d over K such that*

$$\mathfrak{P}_\theta(K) \simeq \mathsf{U}_y \tag{2.0.17}$$

Proof: We first give a different description of the elements of U_y. By (1.0.2) a (Γ_y, G)–bundle on Y gets a Γ_y-equivariant trivialization; in other words, the Γ_y-action on $N_y \times G$ is given by a representation $\rho : \Gamma_y \longrightarrow G$

$$\gamma \cdot (u, g) = (\gamma u, \rho(\gamma) g),\ u \in N_y,\ \gamma \in \Gamma_y. \tag{2.0.18}$$

Let $\phi_0 \in \mathsf{U}_y$. Then we see that the map

$$\phi_0 : N_y \times G \longrightarrow N_y \times G. \tag{2.0.19}$$

is equivariant for the Γ_y–action. This implies that

$$\phi_0(u, g) = (u, \phi(u) g)$$

where $\phi : N_y \longrightarrow G$ is a regular map satisfying the following:

$$\phi(\gamma \cdot u) \ = \ \rho(\gamma)\phi(u)\rho(\gamma)^{-1}, u \in N_y. \tag{2.0.20}$$

We may thus identify U_y with the following:

$$\mathsf{U}_y = \{\phi : N_y \to G \mid (2.0.20)\ holds\} \tag{2.0.21}$$

Observe that we can view $\mathsf{U}_y \subset G(B) \subset G(L)$.

Let Δ be as in (2.0.14). Consider the inner automorsphism defined by Δ:

$$i_\Delta : G(L) \to G(L) \tag{2.0.22}$$

given by $i_{\Delta}(\eta) = \Delta^{-1}.\eta.\Delta$. Define

$$\mathsf{U}'_y := i_{\Delta}(\mathsf{U}_y) \tag{2.0.23}$$

Let $\psi = i_{\Delta}(\phi) = \Delta^{-1}.\phi.\Delta$ with $\phi \in \mathsf{U}_y$. Then we observe that

$$\psi(\gamma u) = \psi(u)$$

so that $\psi \in G(L)^{\Gamma_y}$. That is, it *descends* to a rational function $\tilde{\psi} : D_x \longrightarrow G$, where $\tilde{\psi}(z) := \psi(\omega)$. In other words, we get

$$\mathsf{U}'_y \subset G(K) = G(L)^{\Gamma_y} \tag{2.0.24}$$

Then we claim the following:

$$\mathsf{U}'_y = \mathfrak{P}_{\theta_\Delta}(K) \tag{2.0.25}$$

where $\theta_{\Delta} \in Y(T) \otimes \mathbb{Q}$ is as in (2.0.16). Recall the definition of the parahoric subgroup:

$$\mathfrak{P}_{\theta_\Delta}(K) = \langle T(A),\ U_r(z^{m_r(\theta_\Delta)}A),\ r \in R\rangle \tag{2.0.26}$$

Let $\psi \in \mathsf{U}'_y$ and let $\psi = i_{\Delta}(\phi)$, with $\phi \in \mathsf{U}_y$. Thus,

$$\phi = \Delta\psi\Delta^{-1}.$$

We can describe $\phi = \big(\phi_r(u)\big)_{r\in R}$ and $\psi = \big(\psi_r(u)\big)_{r\in R}$, where the

$$\{\phi_r, \psi_r : \mathbb{G}_{a,L} \to G \mid r \in R\}$$

satisfy the following:

$$\phi_r(\omega) = \Delta\psi_r(\omega)\Delta^{-1}. \tag{2.0.27}$$

i.e

$$\phi_r(\omega) = \psi_r(\omega)\omega^{r(\Delta)} \tag{2.0.28}$$

In terms of $\tilde{\psi}$, this gives:

$$\phi_r(\omega) = \tilde{\psi}_r(z)z^{\frac{r(\Delta)}{d}} \tag{2.0.29}$$

Now interpreting the condition that $\tilde{\psi}$ should satisfy so that the ϕ's are regular functions in the variable ω at $\omega = 0$, we see that the *order of pole* for $\psi_r(z)$ at $z = 0$, is bounded above by $[\frac{r(\Delta)}{d}]$ (the biggest integer smaller than $\frac{r(\Delta)}{d}$) . In other words $\forall r \in R$,

$$\tilde{\psi}_r(z) \in U_r(z^{-[r(\theta_\Delta)]}A) = U_r(z^{m_r(\theta_\Delta)}A) \tag{2.0.30}$$

and hence $\tilde{\psi} \in \mathfrak{P}_{\theta_\Delta}(K)$. This proves the claim (2.0.25).

Conversely, we show that any parahoric subgroup of $G(K)$ can be identified, upto conjugation by a $g \in G(K)$, with a *unit group* U_y. Let $\theta \in \mathbb{E}$ and let $\mathfrak{P}_\theta(K)$ be a parahoric subgroup. We would like to modify θ to a $\theta_{_\Delta}$ for a suitable $\Delta \in Y(T)$ so that, interpreted as unit groups we get $\mathfrak{P}_{_\theta}(K) \simeq \mathfrak{P}_{_{\theta_\Delta}}(K) \simeq \mathsf{U}_y$.

We observe firstly that the parahoric subgroup $\mathfrak{P}_{_\theta}(K)$ given by $\theta \in \mathbb{E}$ remains the same when another choice of θ is made in a neighbourhood. In other words, we may assume without loss of generality that $\theta \in Y(T) \otimes \mathbb{Q}$. Expressing it in terms of generators and clearing denominators, we see that there exists a positive integer d so that $d.\theta \in Y(T)$. Then the obvious choice is $\Delta = d.\theta$ which therefore forces $\Delta \in Y(T)$.

Now we view Δ as a "rational" map $\Delta : N_{_y} \to T$ and hence can be expressed as in (2.0.14), the a_i's being determined by the following considerations: for $r \in R$ be any root we define

$$r(\Delta) = d.(\theta, r)$$

By Lemma 2.1 and Remark 2.2, the numbers $\{a_1, a_2, \ldots, a_\ell\}$ are determined uniquely modulo d. Once this is done, then we may define the representation ρ in terms of these a_i's by the Lemma 2.1. More precisely, the representation ρ is determined by its action on the root groups $U_r(B) \subset G(B)$ which are given by (see 2.0.3):

$$\rho(\gamma).U_r(B).\rho(\gamma)^{-1} = U_r(\zeta^{r(\Delta)}B) \tag{2.0.31}$$

Retracing the steps in the first half of the proof, it is easy to see that $\mathfrak{P}_{_\theta}(K) \simeq \mathsf{U}_y$ completing the proof of the theorem.

Q.E.D

2.4. *Remark.* In the next section we will see the entire discussion carried out above in the more general setting of Bruhat-Tits theory. In fact, since $\Delta \in Y(T)$, the parahoric subgroup $\mathfrak{P}_{_\Delta}(L) \subset G(L)$ can be identified with $G(B)$, upto conjugation by an element $t \in T(L)$ (see (2.0.8)). The Galois group $\Gamma_{_y}$ acts on the Bruhat-Tits building as well as the parahoric subgroups and we see that $\mathfrak{P}_{_{\theta_\Delta}}(K) = \mathfrak{P}_{_\Delta}(L)^{\Gamma_y}$. The identification of the elements of U_y as ψ's forces the containment $\psi \in \mathfrak{P}_{_\Delta}(L)^{\Gamma_y}$ and hence the identification $\mathsf{U}_y = \mathfrak{P}_{_{\theta_\Delta}}(K)$.

2.5. Definition. *Let $\theta \in Y(T) \otimes \mathbb{Q}$. Let $\Delta = d.\theta$ as above. To this data, we associate a representation $\rho_{_\theta} : \Gamma_{_y} \to G$ defined by the a_i's and given by Lemma 2.1 and which acts on the root groups by* (2.0.31)*. In particular, let $\alpha \in S$ be a simple root. Let $\theta_{_\alpha} = \frac{\alpha^*}{c_\alpha}$ (see* (3.0.7) *below). We denote the representation $\rho_{_{\theta_\alpha}}$ by $\rho_{_\alpha}$.*

2.6. *Remark.* It is remarked in [28, Page 8] that it was not clear whether the unit group in case III considered there is a parahoric subgroup at all. In fact, this is *the case* as can be seen by Theorem 2.3. Moreover, it is not too hard to check by some elementary computations that in the case considered there the unit group contains the standard Iwahori but after a conjugation by a suitable element of $G(K)$.

2.7. *Example.* Let us now take $G = GL(m)$. We invite the reader to compare this discussion with the one in Weil ([33, page 56]). Then we can write $\phi = ||\phi_{ij}(\omega)||$, $\tilde{\psi} = ||\tilde{\psi_{ij}}(z)||$, $1 \leq i, j \leq m$ (as matrices). Then (2.0.29) takes the form

$$\phi_{ij}(\omega) = \tilde{\psi_{ij}}(z) z^{\alpha_i - \alpha_j}. \tag{2.0.32}$$

We can suppose that $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m < 1$. Since $|\alpha_i - \alpha_j| < 1$, we deduce easily that $\tilde{\psi_{ij}}$ are regular i.e. $\mathsf{U}_y \subset G(A)$. To see this suppose that $\tilde{\psi_{ij}}$ is *not* regular. Then considered as a function in ω ($z = \omega^d$), ψ_{ij} has a pole of order $\geq d$, whereas $z^{\alpha_i - \alpha_j}$ could have only a pole of order d (as a function in ω). But $\phi_{ij}(\omega)$ is regular, which leads to a contradiction.

3. Relationship with Bruhat-Tits theory

Let the notations be as in the beginning of §2. Recall from §2 (2.0.4) the notations $\mathbb{E}$ and $\mathbb{E}'$. Denote by Q(resp $Q^\vee$) the lattice of R (resp $R^\vee$), i.e the subgroup of $X(T)$ (resp $Y(T)$) generated by R (resp $R^\vee$), i.e, the root lattice and the coroot lattice respectively. Denote by

$$P = \left\{ x \in \mathbb{E}' \mid (\beta^\vee, x) \in \mathbb{Z} \ \forall \ \beta^\vee \in R^\vee \right\} \tag{3.0.1}$$

the weight lattice and

$$P^\vee = \left\{ x \in \mathbb{E} \mid (x, \beta) \in \mathbb{Z} \ \forall \ \beta \in R \right\} = \oplus_{\alpha \in S} \mathbb{Z} \alpha^* \tag{3.0.2}$$

We have the inclusions $Q^\vee \subset Y(T) \subset P^\vee$. It is known that the quotient $P^\vee / Q^\vee \simeq$ *centre*(G). *If we assume G to be simply connected, then we have*

$$Q^\vee = Y(T).$$

Let α_{max} denote the highest root. Then

$$\alpha_{max} = \sum_{\alpha \in S} c_\alpha \cdot \alpha \tag{3.0.3}$$

with $c_\alpha \in \mathbb{Z}^+$.

3.1. Definition. *Define*

$$d_1 = exponent(P^\vee / Q^\vee) \tag{3.0.4}$$

and

$$d_2 = lcm_{\alpha \in S}\{c_\alpha\} \tag{3.0.5}$$

3.0.4. **Parahoric subgroups.** Let $\Omega \subset \mathbb{E}$ be a nonempty subset of $\mathbb{E}$. Denote by $\mathfrak{P}_{\Omega}(K) \subset G(K)$ the subgroup generated by $T(A)$ and the root groups $U_r(z^{m_r}A)$ for all the roots $r \in R$, where

$$m_r = m_r(\Omega) = -[inf_{\theta \in \Omega}(\theta, r)] \tag{3.0.6}$$

The group $\mathfrak{P}_{\Omega}(K)$ is a parahoric subgroup. The precise relationship of these parahoric subgroups with the general ones occurring in Bruhat-Tits theory has been given in the paragraph following(2.0.7) and (2.0.8).

3.0.5. **Hyperspecial Parahorics.** In Bruhat-Tits theory, we encounter the so-called *hyperspecial* maximal parahorics which have the following characterizing property: each parahoric group $\mathfrak{P}_{\Omega}(K)$ is identified with $\mathcal{G}_{\Omega}(A)$, the A–valued points of a certain canonically defined smooth group scheme $\mathcal{G}_{\Omega}$ defined over A. It is a fact that the parahoric subgroup $\mathfrak{P}_{\theta_\alpha}(K)$ is hyperspecial if and only if $c_\alpha = 1$ in the description of the long root α_{max}. This can be checked by an inspection of the tables and some easy computations. In particular, type-wise we have the following description upto conjugation by $G(K)$:

(1) In type A_n, all the $n+1$ maximal parahoric subgroups are hyperspecial parahorics.
(2) In types B_n, C_n we have exactly 2 hyperspecial maximal parahoric subgroups.
(3) Type D_n, has exactly 4 hyperspecials maximal parahoric subgroups.
(4) Type E_6 has exactly 3 maximal parahoric subgroups.
(5) Type E_7 has exactly 2 maximal parahoric subgroups.
(6) In types G_2, F_4, E_8, we have only one hyperspecial maximal parahoric subgroup each.

