BRAIDED SYMMETRIC ALGEBRAS OF SIMPLE $U_q(sl_2)$-MODULES AND THEIR GEOMETRY

SEBASTIAN ZWICKNAGL

Abstract. In the present paper we prove decomposition formulae for the braided symmetric powers of simple $U_q(sl_2)$-modules, natural quantum analogues of the classical symmetric powers of a module over a complex semisimple Lie algebra. We show that their point modules form natural non-commutative curves and surfaces and conjecture that braided symmetric algebras give rise to an interesting non-commutative geometry, which can be viewed as a flat deformation of the geometry associated to their classical limits.

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1. Introduction

Braided Symmetric and Exterior Algebras were introduced by A. Berenstein and the author in [2] as quantum analogs for the symmetric and exterior algebras of finite-dimensional modules over a quantized enveloping algebra $U_q(\mathfrak{g})$. One motivation for this project was to develop a new, quantum, approach to the following classical problem which is still open, and not completely settled even in the case of $\mathfrak{g} = sl_2(\mathbb{C})$:

Problem 1.1. *Let V be a finite-dimensional module over a semisimple complex Lie algebra $\mathfrak{g}$. Find a decomposition formula for the symmetric and exterior powers of V.*

The main result of the present paper is such a decomposition formula for the simple $U_q(sl_2)$-modules (Theorem 1.2 and Theorem 3.1). Let us briefly explain the definition of the braided powers and algebras, and outline their relation to the classical counterparts and some other objects in algebra and geometry. Then we will briefly outline the proof and explain some of the consequences in non-commutative geometry and the relation to Poisson geometry.

Recall that the category of finite-dimensional $U_q(\mathfrak{g})$-modules is braided monoidal, with braiding $\mathcal{R}_{U,V}: U\otimes V \to V\otimes U$ given by the permutation of factors composed with the action of the uiversal R-matrix. We define for each finite-dimensional $U_q(\mathfrak{g})$-module V, the braided exterior square $\Lambda_q^2 V \subset V\otimes V$ to be the span of all "negative" eigenvectors of $\mathcal{R}_{V,V}$ (i.e. of the form $-q^r$, $r\in\mathbb{Z}$), and the braided symmetric square $S_q^2 V\subset V\otimes V$ to be the span of all positive eigenvectors of $\mathcal{R}_{V,V}$ (i.e. of the form q^r, $r\in\mathbb{Z}$). Note that

$$V\otimes V = \Lambda_q^2 V \oplus S_q^2 V \ ,$$

since $\mathcal{R}_{V,V}^2$ is a diagonalizable endomorphism whose eigenvalues are powers of q. We define the braided symmetric algebra $S_q(V)$ to be the quotient of the tensor algebra $T(V)$ by the ideal generated by $\Lambda_q^2 V$, and analogously for the braided exterior algebra $\Lambda_q(V)$. Note that these algebras are naturally $\mathbb{Z}_{\geq 0}$ graded. Additionally we constructed braided symmetric and exterior powers and were able to show that they are isomorphic as $U_q(\mathfrak{g})$-modules to the respective graded components of the braided symmetric and exterior algebras. We show in [2] that the algebras $S_q(V)$ are deformations of the classical symmetric algebras $S(\overline{V})$ of the semiclassical limit $\overline{V}$ of V (see Section 2.3). and a complete classification of the cases in which this deformation is flat was obtained by the author in [16]. The first examples of non-flat quantum symmetric algebras were obtained much earlier by Vancliff [14] and Rossi-Doria [13].

The list of simple modules with this property almost coincides with the list of maximal parabolic subalgebras with Abelian nilradical, and they play an important role in classical invariant theory (see e.g. Howe's article [8]), and provide many known examples of quantized coordinate spaces and exterior powers (see e.g. [6] and [10]). This explains why braided symmetric algebras have been recently used by Lehrer, Zhang and Zhang in [11] to study non-commutative classical invariant theory.

