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THE ROLE OF THE CENTRAL LIMIT THEOREM IN DISCOVERING SHARP RATES OF CONVERGENCE TO EQUILIBRIUM FOR THE SOLUTION OF THE KAC EQUATION

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In Dolera, Gabetta and Regazzini [*Ann. Appl. Probab.* **19** (2009) 186–201] it is proved that the total variation distance between the solution $f(\cdot, t)$ of Kac's equation and the Gaussian density $(0, \sigma^2)$ has an upper bound which goes to zero with an exponential rate equal to $-1/4$ as $t \to +\infty$. In the present paper, we determine a lower bound which decreases exponentially to zero with this same rate, provided that a suitable symmetrized form of f_0 has nonzero fourth cumulant κ_4. Moreover, we show that upper bounds like $\overline{C}_\delta e^{-(1/4)t}\rho_\delta(t)$ are valid for some ρ_δ vanishing at infinity when $\int_{\mathbb{R}} |v|^{4+\delta} f_0(v)\,dv < +\infty$ for some δ in $[0, 2[$ and $\kappa_4 = 0$. Generalizations of this statement are presented, together with some remarks about non-Gaussian initial conditions which yield the insuperable barrier of -1 for the rate of convergence.

1. Introduction.

In order to determine the rates of relaxation to equilibrium in kinetic theory, Kac derived the following Boltzmann-like equation, commonly known as *the Kac equation*:

$$\begin{aligned}\frac{\partial f}{\partial t}(v,t) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} &[f(v\cos\theta - w\sin\theta, t) \\ &\times f(v\sin\theta + w\cos\theta, t) \\ &- f(v,t)\cdot f(w,t)]\,dw\,d\theta \qquad (v \in \mathbb{R}, t > 0)\end{aligned} \tag{1}$$

with some specific probability density function f_0 as initial datum. The resulting Cauchy problem admits a unique solution within the class of all probability density functions on $\mathbb{R}$. Such a solution provides the probability

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distribution at any time of the velocity of a single particle in a chaotic bath of like molecules *moving on the real line*; see Kac (1956, 1959) and McKean (1966). It is well known that the probability measure $\mu(\cdot,t)$ determined by $f(\cdot,t)$ converges to a distinguished Gaussian law in the *variational metric*, namely

$$d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) := \sup_{B\in\mathscr{B}(\mathbb{R})} |\mu(B,t)-\gamma_\sigma(B)| \to 0 \qquad (t\to+\infty) \tag{2}$$

where γ_σ denotes the Gaussian distribution with zero mean and variance σ^2 and, for any metric space S, $\mathscr{B}(S)$ stands for the Borel class on S. It should be recalled that (2) holds true if and only if the initial datum has finite second moment and σ^2 is the value of this moment. The proof of the "if" part of this assertion is given in Dolera (2007) by adapting arguments explained in Carlen and Lu (2003), whereas the proof of the "only if" part is contained in Gabetta and Regazzini (2008).

In regard to the speed of approach to equilibrium, it has been proven that

$$d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) \le C_* e^{-(1/4)t} \qquad (t\ge 0) \tag{3}$$

holds, with C_* being some suitable constant depending only on the behavior of f_0, when f_0 has finite fourth moment and

$$\varphi_0(\xi) := \int_{\mathbb{R}} e^{i\xi x} f_0(x)\,dx = o(|\xi|^{-p}) \qquad (|\xi|\to+\infty) \tag{4}$$

is valid for some $p>0$; see Dolera, Gabetta and Regazzini (2009). This work will be refered to as DGR throughout the rest of the present paper. Inequality (3) is known as *McKean's conjecture* and the above statement constitutes the first satisfactory support of this conjecture. Other bounds with respect to weak metrics have been given in Gabetta and Regazzini (2010).

At the end of Section 2.2 of DGR, the question of whether the upper bound in (3) can be improved is posed. To the best of the authors' knowledge, this problem has not yet been tackled, except for a hint on page 370 of Carlen, Carvalho and Gabetta (2005). The main proposition in the present paper states that the answer is in the affirmative only in the rather peculiar case in which the *fourth cumulant* of the density $\tilde{f}_0(x) := \{f_0(x)+f_0(-x)\}/2$ is zero. The term "fourth cumulant" of a probability distribution Q on $\mathscr{B}(\mathbb{R})$ refers to the quantity

$$\kappa_4(\mathsf{Q}) := \int_{\mathbb{R}} (x-\overline{Q})^4 \mathsf{Q}(dx) - 3\left(\int_{\mathbb{R}} (x-\overline{Q})^2 \mathsf{Q}(dx)\right)^2,$$

with $\overline{Q} := \int_{\mathbb{R}} x\mathsf{Q}(dx)$, under the assumption that the fourth moment is finite. This cumulant is zero, for example, when Q is Gaussian.

In view of this fact, one could comment on the main proposition by noting that improvements of the rate expressed by (3) turn out to be impossible when f_0 is dissimilar to all of the members in the class of all Gaussian probability density functions. For the sake of completeness, we recall that, given the Fourier–Stieltjes transform q of Q, the rth cumulant of Q is defined to be the coefficient of $(i\xi)^r/r!$ in the Taylor expansion of $\log(q(\xi))$; see, for example, Sections 3.14–3.15 of Stuart and Ord (1987).

As a further remark on the aforementioned proposition, it is worth noting its resemblance to well-known facts related to the approximation of the distribution function F_n of the "standardized" sum of n independent and identically distributed random variables with finite variance, by the standard Gaussian distribution Φ. Indeed, in general, F_n is approximated by Φ, except for terms of order $1/\sqrt{n}$. However, higher orders of approximation hold when the skewness and kurtosis of the common distribution of each summand are zero. Lyapounov (1901) was the pioneer of these kinds of problems, followed by Cramér (1937), Esseen (1945) and others.

The structure of the paper is as follows. Section 2 contains the presentation of the main results. Section 3 deals with the basic preliminary facts which pave the way for proofs of the main results. It is split into two subsections. The former consists of a brief description of the probabilistic interpretation, according to which $\mu(\cdot, t)$ can be seen as distribution of a random weighted sum of random variables. The latter is devoted to the analysis of the error associated with the approximation of the law of certain weighted sums of independent random variables to the Gaussian distribution. Section 4 contains the proofs of the main results stated in Section 2. Finally, some purely technical details are deferred to the Appendix, together with the proofs of two lemmas formulated in Section 3.

2. Presentation of the new results.

In order to present the main results we intend to prove in this paper, it is worth mentioning the following weak version of Kac's problem (1) proposed in Bobylev (1984). Taking the Fourier transform of both sides of (1) yields

$$\frac{\partial \varphi}{\partial t}(\xi, t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\xi\cos\theta, t)\cdot\varphi(\xi\sin\theta, t)\,d\theta - \varphi(\xi, t) \tag{5}$$

with initial datum $\varphi_0(\xi) := \int_{\mathbb{R}} e^{i\xi x} f_0(x)\,dx$. It should be noted that if φ_0 is the Fourier–Stieltjes transform of any (not necessarily absolutely continuous) probability distribution μ_0 on $\mathscr{B}(\mathbb{R})$, then (5) can be thought of as a new problem which generalizes (1). In any case, (5) admits a unique solution $\varphi(\cdot, t)$, which characterizes—in the form of a Fourier–Stieltjes transform—a probability distribution $\mu(\cdot, t)$ which, throughout the paper, will be said to be a solution of (5). Obviously, in problem (1), one has $\mu_0(B) := \int_B f_0(v)\,dv$ and $\mu(B, t) := \int_B f(v, t)\,dv$ for every B in $\mathscr{B}(\mathbb{R})$.

In order to formulate the new results exhaustively, let $\mathfrak{m}_r$ and $\overline{\mathfrak{m}}_r$ denote the rth moment and the absolute rth moment of μ_0, respectively, and let $\tilde{\mu}_0$ be the *symmetrized form of* μ_0 defined by

$$\tilde{\mu}_0(B) := \{\mu_0(B) + \mu_0(-B)\}/2, \qquad B \in \mathscr{B}(\mathbb{R}), \tag{6}$$

where $-B$ denotes the set $\{x | -x \in B\}$.

A precise statement of the fact that the rate $-1/4$ may be the best possible one is contained in the following theorem.

THEOREM 2.1. *Suppose that* μ_0 *possesses finite fourth moment* $\mathfrak{m}_4$ *and that* $\kappa_4(\tilde{\mu}_0) \neq 0$. *Moreover, let* σ^2 *be the value of* $\mathfrak{m}_2$. *There then exists a strictly positive constant* C, *depending only on the behavior of* μ_0, *for which*

$$d_{\mathrm{TV}}(\mu(\cdot, t); \gamma_\sigma) \geq C e^{-(1/4)t} \tag{7}$$

holds true for every $t \geq 0$.

The proof of this theorem, deferred to Section 4, also contains a precise quantification of C. Since

$$\sup_{B \in \mathscr{B}(\mathbb{R})} |\mathsf{P}(B) - \mathsf{Q}(B)| = \frac{1}{2} \int_{\mathbb{R}} |p(x) - q(x)| \, dx$$

is valid whenever P and Q are absolutely continuous probability distributions with densities p and q, respectively, as an immediate consequence of Theorem 2.1, it follows that

$$\frac{1}{2} \int_{\mathbb{R}} \left| f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-v^2/(2\sigma^2)} \right| dv \geq C e^{-(1/4)t} \qquad (t \geq 0) \tag{8}$$

is true for the solution $f(\cdot, t)$ of (1), provided that the initial datum f_0 yields a probability measure μ_0 with the same properties as in Theorem 2.1. From (8), it plainly follows that any inequality such as

$$\int_{\mathbb{R}} \left| f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-v^2/(2\sigma^2)} \right| dv \leq C_* e^{-(1/4)t} \rho(t) \qquad (t \geq 0)$$

is *not* valid when ρ vanishes at infinity. This clarifies why inequality (3) can be viewed as sharp.

We now analyze the effect of assuming that $\kappa_4(\tilde{\mu}_0) = 0$.

THEOREM 2.2. *Consider Kac's equation (1) with initial datum* f_0 *such that* $\overline{\mathfrak{m}}_{4+\delta} < +\infty$ *for some* δ *in* $[0, 2[$ *and* $\kappa_4(\tilde{\mu}_0) = 0$. *Further, let* φ_0, *the Fourier transform of* f_0, *satisfy the usual tail condition (4) for some strictly* *positive p. There then exist a strictly positive constant $\overline{C}_\delta = \overline{C}_\delta(f_0; p)$ and a function $\rho_\delta : [0, +\infty[\to [0, +\infty[$ which vanishes at infinity, for which*

$$\int_{\mathbb{R}} \left| f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-v^2/(2\sigma^2)} \right| dv \le \overline{C}_\delta e^{-(1/4)t} \rho_\delta(t) \qquad (t \ge 0). \tag{9}$$

In particular, if δ belongs to $]0, 2[$, one can take

$$\rho_\delta(t) = \exp\{(-3/4 + 2\alpha_{4+\delta})t\} \tag{10}$$

with $\alpha_s := \frac{1}{2\pi}\int_0^{2\pi} |\sin\theta|^s \, d\theta$.

Useful information for quantifying $\overline{C}_\delta$ can be found in Sections 4.3, 4.4 and Appendices A.2 and A.4.

Since even cumulants κ_{2m} of the Gaussian distribution $(0, \sigma^2)$ vanish for $m \ge 2$ and $\sup_{\xi \in \mathbb{R}} |\varphi(\xi, t) - \mathrm{Re}\, \varphi(\xi, t)| \le 2e^{-t}$, one is led to think that the approach to equilibrium of $\mu(\cdot, t)$ might become faster when the symmetrized form of the initial datum gives an increasing number of zero even cumulants.

Theorem 2.3. *Consider problem (1) and maintain the same notation as before for f_0, μ_0, $\tilde{\mu}_0$, φ_0 and α_s. Further, assume that there exist an integer χ greater than 2 and a number δ in $[0, 2[$ for which:*

- (i) $\int_{\mathbb{R}} |v|^{2\chi+\delta} f_0(v) \, dv < +\infty$;
- (ii) *the cumulants κ_{2m} of $\tilde{f}_0$ vanish for $m = 2, \ldots, \chi$;*
- (iii) *φ_0 meets (4) for some strictly positive p.*

There then exists a strictly positive constant $\overline{C}_{\chi,\delta} = \overline{C}_{\chi,\delta}(f_0; p)$ for which

$$\int_{\mathbb{R}} \left| f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-v^2/(2\sigma^2)} \right| dv \le \overline{C}_{\chi,\delta} e^{-(1-2\alpha_{2\chi+\delta})t} \qquad (t \ge 0) \tag{11}$$

holds true.

Useful information for quantifying $\overline{C}_{\chi,\delta}$ can be found in Section 4.4 and Appendix A.2.

It should be noted that, except for the centered Gaussian law, the most common distributions do not share condition (ii), at least for large values of χ. Therefore, it is reasonable to believe that Theorem 2.1 covers the usual applications.

