On Affine Motions and Bar Frameworks in General Position

A. Y. Alfakih *†
Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario N9B 3P4
Canada

Yinyu Ye ‡§
Department of Management Science and Engineering
Stanford University
Stanford, California 94305
USA

September 20, 2010

AMS classification: 52C25, 05C62.
Keywords: Bar frameworks, universal rigidity, stress matrices, points in general position, Gale transform.

Abstract

A configuration p in r-dimensional Euclidean space is a finite collection of points $(p^1, \ldots, p^n)$ that affinely span $\mathbb{R}^r$. A bar framework, denoted by $G(p)$, in $\mathbb{R}^r$ is a simple graph G on n vertices together with a configuration p in $\mathbb{R}^r$. A given bar framework $G(p)$ is said to be universally rigid if there does not exist another configuration q in any Euclidean space, not obtained from p by a rigid motion, such that $||q^i - q^j|| = ||p^i - p^j||$ for each edge (i, j) of G.

It is known [2, 6] that if configuration p is generic and bar framework $G(p)$ in $\mathbb{R}^r$ admits a positive semidefinite stress matrix S of rank ($n - r - 1$), then $G(p)$ is universally rigid. Connelly asked [8] whether the same result holds true if the genericity assumption of p is replaced by the weaker assumption of general position. We answer this question in the affirmative in this paper.

*E-mail: alfakih@uwindsor.ca
†Research supported by the Natural Sciences and Engineering Research Council of Canada.
‡E-mail:yinyu-ye@stanford.edu
§Research supported in part by NSF Grant GOALI 0800151 and DOE Grant de-sc0002009.

1 Introduction

A *configuration* p in r-dimensional Euclidean space is a finite collection of points $(p^1, \ldots, p^n)$ in $\mathbb{R}^r$ that affinely span $\mathbb{R}^r$. A *bar framework* (or framework for short) in $\mathbb{R}^r$, denoted by $G(p)$, is a configuration p in $\mathbb{R}^r$ together with a simple graph G on the vertices $1, 2, \ldots, n$. For a simple graph G, we denote its node set by $V(G)$ and its edge set by $E(G)$. To avoid trivialities, we assume throughout this paper that graph G is connected and not complete.

Framework $G(q)$ in $\mathbb{R}^r$ is said to be *congruent* to framework $G(p)$ in $\mathbb{R}^r$ if configuration q is obtained from configuration p by a rigid motion. That is, if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $i, j = 1, \ldots, n$, where $\|.\|$ denotes the Euclidean norm. We say that framework $G(q)$ in $\mathbb{R}^s$ is *equivalent* to framework $G(p)$ in $\mathbb{R}^r$ if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $(i, j) \in E(G)$. Furthermore, we say that framework $G(q)$ in $\mathbb{R}^r$ is *affinely-equivalent* to framework $G(p)$ in $\mathbb{R}^r$ if $G(q)$ is equivalent to $G(p)$ and configuration q is obtained from configuration p by an affine motion; i.e., $q^i = Ap^i + b$, for all $i = 1, \ldots, n$, for some $r \times r$ matrix A and an r-vector b.

A framework $G(p)$ in $\mathbb{R}^r$ is said to be *universally rigid* if there does exist a framework $G(q)$ in any Euclidean space that is equivalent, but not congruent, to $G(p)$. The notion of a stress matrix S of a framework $G(p)$ plays a key role in the problem of universal rigidity of $G(p)$.

1.1 Stress Matrices and Universal Rigidity

Let $G(p)$ be a framework on n vertices in $\mathbb{R}^r$. An *equilibrium stress* of $G(p)$ is a real valued function ω on $E(G)$ such that

$$\sum_{j:(i,j)\in E(G)} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n. \tag{1}$$

Let ω be an equilibrium stress of $G(p)$. Then the $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E(G), \\ 0 & \text{if } i \neq j \text{ and } (i,j) \notin E(G), \\ \displaystyle\sum_{k:(i,k)\in E(G)} \omega_{ik} & \text{if } i = j, \end{cases} \tag{2}$$

is called the *stress matrix* associated with ω, or a stress matrix of $G(p)$. The following result provides a sufficient condition for the universal rigidity of a given framework.