Howe studies classical invariant theory from the point of view of multiplicity free-actions, i.e. he considers symmetric algebras of modules, where each graded component splits into a direct sum of pairwise non-isomorphic simple modules. In the present paper, we show that the braided symmetric algebras of simple $U_q(sl_2)$-modules have this property, and prove an explicit decomposition formula, the following Main Theorem, where V_k denotes a $(k+1)$-dimensional simple $U_q(sl_2)$-module.

Main Theorem 1.2. *(a) If ℓ is odd, and $n\geq 3$, then*

$$S_q^n V_\ell \cong \bigoplus_{i=0}^{\frac{\ell-1}{2}} V_{n\ell-4i} \ , \quad \Lambda_q^n V_\ell = 0 \ .$$

(a) If ℓ is even, and $n\geq 3$, $p\geq 4$, then

$$S_q^n V_\ell \cong \bigoplus_{i=0}^{\frac{n\ell}{4}} V_{n\ell-4i} \ , \quad \Lambda_q^3 V_\ell \cong \bigoplus_{i=\frac{\ell}{2}}^{\frac{3\ell-2}{4}} V_{3\ell-4i-2} \ , \quad \Lambda_q^k V_\ell = 0 \ .$$

The results for the exterior powers and the symmetric cubes were already obtained in [2], hence the main accomplishment of the present work is the explicit computation of the higher braided symmetric powers. The main idea of the proof is as follows: First, notice that the classical limit of a braided symmetric algebra $\overline{S_q(V)}$ is a Poisson algebra with the Poisson bracket defined by the standard classical r-matrix (see also Section 2.3). It can be described as a quotient of $S(\overline{V})$ by some ideal. The classical r-matrix defines a skew-symmetric bracket on $S(\overline{V})$ which does not necessarily satisfy the Jacobi identity–hence it need not be Poisson. The obstruction is an ideal generated in degree 3, and it allows us to construct the minimal quotient of $S(\overline{V})$ which is Poisson, the Poisson closure. The generators of the obstruction ideal were computed in [2], and in the present paper we show that $\overline{S_q(V)}$ is isomorphic to the Poisson closure of $S(\overline{V})$ if V is an even-dimensional simple $U_q(sl_2)$-module, by employing results on quantum Howe-duality (see [15]) obtained in [2].

In the case of odd-dimensional simple $U_q(sl_2)$-modules V_{2k} we consider the k-th Veronese algebra $V_q(2,k)$ of $S_q(V_2) \cong \mathbb{C}_q[x_0,x_1,x_2]$. There exists a surjective $U_q(sl_2)$-module algebra map $S_q(V_{2k}) \to A \subset V_q(2,k)$ where A is a certain subalgebra of $V_q(2,k)$. We then easily conclude the assertion of Theorem 3.1.

The previous paragraph suggests a connection to non-commutative geometry. Following Artin, Tate and Van Den Bergh, as well as Polishchuk [12] one defines the following notion of $Proj$ for non-commutative algebras. Let A be a non-commutative algebra. Then, $Proj(A)$ is the quotient of the category of A-modules by the finite-dimensional modules. This allows for the definition of non-commutative point schemes (see e.g. [1]). In this paper we show that the non-commutative point-schemes attached to the braided symmetric algebras for simple $U_q(sl_2)$-modules are obtained from Veronese varieties. Moreover, they are deformations of the point schemes of their respective classical limits (see Section 4.2). In the case $\ell = 3$, this was first noticed by Vancliff in [14]. That the symmetric algebra of the four-dimensional simple sl_2-module allows no flat deformations was first published by Rossi-Doria in [13]. While non-commutative geometry, in general, appears to be very difficult, we therefore conjecture that braided symmetric and exterior algebras give rise to a beautiful and natural geometry, which can be obtained from the geometry of the classical limits.

Finally, while classical symmetric and exterior powers, in general, are given as intersections of highly non-generic subspaces–if $\Lambda^2\overline{V} \subset \overline{V} \otimes \overline{V}$ was generic, then $\Lambda^3\overline{V} = 0$– it appears from the results of [2], [16] and our results that braided symmetric and exterior powers are as generic as the can be while still respecting the weight grading. We therefore believe that these algebras are natural objects which may shed significant insight into Problem 1.1.