3.2. *Remark.* The hyperspecial simple roots correspond precisely to the dual notion of *minuscule coweights*.

3.0.6. **Standard parahorics.** Following the loop group terminology, the *standard parahoric subgroups* of $G(K)$ are parahoric subgroups of the canonical hyperspecial parahoric subgroup $G(A)$. These are realized as inverse images under the evaluation map

$$ev : G(A) \to G(k)$$

of standard parabolic subgroups $P_I \subset G$, where $I \subset S$ is any subset of the simple roots. In particular, the *Iwahori subgroup* $\mathfrak{I}$ is a standard parahoric and indeed, $\mathfrak{I} = ev^{-1}(B)$, $B \subset G$ being the standard Borel subgroup containing the fixed maximal torus T.

Since we have assumed that G is semisimple and simply connected, it is known (see for example [31, Section 3.1, page 50]) that every parahoric subgroup of $G(K)$, upto conjugation by an element of $G(K)$, can be identified with a $\mathfrak{P}_{\theta}(K)$ for a suitable $\theta \in \mathbb{E}$. If furthermore, for every $\alpha \in S$, we define

$$\theta_\alpha = \frac{\alpha^*}{c_\alpha} \in \mathbb{E}, \tag{3.0.7}$$

then in fact, $\{\mathfrak{P}_{\theta_\alpha}(K) \mid \alpha \in S\}$ *and the group* $\mathfrak{P}_0(K)$ *represent the conjugacy classes under* $G(K)$ *of all* maximal parahoric subgroups *of* $G(K)$ (see the last paragraph in [32, Page 662]). In other words, these are indexed precisely by the vertices of the *extended Dynkin diagram.*

Since the *standard parahoric subgroups* of $G(A)$ are also indexed by the subsets of the set of simple roots, to avoid any confusion, we will henceforth denote the *standard parahoric subgroups* of $G(A)$ by $\mathfrak{P}^{st}_{I}(K)$ for every subset $I \subset S$. For instance let $\alpha \in S$. Then $P_\alpha \subset G$ is a maximal parabolic subgroup while $ev^{-1}(P_\alpha) = \mathfrak{P}^{st}_{\alpha}(K)$ is a standard parahoric, and we have the obvious inclusions:

$$\mathfrak{I} \subset \mathfrak{P}^{st}_{\alpha}(K) \subset \mathfrak{P}_{\theta_\alpha}(K) \cap \mathfrak{P}_0(K) \tag{3.0.8}$$

These standard parahorics will play a role when we re-look at Hecke correspondences.

3.0.7. **Galois action on buildings.** For the notion of Bruhat-Tits buildings and their behaviour under field extensions (cf. [31, Page 43]). Let ω be the primitive d-th root of z, where z is the uniformizer of K. Let $L = K^d$ and $B = A[\omega]$ the integral closure of A in L. Let $\mathfrak{G} = Gal(L/K)$, which is a cyclic group of order d.

Thus, $G(K) \subset G(L)$. Consider the multiplication $d : \mathbb{E} \to \mathbb{E}$. Then, if $\Omega \subset \mathbb{E}$ is a subset, denote by $d(\Omega)$ its image in $\mathbb{E}$; then one has the relation

$$\mathfrak{P}_{\Omega}(K) \subset \mathfrak{P}_{d(\Omega)}(L)$$

The choice of T identifies $\mathbb{E}$ with an apartment in the building $\mathcal{B}(G,K)$ as well as one in $\mathcal{B}(G,L)$. Further, it is known that there is a canonical injection of buildings:

$$i_{K,L} : \mathcal{B}(G,K) \subset \mathcal{B}(G,L)$$

which maps an apartment in $\mathcal{B}(G,K)$ into one in $\mathcal{B}(G,L)$ and the image of $\mathcal{B}(G,K)$ is fixed pointwise by $\mathfrak{G}$ (since we are in the tamely ramified case). If $App(G,K)$ denotes an apartment in $\mathcal{B}(G,K)$ associated to T, then every vertex and barycenter of a facet of the apartment $App(G,K)$ becomes, in $App(G,L)$, a translate of the hyperspecial vertex (corresponding to $\mathfrak{P}_0(K)$) by $Y(T)$ and hence a conjugate by an element of $T(L)$. If we fix an origin in $App(G,K)$, then one can identify it with the vector space $\mathbb{E}$. Fix a facet F in $App(G,K) = \mathbb{E}$ and let $\theta \in F$ be a point in general position. Then $d.\theta$ lies in $App(G,L)$.

In fact, we have:

$$\mathfrak{P}_{\theta}(K) = \left[\mathfrak{P}_{d.\theta}(L)\right]^{\mathfrak{G}} \tag{3.0.9}$$

To see this, we now consider the action of $G(L)$ on the apartment $\mathbb{E}$, and let $G(L)_{d.\theta}$ denote *the stabilizer of* $d.\theta$ *by the canonical action of* $G(L) \ltimes \mathfrak{G}$ *on the building* $\mathcal{B}(G, L)$. Then by the general theory of Bruhat-Tits, since $\mathfrak{P}_{d.\theta}(L) = G(L)_{d.\theta}$, we have

$$\left[\mathfrak{P}_{d.\theta}(L)\right]^{\mathfrak{G}} = \left[G(L)_{d.\theta}\right]^{\mathfrak{G}} = \left[G(L)^{\mathfrak{G}}\right]_{\theta} = G(K)_{\theta} = \mathfrak{P}_{\theta}(K) \tag{3.0.10}$$

3.3. *Remark.* Compare this discussion with Remark 2.4.

3.4. Proposition. *Let* $\mathfrak{P}_{\Omega}(K)$ *be any parahoric subgroup of* $G(K)$. *Then there exists a positive integer* d, *a field extension* $L = K(\omega)$ *of degree* d *over* K *and a* $g \in G(L)$ *such that*

$$g.\mathfrak{P}_{\Omega}(K).g^{-1} \subset G(B) \tag{3.0.11}$$

Moreover,if $\mathfrak{G} = Gal(L/K)$, *the action of* $\mathfrak{G}$ *lifts to the Bruhat-Tits building over the field* L *and we have:*

$$\left[g^{-1}.G(B).g\right]^{\mathfrak{G}} = \mathfrak{P}_{\Omega}(K) \tag{3.0.12}$$

Proof: (following [8, Lemma I.1.3.2], [17, Lemma 2.4] and [25, Proposition 8, p. 546]) Each parahoric subgroup containing a fixed Iwahori can be identified with the stabilizer of a facet of the building $\mathcal{B}(G, K)$. Furthermore, the stabilizer for the $G(K)$–action of a facet can be realized as the stabilizer of a general point on the facet (cf. [31, §3.1, page 50]). In particular, we may assume that $\Omega = \{\theta\}$, i.e a singleton and express it as a rational linear combination of the θ_α's, say $\theta = \sum_{\alpha \in S} b_\alpha.\theta_\alpha$. If the parahoric is $\mathfrak{P}_0(K) = G(A)$, there is nothing to check.

Observe that $c_\alpha.\theta_\alpha = \alpha^*$ and therefore $c_\alpha.\theta_\alpha \in P^\vee$. Again, since $d_1 = exponent(P^\vee/Q^\vee)$, it follows that $d_1.P^\vee \subset Q^\vee \subset Y(T)$.

Thus, $d_1.c_\alpha.\theta_\alpha \in Y(T)$ and *a fortiori*, $d_1.d_2.\theta_\alpha \in Y(T)$. Choose, $m =$ lcm of denominators of b_α, then clearly $d.\theta \in Y(T)$, where $d = d_1.d_2.m$.

By (2.0.8), this implies that there exists a $t \in T(L)$ such that $t.\mathfrak{P}_{d.\theta}(K).t^{-1} = \mathfrak{P}_0(L) = G(B)$. Then by going to this d–sheeted ramified cover, with $d = d_1.d_2.m$, we see that $d.\theta$ is actually hyperspecial. Hence, by (3.0.9), we get the required proposition.

Q.E.D

4. The moduli stack of parahoric bundles

We stick to the notations in 1.5 and in Theorem 2.3. For the sake of simplicity we will assume that $|\mathcal{R}_p| = 1$ and we fix $x \in \mathcal{R}_p$ in the set of ramifications.

4.0.8. **The adèlic picture.** We shall now describe (Γ, G) bundles on Y in terms of objects on X, where $\Gamma = Gal(Y/X)$ as before.

Two (Γ, G) bundles on Y are said to be *locally isomorphic at* x if they are isomorphic as (Γ, G) bundles over $p^{-1}(D_x) = V_1$, D_x a sufficiently small analytic nighbourhood of x as above. We can suppose that V_1 is a disjoint union of discs of the form N_y i.e. each disc contains a unique point of Y lying over x. We denote the stabilizer of the Γ–action at $y \in Y$ by Γ_y. Observe that N_y is a Γ_y-invariant neighbourhood of y, y being a point of Y lying over x. We see that two such bundles are locally isomorphic at x if and only if their restrictions to N_y are isomorphic as Γ_y-bundles. Recall (1.0.2), that (Γ_y, G)–bundles are locally given by

$$\gamma \cdot (u, g) = (\gamma u, \rho(\gamma) g),\ u \in N_y,\ \gamma \in \Gamma_y. \tag{4.0.1}$$

Let $E \in Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ be a (Γ, G) bundle on Y of local type $\boldsymbol{\tau}$ (see Definition 1.2). Let

$$X_1 = X - x,\ and\ Y_1 = p^{-1}(X_1) \tag{4.0.2}$$

Now Γ acts freely on Y_1 so that the restriction of E to X_1 goes down to a principal G-bundle on X_1 which is trivial (in the algebraic sense) since G is semi-simple. Hence we have:

$$E|_{Y_1} \simeq Y_1 \times G \text{ with the action of } \Gamma \text{ given by } \gamma \cdot (u, g) = (\gamma u, g),\ \gamma \in \Gamma \text{ and } u \in Y_1. \tag{4.0.3}$$

Let

$$E|_{V_1} = E_1\ and\ E|_{Y_1} = E_2 \tag{4.0.4}$$

We note that $E_1|_{N_y}$ is given by (4.0.1) and $E_2|_{N_y}$ by (4.0.3).

The (Γ, G)–bundle E is given by a (Γ, G)–isomorphism

$$\vartheta : E_2|_{V_1 \cap Y_1} \longrightarrow E_1|_{V_1 \cap Y_1}. \tag{4.0.5}$$

Let s be a Γ-invariant rational section of E over Y_1. Let s_1 be the restriction of s to a section of $E_1|_{V_1 \cap Y_1}$ and s_2 the restriction of s to a section of $E_2|_{V_1 \cap Y_1}$.

Note that s exists, since the restriction of E to Y_1 descends to the G-bundle on X_1 which is trivial in the algebraic sense.

By (4.0.5), the isomorphism ϑ takes s_2 to s_1. We write this in this form:

$$\vartheta s_2 = s_1. \tag{4.0.6}$$

Observe that if Q is any (Γ, G)–bundle in $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$, then $Q|_{V_1} \simeq E_1$ and $Q|_{Y_1} \simeq E_2$ as (Γ, G)–bundles by (4.0.1) and (4.0.3) above since $Q|_{V_1}$ is completely determined by $Q|_{N_y}$. Thus Q is defined by an isomorphism as in (4.0.5) above. Let us denote it by ϕ. Then E is (Γ, G)–isomorphic to Q if and only if we have the following:

$$\lambda\ \vartheta\ \mu = \phi \tag{4.0.7}$$

where λ is a (Γ, G)–automorphism of E_1 and μ a (Γ, G)–automorphism of E_2 .

Observe that by (4.0.3) the map μ is given by a morphism:

$$\begin{aligned} Y_1 \times G &\longrightarrow Y_1 \times G, \\ (u, g) &\to (u, \mu^*(u)g), \end{aligned} \tag{4.0.8}$$

where $\mu^*(\gamma \cdot u) = \mu(u), \gamma \in \Gamma$. In other words, the map μ^* goes down to a morphism $X_1 \longrightarrow G$.

We now trace the various identifications by restricting the above picture to the punctured disc $N_y^* = N_y - (0) \hookrightarrow V_1 \cap Y_1$; note that the (Γ, G)–isomorphism ϑ is completely characterized by its restriction to N_y^*.