Theorem 1.1 (Connelly [5, 6], Alfakih [1]) *Let $G(p)$ be a bar framework in $\mathbb{R}^r$, for some $r \leq n-2$. If the following two conditions hold:*

1. *There exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$.*
2. *There does not exist a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.*

Then $G(p)$ is universally rigid.

Note that $(n-r-1)$ is the maximum possible value for the rank of the stress matrix S. In connection with Theorem 1.1, we mention the following result obtained in So and Ye [11] and Biswas et al. [4]: Given a framework $G(p)$ in $\mathbb{R}^r$, if there does not exist a framework $G(q)$ in $\mathbb{R}^s$ ($s \neq r$) that is equivalent to $G(p)$, then $G(p)$ is universally rigid. Moreover, if $G(p)$ contains a clique of $r+1$ points in general position, then the existence of a rank-$(n-r-1)$ positive semidefinite stress matrix implies that framework $G(p)$ is universally rigid, regardless whether the other non-clique points are in general position or not.

Condition 2 of Theorem 1.1 is satisfied if configuration p is assumed to be generic (see Lemma 2.2 below). A configuration p (or a framework $G(p)$) is said to be *generic* if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, if there does not exist a non-zero polynomial f with integer coefficients such that $f(p^1, \ldots, p^n) = 0$. Thus

Theorem 1.2 (Connelly [6], Alfakih [2]) *Let $G(p)$ be a generic bar framework on n nodes in $\mathbb{R}^r$, for some $r \leq n-2$. If there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$. Then $G(p)$ is universally rigid.*

The converse of Theorem 1.2 is also true.

Theorem 1.3 (Gortler and Thurston [10]) *Let $G(p)$ be a generic bar framework on n nodes in $\mathbb{R}^r$, for some $r \leq n-2$. If $G(p)$ is universally rigid, then there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$.*

Connelly [8] asked whether a result similar to Theorem 1.2 holds if the genericity assumption of $G(p)$ is replaced by the weaker assumption of general position. A configuration p (or a framework $G(p)$) in $\mathbb{R}^r$ is said to be in *general position* if no subset of the points $p^1, \ldots, p^n$ of cardinality $r+1$ is affinely dependent. For example, a set of points in the plane are in general position if no 3 of them lie on a straight line.

In this paper we answer Connelly's question in the affirmative. Thus the following theorem is the main result of this paper.

Theorem 1.4 *Let $G(p)$ be a bar framework on n nodes in general position in $\mathbb{R}^r$, for some $r \leq n-2$. If there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$. Then $G(p)$ is universally rigid.*

The proof of Theorem 1.4 will be given in Section 3. This paper and [3] are first steps toward the study of universal rigidity under the general position assumption. In [3], it was shown that the framework $G(p)$ on n nodes in general position in $\mathbb{R}^r$ for some $r \leq n-2$, where G is the $(r+1)$-lateration graph, admits a rank $(n-r-1)$ positive semi-definite stress matrix.

2 Preliminaries

To develop the ingredients needed for the proof of our main result, we review the necessary background on affine motions, stress matrices, and Gale matrices.

An affine motion in $\mathbb{R}^r$ is a map $f : \mathbb{R}^r \to \mathbb{R}^r$ of the form

$$f(p^i) = Ap^i + b,$$

for all p^i in $\mathbb{R}^r$, where A is an $r \times r$ matrix and b is an r-vector. A rigid motion is an affine motion where matrix A is orthogonal.