The paper is organized as follows: First we recall the notions of braided symmetric algebras and their classical limits in Section 2. The following Section 3 is devoted to Theorem 3.1 and its proof, while Section 4 considers the applications to the theory of Poisson closures and non-commutative geometry. Finally, we provide some facts about classical and quantum Veronese algebras in Appendix A.

Acknowledgements 1.3. *The author would like to thank A. Berenstein for many stimulating discussions about braided symmetric algebras. However, he would have never been able to see the connection to Veronese algebras and surfaces, if M. Vancliff had not told him about the results of her dissertation* [14]*, where she obtained that the point-modules of braided symmetric algebra of the irreducible* 4*-dimensional*

representation form a Veronese curve. She was therefore also the first person to notice that the 4-dimensional irreducible $U_q(sl_2)$-module does not permit a symmetric algebra which is a flat deformation of the classical symmetric algebra.

2. Braided Symmetric Algebras

2.1. **The Quantum Group $U_q(\mathfrak{g})$ and its Modules.** We start with the definition of the quantized enveloping algebra associated with a complex reductive Lie algebra $\mathfrak{g}$ (our standard reference here will be [3]). Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $P(\mathfrak{g})$ the weight lattice and let $A = (a_{ij})$ be the Cartan matrix for $\mathfrak{g}$. Additionally, let $(\cdot, \cdot)$ be the standard non-degenerate symmetric bilinear form on $\mathfrak{h}$.

The *quantized enveloping algebra* U is a $\mathbb{C}(q)$-algebra generated by the elements E_i and F_i for $i \in [1, r]$, and K_λ for $\lambda \in P(\mathfrak{g})$, subject to the following relations: $K_\lambda K_\mu = K_{\lambda+\mu},\ K_0 = 1$ for $\lambda, \mu \in P$; $K_\lambda E_i = q^{(\alpha_i\,,\lambda)} E_i K_\lambda,\ K_\lambda F_i = q^{-(\alpha_i\,,\lambda)} F_i K_\lambda$ for $i \in [1, r]$ and $\lambda \in P$;

$$E_i, F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}} \tag{2.1}$$

for $i, j \in [1, r]$, where $d_i = \frac{(\alpha_i\,,\alpha_i)}{2}$; and the *quantum Serre relations*

$$\sum_{p=0}^{1-a_{ij}} (-1)^p E_i^{(1-a_{ij}-p)} E_j E_i^{(p)} = 0, \quad \sum_{p=0}^{1-a_{ij}} (-1)^p F_i^{(1-a_{ij}-p)} F_j F_i^{(p)} = 0 \tag{2.2}$$

for $i \neq j$, where the notation $X_i^{(p)}$ stands for the *divided power*

$$X_i^{(p)} = \frac{X^p}{(1)_i \cdots (p)_i}, \quad (k)_i = \frac{q^{kd_i} - q^{-kd_i}}{q^{d_i} - q^{-d_i}} . \tag{2.3}$$

The algebra U is a q-deformation of the universal enveloping algebra of the reductive Lie algebra $\mathfrak{g}$, so it is commonly denoted by $U = U_q(\mathfrak{g})$. It has a natural structure of a bialgebra with the co-multiplication $\Delta : U \to U \otimes U$ and the co-unit homomorphism $\varepsilon : U \to \mathbb{Q}(q)$ given by

$$\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i,\ \Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i,\ \Delta(K_\lambda) = K_\lambda \otimes K_\lambda , \tag{2.4}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_\lambda) = 1 . \tag{2.5}$$

In fact, U is a Hopf algebra with the antipode anti-homomorphism $S : U \to U$ given by

$$S(E_i) = -K_{-\alpha_i} E_i,\ S(F_i) = -F_i K_{\alpha_i},\ S(K_\lambda) = K_{-\lambda} . \tag{2.6}$$

Let U^- (resp. U^0; U^+) be the $\mathbb{Q}(q)$-subalgebra of U generated by $F_1, \ldots, F_r$ (resp. by $K_\lambda\,(\lambda \in P)$; by $E_1, \ldots, E_r$). It is well-known that $U = U^- \cdot U^0 \cdot U^+$ (more precisely, the multiplication map induces an isomorphism of vectorspaces $U^- \otimes U^0 \otimes U^+ \to U$).