We observe by (4.0.3) that the restriction of E_2 to N_y^* is the (Γ_y, G)–bundle $N_y^* \times G$ over N_y^* with the action of Γ_y given by

$$\begin{aligned} \gamma : N_y^* \times G \longrightarrow N_y^* \times G, \ \gamma \in \Gamma_y \\ \gamma(u, g) = (\gamma u, g). \end{aligned} \tag{4.0.9}$$

The restriction of E_1 to N_y^* is the (Γ_y, G)–bundle $N_y^* \times G$ on N_y^* with the action of Γ_y given by

$$\begin{aligned} &\gamma : N_y^* \times G \longrightarrow N_y^* \times G \\ &\gamma(u, g) = (\rho u, \rho(\gamma)g), \ \gamma \in \Gamma_y. \end{aligned} \tag{4.0.10}$$

The restriction of ϑ to N_y^* is then a (Γ_y, G)–isomorphism of the bundle in (4.0.8) with the one of (4.0.7). We see easily that ϑ is defined by the map:

$$\begin{aligned} N_y^* \times G &\longrightarrow N_y^* \times G \\ (u, g) &\longrightarrow (u, \vartheta^*(u)g) \end{aligned} \tag{4.0.11}$$

where $\vartheta^* : N_y^* \to G$ is such that $\vartheta^*(\gamma \cdot u) = \rho(\gamma)\vartheta(u)$.

Recall that the map Δ as in (2.0.15) is a morphism $N_y^* \longrightarrow G$ and has similar properties. Thus we can write

$$\vartheta^* = \Delta\vartheta^{**} \text{ such that } \vartheta^{**}(\gamma u) = \vartheta^{**}(u) \tag{4.0.12}$$

i.e. ϑ^{**} descends to a regular map $D_x^* \longrightarrow G$, $D_x^* = D_x - (0)$.

Note the following

(4.0.13) ϑ^{**} extends to a meromorphic map $N_y \longrightarrow G$. It is furthermore regular on N_y^* and hence by (4.0.12) descends to a meromorphic map $D_x \longrightarrow G$ which is regular on D_x^*.

This is an easy consequence of (4.0.6) with s_1, s_2 representing a rational section of E over Y (we could do explicit computations for ϑ^*, s_1, s_2 by restriction to N_y etc. as we did above).

Now the (Γ, G)–automorphisms of E_1 can be identified with the (Γ_y, G)–automorphisms of the restriction of E_1 to N_y i.e. of the (Γ_y, G)–bundle on N_y given by (4.0.1). Thus λ identifies with an element λ^* of the unit group at x (i.e. an element as in (4.0.13) above before its identification as a subgroup of $G(K_x)$). The equivalence relation (4.0.7) therefore takes the following form:

$$\lambda^*(\Delta\vartheta^{**})\mu^* = \Delta\phi^{**} \tag{4.0.14}$$

which implies

$$(\Delta^{-1}\lambda^*\Delta)\vartheta^{**}\mu^{**} = \phi^{**}. \tag{4.0.15}$$

Let us denote by $G(K_x^h)$ the set of germs of holomorphic maps $D_x^* \longrightarrow G$ which extend to meromorphic maps $D_x \longrightarrow G$. We have an inclusion $G(K_x^h) \subset G(K_x)$.

Set $\mathfrak{U}_x^h = \mathfrak{U}_x \cap G(K_x^h)$ i.e. we work in the holomorphic and meromorphic setup instead of the "formal" one (see Definition 1.6 and the paragraphs following this for the definition of $\mathfrak{U}_x$). Observe that $\vartheta^{**} \in G(K_x^h)$ and $(\Delta^{-1}\lambda^*\Delta) \in \mathfrak{U}_x^h$. Further μ^{**} is a regular map $X_1 \longrightarrow G$ i.e. $\mu^{**} \in G(X - x)$. Thus from (4.0.15) we deduce the following identification of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ with a double coset space:

$$Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \simeq \left[\mathfrak{U}_x^h \backslash^{G(K_x^h)} / G(X - x)\right] \tag{4.0.16}$$

4.1. *Remark.* Recall that, by Theorem 2.3, any parahoric subgroup of $G(K)$ is isomorphic to a unit group U_y. Let us fix a $\mathfrak{U}_x$ in its conjugacy class for every $x \in \mathcal{R}_p$. So we can talk of the *unit group at* x and denote it by $\mathfrak{U}_x$. The discussion carried out above in the analytic setting carries over without serious difficulty to the "formal" setting. In other words, *the moduli stack* $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ can be identified with

$$\left[\mathfrak{U}_x \backslash^{G(K_x)} / G(X - x)\right] \tag{4.0.17}$$

or in adèlic language with:

$$\prod_{p \in X} G(\mathcal{C}_p) \backslash^{G(\mathfrak{A})} / G(K(X)) \tag{4.0.18}$$

where $\mathfrak{A}$ denotes the adèles, $K(X)$ the function field of X, and $G(\mathcal{C}_p) = G(\mathcal{O}_p)$ for $p \neq x$ and $G(\mathcal{C}_x) = \mathfrak{U}_x$.

On the other hand, a choice of a parahoric subgroup $\mathfrak{P}_\Omega(K) \subset G(K)$ defines canonically the double coset space

$$\mathfrak{M}_X(\mathfrak{P}_\Omega(K)) := \left[\mathfrak{P}_\Omega(K) \backslash^{G(K)} / G(X - x)\right] \tag{4.0.19}$$

The above description immediately allows us to make the following definition in accordance with the notion of a quasi-parabolic structure in [18]. Recall from §2.0.2 the notion of "boundedness" of subsets of $G(K)$.

4.2. Definition. *Fix a parahoric subgroup* $\mathfrak{P}_{\Omega}(K) \subset G(K)$. *A* quasi-parahoric *bundle is an element* $\mathcal{E} \in \mathfrak{M}_{X}(\mathfrak{P}_{\Omega}(K))$; *equivalently, giving* $\mathcal{E}$ *is giving an isomorphism of principal homogeneous spaces:*

$$\psi : E_{K} \simeq G \times Spec(K) \tag{4.0.20}$$

together with the bounded subset $\psi^{*}(\mathfrak{P}_{\Omega}(K)) \subset E_{K}(K)$.

An isomorphism of quasi-parahoric bundles is given by a diagram:

$$\begin{array}{ccc} E_{K} & \xrightarrow{\quad f \quad} & E'_{K} \\ & \searrow^{\psi} \quad \swarrow_{\phi} & \\ & G \times Spec(K) & \end{array} \tag{4.0.21}$$

such that the bounded subsets $\psi^{*}(\mathfrak{P}_{\Omega}(K))$ and $\phi^{*}(\mathfrak{P}_{\Omega}(K))$ get identified by f.

4.3. *Remark.* In fact, if we work in the analytic category, it is not hard to see that in the above arguments, with the parameter $t \in T$ thrown in, we get an identification of a T–valued point of $Bun_{Y}^{\boldsymbol{\tau}}(\Gamma, G)$ with a T–point in the double coset space. One needs a Γ–invariant local trivialization for families and the details can be seen in the proof of Theorem 4.6 below.

4.4. *Remark.* The moduli stack $Bun_{Y}^{\boldsymbol{\tau}}(\Gamma, G)$ clearly depends only on the unit group $\mathfrak{U}_{x}$ and not on the explicit nature of the representation ρ. We can therefore denote $Bun_{Y}^{\boldsymbol{\tau}}(\Gamma, G)$ as $\mathfrak{M}(\mathfrak{U}_{x})$.

4.0.9. *Bruhat-Tits group schemes.* By the main theorem of Bruhat-Tits ([6]), there exist smooth group schemes $\mathcal{G}_{\Omega}$ over $Spec(A)$ such that the group $\mathcal{G}_{\Omega}(A) = \mathfrak{P}_{\Omega}(K)$.

4.5. Definition. (following Pappas and Rapoport [21]) *The Bruhat-Tits group scheme* $\mathcal{G}_{\Omega,X}$ *on the curve* X *is the one obtained by gluing the group scheme* $\mathcal{G}_{\Omega}$ *on* $Spec(A)$ *with the trivial group scheme* $(X - x) \times G$.

Let $Bun_{X}(\mathcal{G}_{\Omega,X})$ denote the moduli stack of principal homogeneous spaces under $\mathcal{G}_{\Omega,X}$. We then have the following obvious set-theoretic identification:

$$\mathfrak{M}_{X}(\mathfrak{P}_{\Omega}(K)) \simeq Bun(\mathcal{G}_{\Omega,X}) \tag{4.0.22}$$

Let $\mathfrak{P}_{\Omega}(K) \subset G(K)$ be a parahoric subgroup. Suppose that we fix the identification $w_{\theta} : \mathfrak{P}_{\Omega}(K) \simeq \mathfrak{P}_{\theta}(K)$ (see (5.0.1) for the notion of weights). Then we have the following:

4.6. Theorem. *There exists a ramified cover* $p : Y \to X$ *ramified at* $x \in X$, *such that we have an isomorphism of the stacks* $Bun(\mathcal{G}_{\Omega,X})$ *and* $Bun_{Y}^{\boldsymbol{\tau}}(\Gamma, G)$.

Proof: The identification $w_\theta : \mathfrak{P}_\Omega(K) \simeq \mathfrak{P}_\theta(K)$ together with Theorem 2.3 immediately gives the ramified cover $p : Y \to X$ and the set-theoretic identification of the points of the respective stacks. We need to show that we have an isomorphism of the corresponding functors. The smooth Artin stack structure on $Bun(\mathcal{G}_{\Omega,X})$ has been given in [11].

Let T be a finite type scheme over $\mathbb{C}$. The question boils down to defining a *family of* (Γ, G)*–bundles of local type* $\boldsymbol{\tau}$. Let $E = E_T \to Y \times T$ be a family of (Γ, G)–bundles. Then by [30, Lemma 2.5], for any $t \in T$, there exists an étale neighbourhood T_t of t and a formal neighbourhood $\tilde{N}_y$ of $y \in Y$, such that the action of Γ on $E|_{\tilde{N}_y \times T_t}$ gets a uniform trivialization by a representation $\rho : \Gamma_y \to G$. Thus, there exists an étale covering $T' \to T$ such that the pull-back $E|_{\tilde{N}_y \times T'}$ has uniform local type $\boldsymbol{\tau}$. By Theorem 2.3 we get a $\mathcal{G}_A$–torsor $\mathcal{E}|_{D_x \times T_t}$, where $D_x = Spec(A)$. The Γ–equivariant triviality of E on $(Y - p^{-1}(x)) \times T'$ gives a trivial $\mathcal{G}$–torsor on $(X - x) \times T'$ (which is the result of Drinfeld-Simpson ([7]). Gluing, we get a $\mathcal{G}$–torsor on $X \times T'$. The converse follows similarly by using [11, Theorem 1] and we get the desired isomorphism of functors.

Q.E.D

4.7. *Remark.* If $\mathfrak{P}_\Omega(K) = \mathfrak{P}_0(K)$ is the hyperspecial parahoric $G(A) \subset G(K)$, then the double coset space $\mathfrak{M}_X(\mathfrak{P}_0(K))$ is simply the usual moduli stack $\mathfrak{M}_X(G)$ of principal G–bundles on X.

4.8. Definition. (see 3.0.6) *Define the standard Iwahori subgroup* $\mathfrak{I} \subset G(K)$ *as*

$$\mathfrak{I} = \mathfrak{P}_S(K)$$

S being the set of simple roots in R.

In what follows, we consider parahoric subgroups $\mathfrak{P}_\Omega(K)$ of $G(K)$ such that $\mathfrak{I} \subset \mathfrak{P}_\Omega(K)$. The next proposition shows that the moduli stack depends only on the conjugacy class of the parahoric subgroup.

4.9. Proposition. *Let* $g \in G(K)$ *and consider parahoric subgroups* $\mathfrak{P}_\Omega(K)_g = g.\mathfrak{P}_\Omega(K).g^{-1}$ *and* $\mathfrak{P}_\Omega(K)$ *of* $G(K)$*. Then there is a natural isomorphism*

$$\phi_g : \mathfrak{M}_X(\mathfrak{P}_\Omega(K)_g) \to \mathfrak{M}_X(\mathfrak{P}_\Omega(K)) \tag{4.0.23}$$

Proof: This is follows easily from the following observation. Given a $g \in G(K)$, define the map

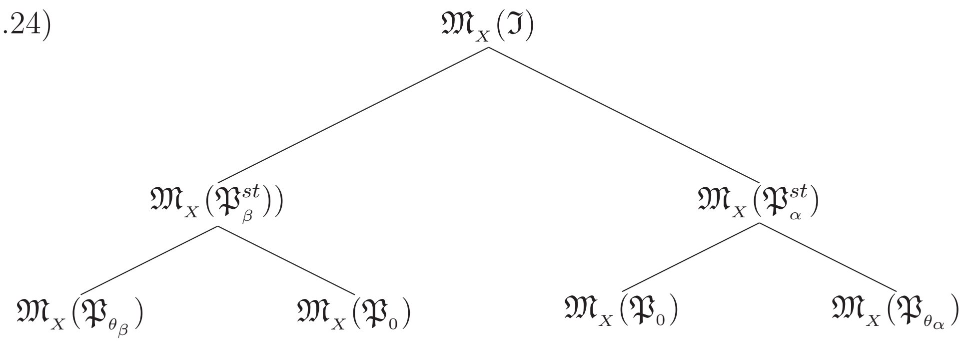
$$\phi_g : \mathfrak{M}_X(\mathfrak{P}_\Omega(K)_g) \to \mathfrak{M}_X(\mathfrak{P}_\Omega(K))$$

by $\phi_g(\theta) = g.\theta$. That this defines an isomorphism of double coset spaces is easy to check.

