

Parity balance of the i -th dimension edges in Hamiltonian cycles of the hypercube

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Abstract

Let $n \geq 2$ be an integer, and let $i \in \{0, \dots, n-1\}$. An i -th dimension edge in the n -dimensional hypercube Q_n is an edge v_1v_2 such that v_1, v_2 differ just at their i -th entries. The parity of an i -th dimension edge v_1v_2 is the number of 1's modulus 2 of any of its vertex ignoring the i -th entry. We prove that the number of i -th dimension edges appearing in a given Hamiltonian cycle of Q_n with parity zero coincides with the number of edges with parity one. As an application of this result it is introduced and explored the conjecture of the inscribed squares in Hamiltonian cycles of the hypercube: Any Hamiltonian cycle in Q_n contains two opposite edges in a 4-cycle. We prove this conjecture for $n \leq 7$, and for any Hamiltonian cycle containing more than 2^{n-2} edges in the same dimension. This bound is finally improved considering the equi-independence number of Q_{n-1} , which is a concept introduced in this paper for bipartite graphs.

Keywords: Hypercube; Hamiltonian cycles; i -th dimension edges; equi-independence number; inscribed square conjecture; bipartite graphs.

1 Introduction

Let $n \geq 1$ be an integer number. The n -dimensional hypercube, denoted Q_n , is the graph having $\{0, 1\}^n$ as set of vertexes, two of them being joined by an edge if they differ just in one of their entries. It is well known that hypercubes are all Hamiltonian graphs. The canonical examples of Hamiltonian cycles in the hypercube are the so called *binary Gray codes* [9].

The study of structural properties of the Hamiltonian cycles of the hypercube have allowed the solution of relevant problems. For instance, the proof of Kreweras's conjecture by Fink in 2007 (see [6, 7]) about the extensibility of any perfect matching on the hypercube allowed Feder and Subi in 2009 to find tight bounds on the number of different Hamiltonian cycles in the hypercube [5]; the same structural property allowed Gregor in 2009 to prove that perfect matchings on subcubes can be extended to Hamiltonian cycles of hypercubes [8]. Something similar has been done with Hamiltonian paths, for example Chen proved in 2006 (see [2]) that any path of length at most $2n-1$ is a sub-path of a Hamiltonian path in Q_n and he used this basic structural property to prove that, for $n \geq 3$, Q_n is $(2n-3)$ -path bipancyclic but it is not $(2n-2)$ -path bipancyclic. Moreover, he proved that a path P of length k , with $2 \leq k \leq 2n-3$, lies in a cycle of length $(2k-2)$ if and only if P contains two edges in the same dimension [2].

Our objective here is to extend the structural knowledge about Hamiltonian cycles of the hypercube by classifying permissible sets of edges in Hamiltonian cycles, in terms of the dimension to which these edges belong and their parities (see Theorem 4.1).

As an application we conjecture (see Conjecture 5.1) that any Hamiltonian cycle in the hypercube contains the opposite edges of a 4-cycle, and we prove some particular instances of this conjecture (see Theorem 4.1). In the 80's Erdős [4] conjectured that any subgraph of Q_n asymptotically containing at most half edges of the whole Q_n is 4-cycle free, recent advances on this conjecture appeared in [1] and [10]. The substantial difference of Erdős conjecture and ours is that Erdős conjecture poses conditions on the

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maximum number of edges in a 4-cycle-free graph and our conjecture is about the existence of 4-cycles with opposite edges in any Hamiltonian cycle of the hypercube.

As a first approach to explore Conjecture 5.1 it is introduced the notion of i -th dimension graph which in turn is isomorphic to Q_{n-1} and allows us to translate our decision problem (whether there is a 4-cycle within any Hamiltonian cycle h in the hypercube) into verifying that the maximum number of edges in a dimension in $E(h)$ is greater than the independence number of Q_{n-1} (see Theorem 5.1). Then the last bound is improved as the equi-independence number of Q_{n-1} (see Corollary 5.1). The equi-independence number of a bipartite graph G is the cardinality of the maximum independent set in G containing the same number of elements in each bipartition class; we prove in Theorem 5.3 that finding the equi-independence number of a graph is polynomial time reducible to the independence number computation problem.