It would be interesting to check when, under suitable conditions for the initial distribution, the value -1 for the rate of relaxation to equilibrium is actually obtained. The following propositions resolve this issue, under the additional condition that all moments of μ_0 are finite. It therefore remains to check whether this moment assumption can actually be recovered from this high order of relaxation to equilibrium. This problem will be tackled in a forthcoming work.

PROPOSITION 2.4. *If μ_0 possesses moments of every order and the solution $\mu(\cdot,t)$ of (5) satisfies*

$$d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) \le Ce^{-t}$$

for some strictly positive constant C, then

$$\mu_0(\cdot) = \gamma_\sigma(\cdot) + o_\sigma(\cdot), \tag{12}$$

where o_σ is a finite signed measure satisfying $o_\sigma(A) = -o_\sigma(-A)$ and $\gamma_\sigma(A) + o_\sigma(A) \ge 0$ for every Borel subset A of $\mathbb{R}$.

Observe that the Wild formula [cf. (13) in Section 3.1] implies that $d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) = |o_\sigma|e^{-t}$ when the initial datum is of the type (12). Therefore, if one assumes there exists some $\rho: [0,+\infty[\ \to [0,+\infty[$ vanishing at infinity so that $d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) \le Ce^{-t}\rho(t)$, then the total variation $|o_\sigma|$ of o_σ satisfies $|o_\sigma| \le C\rho(t)$ for all positive t, which is tantamount to asserting that o_σ is the null measure. This provides a proof for the following result.

COROLLARY 2.5. *If μ_0 has moments of every order and the solution $\mu(\cdot,t)$ of (5) satisfies*

$$d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) \le Ce^{-t}\rho(t)$$

for some ρ vanishing at infinity and for some positive constant C, then $\mu(\cdot,t) = \gamma_\sigma(\cdot)$ for every $t \ge 0$.

Thus, if all of the moments of μ_0 are finite, then the value for the rate of convergence to equilibrium that one cannot sharpen is just -1, unless μ_0 is Gaussian.

3. Preliminaries.

To pave the way for the proofs of the main statements, this section presents some necessary preliminary facts and results. First, it explains the probabilistic meaning of Wild's series, originally pointed out in McKean (1966). Second, it gives new asymptotic expansions for the characteristic function of weighted sums of independent and identically distributed random variables, which complement analogous statements formulated in, for example, Chapter 8 of Gnedenko and Kolmogorov (1954), Chapter 6 of Petrov (1975) and Section 3.2 of DGR.

3.1. *McKean's interpretation of Wild's sums.* Following Wild (1951), one can express the solution $\varphi(\cdot,t)$ of (5) as a time-dependent mixture of characteristic functions, that is,

$$\varphi(\xi,t) = \sum_{n\ge 1} e^{-t}(1-e^{-t})^{n-1}\hat{q}_n(\xi;\varphi_0), \tag{13}$$

where

$$\begin{cases} \hat{q}_1(\xi;\varphi_0) := \varphi_0(\xi), \\ \hat{q}_n(\xi;\varphi_0) = \dfrac{1}{n-1}\displaystyle\sum_{k=1}^{n-1} \hat{q}_k(\xi;\varphi_0) \star \hat{q}_{n-k}(\xi;\varphi_0) \qquad (n \geq 2) \end{cases}$$

and $\star$ denotes the so-called *Wild product* defined by

$$g_1(\xi) \star g_2(\xi) := \frac{1}{2\pi}\int_0^{2\pi} g_1(\xi\cos\theta) \cdot g_2(\xi\sin\theta)\, d\theta.$$

The Wild series, thanks to a symmetry property of the Wild product, yields a useful decomposition of $\mu(\cdot,t)$ which we will use later. Such a decomposition involves the symmetrized form $\tilde{\mu}$ of a probability measure μ defined by $\tilde{\mu}(B) := [\mu(B) + \mu(-B)]/2$ for any B in $\mathscr{B}(\mathbb{R})$. It is well known that if $\mu^{(s)}(\cdot,t)$ denotes the solution of (5) with initial datum $\tilde{\mu}_0$ [see (6)], then one can write

$$\mu(\cdot,t) - \mu^{(s)}(\cdot,t) = o_0(\cdot)e^{-t} \tag{14}$$

with $o_0(\cdot) := \mu_0(\cdot) - \tilde{\mu}_0(\cdot)$.

The next description of the probabilistic reinterpretation of (13) closely follows Section 3.1 of DGR. Accordingly, we introduce, using exactly the same notation adopted therein, the measurable space $(\Omega, \mathscr{F})$ as a product, together with its coordinate random elements ν, τ, $\theta := (\theta_n)_{n\geq 1}$, $\upsilon := (\upsilon_n)_{n\geq 1}$. We then recall the definitions of the random elements δ_j, π_j given in terms of *McKean trees* and put $\beta = (\nu, \tau, \theta)$. Concerning the random variables π_j, recall the fundamental equality

$$\sum_{j=1}^{\nu} \pi_j^2 \equiv 1, \tag{15}$$

which holds true whenever τ belongs to $\mathbb{G}(\nu)$.

Now, for some fixed initial datum μ_0 for problem (5), define a family $(\mathsf{P}_t)_{t\geq 0}$ of probability measures on $(\Omega, \mathscr{F})$ according to (12) in DGR. Next, consider the random variable

$$V = \sum_{j=1}^{\nu} \pi_j \upsilon_j \tag{16}$$

and note, via the Wild formula, that

$$\mu(B,t) = \mathsf{P}_t\{V \in B\} \qquad (B \in \mathscr{B}(\mathbb{R}), t \geq 0)$$

$\mu(\cdot,t)$ being the solution of (5) with μ_0 as initial datum.

Consequently, the random variables υ_n turn out to be *conditionally independent*, given β, with respect to each P_t. Moreover, since β and υ are independent, one can think of the conditional probability distribution of V given β as the distribution of a weighted sum of independent random variables. Indeed, for any fixed elementary case $\overline{\omega}$ in Ω, one can define the random variable

$$\overline{V}(\cdot) := \sum_{j=1}^{\nu(\overline{\omega})} \pi_j(\overline{\omega}) \upsilon_j(\cdot) \tag{17}$$

on $(\Omega, \mathscr{F})$, for which

$$\mathsf{P}_t\{V \leq x | \beta\}(\overline{\omega}) = \mathsf{P}_t\{\overline{V} \leq x\} \qquad (x \in \mathbb{R}, t \geq 0) \tag{18}$$

holds P_t-almost surely in $\overline{\omega}$. This last equality plays a central role in the rest of the paper since it allows us to work on a finite sum of independent random variables using typical tools of the *central limit problem*. In this context, it is important to examine the behavior of the moments of the random variable V. Their evaluation essentially depends on sums of powers of the π_j via the following identity proven in Gabetta and Regazzini (2006):

$$\mathsf{E}_t\left[\sum_{j=1}^{\nu} |\pi_j|^m\right] = e^{-(1-2\alpha_m)t}, \tag{19}$$

α_m being the same as in Section 2.

3.2. *Some asymptotic expansions for the characteristic function of weighted sums of independent random variables.* As in Section 3.2 of DGR, the subject to be investigated here is the behavior of the characteristic function of weighted sums of independent and identically distributed random variables. The expansions given here turn out to be more careful than the analogous ones contained in the aforementioned work since it is now assumed that the common probability law of the summands possesses moments of arbitrarily high order. Cumulants will play a central role in the analysis of the remainder terms. Finally, the study of the convergence of weighted sums will provide appropriate conditions to improve the rate of approach to equilibrium for solutions of (1).

In the rest of this subsection, $(X_j)_{j\geq 1}$ stands for a sequence of independent and identically distributed real-valued random variables on some probability space $(E, \mathscr{E}, \mathsf{Q})$ with common nondegenerate distribution ζ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. It is assumed that ζ is symmetric [that is, $\zeta(B) = \zeta(-B)$ for every Borel set B of $\mathbb{R}$] and possesses finite moments up to order $k + \delta$, where $k = 2\chi$, χ being some integer greater than 1 and δ being an element of the interval $[0, 2[$. Denote the rth moment and the absolute rth moment of ζ by $\mathfrak{m}_r$ and $\overline{\mathfrak{m}}_r$, respectively. Note that the variance σ^2 of ζ coincides with $\mathfrak{m}_2$. Set $\psi(\xi) := \int_{\mathbb{R}} e^{i\xi x} \zeta(dx)$, which turns out to be an even real-valued function, and for every positive integer n, define $\{c_{1,n}, \dots, c_{n,n}\}$ to be an array of real constants such that

$$\sum_{j=1}^{n} c_{j,n}^2 = 1 \tag{20}$$

holds for every n. Now, let V_n be the sum of $Y_{1,n}, \dots, Y_{n,n}$, where

$$Y_{j,n} := \frac{1}{\sigma} c_{j,n} X_{j,n} \qquad (j = 1, \dots, n)$$

and let ψ_n be the characteristic function of V_n. Consider the rth cumulant κ_r and recall that, in general, it can be defined by

$$\kappa_r = r! \sum_{(*)} (-1)^{s-1} \cdot (s-1)! \cdot \prod_{l=1}^{r} \frac{1}{k_l!} \left(\frac{\mathfrak{m}_l}{l!} \right)^{k_l} \qquad (r = 1, \dots, k) \tag{21}$$

where the symbol $(*)$ means that the sum is carried out over all nonnegative integer solutions $(k_1, \dots, k_r)$ of equations

$$k_1 + 2k_2 + \dots + r k_r = r,$$
$$k_1 + k_2 + \dots + k_r = s$$

with the proviso that $0^0 = 1$. Symmetry of ζ implies that existing cumulants of odd order are equal to zero.

From a technical fact proved in the Appendix, Section A.1, after defining $y_0 := \{[-6\sigma^2 + (36\sigma^4 + 12\mathfrak{m}_4)^{1/2}]/\mathfrak{m}_4\}^{1/2}$, one has $\psi(\xi) \geq 1/2$ if $|\xi| \leq y_0$ and

$$\log \psi(\xi) = \sum_{r=1}^{\chi} (-1)^r \frac{\kappa_{2r}}{(2r)!} \xi^{2r} + \xi^k \cdot \epsilon_k(\xi), \tag{22}$$

where ϵ_k is continuous on $[-y_0, y_0]$ and differentiable on $[-y_0, y_0] \setminus \{0\}$. Moreover, this function satisfies $\epsilon_k(0) = 0$ and $\lim_{\xi \to 0} \varrho_k(\xi) = 0$, with $\varrho_k(\xi) := \xi \cdot \epsilon_k'(\xi)$. Consequently, $M_0^{(k)} := \sup_{\xi \in [-y_0, y_0]} |\epsilon_k(\xi)|$ and $M_1^{(k)} := \sup_{\xi \in [-y_0, y_0]} |\varrho_k(\xi)|$ are two finite constants which depend only on the behavior of the common probability law ζ.

Now, following the same line of reasoning as in Chapter 6 of Petrov (1975), we introduce the quantities

$$\tilde{\lambda}_{r,n} := \frac{\kappa_{2r}}{\sigma^{2r}} \sum_{j=1}^{n} c_{j,n}^{2r} \qquad (r = 1, \dots, \chi) \tag{23}$$

and define the polynomials

$$\tilde{P}_{r,n}(\xi) := \sum_{(*)} \left(\prod_{m=1}^{r} \frac{1}{k_m!} \left(\frac{\tilde{\lambda}_{m+1,n}}{(2m+2)!} \right)^{k_m} \right) (-1)^{r+s} \xi^{2(r+s)} \tag{24}$$

for $r = 1, \dots, \chi - 1$. In addition, we introduce another family of functions $\eta_{k,n}$, which will be used to approximate the characteristic functions ψ_n, defined by

$$\eta_{k,n}(\xi) = e^{-\xi^2/2} + \sum_{r=1}^{\chi-1} \tilde{P}_{r,n}(\xi) \cdot e^{-\xi^2/2} \qquad (\xi \in \mathbb{R}). \tag{25}$$

At this stage, we are in a position to state a couple of preliminary results that play an important role in the rest of the paper.

LEMMA 3.1. *Assume that $\chi = 2$ (i.e., $k = 4$) and $\delta = 0$. There then exists a positive constant C_4^*, depending only on the behavior of ζ, such that*

$$|\psi_n(\xi) - \eta_{4,n}(\xi)| \le C_4^* \xi^4 e^{-\xi^2/2} \left[\xi^4 \sum_{j=1}^n c_{j,n}^4 + \sum_{j=1}^n c_{j,n}^4 \left| \epsilon_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| \right], \tag{26}$$

$$|\psi_n(\xi) - \eta_{4,n}(\xi)| \le C_4^* \xi^4 (1 + \xi^4) e^{-\xi^2/2} \left[\sum_{j=1}^n c_{j,n}^6 + \sum_{j=1}^n c_{j,n}^4 \left| \epsilon_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| \right] \tag{27}$$

and

$$\begin{aligned} &|\psi_n'(\xi) - \eta_{4,n}'(\xi)| \\ &\quad \le C_4^* |\xi|^3 (1 + \xi^6) e^{-\xi^2/2} \\ &\qquad \times \left[\sum_{j=1}^n c_{j,n}^6 + \sum_{j=1}^n c_{j,n}^4 \left(\left| \epsilon_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| + \left| \varrho_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| \right) \right] \end{aligned} \tag{28}$$

hold true for every $|\xi| \le A_{4,n} := \sigma y_0 (\sum_{j=1}^n c_{j,n}^4)^{-1/4}$.