Q.E.D

4.0.10. **Hecke Correspondence.** Using (3.0.8), we get $\mathfrak{I} \subset \mathfrak{P}^{st}_{\alpha}(K) \subset \mathfrak{P}_{\theta_\alpha}(K) \cap \mathfrak{P}_0(K)$ and natural maps which are in fact morphisms at the level of stacks and get the following generalized Hecke correspondences. The dimension formulae obtained later (see Theorem 7.5 below) reflect the picture accurately.

(4.0.24)



4.10. *Remark.* It is an interesting and important problem to analyse the above morphisms on the substack of stable (resp semistable) objects and the condition needed in terms of weights for the morphism to descend to a morphism of moduli schemes.

5. Stability and semistability

5.0.11. **Notion of weights.** We will now define the concept of weights on points of the moduli stack staying within the realm of adèlic spaces alone.

Consider the double coset space $\mathfrak{M}_X(\mathfrak{P}_\Omega(K))$. Once we fix a root datum for G, we see that we have a choice of an affine apartment and this identifies a parahoric subgroup $\mathfrak{P}_\Omega(K) \subset G(K)$ as the stabilizer subgroup of $G(K)$ of a *facet* of the affine apartment $App(G,K)$. Again, as we have seen earlier, we could in turn take any point in *general position* in the facet and consider the parahoric as the stabilizer of that point. Thus one can make an identification $\mathfrak{P}_\Omega(K) \simeq \mathfrak{P}_\theta(K)$ for a *general element* θ in the facet determined by Ω.

By a *weight*, we will mean an element $\theta \in Y(T) \otimes \mathbb{Q}$ modulo $Y(T)$. Observe that this automatically gives rise to an identification

$$w_\theta : \mathfrak{P}_\Omega(K) \simeq \mathfrak{P}_\theta(K) \simeq \mathsf{U}_y \tag{5.0.1}$$

of the parahoric subgroup $\mathfrak{P}_\Omega(K)$ with a specific parahoric $\mathfrak{P}_\theta(K)$ (and hence a local unit group U_y and not just the conjugacy class $\mathfrak{U}_x$) (see Theorem 2.3). Hence, in particular, it determines an "underlying" quasi-parahoric structure (see Definition 4.2). One might therefore say that, fixing the identification w_θ endows every point of the double coset space $\mathfrak{M}_X(\mathfrak{P}_\Omega(K)) \simeq \mathfrak{M}_X(\mathfrak{P}_\theta(K))$ with *weights*. Note further that fixing weights w_θ therefore automatically fixes the local type $\boldsymbol{\tau}$ of the (Γ, G)–bundle coming from the identification (4.0.16). Recall the notion of a *quasi-parahoric bundle* from Definition 4.2.

5.1. Definition. *A* parahoric bundle *is the pair* $(\mathcal{E}, w_{\theta})$ *of a quasi-parahoric bundle* $\mathcal{E}$ *together with weights* w_{θ}. *A* parahoric structure *on* $\mathcal{E}$ *is giving a "weight"* $\theta \in Y(T) \otimes \mathbb{Q}$ *mod* $Y(T)$ *or equivalently a map* w_{θ}.

5.2. *Remark.* Recall from Theorem 2.3 that given a $\theta \in Y(T) \otimes \mathbb{Q}$, it automatically gives a parahoric subgroup $\mathfrak{P}_{\theta}(K) \subset G(K)$ as well as an identification with a specific local unit group U_y. In particular, this identifies a Bruhat-Tits group scheme and also the moduli stack of quasi-parahoric bundles.

5.3. *Remark.* This is the precise analogue of the classical weight for a parabolic vector bundle.

5.0.12. **Stability of points on the double coset space.** We begin the discussion with a remark.

5.4. *Remark.* (cf. [24, Remark 3.5.7]) Observe that in the usual situation of principal G–bundles, a reduction of structure group of a principal G–bundle E to P gives canonically a reduction also to $g.P.g^{-1}$. Furthermore, it is easy to see that any character $\chi : P \to \mathbb{G}_m$, gives a character $\chi_g : g.P.g^{-1} \to \mathbb{G}_m$ defined by $\chi_g(g.p.g^{-1}) := \chi(p)$. This therefore gives a line bundle $L|_{\chi_g}$ on $E/g.P.g^{-1}$ such that $deg(L|_{\chi_g}) = deg(L_{\chi})$.

For a parahoric bundle $(\mathcal{E}, w_{\theta}) \in \mathfrak{M}_X(\mathfrak{P}_{\theta}(K))$ defining the notion of *reduction of structure group to a parabolic subgroup* $P \subset G$ is a little tricky. This parallels the notion of an induced parabolic structure on a subbundle (cf. [18]).

Let $\mathfrak{P}_{\theta}(K))$ be a fixed parahoric subgroup of $G(K)$ and observe that we have chosen the θ in the facet determining the parahoric in the fixed affine apartment $App(G, K)$. Choice of this weight entails a choice of a maximal torus $T \subset G$. Let $P \subset G$ be a parabolic subgroup. Then by a conjugation by an element $g \in G$, we can ensure that the maximal torus T chosen for defining $\mathfrak{P}_{\theta}(K))$ is contained in $g.P.g^{-1}$. Hence moving P in its conjugacy class, we now take the intersection $\mathfrak{P}_{\theta}(K)) \cap g.P.g^{-1}$.

Let $(\mathcal{E}, w_{\theta}) \in \mathfrak{M}_X(\mathfrak{P}_{\theta}(K))$. Let $P \subset G$ be a parabolic subgroup. Let $\chi : P \to \mathbb{G}_m$ be a dominant character. Let $\mathcal{E}_P$ be a reduction of structure group of $\mathcal{E}$ on $X - x$. Choose $g \in G$ so that $g.P.g^{-1}$ contains the maximal torus T. Let $\mathcal{E}_{g.P.g^{-1}}$ be the induced reduction to $g.P.g^{-1}$ on $X - x$.

The character $\chi_g : g.P.g^{-1} \to \mathbb{G}_m$ maps the *bounded subgroup* $\mathfrak{P}_{\theta}(K)) \cap g.P.g^{-1}$ to one in $\mathbb{G}_m$. Furthermore, it induces a map $Y(T) \otimes \mathbb{Q} \to Y(\mathbb{G}_m) \otimes \mathbb{Q}$, i.e a *weight.* This datum defines a *parabolic line bundle* L_{χ_g} i.e a line bundle together with weights. We denote this parabolic line bundle by $\mathcal{E}_P(\chi)$. Recall that a parabolic line bundle is merely a rank 1 parabolic vector bundle and, by the general correspondence of Γ–bundles and parabolic bundles, a parabolic line bundle is realizable as an invariant direct image of a Γ–line bundle (see proofs of Theorem 5.6 and Proposition 5.11 below).

We can define stability of $(\mathcal{E}, w_\theta)$ as follows (see also Definition 5.12 and Corollary 5.13):

5.5. Definition. *A parahoric bundle* $(\mathcal{E}, w_\theta)$ *is called* stable (resp. semistable) *if for every maximal parabolic* $P_\beta \subset G$*, and dominant character* $\chi : P_\beta \to \mathbb{G}_m$*, and for every reduction of structure group* $\mathcal{E}_P$ *of* $\mathcal{E}$ *to* P_β*, we have pardeg*$(\mathcal{E}_P(\chi)) > 0$ *(resp.* ≥ 0*).*

5.6. Theorem. *The identification*

$$\mathfrak{M}_X(\mathfrak{P}_\theta(K)) \simeq Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$$

given by Theorem 4.6 identifies stable (resp. semistable) objects in $\mathfrak{M}_X(\mathfrak{P}_\theta(K))$ *with stable (resp. semistable)* (Γ, G)*–bundles of local type* $\boldsymbol{\tau}$ *on the ramified cover* Y*.*

Proof: By Theorem 2.3 and Proposition 3.4 that we have a Galois covering $p : Y \to X$ with Galois group Γ such that $\mathfrak{P}_\theta(K)) \simeq G(B)^\Gamma$. Furthermore, the correspondence shows that we have a (Γ, G)–bundle Q on Y of local type $\boldsymbol{\tau}$ corresponding to E.

Recall that a (Γ, G)–bundle Q on Y is stable if (i) for every parabolic subgroup $P \subset G$, and (ii) dominant character $\chi : P \to \mathbb{G}_m$, and (iii) a Γ–equivariant reduction of structure group Q_P of Q such that the local type given by the representation $\rho : \Gamma_y \to G$ factors through $\rho' : \Gamma_y \to P$, the associated Γ–line bundle $Q_P(\chi)$ has degree > 0.

For a given parabolic $P \subset G$, the representation ρ need not factor via P but if we allow a conjugate by a $g \in G$, then we can realize ρ via a maximal torus sitting inside $g.P.g^{-1}$. By Remark 5.4, the stability of Q can be tested by the reduction to $g.P.g^{-1}$.

The theorem now follows by the simple observation that the parabolic line bundle $E_P(\chi)$ is nothing but the *invariant direct image* $p_*^\Gamma(Q_P(\chi))$ (cf. the proof of Proposition 5.11 below).

Q.E.D

5.0.13. **Parabolic subgroup scheme of Bruhat-Tits group schemes.** Let $\mathcal{G}_{\Omega,X}$ be a Bruhat-Tits group scheme on the curve X which is obtained by gluing the Bruhat-Tits group scheme $\mathcal{G}_\Omega$ for a disc $D = Spec(A)$ around the point $x \in X$ and the split semisimple group scheme $G \times (X - x)$. Following Heinloth [11, Definition 17], we have:

5.7. Definition. *A maximal parabolic subgroup $\mathcal{P} \subset \mathcal{G}_{\Omega,X}$ of the group scheme $\mathcal{G}_{\Omega,X}$ is defined as the flat closure of a maximal parabolic subgroup of the generic fibre $\mathcal{G}_K$ of $\mathcal{G}_{\Omega,X}$.*

Let E be a $\mathcal{G}_{\Omega,X}$–torsor on X. Then we have (Heinloth [11, Lemma 23])

5.8. Lemma. *Let $\mathcal{P}_K \subset \mathcal{G}_K$ be a maximal parabolic subgroup and let E be a $\mathcal{G}_{\Omega,X}$–torsor on X. Any choice of reduction section $s_K \in E_K(\mathcal{G}_K/\mathcal{P}_K) = E_K/\mathcal{P}_K$ defines a maximal parabolic subgroup $\mathcal{P}' \subset \mathcal{G}_{\Omega,X}$ together with a reduction s' of E to $\mathcal{P}'$.*

Proof: This follows immediately from [11].

5.9. *Remark.* Note however that $\mathcal{G}_{\Omega,X}/\mathcal{P}$ need not be a projective scheme over A for all parahorics. It is so for instance if $\mathcal{G}_{\Omega,X}$ is a hyperspecial parahoric.

5.0.14. **Weights and parahoric torsors.** Recall that, locally, the group scheme $\mathcal{G} = \mathcal{G}_{\Omega,X}$ is canonically determined by the parahoric subgroup $\mathcal{G}(A) \subset G(K)$.

Thus, by (4.0.22), giving a $\mathcal{G}$–torsor E is therefore equivalent to giving a *quasi-parahoric bundle* $\mathcal{E}$. If further, we fix an identification

$$w_\theta : \mathcal{G}(A) \simeq \mathfrak{P}_\theta(K) \simeq \mathsf{U}_y \tag{5.0.2}$$

of the parahoric subgroup $\mathcal{G}(A)$ with a specific parahoric $\mathfrak{P}_\theta(K)$, then we endow the torsor E with *weights*. In other words, fixing the identification w_θ endows every point of the double coset space $Bun_X(\mathcal{G}) \simeq \mathfrak{M}_X(\mathfrak{P}_\theta(K))$ with *weights*.

5.10. Definition. *A* parahoric $\mathcal{G}$–torsor *is a pair (E, w_θ), where E is a $\mathcal{G}$–torsor and w_θ a weight as in* (5.0.2).

Recall further (by Theorem 4.6) that fixing w_θ gives the following identification:

$$Bun_X(\mathcal{G}) \simeq Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \tag{5.0.3}$$

for a suitably defined covering $p : Y \to X$ with Galois group Γ.