The outline of the paper is the following: In Section 2 we recall very basic notions of Graph Theory and we introduce some notation, in Section 3 we introduce and study i -th dimension graphs. In Section 4 we introduce the notion of *chromatic vector* and some other operators defined over Hamiltonian cycles in the hypercube, and we prove Theorem 4.1 which is the main result in this paper. In Section 5 we state and discuss the inscribed square conjecture, we introduce the equi-independence concept for bipartite graphs and we report our final progress on this conjecture. In the conclusions, we suggest additional applications to Theorem 4.1 and we pose a list of open conjectures and problems.

2 Preliminaries

Let $G = (V, E)$ be graph with set of nodes V and set of edges E . Let us recall the following elementary notions. A k -cycle in G is a sequence of pairwise different nodes $v_0 v_1 \dots v_{k-1}$ such that for each index i , $v_i v_{i+1}$ is an edge in E (index addition is taken modulus k). A *Hamiltonian cycle* is a k -cycle, where $k = \text{card}(V)$ is the order of the graph. The graph G is *Hamiltonian* if it possesses a Hamiltonian cycle. A non-empty set $I \subset V$ is *independent* if no pair of different elements in I is an edge: $\forall u, v \in I [u \neq v \Rightarrow uv \notin E]$. A *maximal independent set* is an independent set which is maximal with respect to set-inclusion. The *independence number* $\alpha(G)$ of G is the number of vertexes in a largest independent set: $\alpha(G) = \max\{\nu \mid \exists I \subset V : I \text{ independent} \ \& \ \text{card}(I) = \nu\}$. Any independent set consisting of $\alpha(G)$ vertexes is called a *maximum independent set*.

Let $n \geq 2$ be an integer. It may be assumed that $E(Q_n)$ consists of pairs $v(v + e_i)$, where e_i is the i -th canonical basic vector. Clearly, $v(v + e_i) = u(u + e_j)$ if and only if $i = j$ and either $u = v$ or $u = v + e_i$ (addition is integer addition modulus 2).

Any 4-cycle in Q_n has thus the form $v(v + e_i)(v + e_i + e_j)(v + e_j)$, with $v \in V(Q_n)$ and $0 \leq i < j \leq n - 1$.

The order of Q_n is thus 2^n and its independence number is $\alpha(Q_n) = 2^{n-1}$.

The hypercube Q_n is Hamiltonian and the *binary Gray code* [9], Gr_n , is a Hamiltonian cycle. As a sequence, this code is determined recursively by the following recurrence:

$$\text{Gr}_1 = [0, 1] \quad , \quad \text{Gr}_n = \text{join}(0 * \text{Gr}_{n-1}, 1 * \text{rev}(\text{Gr}_{n-1}))$$

(join and rev are respectively list concatenation and list reversing, $*$ is a *prepend* map: $b * \text{list}$ prepends the bit b to each entry at the list). In general, we will follow the notions and notations in Diestel's textbook [3].

3 Graphs along a dimension

Let $n \geq 2$ be an integer. Let i be an integer in $\{0, \dots, n - 1\}$. An i -th dimension edge in Q_n is an edge of the form $u(u + e_i)$ where $u \in V(Q_n)$.

For each $i \in \{0, \dots, n - 1\}$ let \mathcal{D}_{in} be the i -dimension graph whose vertexes are the i -th dimension edges of Q_n , and whose edges are the pairs of i -th dimension edges forming a 4-cycle within Q_n :

$$v(v + e_i)u(u + e_i) \text{ is an edge in } \mathcal{D}_{in} \iff \exists j \neq i : u(u + e_i) = (v + e_j)(v + e_j + e_i). \quad (1)$$

Theorem 3.1. *Let $n \geq 2$ be an integer. For each $i \in \{0, \dots, n - 1\}$ the graph \mathcal{D}_{in} is isomorphic to Q_{n-1} .*

Proof: The map $f_{in} : V(\mathcal{D}_{in}) \rightarrow V(Q_{n-1})$, $v(v + e_i) \mapsto f_{in}(v(v + e_i)) = \rho_{in}(v)$, where ρ_{in} is deletion of the i -th entry, is a graph isomorphism. ■

Hence, being all isomorphic to Q_{n-1} , the i -th dimension graphs \mathcal{D}_{in} are pairwise isomorphic and each \mathcal{D}_{in} is bipartite.