Vectors $v^1, \ldots, v^m$ in $\mathbb{R}^r$ are said to lie on a *quadratic at infinity* if there exists a non-zero symmetric $r \times r$ matrix Φ such that

$$(v^i)^T \Phi v^i = 0, \text{ for all } i = 1, \ldots, m. \tag{3}$$

Lemma 2.1 *(Connelly [7]) Let $G(p)$ be a bar framework on n vertices in $\mathbb{R}^r$. Then the following two conditions are equivalent:*

1. *There exists a framework $G(q)$ in $\mathbb{R}^r$ that is equivalent, but not congruent, to $G(p)$ such that $q^i = Ap^i + b$ for all $i = 1, \ldots, n$,*
2. *The vectors $p^i - p^j$ for all $(i, j) \in E(G)$ lie on a quadratic at infinity.*

Lemma 2.2 *(Connelly [7]) Let $G(p)$ be a generic bar framework on n vertices in $\mathbb{R}^r$. Assume that each node of G has degree at least r. Then the vectors $p^i - p^j$ for all $(i, j) \in E(G)$ do not lie on a quadratic at infinity.*

Therefore, under the genericity assumption, Condition 2 in Lemma 2.1 does not hold. Consequently, Theorem 1.2 follows as a simple corollary of Theorem 1.1.

Note that Condition 2 in Lemma 2.1 is expressed in terms of the edges of G. An equivalent condition in terms of the missing edges of G can also be obtained using Gale matrices. This equivalent condition turns out to be crucial for our proof of Theorem 1.4.

To this end, let $G(p)$ be a framework on n vertices in $\mathbb{R}^r$. Then the following $(r+1) \times n$ matrix

$$\mathcal{A} := \begin{bmatrix} p^1 & p^2 & \ldots & p^n \\ 1 & 1 & \ldots & 1 \end{bmatrix} \tag{4}$$

has full row rank since $p^1, \ldots, p^n$ affinely span $\mathbb{R}^r$. Note that $r \leq n-1$. Let

$$\bar{r} = \text{the dimension of the null space of } \mathcal{A};\ \text{i.e., } \bar{r} = n - 1 - r. \tag{5}$$

Definition 2.1 *Suppose that the null space of $\mathcal{A}$ is nontrivial, i.e., $\bar{r} \geq 1$. Any $n \times \bar{r}$ matrix Z whose columns form a basis of the null space of $\mathcal{A}$ is called a* Gale matrix *of configuration p. Furthermore, the ith row of Z, considered as a vector in $\mathbb{R}^{\bar{r}}$, is called a* Gale transform *of p^i [9].*

Let S be a stress matrix of $G(p)$ then it follows from (2) and (4) that

$$\mathcal{A}S = 0. \tag{6}$$

Thus

$$S = Z \Psi Z^T, \tag{7}$$

for some $\bar{r} \times \bar{r}$ symmetric matrix Ψ, where Z is a Gale matrix of p. It immediately follows from (7) that rank S = rank Ψ. Thus, S attains its maximum rank of $\bar{r} = (n-1-r)$ if and only if Ψ is nonsingular, i.e., rank $\Psi = \bar{r}$.

Let e denote the vector of all 1's in $\mathbb{R}^n$, and let V be an $n \times (n-1)$ matrix that satisfies:

$$V^T e = 0,\ V^T V = I_{n-1}, \tag{8}$$

where I_{n-1} is the identity matrix of order $(n-1)$. Further, let E^{ij}, $i \neq j$, denote the $n \times n$ symmetric matrix with 1 in the (i,j)th and (j,i)th entries and zeros elsewhere, and let $\mathcal{E}(y) = \sum_{(i,j) \notin E(G)} y_{ij} E^{ij}$ where $y_{ij} = y_{ji}$. In other words, the (k,l) entry of matrix $\mathcal{E}(y)$ is given by

$$\mathcal{E}(y)_{kl} = \begin{cases} 0 & \text{if } (k,l) \in E(G), \\ 0 & \text{if } k = l, \\ y_{kl} & \text{if } k \neq l \text{ and } (k,l) \notin E(G). \end{cases} \tag{9}$$

Then we have the following result.

