

CYCLIC POLYGONS IN CLASSICAL GEOMETRY

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ABSTRACT. Formulas about the side lengths, diagonal lengths or radius of the circumcircle of a cyclic polygon in Euclidean geometry, hyperbolic geometry or spherical geometry can be unified.

1. INTRODUCTION

In Euclidean geometry, hyperbolic geometry or spherical geometry, a cyclic polygon is a polygon whose vertexes are on a same circle. In Euclidean geometry, the side lengths and diagonal lengths of a cyclic polygon satisfy some polynomials. Ptolemy's theorem about a cyclic quadrilateral and Fuhrmann's theorem about a cyclic hexagon are examples. The two theorems also hold in hyperbolic geometry, for example, see [S]. It is also observed in [S] that the formulas for hyperbolic geometry are easily obtained by replacing an edge length $l/2$ in Euclidean geometry by $\sinh l/2$. In the paper we will show this is a general principle to translate a result in Euclidean geometry to a result in hyperbolic geometry. Also we get formulas in spherical geometry by replacing an edge length $l/2$ in Euclidean geometry by $\sin l/2$.

In Euclidean geometry, the radius of the circumcircle of a cyclic polygon can be calculated from the side lengths. For more information about the radius of the circumcircle of a cyclic polygon, see, for example, [MRR, FP, P, V]. As a corollary of our main result, the formulas of radius of the circumcircle of a cyclic polygon in Euclidean geometry, hyperbolic geometry or spherical geometry can be unified.

In this paper, we do not give any new theorems about a cyclic polygon in Euclidean geometry. But we show that once there is a formula about sides lengths, diagonal lengths and radius of circumcircle of a cyclic polygon in Euclidean geometry, there is an essentially same formula in hyperbolic geometry or spherical geometry. In other words, we provide a machinery to generate theorems in hyperbolic geometry or spherical geometry.

For recent development of the study of cyclic polygons in Euclidean geometry, see, for example, [P1, P2].

In section 2, 3 and 4, the general principle are illustrated by examples about triangles, cyclic quadrilaterals and cyclic hexagons. In section 5, the main result, Theorem 1, is stated formally. Section 6 establishes a lemma and section 7 proves Theorem 1.

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2. TRIANGLE

For a triangle on the Euclidean plane, assume that it has the side lengths a, b, c and the radius of circumcircle r . Then

$$r^2 = \frac{(abc)^2}{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}.$$

For a triangle on the hyperbolic plane, assume that it has the side lengths a, b, c and the radius of circumcircle r . Then

$$\frac{1}{4} \sinh^2 r = \frac{(\sinh \frac{a}{2} \sinh \frac{b}{2} \sinh \frac{c}{2})^2}{A_h(A_h - 2 \sinh \frac{a}{2})(A_h - 2 \sinh \frac{b}{2})(A_h - 2 \sinh \frac{c}{2})},$$

where $A_h = \sinh \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2}$.

For a triangle on the unit sphere, assume that it has the side lengths a, b, c and the radius of circumcircle r . Then

$$\frac{1}{4} \sin^2 r = \frac{(\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2})^2}{A_s(A_s - 2 \sin \frac{a}{2})(A_s - 2 \sin \frac{b}{2})(A_s - 2 \sin \frac{c}{2})},$$

where $A_s = \sin \frac{a}{2} + \sin \frac{b}{2} + \sin \frac{c}{2}$. The three formulas are essentially the same. They can be unified by introduce the following function:

$$s(x) = \begin{cases} \frac{x}{2} & \text{in Euclidean geometry} \\ \sinh \frac{x}{2} & \text{in hyperbolic geometry} \\ \sin \frac{x}{2} & \text{in spherical geometry} \end{cases}$$

The unified formula is

$$\frac{1}{4} s(2r)^2 = \frac{(s(a)s(b)s(c))^2}{A_3(A_3 - 2s(a))(A_3 - 2s(b))(A_3 - 2s(c))},$$

where $A_3 = s(a) + s(b) + s(c)$.

3. CYCLIC QUADRILATERAL

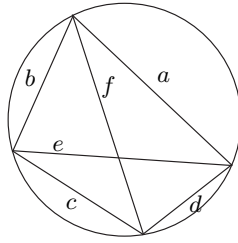


FIGURE 1. Ptolemy's theorem

We can observe the similar phenomenon in the case of a quadrilateral inscribed in a circle which is called a cyclic quadrilateral. That means the formula about the side lengths, diagonal lengths and radius of circumcircle of a cyclic quadrilateral can be unified in Euclidean geometry, hyperbolic geometry and spherical geometry. Assume the side lengths are a, b, c, d , the diagonal lengths are e, f as labeled in Figure 1 and the radius of the circumcircle is r . Then the Ptolemy's theorem says

$$s(e)s(f) = s(a)s(c) + s(b)s(d).$$

For more information about generalization of Ptolemy's theorem in hyperbolic and spherical geometry, see [V1, V2].