Let $par_n : Q_n \rightarrow \{0, 1\}$ be the map such that $par_n(v)$ is the parity of the Hamming weight (number of entries equal to 1) of the vector $v \in Q_n$, and for any index $i \in \{0, \dots, n-1\}$ let $par_{in} = par_{n-1} \circ \rho_{in}$ where ρ_{in} is the map that suppresses the i -th entry at any n -dimensional vector. In other words, $par_{in}(v)$ is the parity of the vector resulting by suppressing the i -th entry in v . We have that the bipartition classes of \mathcal{D}_{in} are realized as:

$$v_0v_1, v_2v_3 \text{ in the same bipartition class} \iff par_{n-1} \circ f_{in}(v_0v_1) = par_{n-1} \circ f_{in}(v_2v_3).$$

Or equivalently,

$$v(v + e_i), u(u + e_i) \text{ in the same bipartition class} \iff par_{in}(v) = par_{in}(u). \quad (2)$$

According to the common value at the right side of relation (2) we will refer to the partition classes as 0- and 1-*bipartition classes* of \mathcal{D}_{in} .

Let $h = h_0 \cdots h_{2^n-1}$ be a Hamiltonian cycle of Q_n . For each $\iota \in \{0, \dots, 2^n - 1\}$ let us associate the dimension i of the edge $h_\iota h_{\iota+1}$ as a color of the starting vertex (in the sense of the cycle) h_ι . Let us denote $\chi_h : V(Q_n) \rightarrow \{0, \dots, n-1\}$ this vertex coloring induced by the Hamiltonian cycle h .

Let $c(h) = (c_i)_{i=0}^{n-1} \in \mathbb{N}^n$ be the vector such that for each i , c_i is the number of vertexes colored i by the Hamiltonian cycle h . The vector $c(h)$ is called the *chromatic vector* of h .

$c : \{\text{Hamiltonian cycles}\} \rightarrow \mathbb{N}^n$ is a well defined map. A full characterization of the image $C \subset \mathbb{N}^n$ of this map is out of the scope of this paper. However, some necessary conditions for chromatic vectors are asserted at the following

Lemma 3.1. *Let $n \geq 2$ be an integer and let $c \in C$ be the chromatic vector of some Hamiltonian cycle h , $c = c(h)$. Then:*

1. All entries of c are even integers.
2. No entry at c is zero.
3. The sum of the entries of c is equal to 2^n .
4. The greatest entry in c is lower or equal than 2^{n-1} .
5. The lowest entry in c is greater or equal than 2.
6. Any permutation of c is the chromatic vector of a Hamiltonian cycle of Q_n .

Proof: The first assertion follows from the fact that on the hypercube Q_n , $\sum_{v \in Q_n} v = 0 \in V(Q_n)$ (sum is addition modulus 2). The second because if $c_i = 0$, for some i , then the nodes at the Hamiltonian h either all lie at the semi-space $v_i = 0$ or at the opposite semi-space $v_i = 1$, which is not possible. Third assertion holds because each vertex has assigned a color. Fourth assertion follows from the fact that no consecutive pair of edges in h can be parallel, or, in other words, if $h_j h_{j+1}$ has the same direction as $h_{j+1} h_{j+2}$, for some $j \in \{0, \dots, 2^n - 1\}$, then necessarily $h_j = h_{j+2}$ but that is impossible. Fifth assertion is a consequence of the first and the second. For the last assertion, let $\pi : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$ be a permutation, then $c(\pi \circ h) = \pi(c(h))$ and $\pi \circ h$ is a Hamiltonian cycle. ■

