A NOTE ON KNOTS WITH H(2)-UNKNOTTING NUMBER ONE

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ABSTRACT. We give an obstruction to unknotting a knot by adding a twisted band, derived from Heegaard Floer homology.

1. INTRODUCTION

Many unknotting operations have been defined and studied in knot theory. For example, as well-known, (a), (b) (cf. [5, 7]) and (c) in Figure 1 are three types of unknotting operations. Especially, (c) was introduced by Hoste, Nakanishi and Taniyama [1], which they called H(n)-move. Here n is the number of arcs inside the circle. Note that an H(n)-move is required to preserve the component number of the diagram. The H(n)-unknotting number of a knot is the minimal number of H(n)-moves needed to change the knot into the unknot. In this note, we focus on the special case when n equals two. Given two knots K and K', when K' is obtained from K by applying an H(2)-move, we also alternatively say that K' is obtained from K by adding a twisted band, as shown in Figure 2. We only choose those bands for which the diagrams before and after represent knots. Following [1], we denote the H(2)-unknotting number of a knot K by $u_2(K)$. In this note, we give a necessary condition for a knot K to have $u_2(K) = 1$, by using a method introduced by Ozsváth and Szabó [12].

The question whether a given knot has H(2)-unknotting number one should be traced back to Riley. He made the conjecture that the figure-eight knot could never be unknotted by adding a twisted band. Lickorish confirmed this conjecture in [4]. Here we give a brief review of his method. Given a knot K, let $\Sigma(K)$ denote the double-branched cover of S^3 along K and let $\lambda : H_1(\Sigma(K)) \times H_1(\Sigma(K)) \to \mathbb{Q}/\mathbb{Z}$ be the linking form of $\Sigma(K)$. Lickorish proved that if the knot K can be unknotted by adding a twisted band, then $H_1(\Sigma(K))$ is cyclic and it has a generator g such that $\lambda(g, g) = \pm 1/\det(K)$, where $\det(K)$ is the determinant of K. For the figure-eight knot 4_1, the linking form has the form $\lambda(g, g) = 2/5$ for some generator $g \in H_1(\Sigma(4_1)) \cong \mathbb{Z}/5\mathbb{Z}$. If there is another generator $g' = xg$ such that $\lambda(g', g') = \pm 1/5$, we have $2x^2 \equiv \pm 1 \pmod 5$. There is no such an integer x satisfing the condition. Therefore Riley's conjecture holds.

Now we turn to the description of our result. Consider a positive-definite symmetric $n \times n$ matrix Q over $\mathbb{Z}$. Suppose $\det(Q)$ is p. Then Q as a presentation determines a group G. A characteristic vector for Q is an element in

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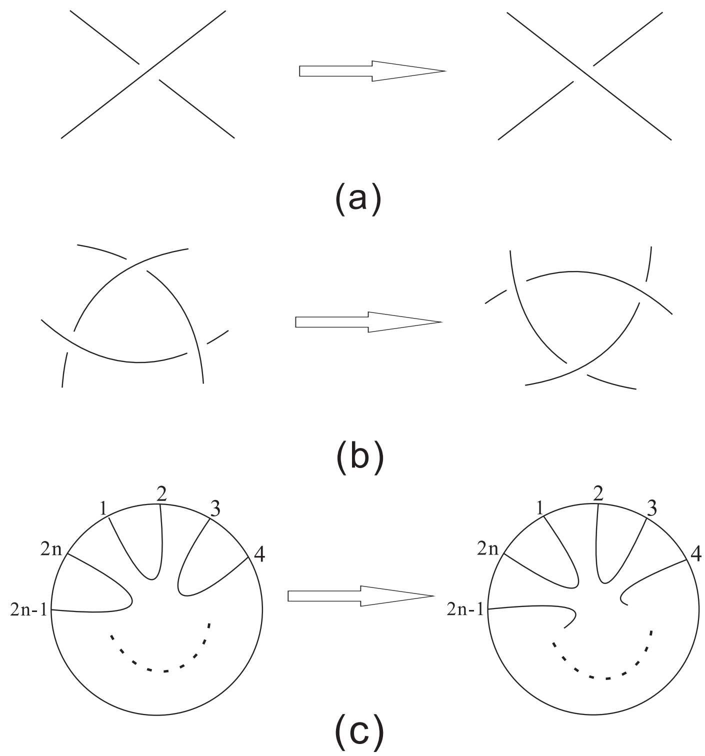

FIGURE 1. Some unknotting operations.

