Exploring Mount Neverest

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September 16, 2010

In one of the columns in the series 'Perplexities' in 1922, Henry Ernest Dudeney formulated the following problem:

> Professor Walkingholme, one of the exploring party, was allotted the special task of making a complete circuit of the base of the mountain at a certain level. The circuit was exactly a hundred miles in length and he had to do it all alone on foot. He could walk twenty miles a day, but he could only carry rations for two days at a time, the rations for each day being packed in sealed boxes for convenience in dumping. He walked his full twenty miles every day and consumed one day's ration as he walked. What is the shortest time in which he could complete the circuit?

This problem can be found in the book '536 Puzzles & Curious Problems' from Henry Ernest Dudeney, edited by Martin Gardner. This far, I did not find an optimal solution to the problem. Albeit Martin Gardner is making fun on it, the right interpretation of the problem is not clear to me at all. Let us first formulate some ways to tackle the problem.

Solution of the problem

One way to make the circuit is doing the same as with a straight distance of one hundred miles. No matter how you interpret this problem, this takes between 82 and 87 days (Dudeney found a solution of 86 days, but it can be done in $82\frac{6097}{6144}$ days). A better approach is to walk two round trips to the 50 miles distance point on the other side of the mountain, since it requires between 41 and 43 days (Dudeney would have found a solution of $42\frac{1}{2}$ days if he had considered this problem). But Dudeney found the following solution:

Algorithm 1. ($23\frac{1}{2}$ days)

1. Dump 5 rations at 90-mile point and return to base (5 days).
2. Dump 1 at 85 and return to 90 (1 day).
3. Dump 1 at 80 and return to 90 (1 day).

4. Dump 1 at 80, return to 85, pick up 1 and dump at 80 (1 day).
5. Dump 1 at 70 and return to 80 (1 day).
6. Return to base (1 day). We have thus left one ration at 70 and one at 90.
7. Dump 1 at 5 and return to base (1 day). If he must walk 20 miles he can do so by going to 10 and returning to base.
8. Dump 4 at 10 and return to base (4 days).
9. Dump 1 at 10 and return to 5; pick up 1 and dump at 10 (1 day).
10. Dump 2 at 20 and return to 10 (2 days).
11. Dump 1 at 25 and return to 20 (1 day).
12. Dump 1 at 30, return to 25, pick up 1 and dump at 30 (1 day).
13. March to 70 (2 days).
14. March to base ($1\frac{1}{2}$ days).

Now let us look at step 7. In my opinion, it only takes half a day. You just throw away half of the content of one of the boxes. Apparently, Dudeney and I have a disagreement on the interpretation of the problem. The problem might be that unsealed boxes can not pass through the night, since the forest ants creep into it. But then, you just can start walking around the mountain in the middle of the day and do step 7 prior to the others. If Dudeney would have allowed not to start at dawn, it is likely that he would have formulated his solution accordingly, so we can add the following items to the analysis of Gardner:

13. If he does not finish his unsealed box(es) before the night, the forest ants will.
14. Having a long lie-in is a waste of time.

We can divide Dudeney's solution in three parts:

Part A: a round trip to the 70 miles point from the base, in which boxes are dumped on positions 70 and 90 for later use (steps 1 to 6, 10 days).

Part B: a one-way trip to the 70 miles point from the base around the mountain (steps 7 to 13, $11\frac{1}{2}$ or 12 days).

Part C: walking from 70 to 100, using the boxes dumped in part A (step 14, $1\frac{1}{2}$ days).

Since it is almost proved that Dudeneys solution can not be optimal if we count 12 days for part B, we allow starting on mid-day. Only for part C, it is immediately clear that is optimal. Later, we will see that part A is optimal as well, but part B can be improved, to $11\frac{1}{7}$ days exactly.

Before trying to find a better solution, it is always a good idea what others did. When I searched the internet with google, I found the homepage of a youngster called Nightvid Cole, who presents better solutions than that of Dudeney. In the first one, he allows throwing away partially used boxes (and this solution can easily be adapted to overcome the ants, starting on mid-day). Albeit only part C of this solution is optimal, it requires $22\frac{10}{11}$ days instead of 23. The improvement is that he dumps boxes on positions $70\frac{10}{11}$ and $90\frac{10}{11}$ rather than 70 and 90. Now part B must reach further, whence it takes 12 days now.