Let $h = h_0 \cdots h_{2^n-1}$ be a Hamiltonian cycle in the hypercube and let $i \in \{0, \dots, n-1\}$. By rotating and reverting, if necessary, the indexes in $\{0, \dots, 2^n - 1\}$, we may assume that i is the color assigned to h_0 and that $h_{0i} = 0$ (indeed, $h_0 h_1$ is an i -th dimension edge). Let us define the following operators:

inverse image index list. $\lambda_i(h) = [j_0, j_1, \dots, j_{c_i-1}]$ ($j_0 = 0$): the list of indexes colored i at the Hamiltonian h . This is the list of indexes $\ell \in \{0, \dots, 2^n - 1\}$ such that $\chi_h(h_\ell) = i$. In other words, it is the list of indexes in h where an i -th dimension edge starts,

inverse image lists. $\eta_i(h) = [h_{j_0}, h_{j_1}, \dots, h_{j_{c_i-1}}]$, and $\xi_i(h) = [h_{j_0}h_{j_0+1}, h_{j_1}h_{j_1+1}, \dots, h_{j_{c_i-1}}h_{j_{c_i-1}+1}]$.
The list $\eta_i(h)$ consists of the i -th colored vertexes at Q_n , and the list $\xi_i(h)$ consists of the i -th dimension edges at Q_n .

Let us study the maps λ_i, η_i, ξ_i .

4 Parity balance

Let $n \geq 2$ be an integer. The n -dimensional hypercube Q_n is bipartite and its bipartite classes are the collections of vertexes of odd and even parity: $P_{0n} = \text{par}_n^{-1}(0)$, $P_{1n} = \text{par}_n^{-1}(1)$. Let $h = h_0 \cdots h_{2^n-1}$ be a Hamiltonian cycle in the n -dimensional hypercube Q_n .

Remark. The sequence $\text{par}_n \circ h$ consists of alternating values 0 and 1.

Theorem 4.1. *Let $n \geq 2$ be an integer, let $i \in \{0, \dots, n-1\}$ and let h be a Hamiltonian cycle in Q_n . Then half of the i -th dimension edges appearing in h , namely at the list $\xi_i(h)$, lie in the 0-bipartition class of \mathcal{D}_{in} and the other half in the 1-bipartition class.*

Proof: The first statement at Lemma 3.1 asserts that $c_i = |\xi_i(h)|$ is indeed an even integer. Without loss of generality we may assume that h_0h_1 is an i -th dimension edge and the i -th entry of the starting vertex h_0 has value 0, i.e. $h_{0i} = 0$.

For a bit $b \in \{0, 1\}$, let Q_{bin} be the subgraph of Q_n induced by those vertexes with value b at their i -th entry. Both Q_{0in} and Q_{1in} are isomorphic to Q_{n-1} and they are bipartite as well.

Let $\lambda_i(h) = [j_0, j_1, \dots, j_{c_i-1}]$ ($j_0 = 0$) be the index list corresponding to i -colored nodes at h . From relation (2) we have

$$\forall k < c_i, b \in \{0, 1\} \quad [\text{par}_{in}(h_{j_k}) = \text{par}_{in}(h_{j_{k+1}}) = b \iff h_{j_k}h_{j_{k+1}} \in b\text{-bipartition class}] \quad (3)$$

Let us consider the parity list $[\text{par}_{in}(h_{j_k})]_{k=0}^{c_i-1}$. By relation (3), the theorem will be proved by showing that half values in this list are 0 and half are 1.

Since $h_{0i} = 0$, for any even $k \in \{0, 1, \dots, c_i-1\}$ the path $H_k = h_{j_{k+1}} \cdots h_{j_{k+1}}$ lies entirely in Q_{1in} . Let $V(H_k)$ be the collection of vertexes appearing in H_k . Indeed,

$$\{V(H_{2k_1})\}_{k_1=0}^{\frac{c_i}{2}-1} \text{ is a partition of } V(Q_{1in}). \quad (4)$$

Similarly,

$$\{V(H_{2k_1+1})\}_{k_1=0}^{\frac{c_i}{2}-1} \text{ is a partition of } V(Q_{0in}). \quad (5)$$