The formulas representing diagonal lengths in terms of side lengths are

$$s(e)^2 = (s(a)s(c) + s(b)s(d)) \frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)},$$

$$s(f)^2 = (s(a)s(c) + s(b)s(d)) \frac{s(a)s(b) + s(c)s(d)}{s(a)s(d) + s(b)s(c)}.$$

Ptolemy's theorem is a corollary of the two formulas.

The formula involving the radius is

$$\frac{1}{4}s(2r)^2 = \frac{(s(a)s(b) + s(c)s(d))(s(a)s(c) + s(b)s(d))(s(a)s(d) + s(b)s(c))}{(A_4 - 2s(a))(A_4 - 2s(b))(A_4 - 2s(c))(A_4 - 2s(d))}$$

where $A_4 = s(a) + s(b) + s(c) + s(d)$.

4. CYCLIC HEXAGON

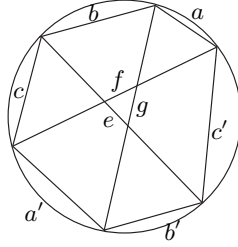


FIGURE 2. Fuhrmann's theorem

One more example is Fuhrmann's theorem about a cyclic hexagon in Euclidean geometry. This theorem also holds in hyperbolic and spherical geometry. And the formula can be unified in the three cases. Assume that a convex cyclic hexagon in Euclidean geometry, hyperbolic geometry and spherical geometry have the side lengths a, a', b, b', c , and c' , and diagonal lengths e, f , and g as labeled in Figure 2, then

$$s(e)s(f)s(g) = s(a)s(a')s(e) + s(b)s(b')s(f) + s(c)s(c')s(g) + s(a)s(b)s(c) + s(a')s(b')s(c').$$

The generalization of Fuhrmann's theorem into hyperbolic geometry is treated in Wilson Stothers' web page about hyperbolic geometry [S].

5. CYCLIC POLYGON

We can expect that the similar phenomenon holds for general polygon inscribed in a circle which is called the cyclic polygon. In this paper we show that the same relationships about the side lengths, diagonal lengths and the radius of the circumcircle hold in Euclidean geometry, hyperbolic geometry and spherical geometry. To make the statement formal, we introduce the following notations.

Fix an integer $n \geq 3$.

The set \mathcal{E}_n of polynomials is defined as follows. A polynomial f of $\frac{n(n+1)}{2}$ variables is in the set \mathcal{E}_n if $f(|P_i P_j|_e, r_e) = 0$ for any n points P_1, P_2, \dots, P_n on a circle of radius r_e on the Euclidean plane, where $|P_i P_j|_e$ denotes the Euclidean distance between the two points P_i and P_j .

The set \mathcal{H}_n of polynomials is defined as follows. A polynomial f of $\frac{n(n+1)}{2}$ variables is in the set \mathcal{H}_n if $f(\sinh \frac{|P_i P_j|_h}{2}, \frac{1}{2} \sinh r_h) = 0$ for any n points P_1, P_2, \dots, P_n on a circle of radius r_h on the hyperbolic plane, where $|P_i P_j|_h$ denotes the hyperbolic distance between the two points P_i and P_j .

The set \mathcal{S}_n of polynomials is defined as follows. A polynomial f of $\frac{n(n+1)}{2}$ variables is in the set \mathcal{S}_n if $f(\sin \frac{|P_i P_j|_s}{2}, \frac{1}{2} \sin r_s) = 0$ for any n points P_1, P_2, \dots, P_n on a circle of radius r_s on the unit sphere, where $|P_i P_j|_s$ denotes the spherical distance between the two points P_i and P_j .

Theorem 1. $\mathcal{E}_n = \mathcal{H}_n = \mathcal{S}_n$.

R. J. Gregorac [G] generalized Ptolemy's theorem to a convex cyclic polygon on the Euclidean plane and generalized Fuhrmann's theorem to a convex cyclic $2n$ -gon on the Euclidean plane. By Theorem 1, Gregorac's two results can be easily generalized into hyperbolic and spherical geometry and the formulas can be unified. For example, we consider the later one.