But probably, he reasoned the other way around: he reserved 12 days for part B and then thought out an according solution. That was a very good idea, but since he failed to optimize parts A and B, his solution is not optimal. So I optimized parts A and B, which resulted in the following solution, a solution that turned out to be optimal later on:

Algorithm 2. ($22\frac{9}{16}$ days)

1. Dump one ration at $98\frac{3}{4}$ point and return to base ($\frac{1}{8}$ day).
2. Dump one ration at $97\frac{1}{2}$, return to $98\frac{3}{4}$, pick up one, dump at $91\frac{1}{4}$ and return to base (1 day).
3. Dump one ration at $93\frac{3}{4}$, return to $97\frac{1}{2}$, pick up one, dump at $93\frac{3}{4}$ also and return to base (1 day).
4. Dump two rations at 90 and return to base (2 days).
5. Dump one ration at $86\frac{7}{8}$ and return to $93\frac{3}{4}$ (1 day).
6. Dump one ration at $82\frac{1}{2}$, return to $86\frac{7}{8}$, pick up one, dump at $86\frac{1}{4}$ and return to 90 (1 day).
7. Dump one ration at 80, return to $86\frac{1}{4}$, pick up one and get to $82\frac{1}{2}$ (1 day).
8. Dump one ration at $71\frac{1}{4}$ and return to 80 (1 day).
9. Return to base (1 day). We have thus left one ration at $71\frac{1}{4}$ and one at $91\frac{1}{4}$.
10. Dump five rations at 10 and return to base (5 days).
11. Dump one ration at $12\frac{1}{2}$, return to 10, pick up one, dump at $12\frac{1}{2}$ also and return to 10 (1 day).
12. Dump one ration at 20 and return to 10 (1 day).
13. Dump one ration at $20\frac{5}{8}$, return to 20, pick up one, dump at $20\frac{5}{8}$ also and return to $12\frac{1}{2}$ (1 day).

14. Dump one ration at $26\frac{9}{16}$ and return to $20\frac{5}{8}$ (1 day).
15. Dump one ration at $31\frac{1}{4}$, return to $26\frac{9}{16}$, pick up one and get to $31\frac{1}{4}$ (1 day).
16. March to $71\frac{1}{4}$ (2 days).
17. March to base ($1\frac{7}{16}$ day).

If you look at the above algorithm, then one thing immediately strikes: it would have been nicer if the circuit would have been 160 kilometers, with a unit distance of 32 kilometers a day. Dudeney only considered solutions from which the eating and turning points were a multiple of 5 miles. This is however impossible for a solution of $22\frac{9}{16}$ days (for $\frac{20}{5} \cdot 22\frac{9}{16}$ is not integral).

If professor Walkingholme must start at dawn, then the extra $\frac{7}{8}$ days must be used to increase the points where boxes are dumped in part A for use in part C. Nightvid Cole found a solution of $23\frac{1}{3}$ days in this context, but again, parts A and B are not optimal. The following optimal solution not only has ugly positions, but also both part A and part B are partially done on the first day.

Algorithm 3. ($23\frac{25}{116}$ days)