Figure 1 displays a diagram of this situation. According to relation (3), the following equivalences hold:

$$\text{par}_{in}(h_{j_{k+1}}) \neq \text{par}_{in}(h_{j_k}) \iff \text{the length of } H_k \text{ is odd}, \quad (6)$$

$$\text{par}_{in}(h_{j_{k+1}}) = \text{par}_{in}(h_{j_k}) \iff \text{the length of } H_k \text{ is even}. \quad (7)$$

If the inequation at the left side of (6) holds then the endpoints of H_k have different parities and they does not introduce any imbalance on the number of zeros and ones in $[\text{par}_{in}(h_{j_k})]_{k=0}^{c_i-1}$ (recall that by definition $\text{par}_{in}(h_{j_{k+1}}) = \text{par}_{in}(h_{j_k})$). If the equation at the left side of (7) holds with value $b \in \{0, 1\}$ for some even number $k = k_1$, then h_{k_1} has an odd number of vertexes and in consequence one vertex less in the \bar{b} -bipartition class of Q_{1in} than in the b -bipartition class. Since relation (4) is true this deficit must be compensated with the existence of some even value $k = k_2$ such that $\text{par}_{in}(h_{j_{k_2+1}}) = \text{par}_{in}(h_{j_{k_2+1}}) = \bar{b}$. In other words

$$\begin{aligned} \text{card}(\{k \in \{0, 2, \dots, c_i-2\} | \text{par}_{in}(h_{j_{k+1}}) = 0 = \text{par}_{in}(h_{j_{k+1}})\}) &= \\ \text{card}(\{k \in \{0, 2, \dots, c_i-2\} | \text{par}_{in}(h_{j_{k+1}}) = 1 = \text{par}_{in}(h_{j_{k+1}})\}) & \end{aligned}$$

Consequently $[\text{par}_{in}(h_{j_k})]_{k=0}^{c_i-1}$ shall have the same number of 0's and 1's. ■

The above result entails some necessary conditions for a vector to be of the form $c(h)$ for some Hamiltonian cycle h .

For a Hamiltonian cycle h on Q_n and an index $i \in \{0, \dots, n-1\}$ let us define:

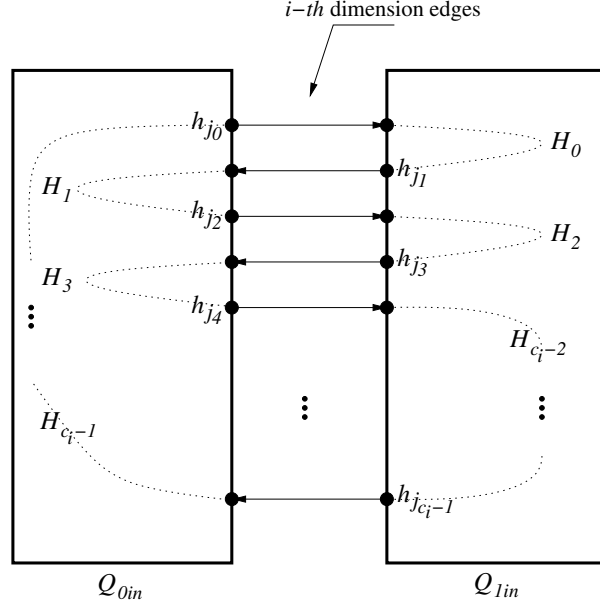


Figure 1: The traces of a Hamiltonian cycle on the parallel semi-hypercubes with constant i -th entry.

chromatic-segments list. $\delta_i(h) = [j_1 - j_0, j_2 - j_1, \dots, 2^n - j_{c_i-1}]$: the difference among the right shift of $\lambda(h)$ and $\lambda(h)$ itself,

parity list. $\beta_i(h) = [b_0, b_1, \dots, b_{c_i-1}]$: bit list defined recursively as follows:

$$b_0 = \text{par}_{in}(h_0) \ \& \ \forall k \in \{1, \dots, c_i - 1\} : b_k = (b_{k-1} + j_k - j_{k-1} + 1) \bmod 2.$$

Corollary 4.1. Let $n \geq 2$ be an integer and $i \in \{0, \dots, n - 1\}$, and let h be a Hamiltonian cycle on Q_n . Then the i -th parity list of h , $\beta_i(h)$, is a balanced list, i.e. the number of 0 and 1 values coincide.