Let $\chi : \mathcal{P}_K \to \mathbb{G}_{m,K}$ be a dominant character of the parabolic subgroup $\mathcal{P}_K$. Then one knows that this defines an ample line bundle L_χ on $\mathcal{G}_K/\mathcal{P}_K$. Of course, the quotient $\mathcal{G}_{\Omega,X}/\mathcal{P}$ for a flat closure of $\mathcal{P}_K$ is not projective over X but $\mathcal{G}_K/\mathcal{P}_K$ is projective over K. Following Ramanathan, we see that χ defines a line bundle L_χ on $E_K/\mathcal{P}_K$ as well and using a reduction section s_K, we therefore get a line bundle $s_K^*(L_\chi)$ on $X - x$.

5.11. Proposition. *Suppose that we are given the Bruhat-Tits group scheme $\mathcal{G} = \mathcal{G}_{\Omega,X}$ extending the generic group $\mathcal{G}_K$. Suppose further that we are given weights i.e $w_\theta : \mathcal{G}(A) \simeq \mathfrak{P}_\theta(K)$ with $\theta \in Y(T) \otimes \mathbb{Q}$, a point in general position in the facet determined by the parahoric subgroup $\mathcal{G}(A)$. Let s_K be a generic reduction of structure group of E_K to $\mathcal{P}_K$. Then the line bundle $s_K^*(L_\chi)$ on $X - x$ has a canonical extension L_χ^θ to X as a* parabolic line bundle.

Proof: By Theorem 2.3 and Proposition 3.4, once the identification w_θ is fixed along with the choice of $\theta \in Y(T) \otimes \mathbb{Q}$, we have a ramified cover $p : Y \to X$ with $\Gamma = Gal(Y/X)$ so that $\mathcal{G}_{\Omega,X}(A) = G(B)^\Gamma$. The data (E, w_θ), of a $\mathcal{G}$–torsor *together with weights* is therefore equivalent to giving a (Γ, G)–principal bundle F on Y.

The maximal parabolic subgroup $\mathcal{P}_K \subset \mathcal{G}_K$ immediately gives a maximal parabolic $Q \subset G$ and the reduction s_K gives in turn a Γ–equivariant reduction of structure group t_L of F_L/Q_L, where L denotes the quotient field of B the local ring in Y over $x \in X$. By virtue of the projectivity of Y, the reduction section t_L extends to a Γ–equivariant reduction of structure group $t \in F/Q$. The dominant character χ gives a dominant character η of Q and the section t gives a Γ–line bundle $t^*(L_\eta)$.

Now observe that the GIT quotient of F/Q by the finite group Γ gives a natural *compactification* of $\mathcal{G}_{\Omega,X}/\mathcal{P}'$. It is easy to see that the invariant direct image

$$p_*^\Gamma(t^*(L_\eta))$$

gives the required extension of $s_K^*(L_\chi)$. This is by the definition of the invariant direct image, a *parabolic line bundle*.

Q.E.D

5.12. Definition. *Let* $\mathcal{G} = \mathcal{G}_{\Omega,X}$. *A parahoric* $\mathcal{G}$*–torsor* (E, w_θ) *is called* stable (resp. semistable) *if for every maximal parabolic* $\mathcal{P}_K \subset \mathcal{G}_K$*, for the dominant character* χ *as above, for every reduction of structure group* s_K*, we have:*

$$pardeg(L_\chi^\theta) > 0 (resp. \geq 0)$$

5.13. Corollary. *The notions of stability of a parahoric* $\mathcal{G}_{\Omega,X}$*–torsor* (E, w_θ) *and a parahoric bundle* $(\mathcal{E}, w_\theta)$ *given by Definition 5.5 and Definition 5.12 are equivalent.*

Proof: Follows immediately from the above discussions.

6. Unitary representations of π

6.0.15. **Manifold of irreducible unitary representations of π.** Notations in this section are as in the introduction 1.1 and 1.3 (see also (1.0.5) and (1.0.6)). Recall also the notion of local type of unitary representations $\rho : \pi \to K_G$ from Definition 1.4.

We now recall the following result from [34, Page 157].

6.1. Proposition. *Let* ρ *be a representation of* π *on a vector space* V *(over* $\mathbb{R}$*) such that* $d = \dim V$ *and* ρ *is unitary (or more generally leaving invariant a non-degenerate bilinear form on* V*). Then we have*

$$\dim_{\mathbb{R}} H^1(\pi, \rho) = 2d(g-1) + 2\dim_{\mathbb{R}} H^0(\pi, \rho) + \sum_{\nu=1}^{m} e_\nu$$

where e_ν is the rank of the endomorphism $(I - \rho(C_\nu))$ of V.

Let K_G be the maximal compact subgroup of G and κ_{G} *its Lie algebra* which is a real vector space of dimension d, where $d = dim(G)$.

As in the introduction, we assume that $X = \mathbb{H}/\pi$, with $x \in X$ corresponding to $z \in \mathbb{H}$. Let π_z be the stabilizer at z (cyclic of order n_x) and let γ be a generator of π_z and let $\rho : \pi \to K_G$ be a unitary representation of π.

Let $\alpha \in S$ and let ρ_{α} be as in Definition 2.5. Let $\rho_{\alpha}(\gamma) \in K_G$ be the image of the generator γ of π_z. Note that the choice of the simple root α and identification of the representation ρ with ρ_{α} amounts to *fixing the local type* of the representation $\rho : \pi \to K_G$, i.e the *conjugacy class* of $\rho(\gamma)$ in K_G.

We denote by Ad ρ_α, the adjoint transformation on κ_{G}, namely if $M \in \kappa_{G},\ M \mapsto \rho_{\alpha}(\gamma)M\rho_{\alpha}(\gamma)^{-1}$.

Then we have:

6.2. Proposition. *Let $e(\alpha)$ denote the rank of (Id - Ad ρ_α) on κ_{G}. Then*

$$e(\alpha) = dim_{\mathbb{R}}(K_G) - 2\mu(\alpha) - 2\nu(\alpha) - \ell = 2.(dim_{\mathbb{C}}(G/P_{\alpha})) - \mu(\alpha)) \tag{6.0.1}$$

where P_{α} is the maximal parabolic subgroup of G associated to α and

$$\mu(\alpha) = \#\{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta \neq \alpha} x_\beta.\beta\} \tag{6.0.2}$$

$$\nu(\alpha) = \#\{r \in R^- \mid r \textit{ involves simple roots} \neq \alpha\} \tag{6.0.3}$$

and $\ell = \#S$.

Proof. Make K_G operate on itself by inner conjugation. Then, rank of (Id-Ad ρ_{α}) on κ_{G} = the dimension of the orbit through $\rho_{\alpha}(\gamma)$ for the action of K_G on itself by inner conjugation.

We may assume for the purpose of this computation that $\rho_{\alpha}(\gamma)$ lies in the maximal torus. We firstly compute the number roots $r \in R$ so that the corresponding root group $U_r(B)$ is centralized by $\rho_{\alpha}(\gamma)$. Recall from Definition 2.5 that the action of $\rho_{\alpha}(\gamma)$ on U_r is given as follows:

$$\rho_{\alpha}(\gamma).U_r(B).\rho_{\alpha}(\gamma)^{-1} = U_r(\zeta^{r(\Delta_\alpha)}B) \tag{6.0.4}$$

where as seen earlier, $r(\Delta_{\alpha}) = d.(\theta_{\alpha}, r)$. Since ζ is a primitive d^{th}–root of unity, we need to compute the # $\{r \in R \mid (\theta_{\alpha}, r) = \pm 1 \textit{ or } 0\}$. It is easy to see that

$$\{r \in R \mid (\theta_{\alpha}, r) = \pm 1 \textit{ or } 0\} = \bigcup_{i=1}^{4} A_i(\alpha) \tag{6.0.5}$$

where for $i = 1, 2$,

$$A_i(\alpha) = \{r \in R^{\pm} \mid r = \pm c_\alpha.\alpha + \sum_{\beta \neq \alpha} \pm x_\beta.\beta\} \tag{6.0.6}$$

$$A_3(\alpha) = \{r \in R^- \mid r \textit{ involves simple roots} \neq \alpha\} \tag{6.0.7}$$

and

$$A_4(\alpha) = \{r \in R^+ \mid r \textit{ involves simple roots} \neq \alpha\} \tag{6.0.8}$$

Since the maximal torus centralizes $\rho_{\alpha}(\gamma)$, we see that the dimension of the centralizer of $\rho_{\alpha}(\gamma)$ is

$$\#\{r \in R \mid (\theta_{\alpha}, r) = \pm 1 \textit{ or } 0\} + \#S \tag{6.0.9}$$

Observe that $|A_4| = |A_3|$ and $|A_1| = |A_2|$. To compute the rank of (Id - Ad ρ_{α}), we simply subtract the above number (6.0.9) from the $dim_{\mathbb{R}}(K_G)$ to get the first expression for $e(\alpha)$. We see that

$$\nu(\alpha) = dim_{\mathbb{C}}(P_{\alpha}/B) \tag{6.0.10}$$

where P_{α} is the maximal parabolic subgroup of G defined by the simple root $\alpha \in S$. Thus,

$$dim_{\mathbb{R}}(K_G) - 2.\nu(\alpha) - \ell = dim_{\mathbb{C}}(G) - 2.\nu(\alpha) - \ell = 2.dim_{\mathbb{C}}(G/P_{\alpha}).$$

since $2.dim(B) - \ell = dim(G)$.

Hence, $e(\alpha) = 2.(dim_{\mathbb{C}}(G/P_{\alpha})) - \mu(\alpha))$ and the proposition now follows.

Q.E.D

6.3. *Remark.* Fix a simple root $\alpha \in S$. Recall from Definition 1.4 that a representation ρ *of* $\pi \to K_G$ *is said to be of type* $\boldsymbol{\tau}(\alpha)$, if the *conjugacy classes of* $\rho(C)$ *are fixed* and given by $\rho(C) = \rho_{\alpha}(\gamma)$ for some $\alpha \in S$, where ρ_{α} is as in Definition 2.5. *Let* $R^{\boldsymbol{\tau}(\alpha)}(\pi, K_G)$ *be the set all such representations of type* $\boldsymbol{\tau}(\alpha)$.

6.4. Theorem. *The subset* $R_o \subset R^{\boldsymbol{\tau}(\alpha)}(\pi, K_G)$, *of* irreducible representations *is open and non-empty and is further smooth of real dimension equal to*

$$(2g - 1)dim(K_G) + e(\alpha). \tag{6.0.11}$$

Let K_G *act on* $R^{\boldsymbol{\tau}(\alpha)}(\pi, K_G)$ *by inner conjugation. Let* $\overline{K}_G = K_G/centre$. *Then the equivalence classes of irreducible representations corresponds to the quotient space* $R_o/\overline{K}_G$ *and has the natural structure of a* complex analytic orbifold *of dimension*

$$dim_{\mathbb{C}}(R_o/\overline{K}_G) = dim_{\mathbb{C}}(G)(g - 1) + \frac{1}{2}e(\alpha) \tag{6.0.12}$$

Proof: This follows in much the same fashion as in [27, Page 180] and is an immediate consequence of Proposition 6.2.

Q.E.D

6.5. *Remark.* One can obtain similar dimension formulas for the case when we work with representation of type $\boldsymbol{\tau}(I)$, for any subset $I \subset S$. It is given by the following:

$$dim_{\mathbb{C}}(R_o(I)/\overline{K}_G) = dim_{\mathbb{C}}(G)(g-1) + \frac{1}{2}e(I) \tag{6.0.13}$$

where $e(I) = 2.\big(dim(G/P_{_I}) - \mu(I)\big)$ and $\mu(I) = \#\{r \in R^+ = \sum_{\alpha \in I} c_\alpha.\alpha + \ldots\}$.

7. The moduli space of parahoric bundles

In this section we study the moduli space of (Γ, G)–bundles on Y and prove the basic geometric properties of this space. We use these to conclude similar facts about the stack of parahoric bundles. Certain parts of the proofs in [2] needed the developments in this paper to be made complete. A few of the arguments from [2] therefore have been repeated to make this article self-contained.

The notions of *stability and semistability* of (Γ, G)–bundles is defined in the introduction and follows the one given by A.Ramanathan ([23]). It is shown in [2, Theorem 5.8] that the moduli space $M_{_Y}^{\boldsymbol{\tau}}(\Gamma, G)$ of semistable (Γ, G)–bundles of local type $\boldsymbol{\tau}$ is realized as a good quotient $Q_G//\mathcal{G}$ of a suitably defined Quot scheme Q_G. In this paper we show that this scheme $M_{_Y}^{\boldsymbol{\tau}}(\Gamma, G)$ is normal and projective.