In (26)–(28), recall that ϵ_4 is defined by (22) with $k = 4$, and $\rho_4(\xi) := \xi \epsilon_4'(\xi)$.

For general $k = 2\chi$ and δ, we have the following result.

LEMMA 3.2. *If $|\xi| \le A_{k,\delta,n} := \sigma y_0 (\sum_{j=1}^n c_{j,n}^4)^{-1/(k+\delta)}$, then*

$$|\psi_n(\xi) - \eta_{k,n}(\xi)| \le C_{k,\delta}^* p_{0,k}(\xi) |\xi|^{k+\delta} e^{-\xi^2/2} \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta} \right) \tag{29}$$

and

$$|\psi_n'(\xi) - \eta_{k,n}'(\xi)| \le C_{k,\delta}^* p_{1,k}(\xi) |\xi|^{k-1+\delta} e^{-\xi^2/2} \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta} \right), \tag{30}$$

where $C_{k,\delta}^*$ is a constant depending only on the behavior of ζ and $p_{0,k}(\xi)$, $p_{1,k}(\xi)$ are polynomials whose coefficients depend only on k.

The proofs of these lemmata are deferred to Section A.2, in which one can also find instructions for the evaluation of C_4^*, $C_{k,\delta}^*$, $p_{0,k}(\xi)$ and $p_{1,k}(\xi)$. Inequalities (29) and (30) immediately entail that

$$\left(\int_{-A_{k,\delta,n}}^{A_{k,\delta,n}} |\psi_n(\xi) - \eta_{k,n}(\xi)|^2\, d\xi\right)^{1/2} \le C_{k,\delta}^* a_k \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta}\right) \tag{31}$$

and

$$\left(\int_{-A_{k,\delta,n}}^{A_{k,\delta,n}} |\psi_n'(\xi) - \eta_{k,n}'(\xi)|^2\, d\xi\right)^{1/2} \le C_{k,\delta}^* a_k \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta}\right), \tag{32}$$

where a_k is the maximum between $(\int_{\mathbb{R}} \xi^{2k}(1+\xi^2)^2 p_{0,k}^2(\xi) e^{-\xi^2}\, d\xi)^{1/2}$ and $(\int_{\mathbb{R}} \xi^{2k-2}(1+\xi^2)^2 p_{1,k}^2(\xi) e^{-\xi^2}\, d\xi)^{1/2}$.

4. Proofs of the main results. We first prove Theorem 2.1 and then focus on Proposition 2.4. In fact, they rest on similar arguments. We will then provide proofs for Theorems 2.2 and 2.3 by adapting methods used in Section 4 of DGR.

Before starting, it is worth introducing some new symbols which will be used hereafter. First, choose a version of the conditional distribution function $\mathsf{P}_t\{V \le x|\beta\}$ and call it $\mathsf{F}^*(x)$. In view of (18), it does not depend on t. $\mathsf{F}^*(x)[\overline{\omega}]$ will indicate dependence of $\mathsf{F}^*(x)$ on a specific sample point $\overline{\omega}$ in Ω. The Fourier–Stieltjes transform of $\mathsf{F}^*(\cdot)[\overline{\omega}]$ will be designated by $\varphi^*(\cdot)[\overline{\omega}]$. Moreover, an integral over a measurable subset S of Ω will often be denoted by $\mathsf{E}[\cdot; S]$. Symbols $\mathfrak{m}_r$ and $\overline{\mathfrak{m}}_r$ for $\int x^r \mu_0(dv)$ and $\int |x|^r \mu_0(dx)$, respectively, will continue to be used and σ^2 will designate the value of $\mathfrak{m}_2$, while y_0 will stand for the quantity $\{[-6\sigma^2 + (36\sigma^4 + 12\mathfrak{m}_4)^{1/2}]/\mathfrak{m}_4\}^{1/2}$.

4.1. *Proof of Theorem 2.1.* Assume, initially, that μ_0 is symmetric. For simplicity, introduce the *rescaled solution* $\mu_\sigma(\cdot, t)$, defined by $\mu_\sigma(B, t) := \mu(\sigma B, t)$, where $\sigma B := \{y = \sigma x | x \in B\}$ for every B in the Borel class of $\mathbb{R}$. By the homogeneity of the total variation distance, we have $d_{\mathrm{TV}}(\mu(\cdot, t); \gamma_\sigma) = d_{\mathrm{TV}}(\mu_\sigma(\cdot, t); \gamma)$, where γ is shorthand for the standard normal law γ_1. Now, thanks to the elementary inequality

$$d_{\mathrm{TV}}(\mu_\sigma(\cdot, t); \gamma) \ge \frac{1}{2} \sup_{\xi \in \mathbb{R}} |\varphi(\xi/\sigma, t) - e^{-\xi^2/2}|, \tag{33}$$

one can employ the expansions given in Section 3.2. First, observe that for any small ε in $]0, \sigma y_0]$, one has

$$\sup_{\xi \in \mathbb{R}} |\varphi(\xi/\sigma, t) - e^{-\xi^2/2}|$$

$$
\begin{aligned}
&\geq |\varphi(\varepsilon/\sigma, t) - e^{-\varepsilon^2/2}| \\
&= |\mathsf{E}_t\{\mathsf{E}_t[e^{i\varepsilon V/\sigma}|\beta] - e^{-\varepsilon^2/2}\}| \\
&= \left|\int_\Omega \{\varphi^*(\varepsilon/\sigma)[\overline{\omega}] - e^{-\varepsilon^2/2}\}\mathsf{P}_t(d\overline{\omega})\right|.
\end{aligned} \tag{34}
$$

Next, after fixing any $\overline{\omega}$ in Ω, substitute $\nu(\overline{\omega})$ for n and $\pi_j(\overline{\omega})$ for $c_{j,n}$ ($j = 1, 2, \ldots, n$) in Lemma 3.1. This way, $\psi_n(\xi)$ changes into $\varphi^*(\xi/\sigma)$ and the restriction that Lemma 3.1 imposes on ε becomes $|\varepsilon| \leq \sigma y_0\ (\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^4(\overline{\omega}))^{-1/4}$. Clearly, this bound holds P_t-almost surely for every t, whenever ε is not greater then σy_0. Hence, (26) can be applied with

$$
\eta_4(\xi)[\overline{\omega}] := e^{-\xi^2/2} + \frac{\kappa_4}{4!\sigma^4}\left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^4(\overline{\omega})\right)\xi^4 e^{-\xi^2/2}
$$

in place of $\eta_{4,n}(\xi)$. If $R_4^*(\xi)[\overline{\omega}]$ stands for $\varphi^*(\varepsilon/\sigma)[\overline{\omega}] - \eta_4(\xi)[\overline{\omega}]$, then the last member in (34) can be written as

$$
\begin{aligned}
&\left|\int_\Omega R_4^*(\varepsilon)[\overline{\omega}]\mathsf{P}_t(d\overline{\omega}) + \frac{\kappa_4}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}\int_\Omega \left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^4(\overline{\omega})\right)\mathsf{P}_t(d\overline{\omega})\right| \\
&\qquad = \left|\int_\Omega R_4^*(\varepsilon)\, d\mathsf{P}_t + \frac{\kappa_4}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2} e^{-(1/4)t}\right| \\
&\qquad \geq \left|\frac{|\kappa_4|}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2} e^{-(1/4)t} - \left|\int_\Omega R_4^*(\varepsilon)\, d\mathsf{P}_t\right|\right|,
\end{aligned} \tag{35}
$$

where the equality follows from (19) and the inequality follows from $|a+b| \geq ||a| - |b||$. Now, the claim is that there exists an ε independent of t and small enough to have

$$
\left|\int_\Omega R_4^*(\varepsilon)\, d\mathsf{P}_t\right| \leq \frac{|\kappa_4|}{2 \cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2} e^{-(1/4)t} \tag{36}
$$

for every nonnegative t. To this end, recall the following: that ϵ_4 [see (22)] is a continuous function depending only on the initial datum μ_0 so that $\epsilon_4(0) = 0$; that $|\kappa_4|$ is strictly positive; that the constant $C_4^* = C_4^*(\mu_0)$ can never be chosen equal to zero. The inequality

$$
|\epsilon_4(x)| \leq \frac{|\kappa_4|}{4 \cdot 4!\sigma^4 C_4^*}
$$

is surely satisfied for every x belonging to a suitable nondegenerate interval $[-\overline{x}, \overline{x}]$ included in $[-y_0, y_0]$. Thus, taking (26) into account, one can write

$$
\int_\Omega \left[C_4^* \varepsilon^4 e^{-\varepsilon^2/2} \sum_{j=1}^{\nu(\overline{\omega})} \pi_j^4(\overline{\omega}) \left|\epsilon_4\left(\frac{\pi_j^4(\overline{\omega})\varepsilon}{\sigma}\right)\right|\right] \mathsf{P}_t(d\overline{\omega})
$$

(37)
$$\leq \frac{|\kappa_4|}{4\cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-(1/4)t}$$

for every ε in $]0,\sigma\overline{x}]$ and $t \geq 0$. Moreover,

$$C_4^*\varepsilon^8 e^{-\varepsilon^2/2}e^{-(1/4)t} \leq \frac{|\kappa_4|}{4\cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-(1/4)t}$$

is valid for every nonnegative t, provided that ε is chosen not greater than $\overline{\overline{x}} := (\frac{|\kappa_4|}{4\cdot 4! C_4^*\sigma^4})^{1/4}$. Thus, in view of (26), (36) is satisfied for ε in $]0,\min\{\sigma\overline{x};\overline{\overline{x}}\}]$.

To conclude the proof in the symmetric case, fix ε as above in order to have (36) and use the following elementary fact: if $|b| \leq |a|/2$, then $||a|-|b|| = |a|-|b| \geq |a|/2$. Applying this to (35), we get

$$\left|\frac{|\kappa_4|}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-(1/4)t} - \left|\int_\Omega R_4^*(\varepsilon)\,d\mathsf{P}_t\right|\right| \geq \frac{|\kappa_4|}{2\cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-(1/4)t},$$

which, in view of (34), provides a lower bound for $d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma)$. When μ_0 is symmetric, the constant $\tilde{C}$, which appears in Theorem 2.1, can be taken to be equal to $\frac{|\kappa_4|}{4\cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}$ with ε in $]0,\min\{\sigma\overline{x};\overline{\overline{x}}\}]$.

When μ_0 is not symmetric, we employ its symmetrized form $\tilde{\mu}_0$ and recall (14) to obtain

$$\begin{aligned}|\mu^{(s)}(B,t)-\gamma_\sigma(B)| &= |\mu(B,t)-o_0(B)e^{-t}-\gamma_\sigma(B)|\\ &\leq |\mu(B,t)-\gamma_\sigma(B)| + 2e^{-t}\\ &\leq d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) + 2e^{-t}(B\in\mathscr{B}(\mathbb{R})),\end{aligned}$$

which plainly entails

$$d_{\mathrm{TV}}(\mu^{(s)}(\cdot,t);\gamma_\sigma) \leq d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) + 2e^{-t}. \tag{38}$$

From the first part of the proof, one can find a constant $\tilde{C}(\tilde{\mu}_0) \leq 2$ for which

$$d_{\mathrm{TV}}(\mu^{(s)}(\cdot,t);\gamma_\sigma) \geq \tilde{C}(\tilde{\mu}_0)e^{-(1/4)t}.$$

Hence,

$$\begin{aligned}d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) &\geq d_{\mathrm{TV}}(\mu^{(s)}(\cdot,t);\gamma_\sigma) - 2e^{-t}\\ &\geq \tilde{C}(\tilde{\mu}_0)e^{-(1/4)t} - 2e^{-t} \geq \tfrac{1}{2}\tilde{C}(\tilde{\mu}_0)e^{-(1/4)t}\end{aligned}$$

holds, provided that $t \geq \hat{t} := -\log[(\tilde{C}(\tilde{\mu}_0)/4)^{4/3}]$, where $\hat{t}$ is strictly positive. To conclude the proof, observe that (7) is valid, taking, for example,

$$\tilde{C} = \tilde{C}(\mu_0) := \min\left\{\frac{1}{2}\tilde{C}(\tilde{\mu}_0); \inf_{t\in[0,\hat{t}]} d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma)\right\}.$$

Finally, $\inf_{t\in[0,\hat{t}]} d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma)$ is strictly positive in view of the existence of the minimum combined with the uniqueness of the solution of Kac's equation. This point is clarified in Appendix A.3.