Proof: A simple induction proves that $\beta_i(h) = [\text{par}_{in}(h_{j_k})]_{k=0}^{c_i-1}$. The conclusion follows from the proof of Theorem 4.1. ■

Corollary 4.2. Let $n \geq 2$ be an integer and $i \in \{0, \dots, n - 1\}$. The addition of the even numbered entries at the chromatic-segments list $\delta_i(h)$ coincides with the addition of its odd numbered entries and both are equal to 2^{n-1} :

$$\sum_{k_1=0}^{\frac{c_i}{2}-1} (\delta_i(h))_{2k_1} = 2^{n-1} = \sum_{k_1=0}^{\frac{c_i}{2}-1} (\delta_i(h))_{2k_1+1} \quad (8)$$

Proof: For any $k \in \{0, \dots, c_i - 1\}$ the path H_k in the proof of Theorem 4.1 has $\delta_k(h)$ vertexes. The first (resp. last) expression at relation (8) corresponds to the sum of the lengths of paths h_k with even (resp. odd) index and the result follows from relations (4) and (5). ■

5 Inscribed squares in Hamiltonian cycles

We introduce here an application of Theorem 4.1. We will establish sufficient conditions over Hamiltonian cycles, in terms of their chromatic vectors, to guarantee that there are 4-cycles within the Hamiltonian cycles.

Let $n \geq 2$ be an integer and let $h = h_0 \dots h_{2^n-1}$ be a Hamiltonian cycle of Q_n . Let $v_0 v_1 v_2 v_3$ be a 4-cycle in the hypercube Q_n . Let us introduce the following definitions:

1. $v_0 v_1 v_2 v_3$ is a *straight square* within h if there exist indexes $i, j \in \{0, \dots, 2^n - 1\}$ such that

$$h_i h_{i+1} h_j h_{j+1} = v_0 v_1 v_2 v_3.$$

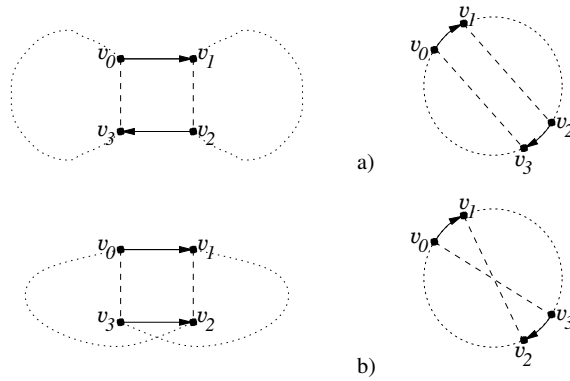


Figure 2: Hamiltonian cycles possessing squares. (a) Straight squares. (b) Twisted squares.

2. $v_0v_1v_2v_3$ is a *twisted square* in h if there exist indexes $i, j \in \{0, \dots, 2^n - 1\}$ such that

$$h_i h_{i+1} h_j h_{j+1} = v_0 v_1 v_3 v_2.$$

3. We will say that the edges v_0v_1 and v_2v_3 are the *rims* and the edges v_1v_2 and v_3v_0 are the *rays*.

In figure 2 we illustrate those notions, the rays are displayed as dashed lines, the rims as continuous lines since they actually “are part of” the Hamiltonian cycle. We will say that the Hamiltonian cycle h contains a square, or that a square is *inscribed* in h , if a straight or twisted square is contained in h . A *square free Hamiltonian cycle* contains no squares.

Conjecture 5.1 (Inscribed squares in Hamiltonian cycles.). *Let $n \geq 2$ be an integer. No Hamiltonian cycle of Q_n is square free, or equivalently any Hamiltonian cycle has inscribed a square.*

Since any 4-cycle in Q_n contains two alternating dimensions, the rims (or rays) in each square inscribed in a Hamiltonian cycle of Q_n belong to the same dimension.

Now let us establish some conditions in order to have edges of the same dimension in a square contained within a Hamiltonian cycle of Q_n .