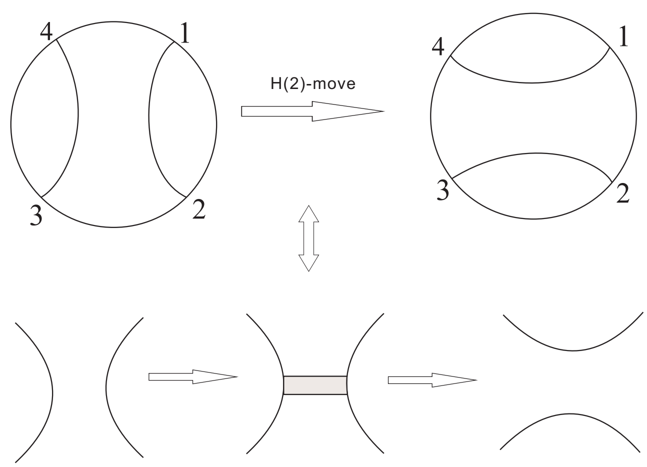

FIGURE 2. Adding a twisted band to a knot diagram.

$$\begin{aligned}\operatorname{char}(Q) &= \left\{\xi \in \mathbb{Z}^n | \xi^t v \equiv v^t Q v \pmod 2 \text{ for any } v \in \mathbb{Z}^n\right\} \\ &= \left\{\xi \in \mathbb{Z}^n | \xi_i \equiv Q_{ii} \pmod 2\right\}.\end{aligned}$$

Two characteristic vectors ξ and ζ are said to be equivalent if $Q^{-1}(\xi - \zeta) \in \mathbb{Z}^n$. Suppose p is odd, and consider the map (cf. [10, 12])

$$M_Q : G \longrightarrow \mathbb{Q}$$

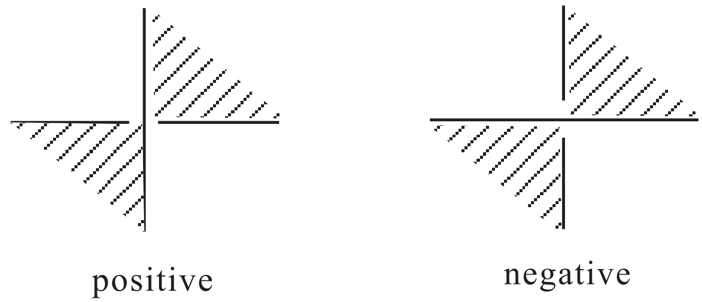

FIGURE 3. The sign convention of a crossing.

defined by

$$M_Q(\alpha) = \min\left\{ \frac{\xi^t Q^{-1}\xi - n}{4} \,\middle|\, \xi \in \mathrm{char}(Q), [\xi] = \alpha \in G \right\}.$$

The map is well-defined up to an automorphism of G.

Now we recall the definition of Goeritz matrix. Given a knot diagram, color this diagram in checkerboard fashion such that the unbounded region has black color. Let $f_0, f_1, \ldots, f_k$ denote the black regions and f_0 correspond to the unbounded one. Define the sign of a crossing as in Figure 3. Then the Goeritz matrix Q is the $k \times k$ symmetric matrix defined as follows

$$(1) \qquad q_{ij} = \begin{cases} \text{the signed count of crossings adjacent to } f_i & \text{if } i = j, \\ \text{minus the signed count of crossings joining } f_i \text{ and } f_j & \text{if } i \neq j \end{cases}$$

for $i, j = 1, 2, \ldots, k$.

Our result about H(2)-unknotting number is as follows:

Theorem 1.1. *Let K be an alternating knot with determinant p, and let Q be the positive-definite Goeritz matrix corresponding to a reduced alternating diagram of K or its mirror image. Suppose G is the group presented by Q. If $u_2(K) = 1$, then there is an isomorphism $\phi : \mathbb{Z}/|p|\mathbb{Z} \longrightarrow G$ and a sign $\epsilon \in \{+1, -1\}$ with the properties that for all $i \in \mathbb{Z}/|p|\mathbb{Z}$:*

$$I_{\phi,\epsilon}(i) := \epsilon \cdot M_Q(\phi(i)) - \frac{1}{4}\left(\frac{1}{|p|}\left(\frac{|p| + (-1)^i|p|}{2} - i\right)^2 - 1\right) = 0 \pmod 2,$$
$$\textit{and} \quad I_{\phi,\epsilon}(i) \leq 0.$$

If one is familar with the work in [12], the proof is immediate. We will give the proof in Section 2. We study the H(2)-unknotting number of the pretzel knot $P(13, 4, 11)$ as an example, to show that the obstruction obtained here works better than other ones that the author knows.

Corollary 1.2. *The pretzel knot $P(13, 4, 11)$ has H(2)-unknotting number 2.*

2. Proofs

2.1. **Proof of Theorem 1.1.** Given a 3-manifold Y and one of its spin^c-structures s, an invariant $d(Y, s)$ called correction term is defined for the pair (Y, s) in [11]. Suppose Y is an oriented rational homology sphere. When $|H^2(Y, \mathbb{Z})|$ is odd, there exists a canonical isomorphism between the space $\mathrm{Spin}^c(Y)$ of spin^c-structures on Y and $H^2(Y,\mathbb{Z})$. In this case, we replace s in $d(Y,s)$ by the corresponding element in $H^2(Y,\mathbb{Z})$. Ozsváth and Szabó studied knots with unknotting number one in [12], and here is an general result they obtained (also refer to [10]).