4.2. *Proof of Proposition 2.4.* To prove this proposition under the assumption that all of the moments of μ_0 are finite, it will suffice to prove that all of the cumulants $\tilde{\kappa}_{2m}$ of even order of $\tilde{\mu}_0$ are zero for $m = 2, 3, \ldots$. Thanks to (38), the inequality, which appears in the statement of Proposition 2.4, can be rewritten as

$$d_{\mathrm{TV}}(\mu^{(s)}(\cdot, t); \gamma_\sigma) \le (C + 2)e^{-t}. \tag{39}$$

In view of this fact, we can assume, without real loss of generality, that μ_0 is symmetric. Then, supposing that $\kappa_{2m} = 0$ for $m = 2, \ldots, s-1$ and $\kappa_{2s} \neq 0$ for some integer s greater than 2, we have contradicted (39).

As in the previous subsection, write

$$\begin{aligned} 2d_{\mathrm{TV}}(\mu(\cdot, t); \gamma_\sigma) &\ge \sup_{\xi \in \mathbb{R}} |\varphi(\xi/\sigma, t) - e^{-\xi^2/2}| \\ &\ge \left| \int_\Omega \{\varphi^*(\varepsilon/\sigma)[\overline{\omega}] - e^{-\varepsilon^2/2}\} \mathsf{P}_t(d\overline{\omega}) \right|, \end{aligned} \tag{40}$$

where ε is any positive constant not greater than σy_0. Following the general lines of Section 3.2, define

$$\eta_{2s}(\xi)[\overline{\omega}] := e^{-\xi^2/2} + (-1)^s \frac{\kappa_{2s}}{(2s)!\sigma^{2s}} \left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^{2s}(\overline{\omega}) \right) \xi^{2s} e^{-\xi^2/2}.$$

After setting $R_{2s}^*(\xi)[\overline{\omega}] := \varphi^*(\varepsilon/\sigma)[\overline{\omega}] - \eta_{2s}(\xi)[\overline{\omega}]$, the last part of (40) becomes

$$\begin{aligned} &\left| \int_\Omega R_{2s}^*(\varepsilon)[\overline{\omega}] \mathsf{P}_t(d\overline{\omega}) + (-1)^s \frac{\kappa_{2s}}{(2s)!\sigma^{2s}} \varepsilon^{2s} e^{-\varepsilon^2/2} \int_\Omega \left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^{2s}(\overline{\omega}) \right) \mathsf{P}_t(d\overline{\omega}) \right| \\ &\quad = \left| \int_\Omega R_{2s}^*(\varepsilon)\, d\mathsf{P}_t + (-1)^s \frac{\kappa_{2s}}{(2s)!\sigma^{2s}} \varepsilon^{2s} e^{-\varepsilon^2/2} e^{-(1-2\alpha_{2s})t} \right| \\ &\quad \ge \left| \frac{|\kappa_{2s}|}{(2s)!\sigma^{2s}} \varepsilon^{2s} e^{-\varepsilon^2/2} e^{-(1-2\alpha_{2s})t} - \left| \int_\Omega R_{2s}^*(\varepsilon)\, d\mathsf{P}_t \right| \right|. \end{aligned} \tag{41}$$

Now, if $|\varepsilon| \le \sigma y_0$, an application of (29), with $k = 2s$ and $\delta = 1$ combined with (19), yields

$$\begin{aligned} \left| \int_\Omega R_{2s}^*(\varepsilon)\, d\mathsf{P}_t \right| &\le \int_\Omega |R_{2s}^*(\varepsilon)|\, d\mathsf{P}_t \\ &\le C_{2s,1}^* |\varepsilon|^{2s+1} [9(1 + |\varepsilon|^{h(s)})] e^{-\varepsilon^2/2} e^{-(1-2\alpha_{2s+1})t} \\ &\le C_{2s,1}^* |\varepsilon|^{2s+1} [9(1 + (\sigma y_0)^{h(s)})] e^{-\varepsilon^2/2} e^{-(1-2\alpha_{2s+1})t} \end{aligned} \tag{42}$$

for every nonnegative t. Here, $h(s) := 2s^2 - s$ and the term $[9(1+|\varepsilon|^{h(s)})]$ is an upper bound for the polynomial $p_{0,k}$ in (29); see also (81) in the Appendix. If ε satisfies the further restriction

$$|\varepsilon| \leq \frac{1}{2C^*_{2s,1}} \cdot \frac{1}{9(1+(\sigma y_0)^{h(s)})} \cdot \frac{|\kappa_{2s}|}{(2s)!\sigma^{2s}},$$

then one can rewrite (42) as

$$\left|\int_\Omega R^*_{2s}(\varepsilon)\, d\mathsf{P}_t\right| \leq \frac{|\kappa_{2s}|}{2\cdot(2s)!\sigma^{2s}} \varepsilon^{2s} e^{-\varepsilon^2/2} e^{-(1-2\alpha_{2s})t}. \tag{43}$$

Hence, inequalities (41) and (43) entail that

$$\frac{|\kappa_{2s}|}{2\cdot(2s)!\sigma^{2s}} \varepsilon^{2s} e^{-\varepsilon^2/2} e^{-(1-2\alpha_{2s})t} \leq 2d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma) \leq 2(C+2)e^{-t}$$

for every nonnegative t, which contradicts the fact that $(1-2\alpha_{2s})$ is strictly smaller than 1. Thus, κ_{2s} must vanish, implying that $\mu_0 = \gamma_\sigma$ since γ_σ is uniquely determined by its moments. Finally, if μ_0 is not symmetric, then $\tilde{\mu}_0 = \gamma_\sigma$.

4.3. *Proof of Theorem 2.2 when $k+\delta = 4$.* We shall closely follow the proof of Theorem 2.1 in DGR. First, let us assume that the condition

(H) $\qquad$ *f_0 and, consequently, $f(\cdot,t)$ are even functions*

holds. This does not limit the generality of subsequent reasoning, thanks to (9)–(10) of DGR. Since $\frac{d}{dv}\mathsf{F}^*(v)$ represents a version of the conditional probability density function of V given β, in view of basic properties of conditional expectation, one has

$$\begin{aligned}
&\int_{\mathbb{R}} \left| f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-v^2/(2\sigma^2)} \right| dv \\
&\quad =: \|f(v,t) - g_\sigma(v)\|_1 \leq \mathsf{E}_t\left[\left\|\frac{d}{dv}\mathsf{F}^*(v) - g_\sigma(v)\right\|_1\right] \\
&\quad = \mathsf{E}_t\left[\left\|\frac{d}{dv}\mathsf{F}^*(\sigma v) - g_1(v)\right\|_1\right],
\end{aligned} \tag{44}$$

where $g_\sigma(v)\, dv = \gamma_\sigma(dv)$. Moreover, from Proposition 2.2 of DGR, which can be applied to f_0, thanks to the hypotheses in Theorem 2.2 and (H), there exist α and λ for which

$$|\varphi_0(\xi)| \leq \left(\frac{\lambda^2}{\lambda^2+\xi^2}\right)^\alpha \tag{45}$$

holds true for every real ξ. In particular, one can set $\alpha = (2 \cdot \lceil 2/p \rceil)^{-1}$, p being the same as in (4) and $\lceil s \rceil$ standing for the least integer not less than s. Define $U \subset \Omega$ by

$$U := \{\nu \le \overline{n}\} \cup \left\{ \prod_{j=1}^{\nu} \pi_j = 0 \right\} \cup \left\{ \sum_{j=1}^{\nu} \pi_j^4 \ge \overline{\delta} \right\} \tag{46}$$

with $\overline{n} = 17 \cdot \lceil 2/p \rceil$ and

$$\overline{\delta} = \min\left\{ \frac{1}{2^{\overline{n}} \overline{n}!}; \frac{\sigma^8}{16 y_0^4 \mathfrak{m}_3^4} \right\} \le \frac{1}{2^{\overline{n}} \overline{n}!}.$$

Next, check that U belongs to $\mathscr{F}$ and rewrite the last term in (44) as

$$\mathsf{E}_t \left[\left\| \frac{d}{dv} \mathsf{F}^*(\sigma v) - g_1(v) \right\|_1 ; U \right] + \mathsf{E}_t \left[\left\| \frac{d}{dv} \mathsf{F}^*(\sigma v) - g_1(v) \right\|_1 ; U^c \right]. \tag{47}$$

By the same arguments as the ones used to prove (22) in DGR, one obtains

$$\mathsf{P}_t\{\nu \le \overline{n}\} \le \overline{n} e^{-t} \quad \text{and} \quad \mathsf{P}_t \left\{ \prod_{j=1}^{\nu} \pi_j = 0 \right\} = 0.$$

As for the third component of the union in the definition of U, one can combine Markov's (with power 2) and Lyapunov's inequalities to get

$$\mathsf{P}_t \left\{ \sum_{j=1}^{\nu} \pi_j^4 \ge \overline{\delta} \right\} \le \frac{1}{\overline{\delta}^2} \mathsf{E}_t \left[\left(\sum_{j=1}^{\nu} \pi_j^4 \right)^2 \right] \le \frac{1}{\overline{\delta}^2} \mathsf{E}_t \left[\sum_{j=1}^{\nu} \pi_j^6 \right] \le \frac{1}{\overline{\delta}^2} e^{-(3/8)t}.$$

The exponent $3/8$ follows from the application of (19) with $m = 6$. Now, combining all of the above computations leads to an estimate for the probability of U under P_t, that is,

$$\mathsf{P}_t(U) \le [\overline{n} + 1/\overline{\delta}^2] e^{-(3/8)t} \qquad (t \ge 0). \tag{48}$$

Inequality (48) leads immediately to the upper bound

$$\mathsf{E}_t \left[\left\| \frac{d}{dv} \mathsf{F}^*(\sigma v) - g_1(v) \right\|_1 ; U \right] \le 2 \mathsf{P}_t(U) \le 2[\overline{n} + 1/\overline{\delta}^2] e^{-(3/8)t}. \tag{49}$$

To control the integral over U^c appearing in (47), we invoke the *Beurling inequality* formulated in Proposition 4.1 of DGR to obtain

$$\begin{aligned} &\mathsf{E}_t \left[\left\| \frac{d}{dv} \mathsf{F}^*(\sigma v) - g_1(v) \right\|_1 ; U^c \right] \\ &\quad \le \frac{1}{\sqrt{2}} \mathsf{E}_t \left[\left\{ \int_{\mathbb{R}} |\Delta|^2 \, d\xi + \int_{\mathbb{R}} |\Delta'|^2 \, d\xi \right\}^{1/2} ; U^c \right], \end{aligned} \tag{50}$$

where $\Delta := \varphi^*(\xi/\sigma) - e^{-\xi^2/2}$ and $\Delta' := \frac{d}{d\xi}\Delta$. Applicability of this result is justified by the fact that the restriction to U^c of the conditional characteristic function $\xi \mapsto \varphi^*(\xi) := \int_{\mathbb{R}} e^{i\xi x}\, d\mathsf{F}^*(x)$ belongs to $H^1(\mathbb{R})$. To see this, note that $\varphi^*(\xi)[\overline{\varpi}] = o(|\xi|^{-34})$ is valid for $|\xi| \to +\infty$ and for $\overline{\varpi}$ in U^c. Indeed, thanks to conditional independence and (45), one has

$$|\varphi^*(\xi)| \le \prod_{j=1}^{\overline{n}} \left(\frac{\lambda^2}{\lambda^2 + \pi_j^2 \xi^2}\right)^{\alpha}$$

and the claimed "tail behavior" of φ^* follows from the definitions of $\overline{n}$ and α, together with the fact that the random numbers π_j do not vanish on U^c. To complete the argument for $H^1(\mathbb{R})$ regularity, use Remark A.2 in Section A.3 of the Appendix of DGR.