1. Dump one ration at $8\frac{18}{29}$ and return to base ($\frac{25}{29}$ days).
2. Dump two rations at $99\frac{9}{29}$ and return to base ($\frac{4}{29}$ days).
3. Dump one ration at $96\frac{26}{29}$, return to $99\frac{9}{29}$, pick up two in turn, dump both at $95\frac{25}{29}$ and return to base (1 day).
4. Dump one ration at 90 and return to base (1 day).
5. Dump one ration at $88\frac{28}{29}$, return to 90, pick up one, dump at $88\frac{28}{29}$ also and return to $95\frac{25}{29}$ (1 day).
6. Dump one ration at $82\frac{12}{29}$ and return to $88\frac{28}{29}$ (1 day).
7. Dump one ration at $75\frac{20}{29}$ and return to $82\frac{12}{29}$ (1 day).
8. Return to $96\frac{26}{29}$, pick up one, dump at $95\frac{20}{29}$ and return to base (1 day). We have thus left one ration at $8\frac{18}{29}$, one at $75\frac{20}{29}$ and another one at $95\frac{20}{29}$.
9. Dump one ration at $9\frac{9}{29}$, return to $8\frac{18}{29}$, pick up one, dump at $9\frac{9}{29}$ also and return to base (1 day).
10. Dump five rations at 10 and return to base (5 days).
11. Dump one ration at $12\frac{19}{58}$, return to 10, pick up one, dump at $12\frac{19}{58}$ also and return to $9\frac{9}{29}$ (1 day).
12. Dump one ration at $19\frac{19}{29}$ and return to 10 (1 day).

13. Dump one ration at $19\frac{24}{29}$, return to $19\frac{19}{29}$, pick up one, dump at $19\frac{24}{29}$ also and return to 10 (1 day).
14. Dump one ration at $21\frac{19}{116}$ and return to $12\frac{19}{58}$ (1 day).
15. Dump one ration at $23\frac{18}{29}$, return to $21\frac{19}{116}$, pick up one, dump at $23\frac{18}{29}$ also and return to $19\frac{24}{29}$ (1 day).
16. Dump one ration at $31\frac{21}{29}$ and return to $23\frac{18}{29}$ (1 day).
17. Dump one ration at $35\frac{20}{29}$, return to $31\frac{21}{29}$, pick up one and get to $35\frac{20}{29}$ (1 day).
18. March to $75\frac{20}{29}$ (2 days).
19. March to base ($1\frac{25}{116}$ days).

Estimates for part A

If professor Walkingholme replaces his unsealed box by a new full box each time he passes the base, then no rations carried in part A are used in part B and vice versa. So we can see part A and part B as separate problems.

The optimality of part B of both solutions is almost proved in [1] and [4]. In both articles, the problem of how far you can get in N days is solved for integers N. But to cross a certain distance, it is very unlikely that you need an integral number of days. The non-integral case is an easy variation, however. We will use the techniques of these articles here.

Since part B is almost done in the above references, we only prove the optimality of part A in full detail. For convenience, we measure the distance in units of 20 miles from now and indicate the positions the other way around.

But before starting, it is always a good idea to determine what must be done. In part A, professor Walkingholme must put boxes on positions γ and $\gamma - 1$, to be used in part C. If $\gamma > 2$, then another box on $\gamma - 2$ and maybe more boxes are needed, but taking $\gamma > 2$ is so bad that remission of the additional costs of getting to the base does not affect the estimate. If on the other hand $\gamma < 1$, then there does not need to be carried a box to $\gamma - 1$, but taking $\gamma < 1$ will turn out to be a bad idea as well.

Suppose for now that $\gamma > 1$ and let m be the moment of the first unsealing after dumping a box on position γ to be used in part C, say at position $r > 0$. Assuming that boxes are not carried back and forth unnecessarily, there is a box on position γ and another box on a position between 0 and $\gamma - 1$ inclusive, to be carried to $\gamma - 1$ at the end.

Let

$$0 < e_l \leq e_{l-1} \leq \cdots \leq e_2 \leq e_1$$

be the positions > 0 where a box is unsealed in part A before dumping a box at γ (prior to m), and define

$$e_{l+1} = e_{l+2} = \cdots = 0$$

Suppose that before moment m, professor Walkingholme unseals the last box on position e_j. Then $(\gamma - e_j) + (\gamma - r) \leq 1$. Together with $e_1 \geq e_j$, we get

$$\gamma \leq \frac{1}{2}e_1 + \frac{1}{2}r + \frac{1}{2} \tag{1}$$

This estimates r to be $2\gamma - e_1 - 1$ at least. In case $\gamma \leq 1$, r might be negative. Therefore we define $r = 0$ in case moment m takes place on a negative position. e_1 is always nonnegative by definition, thus (1) is also valid when $\gamma \leq 1$.