We will consider the full sub-category $\mathcal{O}_f$ of the category $U_q(\mathfrak{g}) - Mod$. The objects of $\mathcal{O}_f$ are finite-dimensional $U_q(\mathfrak{g})$-modules V having a weight decomposition

$$V = \oplus_{\mu \in P} V(\mu) ,$$

where each K_λ acts on each *weight space* $V(\mu)$ by the multiplication with $q^{(\lambda\,|\,\mu)}$ (see e.g., [3, I.6.12]). Note that for convenience we only consider the modules of type I in the terminology of Jantzen [9], but that the results hold of course for any finite-dimensional module over $U_q(\mathfrak{g})$. The category $\mathcal{O}_f$ is semisimple and the irreducible objects V_λ are generated by highest weight spaces $V_\lambda(\lambda) = \mathbb{C}(q) \cdot v_\lambda$, where λ is a *dominant weight*, i.e, λ belongs to $P^+ = \{\lambda \in P : (\lambda \,|\, \alpha_i) \geq 0 \ \forall \ i \in [1, r]\}$, the monoid of dominant weights.

By definition,the universal R-matrix $R \in U_q(\mathfrak{g})\widehat{\otimes} U_q(\mathfrak{g})R$ has can be decomposed as

$$R = R_0 R_1 = R_1 R_0 \tag{2.7}$$

where R_0 is "the diagonal part" of R, and R_1 is unipotent, i.e., R_1 is a formal power series

$$R_1 = 1 \otimes 1 + (q-1)x_1 + (q-1)^2 x_2 + \cdots \ , \tag{2.8}$$

where all $x_k \in {U'}_k^- \otimes_{\mathbb{C}[q,q^{-1}]} {U'}_k^+$, where U'^- (resp. U'^+) is the integral form of U^+, i.e., U'^- is a $\mathbb{C}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by all F_i (resp. by all E_i) and ${U'}_k^-$ (resp. ${U'}_k^+$) is the k-th graded component under the grading $deg(F_i) = 1$ (resp. $deg(E_i) = 1$).

By definition, for any U, V in $\mathcal{O}_f$ and any highest weights elements $u_\lambda \in U(\lambda)$, $v_\mu \in V(\mu)$ we have $R_0(u_\lambda \otimes v_\mu) = q^{(\lambda \,|\, \mu)} u_\lambda \otimes v_\mu$.

Let R^{op} be the opposite element of R; i.e., $R^{op} = \tau(R)$, where $\tau : U_q(\mathfrak{g})\widehat{\otimes} U_q(\mathfrak{g}) \to U_q(\mathfrak{g})\widehat{\otimes} U_q(\mathfrak{g})$ is the permutation of factors. Clearly, $R^{op} = R_0 R_1^{op} = R_1^{op} R_0$.

Following [4, Section 3], define $D \in U_q(\mathfrak{g})\widehat{\otimes} U_q(\mathfrak{g})$ by

$$D := R_0 \sqrt{R_1^{op} R_1} = \sqrt{R_1^{op} R_1} R_0 \ . \tag{2.9}$$

Clearly, D is well-defined because $R_1^{op} R_1$ is also unipotent as well as its square root. By definition, $D^2 = R^{op} R$, $D^{op} R = RD$.

Furthermore, define

$$\widehat{R} := R D^{-1} = (D^{op})^{-1} R = R_1 \left(\sqrt{R_1^{op} R_1}\right)^{-1} \tag{2.10}$$

It is easy to see that

$$\widehat{R}^{op} = \widehat{R}^{\,-1} \tag{2.11}$$

According to [4, Proposition 3.3], the pair $(U_q(\mathfrak{g}), \widehat{R})$ is a *coboundary* Hopf algebra. For definition and properties of coboundary Hopf algebras see e.g. [17].