7.0.16. **The irreducibility of the moduli space.**

7.1. Theorem. *Let G be semisimple and simply connected. Then the moduli stacks $\mathfrak{M}_{_X}(\mathfrak{P}_{_\Omega}(K))$ and $Bun(\mathcal{G}_{_{\Omega,X}})$ are* irreducible *for any Ω.*

Proof: Observe that from Ramanathan [23] (see also [7]), it follows that for the standard parahoric $\mathfrak{P}_{_0}(K) = G(A)$, the moduli stack $\mathfrak{M}_{_X}(\mathfrak{P}_{_0}) \simeq Bun_{_X}(G)$ is irreducible because G is simply connected. Further, the morphism $\mathfrak{M}_{_X}(\mathfrak{I}) \to \mathfrak{M}_{_X}(\mathfrak{P}_{_0})$ is surjective and has fibre G/B, B being the Borel subgroup. Hence, $\mathfrak{M}_{_X}(\mathfrak{I})$ is irreducible. Now observe that the map $\mathfrak{M}_{_X}(\mathfrak{I}) \to \mathfrak{M}_{_X}(\mathfrak{P}_{_\Omega}(K))$ given by (4.0.24) is obviously set-theoretically surjective since it comes from the inclusion $\mathfrak{I} \subset \mathfrak{P}_{_\Omega}(K)$.

By Theorem 2.3, we now make the identification $\mathfrak{P}_{_\Omega}(K) \simeq \mathfrak{U}_x$ for $x \in X$. Hence the stack structure on $\mathfrak{M}_{_X}(\mathfrak{P}_{_\Omega}(K))$ comes from the identification $\mathfrak{M}_{_X}(\mathfrak{P}_{_\Omega}(K)) \simeq Bun_{_Y}^{\boldsymbol{\tau}}(\Gamma, G)$, where Γ is the Galois group of the cover $Y \to X$ associated to $\mathfrak{U}_x$.

Let $E_i \in Bun_{_Y}^{\boldsymbol{\tau}}(\Gamma, G)$, $i = 1, 2$ be two (Γ, G)–bundles of local type $\boldsymbol{\tau}$. Let the images of E_i in $\mathfrak{M}_{_X}(\mathfrak{P}_{_\Omega}(K))$ be denoted by $\mathcal{E}_i$. By the set-theoretic surjection $\mathfrak{M}_{_X}(\mathfrak{I}) \to \mathfrak{M}_{_X}(\mathfrak{P}_{_\Omega}(K))$, we can lift $\mathcal{E}_i$ to two points $\mathcal{F}_i \in \mathfrak{M}_{_X}(\mathfrak{I})$. Hence by the connectedness we get a curve $T \to \mathfrak{M}_{_X}(\mathfrak{I})$ whose image contains $\mathcal{F}_i$. In other words, we have a family $\{\mathcal{F}_t\}_{t \in T}$. The inclusion $\mathfrak{I} \subset \mathfrak{P}_{_\Omega}(K)$ gives map $T \to \mathfrak{M}_{_X}(\mathfrak{P}_{_\Omega}(K))$. Now we observe that the proof on Theorem 2.3 simply goes through with a parameter

t and we get a holomorphic map $T \to Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ whose image contains E_i. This proves the connectedness. Smoothness is immediate from standard deformation theory and we get the required irreducibility. The irreducibility of $Bun(\mathcal{G}_{\Omega,X})$ follows from Theorem 4.6.

Q.E.D

7.2. *Remark.* If we stick to the analytic category and use the language of loop groups, we observe that the connectedness of the stack $Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ can be derived more easily as follows: let E_1, E_2 be two points in $Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$. By Theorem 2.3 we see that they give points in the double coset space $\mathfrak{M}_X(\mathfrak{U}_x(K))$ associated to the unit group $\mathfrak{U}_x$. They give immediately lifts $s_1, s_2 \in G(K_x)$ which can be considered points of the loop group $G(K_x)$.

Since G is assumed to be simply connected, we see that the loop group $G(K_x)$ has the structure of a *connected* Banach Lie group (see [15, Page 511] for an argument due to V. Drinfeld). We can connect s_i be a family in $G(K_x)$. We now remark that a holomorphic map $T \to G(K_x)$ (which by definition can be taken to be a holomorphic map $T \times Spec(K_x) \to G$), gives a family of bundles in $Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ by making a choice of the unit group U_y. This follows from Remark 4.3. Since the s_i are lifts of the E_i we get a family of bundles in $Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ connecting the E_i's.

7.3. *Remark.* Geometric statements such as the *unirationality* of these moduli stacks will follow from these morphisms. This is because, by [16] one knows that the standard moduli stack $\mathfrak{M}_X(\mathfrak{P}_0)$ is *unirational*. It follows immediately that the Iwahori moduli stack $\mathfrak{M}_X(\mathfrak{I})$ is unirational and hence all the remaining ones.

7.0.17. **Some remarks related to Heinloth's work.** The above irreducibility result is proved by Heinloth (cf.[11, Theorem 2]); more precisely, he obtains it as a consequence of a certain "uniformization theorem" and also over base fields of arbitrary characteristics. Using the Hecke correspondence (4.0.10) and the properties of $Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$, we get a different approach (when the ground field is $\mathbb{C}$) for [11, Theorem 1,2] as well as [11, Corollary 20]. By [7], it is immediate that the uniformization statement follows from the Γ–equivariant uniformization (cf. [30] for this). Connectedness has been shown in Theorem 7.1. Properness of the semistable stack follows from the compactness shown in Theorem 7.15.

7.4. Proposition. *The moduli stack $Bun^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ of (Γ, G)–bundles on Y of fixed local type $\boldsymbol{\tau}$ is irreducible when the group G is semisimple and simply connected.*

Proof: This is immediate from Theorem 4.6 and Theorem 7.1.

Q.E.D

7.0.18. **The dimension formula.** Since $\mathfrak{M}_X(\mathfrak{P}_0(K)) \simeq \mathfrak{M}_X(G)$ by [24] we see that $dim(\mathfrak{M}_X(\mathfrak{P}_0(K)) = dim(\mathfrak{M}_X(G)) = dim(G)(g-1)$. The aim of this subsection is to prove the following theorem:

7.5. Theorem. *Let $\mathfrak{P}_{\theta_\alpha}(K)$ be the maximal parahoric subgroup associated to the simple root $\alpha \in S$. The formal dimension of the moduli stack $\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K))$ is*

$$dim(\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K)) = dim(G)(g-1) + dim(\frac{G}{P_\alpha}) - \mu(\alpha) \tag{7.0.1}$$

where $\mu(\alpha) = \#\{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta \neq \alpha} x_\beta.\beta\}$.

Moreover,

$$dim(\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K)) = dim(R_o/\overline{K}_G) \tag{7.0.2}$$

where $R_o \subset R^{\boldsymbol{\tau}(\alpha)}(\pi, K_G)$ is the subspace of irreducible representations of $\pi \to K_G$, which are of type "$\boldsymbol{\tau}(\alpha)$" (see Definition 1.4 and Remark 6.3) and K_G now being the fixed maximal compact subgroup of G acting on R_o by inner conjugation.

Proof: To compute the formal dimension of the moduli stack, we first observe that if $\mathfrak{I}$ is the standard Iwahori subgroup in $G(A) = \mathfrak{P}_0(K)$, then the moduli stack $\mathfrak{M}_X(\mathfrak{I})$ is formally a G/B–fibration over $\mathfrak{M}_X(\mathfrak{P}_0(K))$. In particular, its formal dimension is $dim(\mathfrak{M}_X(\mathfrak{P}_0(K)) + dim(G/B)$. Therfore,

$$dim(\mathfrak{M}_X(\mathfrak{I})) = dim(G)(g-1) + dim(G/B) \tag{7.0.3}$$

On the other hand, the quotient $\frac{\mathfrak{P}_{\theta_\alpha}(K)}{\mathfrak{I}}$ is supported on the residue field $\mathbb{C}$ and is finite dimensional. In fact, its dimension is

$$dim\Big(\frac{\mathfrak{P}_{\theta_\alpha}(K)}{\mathfrak{I}}\Big) = \nu(\alpha) + \mu(\alpha) \tag{7.0.4}$$

where $\nu(\alpha)$ and $\mu(\alpha)$ are given by (6.0.2) and (6.0.3).

To see this, recall the definition of

$$\mathfrak{P}_{\theta_\alpha}(K) = \langle T(A), U_r(z^{-[(\theta_\alpha, r)]}.A) \mid r \in R\rangle$$

and

$$\mathfrak{I} = \langle T(A), U_r(z^{m_r}.A) \mid r \in R\rangle$$

where

$$m_r = \begin{cases} 1 & \text{if } r \in R^- \\ 0 & \text{if } r \in R^+ \end{cases}$$

Now, if $r \in R^+$, then

$$[(\theta_\alpha, r)] = 1 \iff \{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta \neq \alpha} x_\beta.\beta\}$$

Hence,

$$\frac{U_r(z^{-[(\theta_\alpha,r)]}.A)}{U_r(z^{m_r}.A)} \simeq \mathbb{G}_{a,k},\ if\ \{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta\neq\alpha} x_\beta.\beta\}$$

Again, if $r \in R^-$, then

$$[(\theta_\alpha, r)] = 0 \iff \{r \in R^- \mid r\ involves\ simple\ roots \neq \alpha\}$$

Hence,

$$\frac{U_r(z^{-[(\theta_\alpha,r)]}.A)}{U_r(z^{m_r}.A)} \simeq \mathbb{G}_{a,k},\ if\ \{r \in R^- \mid r\ involves\ simple\ roots \neq \alpha\}$$

Putting together these data, we get (7.0.4). Now since

$$dim_{\mathbb{C}}\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K)) = dim_{\mathbb{C}}\mathfrak{M}_X(\mathfrak{I}) - dim\Big(\frac{\mathfrak{P}_{\theta_\alpha}(K))}{\mathfrak{I}}\Big) \tag{7.0.5}$$

using (7.0.3), we get

$$dim_{\mathbb{C}}\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K)) = dim(G)(g-1) + dim(G/B) - \nu(\alpha) - \mu(\alpha) \tag{7.0.6}$$

By (6.0.10), $\nu(\alpha) = dim(G/B) - dim(G/P_\alpha)$ and we get (7.0.1) .

The equality (7.0.2) now follows by Theorem 6.4.

Q.E.D

7.6. Corollary. *Let $\mathfrak{P}(K)^{hs}$ be a maximal parahoric subgroup in $G(K)$* which is hyperspecial. *Then we have*

$$dim_{\mathbb{C}}\mathfrak{M}_X(\mathfrak{P}(K)^{hs}) = dim(G)(g-1)$$

and conversely.

Proof: By the Bruhat-Tits theory, the hyperspecial parahorics are simply the maximal parahorics $\{\mathfrak{P}_{\theta_\alpha}(K) \mid \forall\alpha \in S, \text{with } c_\alpha = 1\}$ upto conjugacy by $G(K)$. In these cases, the number $\mu(\alpha)$ will now be

$$\mu(\alpha) = \#\{r \in R^+ \mid r \text{ involves } \alpha\}$$

since the largest possible coefficient for such an α in any positive root is 1. Hence for the hyperspecial cases, $dim\Big(\frac{\mathfrak{P}_{\theta_\alpha}(K))}{\mathfrak{I}}\Big) = dim(G/B)$ and we are though by (7.0.5).

Q.E.D

7.7. *Remark.* Although the dimension of the moduli space is computable by the final identification with $(R_o/\overline{K}_G)$, the computation given above is of independent interest since it is formal and works in any characteristic.

7.8. *Remark.* Corollary 7.6 shows that in the case when all maximal parahorics are hyperspecial, all the maximal parahoric moduli stacks have the same dimension. This is so for example when $G = SL(n)$, and the moduli stacks are the various moduli of bundles with *"fixed determinants"*. For all other groups G, we always have non-hyperspecial maximal parahorics which give a varying set of dimensions among the maximal parahoric moduli spaces. These indeed are the new phenomena which arise from this paper and need to be investigated further.

7.0.19. **Properness of the moduli of (Γ, G)-bundles.** Let $H = G/Z(G)$, the associated adjoint group. For such semisimple adjoint type groups we have the following property.

7.9. Lemma. *There exists a faithful irreducible representation $\rho : H \hookrightarrow GL(n)$.*

Proof: We easily reduce to the case when the group is simple (by taking the tensor product representation for the product group). Then one can simply take any fundamental representation for the simple factors and we are done.

Q.E.D

Fix the representation $\rho : H \hookrightarrow GL(n)$ and a maximal compact K_H of H such that $K_H \hookrightarrow U(n)$. Consider the subset $Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s \subset Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)$ consisting of the stable $(\Gamma, GL(n))$-bundles.