Theorem 2.1 (Ozsváth-Szabó[12])**.** *Let Y be a rational homology 3-sphere which is the boundary of a simply-connected positive-definite four-manifold W, with $H^2(Y,\mathbb{Z})$ of odd order. If the intersection form of W is represented in a basis by the matrix A and G_A is the group presented by A, then there exists a group isomorphism $\phi: G_A \to H^2(Y,\mathbb{Z})$ with*

$$\begin{aligned} & d(Y,\phi(\alpha)) \leq M_A(\alpha) \\ \textit{and} \quad & d(Y,\phi(\alpha)) \equiv M_A(\alpha) \pmod 2 \end{aligned} \tag{2}$$

for all $\alpha \in G_A$.

When K is an alternating knot in S^3, the correction terms for $\Sigma(K)$ have an extremely easy combinatorial description as follows.

Theorem 2.2 (Ozsváth-Szabó[12, 13])**.** *If K is an alternating knot and Q denotes a Goeritz matrix associated to a reduced alternating projection of K, and G is the group presented by Q, then there is an isomorphism $\varphi: G \to H^2(\Sigma(K),\mathbb{Z})$, with the property that*

$$d(\Sigma(K),\varphi(\alpha)) = M_Q(\alpha)$$

for all $\alpha \in G$.

Proof of Theorem 1.1. If the H(2)-unknotting number of K is equal to one, then by Montesinos's trick [6] we have $\Sigma(K) = \epsilon \cdot S^3_{|p|}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1,-1\}$. Here p is equal to $\det(K)$. The manifold $-S^3_{|p|}(C)$ represents the manifold with reversed orientation. Therefore $\epsilon \cdot \Sigma(K) = S^3_{|p|}(C)$ bounds a four-manifold W, which is obtained by attaching a 2-handle to a four-ball along C with framing $|p|$. The intersection form of W is $A = (|p|)$. In this case we have that $G_A = \mathbb{Z}/|p|\mathbb{Z}$, that W is a simply-connected 4-manifold and that $H^2(S^3_{|p|}(C),\mathbb{Z}) \cong \mathbb{Z}/|p|\mathbb{Z}$.

By Theorem 2.1, there exists a group isomorphism $\phi: \mathbb{Z}/|p|\mathbb{Z} \to H^2(S^3_{|p|}(C),\mathbb{Z})$ with

$$\begin{aligned} d(\epsilon\cdot\Sigma(K),\phi(i)) = \epsilon\cdot d(\Sigma(K),\phi(i)) &\leq M_A(i) \\ \text{and} \quad \epsilon\cdot d(\Sigma(K),\phi(i)) &\equiv M_A(i) \pmod 2 \end{aligned} \tag{3}$$

for all $i \in \mathbb{Z}/|p|\mathbb{Z}$. It is easy to check that $M_A(i) = \frac{1}{4}(\frac{1}{|p|}(\frac{|p|+(-1)^i|p|}{2} - i)^2 - 1)$. Now Theorem 1.1 follows from Theorem 2.2.

□

2.2. **An example.** The pretzel knot $K = P(13,4,11)$ is a knot as shown in Figure 4. A Goeritz matrix associated to this diagram is

$$Q = \begin{pmatrix} 17 & -4 \\ -4 & 15 \end{pmatrix},$$

and the determinant is $\det(Q) = \det(K) = 239$. Suppose G is the group presented by Q. In fact, the group G is isomorphic to $\mathbb{Z}/239\mathbb{Z}$. In the following calculation, we

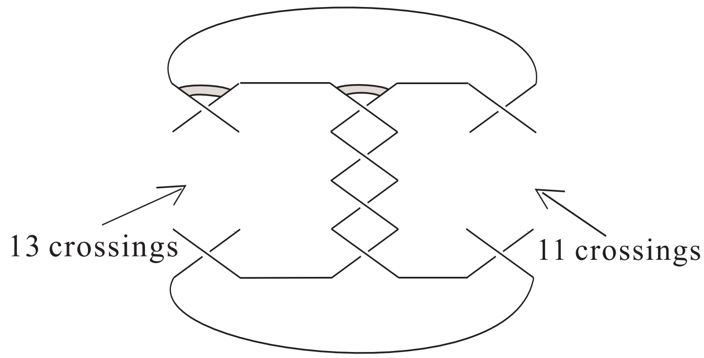

FIGURE 4. The pretzel knot $P(13,4,11)$.