From the construction of graph \mathcal{D}_{in} , specially the edge definition at relation (1), and the Theorem 3.1, if the chromatic vector of some Hamiltonian cycle h of Q_n contains entries greater than the independence number of Q_{n-1} then h must contain a square. Since $\alpha(Q_{n-1}) = 2^{n-2}$ the following theorem results.

Theorem 5.1. *Let $n \geq 2$ be an integer, and let h be a Hamiltonian cycle of Q_n . If some entry i in the chromatic vector $c(h)$ is greater than 2^{n-2} then h contains a square whose rims are i -th dimension edges.*

Theorem 5.1 is a general result aiming to prove Conjecture 5.1 for Hamiltonian cycles whose chromatic vectors have entries bigger than 2^{n-2} . Let us reduce further this lower bound.

Let G be a bipartite graph. The *equi-independence* number of G , denoted $\alpha_{=}(G)$, is the cardinality of the maximum independent set in G having half of its vertexes in one bipartition class of G and half in the other.

According to the proofs of Theorem 5.1 and Theorem 4.1 we have:

Corollary 5.1. *Let $n \geq 2$ be an integer, and let h be a Hamiltonian cycle of Q_n . If some entry c_i in $c(h)$ is greater than $\alpha_{=}(Q_{n-1})$ then h contains a square whose rims are i -th dimension edges.*

Let us estimate a lower bound for the equi-independence number of the n -dimensional hypercube.

Theorem 5.2. *Let $n \geq 3$ be an integer. Then $\alpha_{=}(Q_n) \geq 2^{n-2}$.*

Proof: For any two bits $b_0, b_1 \in \{0, 1\}$ let $Q_{nb_0b_1}$ be the subgraph of Q_n induced by the vertexes in $V(Q_n)$ having values b_0 and b_1 at their first two coordinates. The resulting four graphs are isomorphic to Q_{n-2} .

Let $b \in \{0, 1\}$, and let $I_b = V(Q_{nbb}) \cap (b\text{-bipartition class of } Q_n)$. No vertex in I_0 is adjacent to a vertex of I_1 and in fact $I = I_0 \cup I_1$ is an independent set of Q_n . Moreover, each vertex in $V(Q_{nbb})$ not contained in I is adjacent to one vertex in Q_{nbb} , and each vertex in Q_{n01} (or in Q_{n10}) is adjacent to a vertex in I_0 (or a vertex in I_1 .)

In order to illustrate the last claim let $v = (0, 1, b_2, \dots, b_{n-1})$ be an arbitrary vertex in $Q_{n,0,1}$, if $\text{par}_n(b_2, \dots, b_{n-1}) = 0$ then v is adjacent to $(0, 0, b_2, \dots, b_{n-1})$ which is in I_0 , otherwise v is adjacent to $(1, 1, b_2, \dots, b_{n-1})$ which is in I_1 .

The final conclusion is that I is a maximal independent set in Q_n . Since $\text{card}(I_0) = \text{card}(I_1) = 2^{n-3}$ and

$$\text{card}(I) = \text{card}(I_0) + \text{card}(I_1) = 2^{n-3} + 2^{n-3} = 2^{n-2}$$

the Theorem follows. \blacksquare

Now let us check that finding the equi-independence number of a bipartite graph G is not more difficult than evaluating the independence number of a graph G' derived from G .

Theorem 5.3. *The problem of finding the equi-independence number of a bipartite graph is polynomially-time reducible to the problem of finding the independence number of a graph.*

Proof: Let G be a bipartite graph, and let V_0, V_1 be the bipartition classes of G . The graph $G' = (V', E')$, with set of vertexes $V' = \{(v_0, v_1) \in V_0 \times V_1 | (v_0, v_1) \notin E(G)\}$ and set of edges

$$E' = \{((v_0, v_1), (v'_0, v'_1)) \in V(G') \times V(G') | (v_0, v'_1) \in E(G) \text{ or } (v_1, v'_0) \in E(G) \text{ or } v_0 = v'_0 \text{ or } v_1 = v'_1\},$$

has an independent set I' if and only if the set $I = \{v \in V(G) | v \text{ is an element in some vertex of } I'\}$ is an equi-independent set of G . In this construction $\text{card}(I) = 2 \text{card}(I')$. Since G' can be built in polynomial time, the proposition follows. \blacksquare

Theorem 5.3 was used to compute the equi-independence number of the hypercubes of dimension three to seven. The results are summarized in Table 1. Vertexes in the maximal equi-independence sets in Table 1 are coded in binary. The equi-independence numbers for n equals to four and six reach the lower bound in Theorem 5.2 and so this bound is tight. The graph Q'_n grows very fast as n is increased, thus for $n = 8$ the corresponding value is not included.