7.10. Lemma. *Let $\phi : Bun_Y^{\boldsymbol{\tau}}(\Gamma, H) \longrightarrow Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)$ be the morphism induced by the representation ρ. Then the inverse image of the stable points $\phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s) = Bun_Y^{\boldsymbol{\tau}}(\Gamma, H)^s$, (when nonempty), consists of unitary (Γ, H)-bundles.*

Proof: Let $\mathfrak{h} = Lie(H)$. Consider the adjoint representation $ad : H \to GL(\mathfrak{h})$. Then we claim that a principal (Γ, H) bundle E is unitary if and only if the associated $(\Gamma, GL(\mathfrak{h}))$–bundle $E(\mathfrak{h})$ is so. One way is obvious. Conversely, if $E(\mathfrak{h})$ comes from a unitary representation of π, then we take the Lie bracket morphism $E(\mathfrak{h}) \otimes E(\mathfrak{h}) \to E(\mathfrak{h})$. Either side comes from unitary representations of π and by *local constancy* ([18, Proposition 1.2]) it follows that $E(\mathfrak{h})$ gets a reduction of structure group to the group $A(\mathfrak{h}) = Aut(\mathfrak{h})$. Note that we realize $A(\mathfrak{h})$ as the stabilizer of the $GL(\mathfrak{h})$–action on the tensor space $\mathfrak{h}^* \otimes \mathfrak{h}^* \otimes \mathfrak{h}$ at the point $[,]$, the Lie bracket.

Since H is an adjoint group we have a short exact sequence:

$$1 \to H \to A(\mathfrak{h}) \to F \to 1$$

since H is the component of identity of $A(\mathfrak{h})$. Again we have a similar exact sequence of compact groups:

$$1 \to K_H \to K_{A(\mathfrak{h})} \to F \to 1$$

The bundle E is therefore such that $E(A(\mathfrak{h}))$ is a unitary bundle and comes from a representation $\bar{\chi} : \pi \to K_{A(\mathfrak{h})}$. Furthermore, the extended bundle $E(A(\mathfrak{h}))(F)$ is trivial since it comes with a section (giving E). By composing the representation $\bar{\chi}$ with the map $K_{A(\mathfrak{h})} \to F$, we see that the triviality of $E(A(\mathfrak{h}))(F)$ forces the composite to be the trivial homomorphism, implying that $\bar{\chi}$ factors via $\chi : \pi \to K_H$ to give the bundle E (cf.[1, Lemma 10.12]).

More generally, if $\rho : H \hookrightarrow GL(n)$ is any faithful finite dimensional representation we claim that a H-bundle E is unitary if and only if the associated bundle $E(\rho)$ is so. This assertion will be proved now.

One way is obvious. So assume that $E(\rho)$ is unitary. Then by the earlier remark, if we show that $E(\mathfrak{h})$ is unitary then we are done. First observe that since ρ is a faithful representation of H, the adjoint representation can be realized as a H-submodule of a direct sum of $T^{m,n}(\rho)$. Further, since $E(\rho)$ is a unitary vector bundle, the vector bundle $E(T^{m,n}(\rho))$ is also unitary. Hence $E(T^{m,n}(\rho))$ is a polystable vector bundle of degree zero. Since $E(\mathfrak{h})$ is a degree zero subbundle of a polystable bundle of degree zero, $E(\mathfrak{h})$ must be polystable. Therefore, $E(\mathfrak{h})$ is a unitary vector bundle. This proves the above assertion.

Now using the main theorem of [27] we see that points of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s$, being stable bundles, are all unitary. Hence by the claim above the bundles in the inverse image $\phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ are also unitary.

Q.E.D

7.11. Proposition. *Let ρ be the faithful irreducible representation of H, as obtained above in Lemma 7.9. Then the inverse image of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s$ by the induced morphism ϕ is nonempty.*

Proof: Let π denote the group of holomorphic automorphisms of the universal cover $\mathbb{H}$ of Y, which commute with the composition map $\mathbb{H} \longrightarrow Y \longrightarrow Y/\Gamma$. Then one knows that Γ is the quotient of π by a normal subgroup π_o which acts freely on Y and by [18] a Γ-bundle is stable if and only if it arises from a unitary representation of π. The group π can be identified with the free group on the letters $A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_m$ modulo the relations (1.0.5) and (1.0.6).

So to prove that the inverse image $\phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ is nonempty, we need to exhibit a representation

$$\chi : \pi \to K_H$$

such that the composition

$$\rho \circ \chi : \pi \to U(n) \text{ is } \textit{irreducible.}$$

Choose elements $h_1, \cdots, h_m \in K_H$ so that they are elements of order n_i, where $i = 1, \cdots, m$ (these correspond to fixing the *local type* $\boldsymbol{\tau}$ of our bundles).

It is a well–known fact that every element of a compact connected real semisimple Lie group is a commutator. Further, by [29, Lemma 3.1] there exists a dense subgroup of K_H generated by two general elements (α, β). Now define the representation χ as follows :

$$\chi : \pi \to K_H$$
$$\chi(A_1) = \alpha,\ \chi(B_1) = \beta,\ \chi(A_2) = \beta,\ \chi(B_2) = \alpha$$
$$\chi(A_i) = a_i,\ \chi(B_i) = b_i,\ \chi(C_j) = h_j,\ i = 3, \cdots, g, \text{ and } j = 1, \cdots, m$$

It is clear that χ gives a representation of the group π. Now, since ρ is irreducible, and the image of χ contains a dense subgroup, the composition $\rho \circ \chi$ gives an irreducible representation of π in the unitary group $U(n)$. Therefore, it gives a *stable* Γ-linearized vector bundle, which comes as the extension of structure group of a H-bundle. This completes the proof of the Proposition.

Q.E.D

7.12. Corollary. *In the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, H)$ for the H-bundles, there is a non-empty Zariski open substack consisting of unitary bundles of local type $\boldsymbol{\tau}$.*

Proof: This follows immediately from the Lemma 7.10 and Proposition 7.11.

Q.E.D

We now return to G which is as before a semisimple, simply connected algebraic group.

7.13. Proposition. *In the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ for the G-bundles, there is a non-empty Zariski open substack consisting of stable unitary bundles of local type $\boldsymbol{\tau}$.*

Proof: Let $\psi : Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \to Bun_Y^{\boldsymbol{\tau}}(\Gamma, H)$ be the morphism induced by the quotient map $G \to H$. We claim that the required open subset of $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is $(\phi \circ \psi)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$.

Let E be a (Γ, G)–bundle in $(\phi \circ \psi)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$. It follows that $E(H) \in \phi^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$. By Lemma 7.10 the H–bundle $E(H)$ comes from a unitary representation $\rho : \pi \to K_H$.

Recall that, by the structure of π described above, there is a central extension

$$1 \to Z_{\tilde{\pi}} \to \tilde{\pi} \to \pi \to 1 \tag{7.0.7}$$

where $\tilde{\pi}$ is generated by $A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_m$ together with a central element J satisfying the extra relation

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_1 \cdots C_m = J. \tag{7.0.8}$$

It is easy (as in [19]), by adding an extra lasso around a dummy point (other than the parabolic points) to choose a lift of ρ to a representation $\eta : \tilde{\pi} \to K_G$ so that the associated (Γ, G)–bundle $E(\eta)$ also maps to $E(H)$. Thus, both E and $E(\eta)$ give $E(H)$ under the quotient map $G \to H$.

Therefore, by twisting by a central character of $\tilde{\pi}$, we get a representation $\tilde{\pi} \to K_G$ which gives the (Γ, G)–bundle E (cf. [23, Page 148]).

We claim that this representation $\tilde{\pi} \to K_G$ in fact descends to a representation $\pi \to K_G$. This follows from the fact that the local type of E at the dummy point is trivial.

Thus, all bundles in $(\phi \circ \psi)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ are unitary (cf. [1, Lemma 10.12]).

Furthermore, it is easy to see that a (Γ, G)–bundle is stable if and only if the associated (Γ, H)–bundle is so (cf. [23, Proposition 7.1]). It follows that all points of $(\phi \circ \psi)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ are also stable (Γ, G)–bundles.

Q.E.D

By the categorical quotient property of the moduli space $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ it can be shown that there is a continuous map $\psi : R^{\boldsymbol{\tau}}(\pi, K_G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. Let $f : Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^{ss} \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ be the canonical quotient map.

7.14. *Remark.* In fact, this open substack obtained in Proposition 7.13 gets identified with the open subspace of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ of (Γ, G)–bundles with *full holonomy* and is hence *smooth* since such bundles have trivial automorphism groups.

7.15. THEOREM. *The above map ψ is surjective and hence $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is compact. Thus the variety $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ gets a structure of a projective variety. Moreover, this implies that the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^{ss}$ is* proper.

Proof: Consider the canonical categorical quotient map $f : Bun_Y^{\boldsymbol{\tau}}(\Gamma, G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. Let

$$Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s := (\phi \circ \psi)^{-1}(Bun_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$$

Since f is surjective (and hence dominant), by Chevalley's lemma, the image $f(Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s)$ in $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ contains a Zariski open subset.

By the Proposition 7.13 above the subset $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s$ is nonempty and consists entirely of unitary bundles. That is, the image $f(Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^s)$ is a subset of the image $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ in $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. Thus, it follows that $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ contains a Zariski open subset of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$. But then, since $R^{\boldsymbol{\tau}}(\pi, K_G)$ is compact the image $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ is closed in $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ and contains a dense subset, and is therefore the whole of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$, since these moduli spaces $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ are *irreducible* (by Proposition 7.4).

This proves that $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is topologically compact and hence by GAGA a projective variety.

That this implies the properness of the stack $Bun_Y^{\boldsymbol{\tau}}(\Gamma, G)^{ss}$ follows for instance from [4, Lemma 3.1].

Q.E.D

7.16. Corollary. *Let $\mathfrak{P}_\theta(K) \subset G(K)$ be a parahoric subgroup. Then the moduli stack $\mathfrak{M}_X^{ss}(\mathfrak{P}_\theta(K))$, of semistable parahoric bundles is* irreducible and proper.

Proof: This is immediate from Theorem 7.1, Theorem 4.6, Theorem 5.6, Proposition 7.4 and Theorem 7.15.

Q.E.D

7.17. Corollary. *The map $\psi : R^{\boldsymbol{\tau}}(\pi, K_G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ defined above descends to a map*

$$\psi^* : R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G \to M_Y^{\boldsymbol{\tau}}(\Gamma, G)$$

which gives a homemorphism of topological spaces. Further, the subset $R_o/\overline{K}_G$ of equivalence classes of irreducible unitary representations maps bijectively onto the subset of stable (Γ, G)-bundles.

Proof: Follows from the above discussions. The fact that irreducible representations give stable bundles follows exactly as in [23].

Q.E.D

7.18. Corollary. *Fix the local type $\boldsymbol{\tau} = \boldsymbol{\tau}(\alpha)$ (see Definition 1.4). Then the dimension of the moduli space $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is given by $dim(G)(g-1) + dim(\frac{G}{P_\alpha}) - \mu(\alpha)$ where $\mu(\alpha) = \#\{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta \neq \alpha} x_\beta.\beta\}$ and $P_\alpha \subset G$ the standard parabolic subgroup associated to the simple root α.*

Proof: This follows from Theorem 7.5 and the irreducibility of the stack, or equivalently from Theorem 6.4 and the Corollary 7.17 .

We get as an immediate consequence of Corollary 7.17 the following important conclusion:

7.19. Corollary. *There is an equivalence of categories between stable parahoric $\mathcal{G}_{\Omega,X}$–torsors, (E, w_θ) and stable (Γ, G)–bundles of local type $\boldsymbol{\tau}$ and hence with* irreducible *unitary representations of type $\boldsymbol{\tau}$ of the group π. This induces an isomorphism of the moduli stack $\mathfrak{M}_X(\mathfrak{P}_\theta(K))^s$ of* stable *parahoric torsors with the moduli space $M_Y^{\boldsymbol{\tau}}(\Gamma, G)^s$ of* stable *(Γ, G)–bundles of local type $\boldsymbol{\tau}$ as well as a bijection with the set $R_o/\overline{K}_G$ of equivalence classes of irreducible unitary representations.*

7.20. *Remark.* Following A. Ramanathan, it is natural to expect that the points of the moduli space $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ are precisely certain S–equivalence classes of bundles. This together with a suitable interpretation of this equivalence for $\mathcal{G}_{\Omega,X}$–torsors on X can be given with suitable modifications of the classical ones.

7.21. *Remark.* A proof of properness of the functor of (Γ, G)-semistable bundles along the lines of [4] or [10] can also be given.

7.22. **Extension to the case when the structure group is reductive.**: Let H be a connected reductive algebraic group. Let $S = [H, H]$ be the derived group, i.e *the maximal connected semisimple subgroup* of H. Let Z_0 be the connected component of the centre of H (which is a torus) and one know that S and Z_0 together generate H. In fact, $Z_0 \times S \to H$ is a finite covering map. It is easy to see (following [23, page 145]) that (Γ, G)–bundles gives rise to (Γ, H)–bundles and the stability and semistability of the associated (Γ, H)–bundles follows immediately from that of the (Γ, G)–bundles.

The problem of handling the reductive group G reduces to the problem of handling the semisimple group H but which is *not simply connected.* Let G be the semisimple, simply connected algebraic group which is the covering group of H.