The category $\mathcal{O}_f$ is naturally braided, with braiding defined by $\mathcal{R}_{U,V} : U \otimes V \to V \otimes U$, where

$$\mathcal{R}_{U,V}(u \otimes v) = \tau R(u \otimes v)$$

for any $u \in U$, $v \in V$, with $\tau : U \otimes V \to V \otimes U$ being the ordinary permutation of factors.

Denote by $C \in Z(\widehat{U_q(\mathfrak{g})})$ the *quantum Casimir* element which acts on any irreducible $U_q(\mathfrak{g})$-module V_λ in $\mathcal{O}_f$ by the scalar multiple $q^{(\lambda \,|\, \lambda + 2\rho)}$, where 2ρ is the sum of positive roots.

The following fact is well-known.

Lemma 2.1. *One has $\mathcal{R}^2 = \Delta(C^{-1}) \circ (C \otimes C)$. In particular, for each $\lambda, \mu, \nu \in P_+$ the restriction of $\mathcal{R}^2$ to the ν-th isotypic component $I_{\lambda,\mu}^\nu$ of the tensor product $V_\lambda \otimes V_\mu$ is scalar multiplication by $q^{(\lambda \,|\, \lambda) + (\mu \,|\, \mu) - (\nu \,|\, \nu) + (2\rho \,|\, \lambda + \mu - \nu)}$.*

We now define the diagonalizable $\mathbb{C}(q)$-linear map $D_{U,V} : U \otimes V \to U \otimes V$ by $D_{U,V}(u \otimes v) = D(u \otimes v)$ for any objects U and V of $\mathcal{O}_f$. It is easy to see that the operator $D_{V_\lambda,V_\mu} : V_\lambda \otimes V_\mu \to V_\lambda \otimes V_\mu$ acts on the ν-th isotypic component $I^\nu_{\lambda,\mu}$ in $V_\lambda \otimes V_\mu$ by the scalar multiplication with $q^{\frac{1}{2}((\lambda\,|\,\lambda)+(\mu\,|\,\mu)-(\nu\,|\,\nu))+(\rho\,|\,\lambda+\mu-\nu)}$.

The coboundary Hopf algebra $(U_q(\mathfrak{g}), \widehat{R})$ provides $\mathcal{O}_f$ with the structure of a coboundary or cactus category in the following way. For any U and V in $\mathcal{O}_f$ define the *normalized braiding* $\sigma_{U,V}$ by

$$\sigma_{U,V}(u \otimes v) = \tau \widehat{R}(u \otimes v) \ , \tag{2.12}$$

Therefore, we have by (2.10):

$$\sigma_{U,V} = D^{-1}_{V,U} \mathcal{R}_{U,V} = \mathcal{R}_{U,V} D^{-1}_{U,V} \ . \tag{2.13}$$

We will sometimes write $\sigma_{U,V}$ in a more explicit way:

$$\sigma_{U,V} = \sqrt{\mathcal{R}^{-1}_{V,U}\mathcal{R}^{-1}_{U,V}}\,\mathcal{R}_{U,V} = \mathcal{R}_{U,V}\sqrt{\mathcal{R}^{-1}_{U,V}\mathcal{R}^{-1}_{V,U}} \tag{2.14}$$

The following fact is an obvious corollary of (2.11).

Lemma 2.2. *$\sigma_{V,U} \circ \sigma_{U,V} = id_{U\otimes V}$ for any U, V in $\mathcal{O}_f$. That is, σ is a symmetric commutativity constraint.*

Remark 2.3. *If one replaces the braiding $\mathcal{R}$ of $\mathcal{O}_f$ by its inverse $\mathcal{R}^{-1}$, the symmetric commutativity constraint σ will not change.*

2.2. **Braided Symmetric and Exterior Powers.** In this section we will recall the definitions and some basic properties of the braided symmetric and exterior powers and algebras introduced in [2].