Now, the expectation in the right-hand side of (50) is dominated by

$$\begin{aligned}\mathsf{E}_t\left[\left(\int_{\{|\xi|\le \mathrm{A}\}} |\Delta|^2\, d\xi\right)^{1/2}; U^c\right] + \mathsf{E}_t\left[\left(\int_{\{|\xi|\ge \mathrm{A}\}} |\Delta|^2\, d\xi\right)^{1/2}; U^c\right] \\ + \mathsf{E}_t\left[\left(\int_{\{|\xi|\le \mathrm{A}\}} |\Delta'|^2\, d\xi\right)^{1/2}; U^c\right] + \mathsf{E}_t\left[\left(\int_{\{|\xi|\ge \mathrm{A}\}} |\Delta'|^2\, d\xi\right)^{1/2}; U^c\right]\end{aligned} \tag{51}$$

with

$$\mathrm{A} = \mathrm{A}(\beta) := \frac{\sigma y_0}{(\sum_{j=1}^{\nu} \pi_j^4)^{1/4}}.$$

At this stage, we apply (27) to the evaluation of the first integral in (51) after observing that the function $\eta_{4,n}(\xi)$ here equals $e^{-\xi^2/2}$ almost surely since $\kappa_4 = 0$. This leads to

$$\begin{aligned}&\left(\int_{\{|\xi|\le \mathrm{A}\}} |\Delta|^2\, d\xi\right)^{1/2} \\ &\quad\le 2\sqrt{2\Gamma(17/2)}\, C_4^* \left(\sum_{j=1}^{\nu} \pi_j^6\right) \\ &\qquad + \sqrt{2}\, C_4^* \left[\int_{\mathbb{R}} \xi^8 (1+\xi^4)^2 e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left|\tilde{\epsilon}_4\left(\frac{\pi_j \xi}{\sigma}\right)\right|\right)^2 d\xi\right]^{1/2}\end{aligned} \tag{52}$$

with

$$\tilde{\epsilon}_4(x) := \begin{cases} \dfrac{\log \varphi_0(x) + (\sigma^2/2)x^2 - (\kappa_4/4!)x^4}{x^4}, & \text{if } 0 < |x| \le \sigma y_0, \\ \tilde{\epsilon}_4(\sigma y_0), & \text{if } |x| > \sigma y_0, \\ 0, & \text{if } x = 0. \end{cases}$$

Note that $\tilde{\epsilon}_4$ is a bounded continuous function. Take expectations of both sides of (52) and recall (19) to obtain

$$\begin{aligned}
&\mathsf{E}_t\left(\int_{\{|\xi|\leq \mathrm{A}\}}|\Delta|^2\,d\xi\right)^{1/2}\\
&\quad\leq 2\sqrt{2\Gamma(17/2)}C_4^* e^{-(3/8)t}\\
&\qquad+\sqrt{2}C_4^*\mathsf{E}_t\left[\int_{\mathbb{R}}\xi^8(1+\xi^4)^2e^{-\xi^2}\left(\sum_{j=1}^{\nu}\pi_j^4\left|\tilde{\epsilon}_4\left(\frac{\pi_j\xi}{\sigma}\right)\right|\right)^2d\xi\right]^{1/2}.
\end{aligned}\tag{53}$$

In view of Section A.4,

$$\lim_{t\to+\infty}\rho_0^{(1)}(t)=0,\tag{54}$$

where

$$\rho_0^{(1)}(t):=e^{(1/4)t}\mathsf{E}_t\left[\int_{\mathbb{R}}\xi^8(1+\xi^4)^2e^{-\xi^2}\left(\sum_{j=1}^{\nu}\pi_j^4\left|\tilde{\epsilon}_4\left(\frac{\pi_j\xi}{\sigma}\right)\right|\right)^2d\xi\right]^{1/2}.$$

Similarly, apply (28) to evaluate the second integral in (51) as follows:

$$\begin{aligned}
&\left(\int_{\{|\xi|\leq \mathrm{A}\}}|\Delta'|^2\,d\xi\right)^{1/2}\\
&\quad\leq 4\sqrt{\Gamma(19/2)}C_4^*\left(\sum_{j=1}^{\nu}\pi_j^6\right)\\
&\qquad+2\sqrt{2}C_4^*\left[\int_{\mathbb{R}}\xi^6(1+\xi^{12})e^{-\xi^2}\left(\sum_{j=1}^{\nu}\pi_j^4\left|\tilde{\epsilon}_4\left(\frac{\pi_j\xi}{\sigma}\right)\right|\right)^2d\xi\right]^{1/2}\\
&\qquad+2\sqrt{2}C_4^*\left[\int_{\mathbb{R}}\xi^6(1+\xi^{12})e^{-\xi^2}\left(\sum_{j=1}^{\nu}\pi_j^4\left|\tilde{\varrho}_4\left(\frac{\pi_j\xi}{\sigma}\right)\right|\right)^2d\xi\right]^{1/2}
\end{aligned}\tag{55}$$

with

$$\tilde{\varrho}_4(x):=\begin{cases} x\dfrac{d}{dx}\tilde{\epsilon}_4(x), & \text{if } 0<|x|<\sigma y_0,\\ l:=\lim\limits_{u\uparrow\sigma y_0}\tilde{\varrho}_4(u), & \text{if } |x|\geq\sigma y_0,\\ 0, & \text{if } x=0.\end{cases}$$

Once again, take expectations of both sides of (55) and use (19) to get

$$\mathsf{E}_t\left(\int_{\{|\xi|\leq \mathrm{A}\}}|\Delta'|^2\,d\xi\right)^{1/2}$$

$$
\begin{aligned}
&\le 4\sqrt{\Gamma(19/2)}C_4^* e^{-(3/8)t} \\
&\quad + 2\sqrt{2}C_4^* \mathsf{E}_t \left[\int_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\epsilon}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 d\xi \right]^{1/2} \\
&\quad + 2\sqrt{2}C_4^* \mathsf{E}_t \left[\int_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\varrho}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 d\xi \right]^{1/2}.
\end{aligned} \tag{56}
$$

Another application of Section A.4 leads us to state the following important facts:

$$
\lim_{t\to+\infty} \rho_0^{(2)}(t) = \lim_{t\to+\infty} \rho_0^{(3)}(t) = 0, \tag{57}
$$

where

$$
\rho_0^{(2)}(t) := e^{(1/4)t} \mathsf{E}_t \left[\int_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\epsilon}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 d\xi \right]^{1/2}
$$

and

$$
\rho_0^{(3)}(t) := e^{(1/4)t} \mathsf{E}_t \left[\int_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\varrho}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 d\xi \right]^{1/2}.
$$

After determining upper bounds for integrals of the type $\int_{\{|\xi|\le \mathrm{A}\}}$, it remains to examine the remaining summands in (51). Minkowski's inequality yields

$$
\left(\int_{\{|\xi|\ge \mathrm{A}\}} |\Delta|^2 \, d\xi \right)^{1/2} \le \left(\int_{\{|\xi|\ge \mathrm{A}\}} |\varphi^*(\xi/\sigma)|^2 \, d\xi \right)^{1/2} + \left(\int_{\{|\xi|\ge \mathrm{A}\}} |e^{-\xi^2/2}|^2 \, d\xi \right)^{1/2}
$$

and

$$
\begin{aligned}
&\left(\int_{\{|\xi|\ge \mathrm{A}\}} |\Delta'|^2 \, d\xi \right)^{1/2} \\
&\qquad \le \left(\int_{\{|\xi|\ge \mathrm{A}\}} \left| \frac{d}{d\xi} \varphi^*(\xi/\sigma) \right|^2 d\xi \right)^{1/2} + \left(\int_{\{|\xi|\ge \mathrm{A}\}} |\xi e^{-\xi^2/2}|^2 \, d\xi \right)^{1/2}.
\end{aligned}
$$

From a well-known inequality, proved in, for example, Lemma 2 of VII.1 in Feller (1968), and since $\max_{x\ge 0} x^k e^{-\alpha x^2} = [k/(2e\alpha)]^{k/2}$, one obtains

$$
\left(\int_{\{|\xi|\ge \mathrm{A}\}} e^{-\xi^2} \, d\xi \right)^{1/2} \le \left(\frac{15}{2} \right)^{15/4} e^{-15/4} (\sigma y_0)^{-8} \sum_{j=1}^{\nu} \pi_j^6
$$

and

$$
\left(\int_{\{|\xi|\ge \mathrm{A}\}} \xi^2 e^{-\xi^2} \, d\xi \right)^{1/2} \le \frac{2+\sqrt{2}}{2} \left(\frac{17}{2} \right)^{17/4} e^{-15/4} (\sigma y_0)^{-8} \sum_{j=1}^{\nu} \pi_j^6.
$$

Equation (19) can then be applied to obtain

$$\mathsf{E}_t\left(\int_{\{|\xi|\geq \mathrm{A}\}} e^{-\xi^2}\,d\xi\right)^{1/2} \leq \left(\frac{15}{2}\right)^{15/4} e^{-15/4}(\sigma y_0)^{-8} e^{-(3/8)t} \tag{58}$$

and

$$\mathsf{E}_t\left(\int_{\{|\xi|\geq \mathrm{A}\}} \xi^2 e^{-\xi^2}\,d\xi\right)^{1/2} \leq \frac{2+\sqrt{2}}{2}\left(\frac{17}{2}\right)^{17/4} e^{-15/4}(\sigma y_0)^{-8} e^{-(3/8)t}. \tag{59}$$

At this point, to control the remaining integrals over $\{|\xi|\geq \mathrm{A}\}$, we proceed as in formula (30) of DGR to write

$$\begin{aligned}&\left[\left(\int_{\{|\xi|\geq \mathrm{A}\}} |\varphi^*(\xi/\sigma)|^2\,d\xi\right)^{1/2} + \left(\int_{\{|\xi|\geq \mathrm{A}\}} \left|\frac{d}{d\xi}\varphi^*(\xi/\sigma)\right|^2 d\xi\right)^{1/2}\right]\cdot \mathbb{1}_{U^c} \\ &\qquad \leq 2\sqrt{2}\left(\int_{\mathrm{A}}^{+\infty} |\varphi^*(\xi/\sigma)|\,d\xi\right)^{1/2}\cdot \mathbb{1}_{U^c} + \sqrt{2|\varphi^*(\mathrm{A}/\sigma)|}\cdot \mathbb{1}_{U^c}.\end{aligned} \tag{60}$$

For $\overline{\omega}$ in U^c, the bound

$$\mathrm{A}(\overline{\omega}) \leq \frac{\sigma^3}{2\overline{\mathfrak{m}}_3 \sum_{j=1}^{\nu(\overline{\omega})} |\pi_j(\overline{\omega})|^3}$$

holds true, thanks to the definition of $\overline{\delta}$ and the Lyapunov inequality. Thus, Lemma 12 in Chapter 6 of Petrov (1975) can be applied to the characteristic function $\varphi^*(\xi/\sigma)$ with $b = 1/2$ to deduce

$$\begin{aligned}\sqrt{2|\varphi^*(\mathrm{A}/\sigma)|} &\leq \sqrt{2}e^{-A^2/12} \leq \sqrt{2}(48/e)^4 \mathrm{A}^{-8} \\ &= \sqrt{2}(48/e)^4(\sigma y_0)^{-8}\sum_{j=1}^{\nu} \pi_j^6,\end{aligned}$$

which entails that

$$\mathsf{E}_t\sqrt{2|\varphi^*(\mathrm{A}/\sigma)|} \leq \sqrt{2}(48/e)^4(\sigma y_0)^{-8} e^{-(3/8)t}. \tag{61}$$

It remains to analyze

$$\left(\int_{\mathrm{A}}^{+\infty} |\varphi^*(\xi/\sigma)|\,d\xi\right)^{1/2}\cdot \mathbb{1}_{U^c} = \left(\int_{\mathrm{A}}^{+\infty} \prod_{j=1}^{\nu}\left|\varphi_0\left(\frac{\pi_j \xi}{\sigma}\right)\right| d\xi\right)^{1/2}\cdot \mathbb{1}_{U^c}.$$

An estimate of this term is made using Proposition 2.2 in DGR, together with (33), (34) and (35) therein, with $\overline{\varepsilon} = 1/(2\overline{n}!)$. We then have

$$\begin{aligned}\left(\int_{\mathrm{A}}^{+\infty} |\varphi^*(\xi/\sigma)|\,d\xi\right)^{1/2}\cdot \mathbb{1}_{U^c} &\leq \left[\lambda\sigma\int_{\mathrm{A}/\lambda\sigma}^{+\infty}\left(\frac{1}{\overline{\varepsilon}\eta^{2\overline{n}}}\right)^{\alpha} d\eta\right]^{1/2} \\ &= D\left(\sum_{j=1}^{\nu}\pi_j^4\right)^{(2\overline{n}\alpha-1)/8}.\end{aligned} \tag{62}$$

The definition of $\overline{n}$ in (46) yields $(2\alpha\overline{n}-1)/8 = 2$. Moreover,

$$D := \frac{1}{4\overline{\varepsilon}^{\alpha/2}} \frac{(\lambda\sigma)^{17/2}}{(\sigma y_0)^8} \tag{63}$$
$$\le 2^{13/4}\left[\left(\frac{3}{2\sigma^2}\right)^{17/4} + \left(\frac{2}{1-M}\right)^{17/4}(L_p)^{17/2p}\right]$$

with

$$L_p := \sup_{\xi\in\mathbb{R}}[|\xi|^p \cdot |\varphi_0(\xi)|]$$

and

$$M = \exp\left\{-\frac{3\pi^2}{64(3+(L_p)^{4/p})^2}\left(\frac{\sqrt{2}\sigma}{8\lceil 2/p\rceil\sigma^3 + 40\pi\sqrt{\lceil 2/p\rceil \mathfrak{m}_4}}\right)^2\right\}.$$

Taking expectation in (62) gives

$$\mathsf{E}_t\left[\left(\int_{\mathrm{A}}^{+\infty}|\varphi^*(\xi/\sigma)|\,d\xi\right)^{1/2}\cdot \mathbb{1}_{U^C}\right] \le De^{-(3/8)t}. \tag{64}$$

The claimed upper bound (9) follows from (49), (53), (56), (58), (59), (61) and (64).