Lemma 2.3 *(Alfakih [2]) Let $G(p)$ be a bar framework on n vertices in $\mathbb{R}^r$ and let Z be any Gale matrix of p. Then the following two conditions are equivalent:*

1. *The vectors $p^i - p^j$ for all $(i,j) \in E(G)$ lie on a quadratic at infinity.*
2. *There exists a non-zero $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that:*

$$V^T \mathcal{E}(y) Z = \mathbf{0}, \tag{10}$$

where $\bar{m}$ is the number of missing edges of G, V is defined in (8), and $\mathcal{E}(y)$ is defined in (9). **0** *here is the zero matrix of dimension $(n-1) \times \bar{r}$.*

Condition 2 of Lemma 2.3 can be easily understood if a projected Gram matrix approach is used for the universal rigidity of bar frameworks (see [2] for details).

3 Proof of Theorem 1.4

The main idea of the proof is to show that Condition 2 of Lemma 2.3 does not hold under the general position assumption, and under the assumption that $G(p)$ admits a positive semidefinite stress matrix of rank $(n-r-1)$. The choice of the particular Gale matrix to be used in equation (10) is critical in this regard. We begin with a few necessary lemmas.

Lemma 3.1 *Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$ and let Z be any Gale matrix of configuration p. Then any $\bar{r} \times \bar{r}$ submatrix of Z is nonsingular.*

Proof. For a proof see e.g., [1].

□

Let $\bar{N}(i)$ denote the set of nodes of graph G that are non-adjacent to node i; i.e.,

$$\bar{N}(i) = \{j \in V(G) : j \neq i \text{ and } (i,j) \notin E(G)\}, \tag{11}$$

Lemma 3.2 *Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$. Assume that $G(p)$ has a stress matrix S of rank $(n-1-r)$. Then there exists a Gale matrix $\hat{Z}$ of $G(p)$ such that $\hat{z}_{ij} = 0$ for all $j = 1, \ldots, \bar{r}$ and $i \in \bar{N}(j+r+1)$.*

Proof. Let $G(p)$ be in general position in $\mathbb{R}^r$ and assume that it has a stress matrix S of rank $\bar{r} = (n-1-r)$. Let Z be any Gale matrix of $G(p)$, then $S = Z\Psi Z^T$ for some non-singular symmetric $\bar{r} \times \bar{r}$ matrix Ψ. Let us write Z as:

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{12}$$

where Z_2 is $\bar{r} \times \bar{r}$. By Lemma 3.1, Z_2 is non-singular. Now let

$$\hat{Z} = (\hat{z}_{ij}) = Z\Psi {Z_2}^T. \tag{13}$$

Then $\hat{Z}$ is a Gale matrix of $G(p)$. This simply follows from the fact that the matrix obtained by multiplying any Gale matrix of $G(p)$ from the right by a non-singular $\bar{r} \times \bar{r}$ matrix, is also a Gale matrix of $G(p)$. Furthermore,

$$S = Z\Psi Z^T = Z\Psi \, [Z_1^T \;\; Z_2^T] = [Z\Psi Z_1^T \;\; \hat{Z}].$$

In other words, $\hat{Z}$ consists of the last $\bar{r}$ columns of S. Thus $\hat{z}_{ij} = s_{i,j+r+1}$. By the definition of S we have $s_{i,j+r+1} = 0$ for all $i \neq j+r+1$ and $(i, j+r+1) \notin E(G)$. Therefore, $\hat{z}_{ij} = 0$ for all $j = 1, \ldots, \bar{r}$ and $i \in \bar{N}(j+r+1)$.

□

Lemma 3.3 *Let the Gale matrix in (10) be $\hat{Z}$ as defined in (13). Then the system of equations (10) is equivalent to the system of equations*

$$\mathcal{E}(y)\hat{Z} = \mathbf{0}. \tag{14}$$

$\mathbf{0}$ *here is the zero matrix of dimension $n \times \bar{r}$.*

Proof. System of equations (10) is equivalent to the following system of equations in the unknowns, y_{ij} ($i \neq j$ and $(i,j) \notin E(G)$) and $\xi \in \mathbb{R}^{\bar{r}}$:

$$\mathcal{E}(y)\hat{Z} = e\,\xi^T, \tag{15}$$

Now for $j = 1, \ldots, \bar{r}$, we have that the $(j+r+1, j)$th entry of $\mathcal{E}(y)\hat{Z}$ is equal to ξ_j. But using (9) and Lemma 3.2 we have

$$(\mathcal{E}(y)\hat{Z})_{j+r+1,j} = \sum_{i=1}^{n} \mathcal{E}(y)_{j+r+1,i}\ \hat{z}_{ij} = \sum_{i:i\in\bar{N}(j+r+1)} y_{j+r+1,i}\ \hat{z}_{ij} = 0.$$

Thus, $\xi = 0$ and the result follows.