Assume that $\gamma > 1$ and let t be the time professor Walkingholme uses for part A. Then professor Walkingholme walks $\frac{1}{2}t$ units in forward direction in part A, whence by elementary logistics

$$\gamma + (\gamma - 1) + r + \sum_{i=1}^{k} e_i \leq \frac{1}{2}t \tag{2}$$

for all k. Since $\gamma - 1 \leq 0$ when $\gamma \leq 1$, the above estimate is also valid when $\gamma \leq 1$. This estimate can only be effective when we show the optimality of an algorithm for part A with $\gamma \geq 1$ and $r \leq 1$. If $r > 1$, then there must be an additional box on position $r - 1$ at least, whence

$$\gamma + (\gamma - 1) + r + (r - 1) + \sum_{i=1}^{k} e_i \leq \frac{1}{2}t$$

for all k. But this estimate is satisfied for $r \leq 1$ as well, since then you just subtract $1 - r$ from the left hand side of (2). It is only not effective for $r < 1$.

If we take the average of the above estimate and (2), we get

$$\gamma + (\gamma - 1) + r + \frac{r - 1}{2} + \sum_{i=1}^{k} e_i \leq \frac{1}{2}t \tag{3}$$

for all k, which can only be effective for $r = 1$. Notice that both (2) and (3) can only be effective if $k \geq l$.

Put $e_0 := r$ and define

$$d_i := \frac{1}{2}e_i + \frac{1}{2}e_{i+1} + \frac{1}{2}$$

for all $i \geq 0$. In order to get more information, we ask the following question: how many units does professor Walkingholme walk within the interval $[\beta, \infty)$, before unsealing the box on position r on moment m, where $\beta \geq \frac{1}{2}$? To get the right idea on this question, we assume that part A must satisfy the Dudeney rule that boxes are only unsealed at dawn, albeit we need a more general result (which is left to the reader, the days are too short to write it down).

To answer the question, we slice the total walk up to moment m into parts w_i, such that w_i ends on the middle of the day that the box at e_i is unsealed (at dawn) and starts on the middle of the day before. w_0 is just the last half-day walk and also the first slice does not need to have length one. Now we can estimate how many units professor Walkingholme walks within the interval $[\beta, \infty)$ in w_i for all $i > 0$:

- *Case $\beta \le e_i$:* At most the length of w_i, i.e. one unit.
- *Case $\beta - \frac{1}{2} \le e_i \le \beta$:* At most $1 - 2(\beta - e_i)$ units, since a round trip from β to e_i is $2(\beta - e_i)$ units.
- *Case $e_i \le \beta - \frac{1}{2}$:* No units.

If $i = 0$, then we get half of the above estimates, since w_0 is only half a mile. Now take $\beta = e_{2k+2} + \frac{1}{2}$. Then $i \ge 2k+2$ implies the last case and $i < 2k+2$ implies one of the first two cases of the above, i.e. $2(e_i + \frac{1}{2} - \max\{e_i, \beta\})$ units. This makes a total of

$$\begin{aligned}
\alpha &:= \left(e_0 + \frac{1}{2} - \max\{e_0, \beta\}\right) + 2\sum_{i=1}^{2k+1}\left(e_i + \frac{1}{2} - \max\{e_i, \beta\}\right) \\
&= \sum_{i=0}^{2k+1}\left(e_i + \frac{1}{2} - \max\{e_i, \beta\}\right) + \sum_{i=1}^{2k+2}\left(e_i + \frac{1}{2} - \max\{e_i, \beta\}\right) \\
&= 2\sum_{i=0}^{2k+1} d_i - \left(\max\{e_0, \beta\} + 2\sum_{i=1}^{2k+2}\max\{e_i, \beta\} - \max\{e_{2k+2}, \beta\}\right) \\
&= 2\sum_{i=0}^{2k+1} d_i - \max\{e_0, \beta\} - 2\sum_{i=1}^{2k+2}\max\{e_i, \beta\} + \beta
\end{aligned}$$

units.