4.4. *Proof of Theorems 2.2 and 2.3 when* $2\chi+\delta > 4$. This proof differs from the previous one only in the choice of the constants. One can start from (44) under hypothesis (H). Thanks to (H) and the hypotheses of the theorems to be proven, one can apply Proposition 2.2 of DGR to get (45) with $\alpha = (2\cdot\lceil 2/p\rceil)^{-1}$.

Now, define U exactly as in (46) with $\overline{n} = [k(k+2)+1]\cdot\lceil 2/p\rceil$ and

$$\overline{\delta} = \min\left\{\frac{1}{2^{\overline{n}}\overline{n}!}; \frac{\sigma^8}{16y_0^4\overline{\mathfrak{m}}_3^4}\right\} \le \frac{1}{2^{\overline{n}}\overline{n}!}.$$

The probability of U is then estimated, under each P_t, using the facts that

$$\mathsf{P}_t\{\nu \le \overline{n}\} \le \overline{n}e^{-t} \quad \text{and} \quad \mathsf{P}_t\left\{\prod_{j=1}^{\nu}\pi_j = 0\right\} = 0,$$

whereas, for the third component of the union in the definition of U, one can combine Markov's (with exponent $k/2$) and Lyapounov's inequalities to get

$$\mathsf{P}_t\left\{\sum_{j=1}^{\nu}\pi_j^4 \ge \overline{\delta}\right\} \le \frac{1}{\overline{\delta}^{k/2}}\mathsf{E}_t\left[\left(\sum_{j=1}^{\nu}\pi_j^4\right)^{k/2}\right]$$
$$\le \frac{1}{\overline{\delta}^{k/2}}\mathsf{E}_t\left[\sum_{j=1}^{\nu}\pi_j^{k+2}\right] \le \frac{1}{\overline{\delta}^{k/2}}e^{-(1-2\alpha_{k+2})t}.$$

Thus,

$$\mathsf{P}_t(U) \leq [\overline{n} + 1/\overline{\delta}^{k/2}] e^{-(1-2\alpha_{k+2})t} \qquad (t \geq 0). \tag{65}$$

Now, split the term $\mathsf{E}_t[\|\frac{d}{dv}\mathsf{F}^*(\sigma v) - g_1(v)\|_1]$ into the sum of two contributions, exactly as in (47), and note that (65) entails that

$$\mathsf{E}_t\left[\left\|\frac{d}{dv}\mathsf{F}^*(\sigma v) - g_1(v)\right\|_1; U\right] \leq 2\mathsf{P}_t(U) \leq 2[\overline{n} + 1/\overline{\delta}^{k/2}] e^{-(1-2\alpha_{k+2})t}. \tag{66}$$

To control the integral over U^c, we once again invoke Beurling's inequality (see Proposition 4.1 in DGR) to write (50). Applicability of this result rests on the same arguments as those provided in Section 4.3. The right-hand side of (50) is split into a sum of four terms, exactly as in (51), with

$$\mathrm{A} = \mathrm{A}(\beta) := \frac{\sigma y_0}{(\sum_{j=1}^{\nu} \pi_j^4)^{1/(k+\delta)}}.$$

Now, apply (31) to the evaluation of the first integral in (51), noting that the function $\eta_{k,n}(\xi)$ equals $e^{-\xi^2/2}$ almost surely since $\kappa_{2r} = 0$ for $r = 2, \ldots, \chi$. This leads to

$$\mathsf{E}_t\left[\left(\int_{\{|\xi| \leq \mathrm{A}\}} |\Delta|^2 \, d\xi\right)^{1/2}\right] \leq C^*_{k,\delta} a_k \cdot e^{-(1-2\alpha_{2\chi+\delta})t} \tag{67}$$

and

$$\mathsf{E}_t\left[\left(\int_{\{|\xi| \leq \mathrm{A}\}} |\Delta'|^2 \, d\xi\right)^{1/2}\right] \leq C^*_{k,\delta} a_k \cdot e^{-(1-2\alpha_{2\chi+\delta})t}. \tag{68}$$

After determining upper bounds for integrals of the type $\int_{\{|\xi| \leq \mathrm{A}\}}$, it remains to examine the remaining summands in (51). Minkowski's inequality gives

$$\left(\int_{\{|\xi| \geq \mathrm{A}\}} |\Delta|^2 \, d\xi\right)^{1/2} \leq \left(\int_{\{|\xi| \geq \mathrm{A}\}} |\varphi^*(\xi/\sigma)|^2 \, d\xi\right)^{1/2} + \left(\int_{\{|\xi| \geq \mathrm{A}\}} |e^{-\xi^2/2}|^2 \, d\xi\right)^{1/2}$$

and

$$\left(\int_{\{|\xi| \geq \mathrm{A}\}} |\Delta'|^2 \, d\xi\right)^{1/2} \leq \left(\int_{\{|\xi| \geq \mathrm{A}\}} \left|\frac{d}{d\xi}\varphi^*(\xi/\sigma)\right|^2 d\xi\right)^{1/2} + \left(\int_{\{|\xi| \geq \mathrm{A}\}} |\xi e^{-\xi^2/2}|^2 \, d\xi\right)^{1/2}.$$

Integrals involving the Gaussian density are controlled as in the previous subsection, giving

$$\begin{aligned} &\mathsf{E}_t\left(\int_{\{|\xi| \geq \mathrm{A}\}} e^{-\xi^2} \, d\xi\right)^{1/2} \\ &\quad \leq \left(\frac{k(k+2)-1}{2e}\right)^{(k(k+2)-1)/4} (\sigma y_0)^{-k(k+2)/2} e^{-(1-2\alpha_{k+2})t} \end{aligned} \tag{69}$$

and

$$\mathsf{E}_t\bigg(\int_{\{|\xi|\geq \mathrm{A}\}} \xi^2 e^{-\xi^2}\,d\xi\bigg)^{1/2} \leq \frac{2+\sqrt{2}}{2}\bigg(\frac{k(k+2)+1}{2e}\bigg)^{(k(k+2)+1)/4}(\sigma y_0)^{-k(k+2)/2}e^{-(1-2\alpha_{k+2})t}. \tag{70}$$

To control the remaining integrals over the region $\{|\xi|\geq \mathrm{A}\}$, we proceed as before, writing (60). For $\overline{\omega}$ in U^c, the bound

$$\mathrm{A}(\overline{\omega}) \leq \frac{\sigma^3}{2\overline{\mathfrak{m}}_3\sum_{j=1}^{\nu(\overline{\omega})}|\pi_j(\overline{\omega})|^3}$$

holds true, thanks to the definition of $\overline{\delta}$ and the Lyapunov inequality. We then set $b = 1/2$ in Lemma 12 from Chapter 6 of Petrov (1975) to deduce that

$$\begin{aligned}
&\sqrt{2|\varphi^*(\mathrm{A}/\sigma)|}\\
&\quad\leq \sqrt{2}e^{-A^2/12}\\
&\quad\leq \sqrt{2}\bigg(\frac{3k(k+2)}{e}\bigg)^{(k(k+2))/4}(\sigma y_0)^{-(k(k+2))/2}\cdot\bigg(\sum_{j=1}^{\nu}\pi_j^4\bigg)^{(k(k+2))/(2(k+\delta))}\\
&\quad\leq \sqrt{2}\bigg(\frac{3k(k+2)}{e}\bigg)^{(k(k+2))/4}(\sigma y_0)^{-(k(k+2))/2}\cdot\bigg(\sum_{j=1}^{\nu}\pi_j^{k+2}\bigg)
\end{aligned}$$

and, therefore,

$$\mathsf{E}_t\sqrt{2|\varphi^*(\mathrm{A}/\sigma)|} \leq \sqrt{2}\bigg(\frac{3k(k+2)}{e}\bigg)^{(k(k+2))/4}(\sigma y_0)^{-(k(k+2))/2} \times e^{-(1-2\alpha_{k+2})t}. \tag{71}$$

Finally, in regard to $(\int_{\mathrm{A}}^{+\infty}|\varphi^*(\xi/\sigma)|\,d\xi)^{1/2}\cdot \mathbb{1}_{U^c}$, one can write

$$\bigg(\int_{\mathrm{A}}^{+\infty}|\varphi^*(\xi/\sigma)|\,d\xi\bigg)^{1/2}\cdot \mathbb{1}_{U^c} = D_k\bigg(\sum_{j=1}^{\nu}\pi_j^4\bigg)^{(2\overline{n}\alpha-1)/8}, \tag{72}$$

where the constant D_k is given by

$$\sqrt{\frac{\lambda\sigma(2\overline{n}!)^{\alpha}}{2\alpha\overline{n}-1}}\bigg(\frac{\lambda}{y_0}\bigg)^{(2\alpha\overline{n}-1)/2}.$$

The definition of $\overline{n}$ given at the beginning of this subsection yields $(2\alpha\overline{n} - 1)/8 > k/2$. Now, taking expectation in (72) entails that

$$\mathsf{E}_t\left[\left(\int_A^{+\infty} |\varphi^*(\xi/\sigma)|\, d\xi\right)^{1/2} \cdot \mathbb{1}_{U^C}\right] \le D_k e^{-(1-2\alpha_{k+2})t}. \tag{73}$$

To obtain (11), it will suffice to combine the previous inequalities.

APPENDIX

This appendix contains all of the elements which are necessary to complete the proofs given in Section 4. It is split into four parts. The first focuses on a quantification of the numbers y_0 such that the Fourier–Stieltjes transform of a symmetric probability law turns out to be greater than 1/2 on $[-y_0, y_0]$. The second presents the proofs of Lemmas 3.1 and 3.2. The third aims to clarify the conclusion of the proof of Proposition 2.4. Finally, the fourth provides a proof for (54) and (57).

A.1. Specification of y_0. *Let ψ be the Fourier–Stieltjes transform of a symmetric probability law ζ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, namely $\psi(\xi) := \int_{\mathbb{R}} e^{i\xi x}\zeta(dx)$ for every real ξ. Assume that $\mathfrak{m}_4 := \int_{\mathbb{R}} x^4\zeta(dx)$ is finite and put $\sigma^2 := \int_{\mathbb{R}} x^2\zeta(dx)$, $y_0 := \{[-6\sigma^2 + (36\sigma^4 + 12\mathfrak{m}_4)^{1/2}]/\mathfrak{m}_4\}^{1/2}$. If $|\xi| \le y_0$, then $\psi(\xi) \ge 1/2$.*

PROOF. By the Taylor expansion for characteristic functions, one can write $\psi(\xi) = 1 - (\sigma^2/2)\xi^2 + R(\xi)$ with $|R(\xi)| \le (\mathfrak{m}_4/24)\xi^4$; see, for example, Section 8.4 in Chow and Teicher (1997). The desired bound is obtained if

$$1 - \frac{\sigma^2}{2}\xi^2 - \frac{\mathfrak{m}_4}{24}\xi^4 \ge \frac{1}{2}$$

holds true for every ξ belonging to some interval. Now, one can note that the biquadratic equation $\mathfrak{m}_4\xi^4 + 12\sigma^2\xi^2 - 12 = 0$ possesses exactly two real solutions, namely $\pm y_0$, and the previous inequality is satisfied for every ξ in $[-y_0, y_0]$. □

A.2. Proofs of Lemmas 3.1 and 3.2.