Corollary 2. *Let $n > 3$. Let $\{V_0, V_1, \dots, V_{2n-1}\}$ be points on a circle in Euclidean geometry, hyperbolic geometry or spherical geometry. Let l_{ij} denote the length of the geodesic segment $V_i V_j$ for $i \neq j$. Then*

$$\det(a_{ij}) = 0$$

where

$$a_{ji} = (-1)^{\delta_{ij}} (s(l_{2i-2, 2j-1}) s(l_{2j-1, 2i}))^{-1}$$

and δ_{ij} is the Kronecker delta.

A cyclic polygon in Euclidean, hyperbolic or spherical geometry is uniquely determined by its side lengths. Therefore any diagonal length or radius of the circumcircle is a function of side lengths. In the case of Euclidean geometry, it is not difficult to see that these functions are algebraic functions. As a corollary of Theorem 1, these functions are unified in Euclidean, hyperbolic or spherical geometry.

Corollary 3. *For a Euclidean, hyperbolic or spherical cyclic polygon with vertices P_1, P_2, \dots, P_n and the side lengths $|P_i P_{i+1}| = l_{i, i+1}$ where $n+1 = 1$, the length l_{ij} of the diagonal $P_i P_j$ ($|j-i| \geq 2$) is*

$$s(l_{ij}) = F_{ij}(s(l_{12}), s(l_{23}), \dots, s(l_{n1}))$$

for some algebraic function F_{ij} which is independent of the three kinds of geometry. The radius r of the circumcircle circle satisfies

$$s(2r) = G(s(l_{12}), s(l_{23}), \dots, s(l_{n1}))$$

for some algebraic function G which is independent of the three kinds of geometry.

6. HOMOGENEITY

To prove Theorem 1, we need the following lemma.

Lemma 4. *If $f \in \mathcal{E}_n \cup \mathcal{H}_n \cup \mathcal{S}_n$, then f is homogeneous.*

Proof. Assume $f \in \mathcal{E}_n$. Then $f(|P_i P_j|_e, r_e) = 0$ for any n points P_1, P_2, \dots, P_n on a circle of any radius r_e on the Euclidean plane. We can assume the center of the circle is the origin. For any $x > 0$, by multiplying the coordinates of P_1, P_2, \dots, P_n by x , we obtain new points P'_1, P'_2, \dots, P'_n with is on the circle of radius xr_e centered at

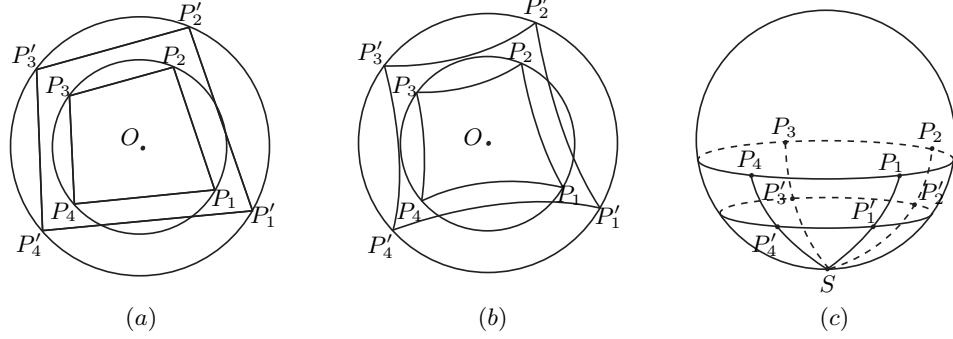


FIGURE 3.

the origin, see Figure 3(a). The distance $|P'_i P'_j|_e = x|P_i P_j|_e$ and the radius xr_e also satisfy the polynomial f , i.e., $f(x|P_i P_j|_e, xr_e) = 0$. Therefore f is homogeneous.

Assume $f \in \mathcal{H}_n$. Then $f(\sinh \frac{|P_i P_j|_h}{2}, \frac{1}{2} \sinh r_h) = 0$ for any points P_1, P_2, \dots, P_n on a circle C of radius r_h on the hyperbolic plane. We use the Poincaré disk model of the hyperbolic plane: $\{z \in \mathbb{C} : |z| < 1\}$. Assume the center of the circle is the origin O . For any $x > 0$, there is another circle C' centered at O with radius r'_h satisfying $\sinh r'_h = x \sinh r_h$. For each i , let P'_i be the intersection point of C' and the geodesic ray starting from O and passing through P_i as in Figure 3(b). By the law of sine of a hyperbolic triangle, we have

$$\frac{\sinh \frac{|P_i P_j|_h}{2}}{\sinh r_h} = \sin \frac{\angle P_i O P_j}{2} = \frac{\sinh \frac{|P'_i P'_j|_h}{2}}{\sinh r'_h} = \frac{\sinh \frac{|P'_i P'_j|_h}{2}}{x \sinh r_h}$$

where $\angle P_i O P_j \in (0, \pi]$. Therefore $\sinh \frac{|P'_i P'_j|_h}{2} = x \sinh \frac{|P_i P_j|_h}{2}$.