□

Now we are ready to prove our main theorem.

Proof of Theorem 1.4

Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$. Assume that $G(p)$ has a positive semidefinite stress matrix S of rank $\bar{r} = n-1-r$. Then $\deg(i) \geq r+1$ for all $i \in V(G)$, i.e., every node of G is adjacent to at least $r+1$ nodes (for a proof see [1, Theorem 3.2]). Thus

$$|\bar{N}(i)| \leq n - r - 2 = \bar{r} - 1. \tag{16}$$

Furthermore, it follows from Lemmas 3.2, 3.3 and 2.3 that the vectors $p^i - p^j$ for all $(i,j) \in E(G)$ lie on a quadratic at infinity if and only if system of equations (14) has a non-zero solution y. But (14) can be written as

$$\sum_{j:\in\bar{N}(i)} y_{ij}\hat{z}^j = 0, \text{ for } i = 1, \ldots, n,$$

where $(\hat{z}^i)^T$ is the ith row of $\hat{Z}$. Now it follows from (16) that $y_{ij} = 0$ for all $(i,j) \notin E(G)$ since by Lemma 3.1 any subset of $\{\hat{z}^1, \ldots, \hat{z}^n\}$ of cardinality $\leq \bar{r} - 1$ is linearly independent.

Thus system (14) does not have a nonzero solution y. Hence the vectors $p^i - p^j$, for all $(i,j) \in E(G)$, do not lie on a quadratic at infinity. Therefore, by Lemma 2.1, there does not exist a framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$. Thus by Theorem 1.1, $G(p)$ is universally rigid.

□

References

[1] A. Y. Alfakih. On dimensional rigidity of bar-and-joint frameworks. *Discrete Appl. Math.*, 155:1244–1253, 2007.

[2] A. Y. Alfakih. On the universal rigidity of generic bar frameworks. *Contrib. Disc. Math.*, 5:7–17, 2010.

[3] A. Y. Alfakih, N. Taheri, and Y. Ye. Toward the universal rigidity of general frameworks, 2010. arXiv/1009.1185.

[4] P Biswas, K-C Toh, and Y. Ye. A distributed SDP approach for large-scale noisy anchor-free graph realization with applications to molecular conformation. *SIAM J. Sci. Comp.*, pages 1251–1277, 2008.

[5] R. Connelly. Rigidity and energy. *Invent. Math*, 66:11–33, 1982.

[6] R. Connelly. Tensegrity structures: Why are they stable? In M. F. Thorpe and P. M. Duxbury, editors, *Rigidity theory and applications*, pages 47–54. Kluwer Academic/Plenum Publishers, 1999.

[7] R. Connelly. Generic global rigidity. *Discrete Comput. Geom.*, 33:549–563, 2005.

[8] R. Connelly. Questions, conjectures and remarks on globally rigid tensegrities. Technical report, Cornell University, 2009.

[9] D. Gale. Neighboring vertices on a convex polyhedron. In *Linear inequalities and related system*, pages 255–263. Princeton University Press, 1956.

[10] S. J. Gortler and D. P. Thurston. Characterizing the universal rigidity of generic frameworks, 2009. arXiv/1001.0172v1.

[11] A. M.-C. So and Y. Ye. Theory of Semidefinite Programming for Sensor Network Localization. *Proceedings of the 17th Annual ACM–SIAM Symposium on Discrete Algorithm (SODA 2005)* 2005, pp. 405–414; and *Mathematical Programming, Series B*, 109:367–384, 2007.