PROOF OF LEMMA 3.1. Set $\psi_{j,n}$ for the characteristic function of $Y_{j,n}$ $(j = 1, 2, \dots, n)$ and use the definition of V_n, combined with independence, to write

$$\psi_n(\xi) = \prod_{j=1}^n \psi_{j,n}(\xi) = \prod_{j=1}^n \psi\left(\frac{c_{j,n}\xi}{\sigma}\right).$$

If $|\xi| \leq A_{4,n}$, then it easily follows that

$$\left|\frac{c_{j,n}\xi}{\sigma}\right| \leq \left|\frac{c_{j,n}\sigma y_0}{\sigma}\left(\sum_{r=1}^{n} c_{r,n}^4\right)^{-1/4}\right| \leq y_0.$$

Now, using elementary properties of the logarithm, one can combine expansion (22) with property (20) of each array $\{c_{1,n}, \ldots, c_{n,n}\}$ to obtain

$$\begin{aligned}\log \psi_n(\xi) &= \sum_{j=1}^{n} \log \psi_{j,n}(\xi) \\ &= \sum_{j=1}^{n}\left[-\frac{1}{2}\sigma^2 \frac{c_{j,n}^2 \xi^2}{\sigma^2} + \frac{1}{4!}\kappa_4 \frac{c_{j,n}^4 \xi^4}{\sigma^4} + \frac{c_{j,n}^4 \xi^4}{\sigma^4}\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right] \\ &= -\frac{1}{2}\xi^2 + \frac{\tilde{\lambda}_{2,n}}{4!}\xi^4 + R_4(\xi),\end{aligned}$$

where

$$R_4(\xi) := \sum_{j=1}^{n} \frac{c_{j,n}^4 \xi^4}{\sigma^4}\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right).$$

Inverting the logarithm, one gets

$$\psi_n(\xi) = e^{-\xi^2/2} \cdot \exp\left\{\frac{\tilde{\lambda}_{2,n}}{4!}\xi^4\right\} \cdot \exp\{R_4(\xi)\}. \tag{74}$$

It is easily verified that the restrictions $|u| := |\tilde{\lambda}_{2,n}\xi^4|/4! \leq \kappa_4 y_0/4!$ and $|R_4(\xi)| \leq M_0^{(4)} y_0^4$ hold true when $|\xi| \leq A_{4,n}$, and that $\tilde{\lambda}_{2,n}\xi^4/4! = \tilde{P}_{1,n}(\xi)$. Finally, set $F(x) := e^x - 1 - x$. At this point, we have all the tools needed to prove (26) and (27). Indeed,

$$\begin{aligned}|\psi_n(\xi) - \eta_{4,n}(\xi)| &= e^{-\xi^2/2}|e^u \exp\{R_4(\xi)\} - 1 - u| \\ &= e^{-\xi^2/2}|e^u \exp\{R_4(\xi)\} - e^u + F(u)| \\ &\leq e^{-\xi^2/2} e^u |\exp\{R_4(\xi)\} - 1| + e^{-\xi^2/2}|F(u)|.\end{aligned}$$

By elementary arguments, if x is any real number satisfying $|x| \leq c$, one has

$$|e^x - 1| \leq e^{|x|} - 1 \leq \left(\frac{e^c - 1}{c}\right)|x|.$$

This fact can be applied to $R_4(\xi)$ to get

$$|\exp\{R_4(\xi)\} - 1| \leq \xi^4 \cdot \left(\frac{e^{M_0^{(4)} y_0^4} - 1}{\sigma^4 y_0^4}\right) \cdot \left(\sum_{j=1}^{n} c_{j,n}^4 \left|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right|\right).$$

Moreover, since the inequality

$$|F(u)| \leq \max_{|x| \leq \kappa_4 y_0/4!}\left[\left|\frac{F(x)}{x^2}\right|\right]\xi^8\left(\sum_{j=1}^n c_{j,n}^4\right)^2$$

holds, one can conclude that

$$\begin{aligned} &|\psi_n(\xi) - \eta_{4,n}(\xi)| \\ &\leq e^{-\xi^2/2}\xi^4 \cdot \exp\left\{\frac{\kappa_4 y_0^4}{4!}\right\}\left(\frac{e^{M_0^{(4)}y_0^4} - 1}{\sigma^4 y_0^4}\right) \cdot \left(\sum_{j=1}^n c_{j,n}^4\left|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right|\right) \qquad (75) \\ &\quad + e^{-\xi^2/2}\max_{|x| \leq \kappa_4 y_0^4/4!}\left[\left|\frac{F(x)}{x^2}\right|\right]\xi^8\left(\sum_{j=1}^n c_{j,n}^4\right)^2. \end{aligned}$$

After setting

$$C_4^{**} := \exp\left\{\frac{\kappa_4 y_0^4}{4!}\right\}\left(\frac{e^{M_0^{(4)}y_0^4} - 1}{\sigma^4 y_0^4}\right) + \max_{|x| \leq \kappa_4 y_0^4/4!}\left[\left|\frac{F(x)}{x^2}\right|\right],$$

the derivation of (26) and (27) follows by rewriting (75) in a more convenient form. To get (26), it is enough to observe that $\sum_{j=1}^n c_{j,n}^4 \leq 1$, while to deduce (27), one can combine the inequality $(\sum_{j=1}^n c_{j,n}^4)^2 \leq \sum_{j=1}^n c_{j,n}^6$ with $\max\{1; \xi^4\} \leq (1 + \xi^4)$.

To prove (28), we start from (74) and take the derivative with respect to ξ. Thus, one obtains

$$\begin{aligned} &|\psi_n'(\xi) - \eta_{4,n}'(\xi)| \\ &\quad \leq \exp\{R_4(\xi)\} \cdot |R_4'(\xi)| \cdot |\eta_{4,n}(\xi) + F(u)e^{-\xi^2/2}| \\ &\qquad + |\eta_{4,n}'(\xi)| \cdot |\exp\{R_4(\xi)\} - 1| \\ &\qquad + \exp\{R_4(\xi)\} \cdot \left|\frac{d}{d\xi}F(u)\right| \cdot e^{-\xi^2/2} + \exp\{R_4(\xi)\} \cdot |F(u)| \cdot |\xi| e^{-\xi^2/2}. \end{aligned}$$

Arguing as in the first part of this proof, we have

$$\begin{aligned} &|\eta_{4,n}'(\xi)| \cdot |\exp\{R_4(\xi)\} - 1| \\ &\quad \leq \left(\frac{e^{M_0^{(4)}y_0^4} - 1}{\sigma^4 y_0^4}\right) \cdot \left(1 + \frac{\kappa_4}{4!\sigma^4}\right)|\xi|^5(1 + \xi^4)e^{-\xi^2/2} \qquad (76) \\ &\qquad \times \left(\sum_{j=1}^n c_{j,n}^4\left|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right|\right) \end{aligned}$$

and

$$\begin{aligned}(77)\qquad &\exp\{R_4(\xi)\}\cdot|F(u)|\cdot|\xi|e^{-\xi^2/2}\\ &\quad\le \max_{|x|\le\kappa_4 y_0/4!}\left[\left|\frac{F(x)}{x^2}\right|\right]\exp\{M_0^{(4)}y_0^4\}|\xi|^9 e^{-\xi^2/2}\left(\sum_{j=1}^n c_{j,n}^4\right)^2.\end{aligned}$$

Moreover,

$$\begin{aligned}(78)\qquad &\exp\{R_4(\xi)\}\cdot|R_4'(\xi)|\cdot|\eta_{4,n}(\xi)+F(u)e^{-\xi^2/2}|\\ &\quad=\exp\{R_4(\xi)\}\cdot|R_4'(\xi)|\cdot e^{-\xi^2/2}e^u\\ &\quad\le \exp\{M_0^{(4)}y_0^4\}\cdot\exp\left\{\frac{\kappa_4 y_0^4}{4!}\right\}4\sigma^{-4}|\xi|^3e^{-\xi^2/2}\cdot\\ &\qquad\times\left[\sum_{j=1}^n c_{j,n}^4\left|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right|+\sum_{j=1}^n c_{j,n}^4\left|\varrho_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right|\right]\end{aligned}$$

and

$$\frac{d}{d\xi}F(u)=\frac{\tilde\lambda_{2,n}}{3!}\xi^3(e^u-1),$$

whence

$$\begin{aligned}(79)\qquad &\exp\{R_4(\xi)\}\cdot\left|\frac{d}{d\xi}F(u)\right|\cdot e^{-\xi^2/2}\\ &\quad\le\exp\{M_0^{(4)}y_0^4\}\frac{\kappa_4^2}{3!4!\sigma^8}\left(\frac{\exp\{\kappa_4y_0^4/4!\}-1}{\kappa_4y_0^4/4!}\right)|\xi|^7e^{-\xi^2/2}\left(\sum_{j=1}^n c_{j,n}^4\right)^2.\end{aligned}$$

Now, set

$$\begin{aligned}C_4^{***}:=&\left(\frac{e^{M_0^{(4)}y_0^4}-1}{\sigma^4y_0^4}\right)\cdot\left(1+\frac{\kappa_4}{4!\sigma^4}\right)+\max_{|x|\le\kappa_4y_0/4!}\left[\left|\frac{F(x)}{x^2}\right|\right]\exp\{M_0^{(4)}y_0^4\}\\ &+\exp\{M_0^{(4)}y_0^4\}\cdot\exp\left\{\frac{\kappa_4y_0^4}{4!}\right\}4\sigma^{-4}\\ &+\exp\{M_0^{(4)}y_0^4\}\frac{\kappa_4^2}{3!4!\sigma^8}\left(\frac{\exp\{\kappa_4y_0^4/4!\}-1}{\kappa_4y_0^4/4!}\right)\end{aligned}$$

and combine (76), (77), (78) and (79), after noting that $|\xi|^5(1+\xi^4)+|\xi|^9+|\xi|^3+|\xi|^7\le 4|\xi|^3(1+\xi^6)$ holds for every ξ. Finally, in order to have the same multiplicative constant in the right-hand sides of (26), (27) and (28), replace C_4^{**} and $4C_4^{***}$ with $C_4^*:=\max\{C_4^{**};4C_4^{***}\}$. □

PROOF OF LEMMA 3.2. In view of the independence of the random variables $X_{j,n}$ and (22), one gets

$$\log\psi_n(\xi) = -\frac{1}{2}\xi^2 + \sum_{r=2}^{\chi}(-1)^r\frac{\tilde{\lambda}_{r,n}}{(2r)!}\xi^{2r} + R_{k+\delta}(\xi),$$

where

$$R_{k+\delta}(\xi) := \sum_{j=1}^{n}\frac{c_{j,n}^k\xi^k}{\sigma^k}\epsilon_{k+\delta}\left(\frac{c_{j,n}\xi}{\sigma}\right),$$

whence

$$\psi_n(\xi) = e^{-\xi^2/2}\cdot\exp\left\{\sum_{r=2}^{\chi}(-1)^r\frac{\tilde{\lambda}_{r,n}}{(2r)!}\xi^{2r}\right\}\cdot\exp\{R_{k+\delta}(\xi)\}. \tag{80}$$

Now, consider the function $z\mapsto f_\xi(z) = \exp\{g_\xi(z)\}$ with

$$g_\xi(z) := \sum_{r=1}^{\chi-1}(-1)^{r+1}\frac{\tilde{\lambda}_{r+1,n}}{(2r+2)!}\xi^{2(r+1)}z^r$$

and its Taylor polynomial of order $(\chi-1)$ at $z=0$, say $p_{\chi-1}(z)$. Then, recall the Faà di Bruno formula, that is,

$$\frac{d^{(\chi)}}{dt^{(\chi)}}\exp\{(y(t))\}$$
$$= \sum_{(*)}\frac{\chi!}{k_1!k_2!\cdots k_\chi!}\exp\{(y(t))\}\left(\frac{y^{(1)}(t)}{1!}\right)^{k_1}\left(\frac{y^{(2)}(t)}{2!}\right)^{k_2}\cdots\left(\frac{y^{(\chi)}(t)}{\chi!}\right)^{k_\chi}$$

with $(*)$ meaning that the sum is carried out over all nonnegative integer solutions $(k_1,\dots,k_\chi)$ of the equation $k_1+2k_2+\cdots+\chi k_\chi=\chi$. An application of this formula entails that

$$p_{\chi-1}(z) = 1 + \sum_{r=1}^{\chi-1}\tilde{P}_{r,n}(\xi)z^r,$$

the functions $\tilde{P}_{r,n}(\xi)$ having been defined in (24). Thus, when $z=1$, the Lagrange remainder can be written with a suitable $u\in[0,1]$ as

$$\frac{1}{\chi!}f_\xi^{(\chi)}(u) = f_\xi(u)\sum_{(*)}\frac{1}{k_1!k_2!\cdots k_\chi!}\left(\frac{g_\xi^{(1)}(u)}{1!}\right)^{k_1}\left(\frac{g_\xi^{(2)}(u)}{2!}\right)^{k_2}\cdots\left(\frac{g_\xi^{(\chi)}(u)}{\chi!}\right)^{k_\chi},$$

which, after repeated application of the multinomial formula, leads to

$$\left|\frac{1}{\chi!}f_\xi^{(\chi)}(u)\right| \le f_\xi(u)\sum_{(*)}\prod_{m=1}^{\chi-1}\sum_{\{l_1+\cdots+l_{\chi-m}=k_m\}}|A_{1,m}^{l_1}(\xi)\cdots A_{\chi-m,m}^{l_{\chi-m}}(\xi)|$$

with $A_{h,m}(\xi) := (-1)^{h+m}\binom{h+m-1}{m}\frac{\tilde{\lambda}_{h+m,n}}{(2(h+m))!}\xi^{2(h+m)}$. We can then introduce the quantity

$$W_\chi := \left[\prod_{s=2}^{\chi}\max\left\{\frac{\kappa_{2s}}{\sigma^{2s}};1\right\}\right]^\chi$$

to obtain, after an application of the Lyapunov inequality,

$$\sum_{\{l_1+\cdots+l_{\chi-m}=k_m\}} |A_{1,m}^{l_1}(\xi)\cdots A_{\chi-m,m}^{l_{\chi-m}}(\xi)| \le \chi^\chi W_\chi \xi^{2mk_m}(\xi^2+\xi^{k-2})^{k_m}\cdot\left(\sum_{j=1}^n c_{j,n}^{k+2}\right)^{2mk_m/k},$$

whence

$$\left|\frac{1}{\chi!}f_\xi^{(\chi)}(u)\right| \le f_\xi(u)\cdot\chi^{\chi^2}W_\chi^{\chi-1}\xi^k[(\xi^2+\xi^{k-2})^2+(\xi^2+\xi^{k-2})^\chi]\cdot\left(\sum_{j=1}^n c_{j,n}^{k+2}\right)$$

and, using the bound $|\xi|\le A_{k,\delta,n}$,

$$|g_\xi(u)| \le \sum_{s=2}^{\chi}\kappa_{2s}y_0^{2s} := B_\chi.$$

Then,

$$\begin{aligned}
&|\psi_n(\xi)-\eta_{k,n}(\xi)| \\
&\quad\le e^{-\xi^2/2}\{[f_\xi(1)-p_{\chi-1}(1)]+[e^{R_{k+\delta}(\xi)}-1]\} \\
&\quad\le e^{-\xi^2/2}\Bigg[e^{B_\chi}\chi^{\chi^2}W_\chi^{\chi-1}\xi^k[(\xi^2+\xi^{k-2})^2+(\xi^2+\xi^{k-2})^\chi]\cdot\left(\sum_{j=1}^n c_{j,n}^{k+2}\right) \\
&\qquad+\left(\frac{\exp\{M_0^{(k+\delta)}y_0^k\}-1}{M_0^{(k+\delta)}y_0^k}\right)\frac{2\overline{\mathfrak{m}}_{k+\delta}}{k!\sigma^{k+\delta}}\left(\sum_{j=1}^n|c_{j,n}|^{k+\delta}\right)|\xi|^{k+\delta}\Bigg].
\end{aligned} \tag{81}$$

After observing that $\xi^k[(\xi^2+\xi^{k-2})^2+(\xi^2+\xi^{k-2})^\chi] \le |\xi|^{k+\delta}(1+\xi^2)[2\xi^2+2\xi^{2k-6}+2^\chi\xi^{k-2}+2^\chi\xi^{\chi k-k-2}]$ for every ξ, one can take $p_{0,k}$ in (29) to be equal to $1+(1+\xi^2)[2\xi^2+2\xi^{2k-6}+2^\chi\xi^{k-2}+2^\chi\xi^{\chi k-k-2}]$.