Since $f \in \mathcal{H}_n$, the equation $f(\sinh \frac{|P'_i P'_j|_h}{2}, \frac{1}{2} \sinh r'_h) = 0$ holds. Thus

$$f(x \sinh \frac{|P_i P_j|_h}{2}, x \frac{1}{2} \sinh r_h) = 0$$

holds for any $x > 0$. Therefore f is homogeneous.

Assume $f \in \mathcal{S}_n$. Then $f(\sin \frac{|P_i P_j|_s}{2}, \frac{1}{2} \sin r_s) = 0$ for any points P_1, P_2, \dots, P_n on a circle C of radius r_s on the unit sphere. Assume the center of the circle is the south pole S . For any $x \in (0, 1)$, there is another circle C' centered at south pole with radius r'_s satisfying $\sin r'_s = x \sin r_s$. For each i , let P'_i be the intersection point of C' and the geodesic ray starting from O and passing through P_i as in Figure 3(c). By the law of sine of a spherical triangle, we have

$$\frac{\sin \frac{|P_i P_j|_s}{2}}{\sin r_s} = \sin \frac{\angle P_i O P_j}{2} = \frac{\sin \frac{|P'_i P'_j|_s}{2}}{\sin r'_s} = \frac{\sin \frac{|P'_i P'_j|_s}{2}}{x \sin r_s}$$

where $\angle P_i O P_j \in (0, \pi]$. Therefore $\sin \frac{|P'_i P'_j|_s}{2} = x \sin \frac{|P_i P_j|_s}{2}$.

Since $f \in \mathcal{S}_n$, the equation $f(\sin \frac{|P'_i P'_j|_s}{2}, \frac{1}{2} \sin r'_s) = 0$ holds. Thus

$$f(x \sin \frac{|P_i P_j|_s}{2}, x \frac{1}{2} \sin r_s) = 0$$

holds for any $x \in (0, 1)$. Therefore f is homogeneous. \square

7. PROOF OF THEOREM 1

7.1. $\mathcal{E}_n \subseteq \mathcal{H}_n$. Let f be a polynomial in \mathcal{E}_n .

Consider n points P_1, P_2, \dots, P_n on a circle of radius r_h on the hyperbolic plane. We use the Poincaré disk model of the hyperbolic plane. Assume that the center of the circle is the origin O . In the hyperbolic triangle $\triangle_h P_i O P_j$ which can be degenerated into a line segment, by the law of sine, we have

$$\sin \frac{\angle P_i O P_j}{2} = \frac{\sinh \frac{|P_i P_j|_h}{2}}{\sinh r_h}$$

where $\angle P_i O P_j \in (0, \pi]$.

On the other hand, this circle is also a Euclidean circle. We consider the Euclidean triangle $\triangle_e P_i O P_j$, by the law of sine, we have

$$\sin \frac{\angle P_i O P_j}{2} = \frac{|P_i P_j|_e}{2r_e}.$$

Hence

$$\sinh \frac{|P_i P_j|_h}{2} = \frac{\sinh r_h}{2r_e} \cdot |P_i P_j|_e.$$

Obviously,

$$\frac{1}{2} \sinh r_h = \frac{\sinh r_h}{2r_e} \cdot r_e.$$

Since $f \in \mathcal{E}_n$, $f(|P_i P_j|_e, r_e) = 0$. By Lemma 4,

$$f\left(\sinh \frac{|P_i P_j|_h}{2}, \frac{1}{2} \sinh r_h\right) = f(x|P_i P_j|_e, x r_e) = 0,$$

where $x = \frac{\sinh r_h}{2r_e}$. Hence $f \in \mathcal{H}_n$.

7.2. $\mathcal{H}_n \supseteq \mathcal{E}_n$. Let f be a polynomial in \mathcal{H}_n .