take the vector $(0,1)^t$ as a generator of G. By calculation, it is easy to see that for any isomorphism $\phi : \mathbb{Z}/239\mathbb{Z} \longrightarrow \mathbb{Z}/239\mathbb{Z}$ there is

$$I_{\phi,\epsilon}(0) = \epsilon \cdot M_Q(\phi(0)) - 119/2 = (\epsilon \cdot 11 - 119)/2.$$

Since $I_{\phi,\epsilon}(0)$ has to be an even number, therefore we have $\epsilon = +1$. Next we obtain that $I_{\phi,+1}(1) = M_Q(\phi(1)) + 119/478$. To guarantee that $I_{\phi,+1}(1)$ is an even number, the isomorphism ϕ has to be either $\phi_1 = 15$ or $\phi_2 = 224$. By calculation, we see that $I_{\phi_1,+1}(1) = I_{\phi_2,+1}(1) = 4$, a positive number, which conflicts with the necessary condition stated in Theorem 1.1. Therefore the H(2)-unknotting number of $P(13,4,11)$ has to be at least two. On the other hand, the knot $P(13,4,11)$ can be changed into the unknot by adding two twisted bands as shown in Figure 4. Hence the H(2)-unknotting number of $P(13,4,11)$ is two. This completes the proof of Corollary 1.2.

2.3. **Comparisons with other criterions.** There have been many criterions and properties which can be used to bound the H(2)-unknotting number of a knot. We want to apply them to the knot $P(13,4,11)$ and compare the results with Corollary 1.2.

The first one is Lickorish's obstruction that we recalled in the beginning. But it does not work for the pretzel knot $K = P(13,4,11)$. It is known that the Goeritz matrix Q is a presentation of $H_1(\Sigma(K),\mathbb{Z})$, and Q^{-1} represents the linking form λ. From Section 2.2, we known that $I_{\phi_1,+1}(1)$ is an integer. This implies that $\lambda(g,g) = 1/239$ over $\mathbb{Q}/\mathbb{Z}$ for $g = (0,15)^t$. The vector g can work as a generator of $H_1(\Sigma(K),\mathbb{Z})$.

There are two invariants of knots which are closely related to H(2)-unknotting number. Given a knot $K \subset S^3$, the crosscap number [8] of K is defined as follows:

$$\gamma(K) = \min\left\{\beta_1(F) \,\middle|\, F \text{ is a non-orientable connected surface in } S^3 \text{ and } \partial F = K\right\}.$$

The four-dimensional crosscap number of K [9], which we denote $\gamma^*(K)$ here, is by name defined as follows:

$$\gamma^*(K) = \min\left\{\beta_1(F) \,\middle|\, \begin{array}{l} F \text{ is a non-orientable connected smooth surface in } B^4 \text{ and} \\ \partial F = K \subset \partial B^4 = S^3 \end{array}\right\}.$$

Their relation with H(2)-unknotting number is as follows. We give a proof here since we have not found any reference of it.

Lemma 2.3. *Given a knot $K \subset S^3$, we have $\gamma^*(K) \leq u_2(K) \leq \gamma(K)$.*

Proof. The knot K can be reconstructed from the unknot by adding $u_2(K)$ twisted bands successively. Let D be a disk bounded by the unknot and $b_1, b_2, \ldots, b_{u_2(K)}$ be the bands added to the boundary of D. Then $F := D \cup \bigcup_{i=1}^{u_2(K)} b_i$ is a non-orientable surface in B^4 with $\partial F = K$. We have $\gamma^*(K) \leq \beta_1(F) = u_2(K)$. The second inequality is proved as follows. Suppose S is a non-orientable surface in S^3 which realizes the crosscap number of K. Namely we have $\beta_1(S) = \gamma(K)$ and $\partial S = K$. Then there are $\gamma(K)$ disjoint essential arcs in S, say $\tau_1, \tau_2, \cdots, \tau_{\gamma(K)}$, such that $S - \tau_i$ has one boundary component for $i = 1, 2, \cdots, \gamma(K)$ and $S - \bigcup_{i=1}^{\gamma(K)} \tau_i$ is a disk. If we add twisted bands to K along τ_i for $i = 1, 2, \cdots, \gamma(K)$, the resulting knot is the unknot. Therefore we have $u_2(K) \leq \gamma(K)$. □

Ichihara and Mizushima [2] calculated the crosscap numbers of pretzel knots. According to their calculation, the crosscap numbers of $P(13, 4, 11)$ is two, but the four-dimensional crosscap number of it is unknown. Therefore the H(2)-unknotting number of $P(13, 4, 11)$ cannot be determined by Lemma 2.3 so far. Kanenobu and Miyazawa [3] introduced some criterions for bounding the H(2)-unknotting number of a knot, but their methods cannot be applied to the knot $P(13, 4, 11)$, either.

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