As for $|\psi_n'-\eta_{k,n}'|$, note that the inequality

$$\begin{aligned}
&|\psi_n'(\xi)-\eta_{k,n}'(\xi)| \\
&\quad\le|\xi|\cdot|\psi_n(\xi)-\eta_{k,n}(\xi)|+e^{-\xi^2/2}\left|\frac{d}{d\xi}f_\xi(1)\right|\cdot|\exp\{R_{k+\delta}(\xi)\}-1|
\end{aligned} \tag{82}$$

$$+ e^{-\xi^2/2}|f_\xi(1)|\exp\{R_{k+\delta}(\xi)\}|R'_{k+\delta}(\xi)|$$
$$+ e^{-\xi^2/2}\left|\frac{d}{d\xi}(f_\xi(1) - p_{\chi-1}(1))\right|$$

obtains. As regards the first summand, it will suffice to multiply the upper bound stated in (81) for $|\psi_n - \eta_{k,n}|$ by $|\xi|$. The latter factor in the second addend of (82) can be dominated by the last addend in (81), while, for the former factor, one has

$$\left|\frac{d}{d\xi}f_\xi(1)\right| \leq \exp\{B_\chi\}\sum_{r=1}^{\chi-1}\frac{\kappa_{2r+2}y_0^{2r+1}}{(2r+1)!\sigma}.$$

As for the third addend, recall that $|f_\xi(1)| \leq \exp\{B_\chi\}$ and $|R_{k+\delta}(\xi)| \leq y_0^k M_0^{(k+\delta)}$. Moreover, $|R'_{k+\delta}(\xi)| \leq \sum_{j=1}^n \sigma^{-k}\xi^{k-1}|c_{j,n}|^k\{k|\epsilon_{k+\delta}(c_{j,n}\xi/\sigma)| + |\xi\sigma^{-1}c_{j,n} \times \epsilon'_{k+\delta}(c_{j,n}\xi/\sigma)|\}$ and, in view of Theorem 1 in Section 8.4 of Chow and Teicher (1997), $(|\epsilon_{k+\delta}(x)| + |x\epsilon'_{k+\delta}(x)|) \leq 4\overline{\mathfrak{m}}_{k+\delta}|x|^\delta/(k-1)!$. It remains to deal with the last summand in (82). Since $\frac{\partial}{\partial\xi}p_{\chi-1}$ is a Taylor polynomial for $\frac{\partial}{\partial\xi}f_\xi$, one can use the Bernstein integral form of the remainder to obtain

$$\begin{aligned}
&\left|\frac{\partial}{\partial\xi}(f_\xi(1) - p_{\chi-1}(1))\right| \\
&\quad\leq \frac{1}{(\chi-1)!}\int_0^1 (1-u)^{\chi-1}\sum_{l=0}^{\chi}\binom{\chi}{l}\left|\frac{\partial^l}{\partial u^l}f_\xi(u)\right|du \\
&\qquad\times \sum_{j=1}^n c_{j,n}^{2(\chi-l+1)}\sum_{r=\chi-l}^{\chi-1}|\xi|^{2r+1}\frac{\kappa_{2(r+1)}}{\sigma^{2(r+1)}} \\
&\quad\leq \sum_{l=0}^{\chi}\frac{1}{(\chi-l)!}e^{B_\chi}\sum_{(*)_l}\prod_{m=1}^{l}\frac{1}{k_m!}\left(\frac{1}{m!}\sum_{r=m}^{\chi-1}\frac{\kappa_{2(r+1)}}{\sigma^{2(r+1)}}|\xi|^{2r+2}\right)^{k_m} \\
&\qquad\qquad\times \sum_{r=\chi-l}^{\chi-1}|\xi|^{2r+1}\frac{\kappa_{2(r+1)}}{\sigma^{2(r+1)}}\left(\sum_{j=1}^n c_{j,n}^{2l+2}\right)\cdot\left(\sum_{j=1}^n c_{j,n}^{2(\chi-l+1)}\right).
\end{aligned}$$

To conclude, think of the last two sums of the $c_{j,n}$'s as moments of order $2l$ and $2(\chi-l)$, respectively, and apply the Lyapunov inequality to each sum to write

$$\left(\sum_{j=1}^n c_{j,n}^{2l+2}\right)\cdot\left(\sum_{j=1}^n c_{j,n}^{2(\chi-l+1)}\right) \leq \sum_{j=1}^n c_{j,n}^{k+2}.$$

□

A.3. A complement to the proof of Theorem 2.1. We clarify why $\inf_{t\in[0,\hat{t}]} d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma)$ must be strictly positive under the hypothesis that $\kappa_4(\tilde{\mu}_0)$ is different from zero. Suppose, on the contrary, that $\inf_{t\in[0,\hat{t}]} d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma)=0$. Then, as $t\mapsto d_{\mathrm{TV}}(\mu(\cdot,t);\gamma_\sigma)$ is continuous on $[0,+\infty[$, by the Wild expansion, there exists t^* in $[0,\hat{t}]$ such that $d_{\mathrm{TV}}(\mu(\cdot,t^*);\gamma_\sigma)=0$. On the one hand, if $t^*=0$, then μ_0 coincides with γ_σ and this contradicts the hypothesis that $\kappa_4(\tilde{\mu}_0)$ is different from zero. On the other hand, if $t^*>0$, then one can conclude, in view of the Wild expansion, that μ_0 possesses moments of every order and is symmetric. A direct consequence of (1) is that $\mathfrak{m}_{2k}(t):=\int_{\mathbb{R}} x^{2k}\mu(dx,t)$ satisfies an ordinary first order differential equation, which admits the constant $\int_{\mathbb{R}} x^{2k}\gamma_\sigma(dx)$ as a stationary solution. Hence, since we are assuming that $\mathfrak{m}_{2k}(t^*)$ is equal to such a constant, the uniqueness of the solutions of the equations under consideration implies that $\mathfrak{m}_{2k}(t)=\int_{\mathbb{R}} x^{2k}\gamma_\sigma(dx)$ for every t in $[0,\infty[$ and every positive integer k. In other words, μ_0 coincides with γ_σ, which once again contradicts the fact that $\kappa_4(\tilde{\mu}_0)$ is different from zero.

A.4. The proofs of (54) and (57). The proofs of (54) and (57) follow from the following proposition. Let $g:\mathbb{R}\to[0,+\infty[$ be an integrable function and $\epsilon:\mathbb{R}\to\mathbb{R}$ be a continuous, bounded function with $\epsilon(0)=0$. Then

$$\lim_{t\to+\infty} H(t):=e^{(1/4)t}\mathsf{E}_t\left\{\left(\int_{\mathbb{R}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right)^{1/2}\right\}=0.$$

PROOFS OF (54) AND (57). We fix an arbitrary small positive δ and show that there exists a value t_δ for which $|H(t)|<\delta$, for every $t>t_\delta$. First, in view of the fact that $\epsilon(\cdot)$ is continuous and $\epsilon(0)=0$, there exists a strictly posityve number $\overline{x}$ such that the inequality

$$|\epsilon(x)|\le\frac{\delta}{3\sqrt{\|g\|_1}}$$

holds for every x in $[-\overline{x},\overline{x}]$ with $\|g\|_1=\int_{\mathbb{R}} g(\xi)\,d\xi$. Set $\overline{\pi}:=\max_{1\le j\le\nu}\pi_j$ and $B:=\overline{x}/|\overline{\pi}|$. B is well defined since, due to (15), $\overline{\pi}\ne 0$. Now,

$$\begin{aligned}&\left\{\int_{\mathbb{R}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right\}^{1/2}\\ &\quad\le\left\{\int_{\{|\xi|\le B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right\}^{1/2}\\ &\qquad+\left\{\int_{\{|\xi|\ge B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right\}^{1/2}.\end{aligned}$$

For the integral over the internal region, one can write

$$\left\{\int_{\{|\xi|\le B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right\}^{1/2} \le \frac{\delta}{3\sqrt{\|g\|_1}}\left(\sum_{j=1}^{\nu}\pi_j^4\right)\sqrt{\|g\|_1}$$

and, taking expectation,

$$e^{(1/4)t}\mathsf{E}_t\left\{\left(\int_{\{|\xi|\le B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right)^{1/2}\right\} \le \delta/3,$$

after a standard application of (19). At this point, we define M to be the maximum of $|\epsilon|$ and determine a positive value $\overline{s}$ such that

$$\int_{\{|\xi|\ge\overline{s}\}} g(\xi)\,d\xi \le \left(\frac{\delta}{3M}\right)^2.$$

Given $S := \{\omega | |\overline{\pi}(\omega)| < \overline{x}/\overline{s}\}$, we write

$$\begin{aligned}
&e^{(1/4)t}\mathsf{E}_t\left\{\left(\int_{\{|\xi|\ge B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right)^{1/2}\right\}\\
&\qquad = e^{(1/4)t}\mathsf{E}_t\left\{\left[\int_{\{|\xi|\ge B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right]^{1/2}; S\right\}\\
&\qquad\quad + e^{(1/4)t}\mathsf{E}_t\left\{\left[\int_{\{|\xi|\ge B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right]^{1/2}; S^c\right\}.
\end{aligned}$$

One can observe that $B(\omega) > \overline{s}$ for ω in S. We then have

$$\begin{aligned}
&e^{(1/4)t}\mathsf{E}_t\left\{\left[\int_{\{|\xi|\ge B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right]^{1/2}; S\right\}\\
&\qquad \le e^{(1/4)t}\left\{\int_{\{|\xi|\ge\overline{s}\}} g(\xi)\,d\xi\right\}^{1/2} M\,\mathsf{E}_t\left[\sum_{j=1}^{\nu}\pi_j^4\right] \le \delta/3.
\end{aligned}$$

For the remaining term,

$$e^{(1/4)t}\mathsf{E}_t\left\{\left[\int_{\{|\xi|\ge B\}} g(\xi)\left[\sum_{j=1}^{\nu}\pi_j^4|\epsilon(\pi_j\xi)|\right]^2 d\xi\right]^{1/2}; S^c\right\} \le e^{(1/4)t}M\sqrt{\|g\|_1}\,\mathsf{P}_t(S^c).$$

An application of Markov's inequality with exponent 6 yields an upper bound for the probability of S^c, that is,

$$\mathsf{P}_t(S^c) \le \mathsf{E}_t[|\overline{\pi}|^6]\cdot\left(\frac{\overline{s}}{\overline{x}}\right)^6 \le \mathsf{E}_t\left[\sum_{j=1}^{\nu}\pi_j^6\right]\cdot\left(\frac{\overline{s}}{\overline{x}}\right)^6 \le e^{-(3/8)t}\cdot\left(\frac{\overline{s}}{\overline{x}}\right)^6.$$

Hence,

$$e^{(1/4)t}M\sqrt{\|g\|_1}\mathsf{P}_t(S^c) \le e^{-(1/8)t}M\sqrt{\|g\|_1}\cdot\left(\frac{\overline{s}}{\overline{x}}\right)^6.$$

Taking $t_\delta = \max\{-8\log[(\delta/3)\cdot(\overline{x}/\overline{s})^6\cdot M^{-1}\|g\|_1^{-1/2}];1\}$ makes the right-hand side of the last inequality smaller than $\delta/3$ for every $t > t_\delta$. This completes the proof. □

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