For any points P_1, P_2, \dots, P_n on a circle of radius r_e centered at the origin on the Euclidean plane. For a number $y \in (0, \frac{1}{r_e})$, consider the circle C' centered at the origin and with radius $y r_e$. For any i , let P'_i be the the intersection point of C' and the ray starting from O and passing through P_i . For any i, j , we consider the Euclidean triangle $\triangle_e P_i O P_j$ which can be degenerated. On the other hand, Since all the points P'_1, P'_2, \dots, P'_n are in the unit disk which is considered as the hyperbolic plane, for any i, j , there is also a hyperbolic triangle $\triangle_h P_i O P_j$. By the law of sine, we have

$$\frac{\sinh \frac{|P'_i P'_j|_h}{2}}{\sinh |O P'_i|_h} = \sin \frac{\angle P_i O P_j}{2} = \frac{|P'_i P'_j|_e}{2y r_e} = \frac{|P_i P_j|_e}{2r_e}.$$

By Lemma 4, $f(x \sinh \frac{|P'_i P'_j|_h}{2}, x \frac{1}{2} \sinh |O P'_i|_h) = 0$ for any $x > 0$. By taking $x = \frac{2r_e}{\sinh |O P'_i|_h}$, we have $f(|P_i P_j|_e, r_e) = 0$. Thus $f \in \mathcal{E}_n$.

To sum up, we have proved that $\mathcal{E}_n = \mathcal{H}_n$.

7.3. $\mathcal{E}_n \subseteq \mathcal{S}_n$. Let f be a polynomial in \mathcal{E}_n .

Consider n points P_1, P_2, \dots, P_n on a circle of radius r_s on the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Assume that the center of the circle is the south pole S . In the spherical triangle $\triangle_s P_i S P_j$ which can be degenerated, by the law of sine, we have

$$\sin \frac{\angle P_i S P_j}{2} = \frac{\sin \frac{|P_i P_j|_s}{2}}{\sin r_s}.$$

For any i , let P'_i be the image of P_i under the stereographic projection from the north pole, i.e., the intersection point of the plane $z = 0$ and the line passing through the north pole and P_i . Then P'_1, P'_2, \dots, P'_n are points on a circle of radius r_e centered at the origin O on the Euclidean plane $z = 0$. In the Euclidean triangle $\Delta_e P_i O P_j$, by the law of sine, we have

$$\sin \frac{\angle P'_i O P'_j}{2} = \frac{|P_i P_j|_e}{2r_e}.$$

Since $\angle P_i S P_j = \angle P'_i O P'_j$, we have

$$\sin \frac{|P_i P_j|_s}{2} = \frac{\sin r_s}{2r_e} \cdot |P'_i P'_j|_e.$$

Obviously,

$$\frac{1}{2} \sin r_s = \frac{\sin r_s}{2r_e} \cdot r_e.$$

Since $f \in \mathcal{E}_n$, $f(|P'_i P'_j|_e, r_e) = 0$. By Lemma 4,

$$f\left(\sin \frac{|P_i P_j|_s}{2}, \frac{1}{2} \sin r_s\right) = f(x |P'_i P'_j|_e, x r_e) = 0,$$

where $x = \frac{\sin r_s}{2r_e}$. Hence $f \in \mathcal{S}_n$.

7.4. $\mathcal{S}_n \subseteq \mathcal{E}_n$. Let g be a polynomial in \mathcal{S}_n .

For any points B_1, B_2, \dots, B_n on a circle of radius r_e centered at the origin O on the Euclidean plane. Consider the stereographic projection from the north pole of the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. The image of B_1, B_2, \dots, B_n under the stereographic projection are points B'_1, B'_2, \dots, B'_n on a circle of radius r_s centered at the south pole S on the unit sphere. By the law of sine, we have

$$\sin \frac{\angle B_i O B_j}{2} = \frac{|B_i B_j|_e}{2r_e},$$

$$\sin \frac{\angle B'_i S B'_j}{2} = \frac{\sin \frac{|B'_i B'_j|_s}{2}}{\sin r_s}.$$

Since $\angle B_i O B_j = \angle B'_i S B'_j$, we have

$$|B_i B_j|_e = \frac{2r_e}{\sin r_s} \cdot \sin \frac{|B'_i B'_j|_s}{2}.$$

Obviously,

$$r_e = \frac{2r_e}{\sin r_s} \cdot \frac{1}{2} \sin r_s.$$

Since $g \in \mathcal{S}_n$, $g\left(\sin \frac{|B'_i B'_j|_s}{2}, \frac{1}{2} \sin r_s\right) = 0$. By Lemma 4,

$$g\left(x \sin \frac{|B'_i B'_j|_s}{2}, x \frac{1}{2} \sin r_s\right) = g(|B_i B_j|_e, r_e) = 0,$$

where $x = \frac{2r_e}{\sin r_s}$. Hence $g \in \mathcal{E}_n$.

To sum up, we have proved that $\mathcal{E}_n = \mathcal{S}_n$.

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