We are in the situation of Proposition 7.13. Recall the central extension (7.0.7). By adding a dummy point other than the parabolic point, the theory of (π, H)–bundles is recovered from that of $(\tilde{\pi}, G)$–bundles. Notice that a homomorphism $\pi \to K_H$ has as many liftings $\tilde{\pi} \to K_G$ as the order of the centre of G. It follows quite easily, following arguments as in Lemma 7.10, that the number of connected components of the moduli space in the non-simply connected case is given by the order of the centre of G. In fact, $Hom(\tilde{\pi}, K_G)$ is a union of spaces labelled by elements of the centre of G. Let $Z_0 = Ker(G \to H)$. Then, there is an action of $H^1(X, Z_0)$ on a specific labelled subset of $Hom(\tilde{\pi}, K_G)$. A component of the moduli space of representations into K_H can be obtained as a quotient of each of these by the action of $H^1(X, Z_0)$. Details of these ideas are again found in [23, page 148] and follow the ideas of Narasimhan and Seshadri [19], where the data over a dummy point is called a "special parabolic structure".

8. More remarks

We begin with some well-known remarks on centralizers of semisimple elements in a simply connected semisimple group G. We have benefitted from a manuscript of V.Drinfeld on this.

Recall that any Levi subgroup $L \subset G$ can be realized as the centralizer $Z_G(g)$ for some semisimple element $g \in G$. If $G = SL(n)$, then the centralizer of any semisimple element is a Levi subgroup but this is *not true* for a general simply connected semisimple G. There are some natural examples which illustrate this.

However, our interest is in centralizers $Z_G(g)$ which are not merely reductive, but are in fact semisimple. A Levi subgroup of G cannot be semisimple unless it equals G, while there are plenty of examples of centralizers which are semisimple and hence not Levi subgroups. Let us assume that G is also simple for the rest of the discussion.

Let $A(G)$ be the following:

$$A(G) = \{\text{conjugacy classes of } g \in G \text{ such that } g \text{ and } Z_G(g) \text{ are semisimple.}\}$$

in particular we note that since g lies in the centre of $Z_G(g)$ it is of finite order. Let

$$B(G) = \{\text{set of vertices of the extended Dynkin diagram}\}$$

We follow the notations in §3. Let α_{max} denote the highest root and c_{α_i} be the coefficients (see (3.0.3)). Choose an isomorphism $e : \mathbb{Q}/\mathbb{Z} \to \{\text{roots of unity in } k^\times\}$.

There is a *bijection*

$$\ell : B(G) \xrightarrow{\sim} A(G) \tag{8.0.1}$$

given as follows: the vertex corresponding to α_{max} goes to $1 \in G$. The vertex corresponding to α_i is sent to g_{α_i}, where

$$g_{\alpha_i} := \check{\omega}_i(e(\frac{1}{c_{\alpha_i}}))$$

and where $\check{\omega}_1, \ldots, \check{\omega}_r$ are the fundamental co-weights i.e $(\alpha_i, \check{\omega}_j) = \delta_{ij}$ and moreover, we view $\check{\omega}_i$ as a morphism $\check{\omega}_i : \mathbb{G}_m \to T$ (cf. Borel-de Siebenthal [5] and Victor Kac [13]) .

8.1. *Remark.* Drinfeld remarks that there are two surprises in this bijection ℓ. Firstly, for each $n \in \mathbb{Z}$, one has the map $f_n : A(G) \to A(G)$ defined by $f_n(g) = g^n$. This map is hard to realize on the side of $B(G)$. Secondly, instead of considering $\check{\omega}_i(e(\frac{1}{c_{\alpha_i}}))$, one could consider $\check{\omega}_i(e(q))$ for $q \in \mathbb{Q}/\mathbb{Z}$. It is not apparent as to why this element comes from $B(G)$. We remark that Theorem 2.3 gives a proper justification of this and places this fact in the context of "weights " developed in §5 of this paper.

8.2. Recall that the set of simple roots in R was denoted by S. Out of this set S one can construct the Dynkin diagram in the usual manner: the vertices of the diagram are the elements of S and the nature of the link viz, empty set, single stroke or oriented double or triple stroke, between two vertices α and β is determined by the value $\check{\alpha}(\beta)$ or $\check{\beta}(\alpha)$. Let $S = S(G)$ also denote the Dynkin diagram of the group G. Let $\alpha_0 = -\alpha_{max}$, the opposite of the highest root, that is, the only root such that, for all $\alpha \in R$, $\alpha - \alpha_0$ is a linear combination with positive coefficients of the elements of S. The set of roots $S \cup \{\alpha_0\}$ also gives rise to a Dynkin diagram, called the *extended Dynkin diagram* of G, or of R and denoted by $\tilde{S}$.

Let $\alpha_* \in S$ (thought of as a vertex of the Dynkin diagram) and set $S' = \tilde{S} - \{\alpha_*\} = S \cup \{\alpha_0\} - \{\alpha_*\}$. Let H be the subgroup of G generated by the root groups U_α and $U_{-\alpha}$ for $\alpha \in S'$. The group H is a *semisimple* subgroup of G containing T and its Dynkin diagram is S'.

Let $\mathcal{G}_{\theta_{\alpha_*}}$ be the Bruhat-Tits group scheme associated to the simple root $\alpha_* \in S$, then recall that the group $\mathcal{G}_{\theta_{\alpha_*}}(A)$ is a *maximal* parahoric subgroup $\mathfrak{P}_{\theta_\alpha}(K)$ of $G(K) = \mathcal{G}_{\theta_{\alpha_*}}(K)$.

It is shown in [32, Page 662] that

$$(\mathcal{G}_{\theta_{\alpha_*}})_k/\text{unipotent radical} \simeq H \tag{8.0.2}$$

In other words, for vertices of the Dynkin diagram the closed fibre of the Bruhat-Tits group schemes modulo the unipotent radical is actually *semisimple* and its Dynkin diagram is given by S'.

Let $Y(T)$, as above, denote the group generated by the coroots $\check{\alpha}_i$. Then one has the identifications:

$$\{\text{Elements of finite order}\}/\mathit{conjugation} \simeq \{\text{Elements of finite order in T}\}/W$$

and this is the same as

$$(Y(T) \otimes \mathbb{Q}/\mathbb{Z})/W \simeq (Y(T) \otimes \mathbb{Q})/W_{aff}$$

where W_{aff} is the affine Weyl group. Further, $(Y(T) \otimes \mathbb{Q})/W_{aff}$ gets identified with the simplex

$$\Delta := \{x \in Y(T) \otimes \mathbb{Q} \mid (x, \alpha_{max}) \leq 1, (x, \alpha_i) \geq 0, 1 \leq i \leq r\}$$

We now tie this with the theory developed in this paper. Recall that each vertex $x \in \Delta$, corresponds to a simple root α in the extended Dynkin diagram $\tilde{S}$ and there is a maximal parahoric subgroup $\mathfrak{P}_{\theta_\alpha}(K) \subset G(K)$. The theory developed in this paper associates to each such maximal parahoric subgroup a moduli stack $\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K))^s$ of *stable* parahoric torsors under a Bruhat-Tits group scheme $\mathcal{G}_{\theta_\alpha}$. We therefore have a set-map $\mathfrak{M}^s_{B(G)} \to B(G)$ with fibres $\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K))^s$.

The "unitary correspondence" established in Theorem 2.0.11 shows that, for every such moduli stack $\mathfrak{M}_X(\mathfrak{P}_{\theta_\alpha}(K))$, there is a corresponding unitary moduli space of irreducible representations $R_o/\overline{K}_G \subset R^{\boldsymbol{\tau}(\alpha)}(\pi, K_G)/\overline{K}_G$ and we have an identification of the moduli spaces given by Corollary 7.19.

Note that to each $g_\alpha \in A(G)$, as a part of the datum in the moduli space of "unitary representations" $R^{\boldsymbol{\tau}(\alpha)}(\pi, K_G)$ the conjugacy class of g_α determines the "type" $\boldsymbol{\tau}(\alpha)$. We again have a set-map $R^{irr}_{A(G)} \to A(G)$ with fibres $R_o/\overline{K}_G$.

The main result of the paper can be interpreted as the "lifting" of the bijection $\ell : B(G) \to A(G)$ to a diagram:

$$\begin{array}{ccc} \mathfrak{M}^s_{B(G)} & \xrightarrow{\bar{\ell}} & R^{irr}_{A(G)} \\ \downarrow & & \downarrow \\ B(G) & \xrightarrow{\ell} & A(G) \end{array} \tag{8.0.3}$$

In fact, as a consequence of the results proved in this paper, one has a larger picture, namely the bijection extends even over the interior points of the simplex Δ. We recall here the classical theorem attributed to Borel-de Siebenthal and Dynkin.

8.3. Theorem. *(Borel-de Siebenthal, Dynkin) The Dynkin diagram of $Z_G(g_{\alpha_i})$ is obtained from the extended Dynkin diagram of G by omitting the i^{th}–vertex.*

The next observation (possibly well-known to experts), which is essentially a piecing together of facts from Bruhat-Tits theory, explains the bijection between $B(G)$ and $A(G)$ somewhat more canonically.

8.4. Theorem.

(1) *Take $\alpha \in B(G)$ and let $\mathcal{G}_{\theta_\alpha}$ be the associated Bruhat-Tits group scheme. For any vertex α , the corresponding centralizer $Z_G(g_\alpha)$ (which is* semisimple*) is realized as*

$$\mathcal{G}^{red}_{\theta_\alpha} \simeq \mathcal{G}_{\theta_\alpha}/(\textit{unipotent radical}) \simeq Z_G(g_\alpha)$$

(2) *For each $\alpha \in \Delta^{hs}$ (the "hyperspecial vertices), we have $Z_G(g_\alpha) = G$.*
(3) *For each $\alpha \notin \Delta^{hs}$ (the non-hyperspecial vertices) we see that $Z_G(g_\alpha) \subsetneq G$ are proper* semisimple *subgroups of G.*

Proof: The group $Z_G(g_{\alpha_i})$ is semisimple and furthermore, by Theorem 8.3, the Dynkin diagram of $Z_G(g_{\alpha_i})$ is obtained from the extended Dynkin diagram of G by omitting the i^{th}–vertex. The first statement now follows from (8.0.2) and from this identification, the last two are easy consequences of the theory of Bruhat-Tits.

8.5. *Remark.*In this context, S. Ramanan told us of a result of his on the moduli space of stable principal G–bundles on curves. The centralizers of various automorphism groups of stable bundles are precisely the groups occurring in the set $A(G)$ described above.

8.6. **Remarks on the variation of weights and stability.** Consider T/W which parametrizes conujugacy classes of elements of G. In each conjugacy class with centralizer semisimple we have an element of finite order. Stratify $\Delta_{\mathbb{R}} = \Delta \otimes \mathbb{R}$ by the real dimension of the conjugacy class above each $x \in \Delta_{\mathbb{R}}$. Let ϕ be the map

$\phi : \mathcal{C} \to \Delta_{\mathbb{R}}$ and we consider $\mathcal{C}$ as the set of conjugacy classes of elements in the maximal compact subgroup $K_{_G} \subset G$. Observe that each fibre

$$\phi^{-1}(x) \simeq K_{_G}/Z_{_K}(g_x)$$

Let

$$\Delta_d = \{x \in \Delta_{\mathbb{R}} \mid dim_{\mathbb{R}}(\phi^{-1}(x)) = d\}$$

8.7. *Remark.*

(1) It seems that the connected components of Δ_d correspond to the non-isomorphic parahoric moduli spaces associated to parahoric groups $\mathfrak{P}_{_x}(K) \subset G(K)$, for $x \in \Delta_d$. The complex dimensions can be computed in a fashion similar to Theorem 7.5.

(2) The strata Δ_d for maximum d are precisely the "hyperspecial moduli". These correspond under the identification ℓ (8.0.1) to the "hyperspecial vertices" in $B(G)$.

(3) If the vertex x corresponds to $\alpha_x = \alpha \in B(G)$ and if $c_{_\alpha} \neq 2$, then the fibre $\phi^{-1}(x)$ gets an almost complex structure which is *not complex*.

(4) A striking example is the case when $G = G_2$ and $\alpha = \alpha_1$, $c_{_\alpha} = 3$ and $\phi^{-1}(x) \simeq S^6$, *the real* 6*–sphere.*

(5) If $c_{_{\alpha_x}} = 2$, then $\phi^{-1}(x)$ admits no invariant almost complex structure. These results follows from the works of Borel-de Siebenthal, Wang and Hermann written in the early 1950's.

(6) In terms of the moduli stacks, these statements reflect the phenomenon that there are no "forget" morphisms from the maximal parahoric moduli to the moduli space of principal G–bundles. In other words, as mentioned in [2], parahoric moduli are not describable as "parabolic G–bundles" as one might erroneously expect from the linear case.

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