Scratching all the $[\beta, \infty)$-parts of the w_i's together gives a walk that starts at β and ends at $\max\{e_0, \beta\}$, of which

$$\frac{\alpha - (\max\{e_0, \beta\} - \beta)}{2}$$

units are in backward direction and

$$\begin{aligned}
\frac{\alpha + (\max\{e_0, \beta\} - \beta)}{2} &= \sum_{i=0}^{2k+1} d_i - \sum_{i=1}^{2k+2}\max\{e_i, \beta\} \\
&\le \sum_{i=0}^{2k+1} d_i - \sum_{i=1}^{k} e_i - (k+2)\beta \qquad (4)
\end{aligned}$$

units are in forward direction. In order to get boxes on $\gamma, e_0, e_1, \ldots$, professor Walkingholme needs to march from β to γ if $\gamma > \beta$ and from β to e_i for all i with $e_i > \beta$, whence at least

$$\begin{aligned}
&(\gamma - \min\{\gamma, \beta\}) + \sum_{i=0}^{\infty}(e_i - \min\{e_i, \beta\}) \\
\ge\ & (\gamma - \beta) + \sum_{i=0}^{k}(e_i - \beta) + \sum_{i=k+1}^{\infty}(e_i - e_i)
\end{aligned}$$

$$= \quad \gamma + r + \sum_{i=1}^{k} e_i - (k+2)\beta \tag{5}$$

units in forward direction within $[\beta, \infty)$ are required. Combining (4), (5) and $e_0 = r$ gives

$$\gamma + r + 2\sum_{i=1}^{k} e_i \leq \sum_{i=0}^{2k+1} d_i \tag{6}$$

Optimality of algorithms 2 and 3

Using (2) for $k = 2, 3, 4$ and (6) for $k = 0, 1$ gives

$$t \geq 12\frac{4}{7}\gamma - 9\frac{1}{7} \tag{7}$$

(add variables $C_k \geq 0$ on the smaller sides of the inequalities to get equations), which proves the optimality of part A of the second solution. Using (3) instead of (2) gives

$$t \geq 14\gamma - 11 \tag{8}$$

which proves the optimality of part A of both Dudeney's solution and the first solution.

Next we sketch the optimality of part B. Let $e_1 \geq e_2 \geq e_3 \geq \cdots$ be the positions where boxes are unsealed in part B before reaching γ from the other side, and define d_i as above. In [1], the last inequality but one reads

$$e_1 + 2\sum_{i=2}^{k} e_i \leq \sum_{i=1}^{2k-1} d_i \tag{9}$$

and the first inequality of lemma B, together with the above definition, looks like

$$e_1 + 2\sum_{i=2}^{k} e_i \leq t - 1 \tag{10}$$

except that the right hand side is $N - 1$ instead of $t - 1$. But an algorithm with N boxes takes N days, thus (10) seems correct if t is the time part B takes in days.

These inequalities can be proved with the techniques of the previous section. Together with $e_1 + 1 = 5 - \gamma$, we get

$$t \quad \geq \quad 13\frac{5}{7}(5-\gamma) - 36\frac{6}{7} \tag{11}$$

$$t \quad \geq \quad 16(5-\gamma) - 45 \tag{12}$$

$$t \quad \geq \quad 19\frac{1}{5}(5-\gamma) - 56\frac{4}{5} \tag{13}$$

using (9) for $k = 1, 2, \ldots, n$, and (10) for $k = n+1, \ldots, 2n$ to prove inequality $(n+6)$.

Notice that part C takes γ days. Both (11) and (12) prove the optimality of part B of algorithm 2 individually, but they need to cooperate to get the optimality of algorithm 2 as a whole. (11) gives the bound $11\frac{1}{7}$ day on getting to $3\frac{1}{2}$ as well.