Table 1: Equi-independence number of Q_n for small values of n .

n	$\alpha_=(Q_n)$	A maximal equi-independent set	$ V(Q'_n) $	$ E(Q'_n) $
3	2	{0, 7}	4	6
4	4	{0, 7, 9, 14}	32	448
5	10	{0, 7, 9, 19, 10, 21, 12, 22, 24, 31}	176	9720
6	16	{0, 7, 9, 19, 33, 21, 10, 22, 34, 28, 36, 56, 43, 31, 45, 55}	882	137536
7	40	{0, 7, 9, 19, 33, 67, 10, 21, 34, 69, 12, 81, 36, 22, 24, 70, 40, 82, 72, 84, 48, 31, 96, 47, 57, 79, 105, 55, 58, 87, 106, 103, 60, 91, 108, 115, 120, 93, 117, 126}	3648	1577184

The study on the inscribed square conjecture in the Hamiltonian cycles of the hypercube concludes as follows.

Theorem 5.4. *Conjecture 5.1 holds for $2 \leq n \leq 7$.*

Proof: From Table 1 we know that $\alpha_=(Q_6) = 16$. It is impossible that the chromatic vector of a Hamiltonian cycle of Q_7 have all its entries lower or equal that 16; otherwise the sum of the seven entries would be at most $16 \cdot 7 = 112$, but we know from Lemma 3.1 that this sum should be 128. Thus some entry should be greater than 16. It follows from Corollary 5.1 that Conjecture 5.1 is true for $n = 7$. The proof is analogous for values of n lower than 7. \blacksquare

Computer experimentation suggests that the inscribed square conjecture in Hamiltonian cycles is true for any value of n .

6 Conclusions

Theorem 4.1 entails several applications, e.g., in [6] it is posed the following question: The partial matchings in Q_n can be extended to Hamiltonian cycles? We are able to submit a partial answer to this question: if the partial matching contains edges in the same dimension that violate the equilibrium condition of the Theorem 4.1, and it is not allowed to incorporate more edges in the same dimension, then no such extension exists. Other application of the Theorem 4.1 consists in a pruning strategy to generate exhaustively all the Hamiltonian cycles of Q_n .

On the other hand several problems and conjectures have remained open, and their study could reveal important structural properties of Q_n . In the following presentation we will refer to the operators introduced at the end of section 3.

As conjectures, besides the already stated Conjecture 5.1, let us state the following:

- Let $n \geq 2$ and let k be an even integer with $2 \leq k \leq 2^{n-1}$. A vector $v \in Q_k$ is the parity vector of a Hamiltonian cycle of Q_n , i.e. $v = \beta_i(h)$ for some i and Hamiltonian cycle h , if and only if it contains the same number of 0's and 1's.

In the Theorem 4.1, the “only if” part of this conjecture was proved.

- The lower bound in Theorem 5.2 is reached for all even values of n .

As open problems, let us state the following:

- Characterize the chromatic vectors of Hamiltonian cycles of the hypercube. Or, in other words, characterize the image of the map $c : \{\text{Hamiltonian cycles}\} \rightarrow \mathbb{N}^n$.

Lemma 3.1 states some conditions that are just necessary.

- Characterize the chromatic-segments vectors of Hamiltonian cycles of Q_n . Or, in other words, characterize, for each i , the image of the map $\delta_i : \{\text{Hamiltonian cycles}\} \rightarrow \mathbb{N}^n$.

A positive answer to the first of the above stated conjectures will give immediately necessary conditions for a vector to be a chromatic-segments vector of a Hamiltonian cycle of the hypercube. But it would be far from a full characterization.

- Calculate $\alpha_=(Q_n)$.

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