If we charge part C for two days, i.e. the number of boxes unsealed in it, which Nightvid Cole preferred, then it is also clear that algorithm 2 is optimal among the solutions with $\gamma > 1$. But if $\gamma \leq 1$, then it follows from (13) and (8) that more than

$$\left(12\frac{4}{7}\gamma - 9\frac{1}{7}\right) + \left(19\frac{1}{7}(5-\gamma) - 57\right) + \gamma = 29\frac{4}{7} - 5\frac{4}{7}\gamma \geq 24$$

days are necessary, so algorithm 2 is optimal in Coles's way of measuring as well. Furthermore, the straight line solution is also proved to be non-optimal now.

Since the time used for part A and B together is optimal in the first solution, the only way to improve it to an optimal solution in Dudeney's way of measuring is to decrease γ. It follows that algorithm 3 is optimal in Dudeney's way of measuring.

At last, the round trips to $2\frac{1}{2}$, or actually to some position γ. The round trip from the other side is in fact a round trip to $5-\gamma$. Let e_1 be the largest position where a box is unsealed. This can be before or after reaching γ, but no other box need to be unsealed in between. Notice that the box for e_1 can be transported on the road to γ, thus e_1 does not need to be counted for dropping.

Let r be the position where the first box after that on e_1 and $e_2 \geq e_3 \geq \cdots$ be the positions where the boxes before that on e_1 are unsealed. The last box before that on e_1 is unsealed on a position $\leq e_2$, after which a walk of two units to r which meets γ follows. Hence

$$\gamma \leq \frac{e_2 + r + 2}{2} \tag{14}$$

Set $d_i := \frac{1}{2}e_i + \frac{1}{2}e_{i+1} + \frac{1}{2}$ for all $i \geq 2$. By way of the techniques for estimating part A, one can get the following inequalities for the round trip to γ.

$$\gamma + r + 2\sum_{i=2}^{k} e_i \leq \frac{e_2 + r + 2}{2} + \sum_{i=2}^{2k} d_i \tag{15}$$

$$\gamma + r + (r-1) + 2\sum_{i=2}^{k} e_i \leq \frac{e_2 + r + 2}{2} + \sum_{i=2}^{2k+1} d_i \tag{16}$$

and

$$\gamma + r + (r-1) + 2\sum_{i=2}^{k} e_i \leq \frac{t}{2} \tag{17}$$

Using (14), (15) for $k = 2, 3, 4$, (16) for $k = 4, 5, 6, 7, 8$, and (17) for $k = 9, 10, \ldots, 18$, we get

$$t \leq 27\gamma - 46\frac{7}{8}$$

whence at least $(27\gamma - 46\frac{7}{8}) + (27(5-\gamma) - 46\frac{7}{8}) = 41\frac{1}{4}$ days are necessary. This bound can be attained with my way of measuring, with two round trips to $2\frac{1}{2}$, but in order to take into account the ants as well, different round trips are necessary, e.g. to $2\frac{1}{2} \pm \frac{1}{72}$, starting $\frac{1}{3}$ way during the day and ending $\frac{7}{12}$ way during another day. The reader may verify this.

If the box on e_1 is not unsealed before reaching γ, then we have $\gamma \geq r+1$, and by adding (14), (15) for $k = 2, 3, \ldots, 8$, and (17) for $k = 9, 10, \ldots, 17$, to the inequality $\gamma \geq r+1$, we can derive

$$t \leq 25\frac{6}{7}\gamma - 44$$

If we use (16) instead of (15) for $k = 8$ (also for $k = 7$ and/or $k = 6$ when desired), we get

$$t \leq 26\frac{1}{7}\gamma - \frac{313}{7}$$

Both bounds on t can be attained simultaneously, if and only if $e_1 = \gamma = 2\frac{1}{2}$.

References

[1] M. de Bondt, The camel-banana problem, *Nieuw Archief voor Wiskunde*, **14** (1996) 415-426.

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[3] H. E. Dudeney, 536 Puzzles & Curious Problems, ed. Martin Gardner Charles Scribner's Sons, New York, 1967.

[4] G. Rote, G. Zhang, Optimal logistics for expeditions: the jeep problem with complete refilling, *Optimierung und Kontrolle*, **71** (1996), Spezialforschungsbereich 003, Karl-Franzens-Universität Graz & Technische Universität Graz.