For any morphism $f : V \otimes V \to V \otimes V$ in $\mathcal{O}_f$ and $n > 1$ we denote by $f^{i,i+1}$, $i = 1, 2, \ldots, n-1$ the morphism $V^{\otimes n} \to V^{\otimes n}$ which acts as f on the i-th and the $(i+1)$st factors. Note that $\sigma^{i,i+1}_{V,V}$ is always an involution on $V^{\otimes n}$.

Definition 2.4. *For an object V in $\mathcal{O}_f$ and $n \geq 0$ define the braided symmetric power $S^n_\sigma V \subset V^{\otimes n}$ and the braided exterior power $\Lambda^n_\sigma V \subset V^{\otimes n}$ by:*

$$S^n_\sigma V = \bigcap_{1\leq i\leq n-1} (Ker\ \sigma_{i,i+1} - id) = \bigcap_{1\leq i\leq n-1} (Im\ \sigma_{i,i+1} + id) \ ,$$

$$\Lambda^n_\sigma V = \bigcap_{1\leq i\leq n-1} (Ker\ \sigma_{i,i+1} + id) = \bigcap_{1\leq i\leq n-1} (Im\ \sigma_{i,i+1} - id),$$

where we abbreviate $\sigma_{i,i+1} = \sigma^{i,i+1}_{V,V}$.

Remark 2.5. *Clearly, $-\mathcal{R}$ is also a braiding on $\mathcal{O}_f$ and $-\sigma$ is the corresponding normalized braiding. Therefore, $\Lambda^n_\sigma V = S^n_{-\sigma} V$ and $S^n_\sigma V = \Lambda^n_{-\sigma} V$. That is, informally speaking, the symmetric and exterior powers are mutually "interchangeable".*

Remark 2.6. *Another way to introduce the symmetric and exterior squares involves the well-known fact that the braiding $\mathcal{R}_{V,V}$ is a semisimple operator $V \otimes V \to V \otimes V$, and all the eigenvalues of $\mathcal{R}_{V,V}$ are of the form $\pm q^r$, where $r \in \mathbb{Z}$. Then positive eigenvectors of $\mathcal{R}_{V,V}$ span $S^2_\sigma V$ and negative eigenvectors of $\mathcal{R}_{V,V}$ span $\Lambda^2_\sigma V$.*

Clearly, $S^0_\sigma V = \mathbb{C}(q)$, $S^1_\sigma V = V$, $\Lambda^0_\sigma V = \mathbb{C}(q)$, $\Lambda^1_\sigma V = V$, and

$$S^2_\sigma V = \{v \in V \otimes V \,|\, \sigma_{V,V}(v) = v\},\ \Lambda^2_\sigma V = \{v \in V \otimes V \,|\, \sigma_{V,V}(v) = -v\} \ .$$

The following fact is obvious.

Proposition 2.7. *For each $n \geq 0$ the association $V \mapsto S_\sigma^n V$ is a functor from $\mathcal{O}_f$ to $\mathcal{O}_f$ and the association $V \mapsto \Lambda_\sigma^n V$ is a functor from $\mathcal{O}_f$ to $\mathcal{O}_f$. In particular, an embedding $U \hookrightarrow V$ in the category $\mathcal{O}_f$ induces injective morphisms*

$$S_\sigma^n U \hookrightarrow S_\sigma^n V,\ \Lambda_\sigma^n U \hookrightarrow \Lambda_\sigma^n V\ .$$

Definition 2.8. *For any $V \in Ob(\mathcal{O})$ define the braided symmetric algebra $S_\sigma(V)$ and the braided exterior algebra $\Lambda_\sigma(V)$ by:*

$$S_\sigma(V) = T(V)/\left\langle \Lambda_\sigma^2 V \right\rangle,\ \Lambda_\sigma(V) = T(V)/\left\langle S_\sigma^2 V \right\rangle\ , \tag{2.15}$$

where $T(V)$ is the tensor algebra of V and $\langle I \rangle$ stands for the two-sided ideal in $T(V)$ generated by a subset $I \subset T(V)$.

Note that the algebras $S_\sigma(V)$ and $\Lambda_\sigma(V)$ carry a natural $\mathbb{Z}_{\geq 0}$-grading:

$$S_\sigma(V) = \bigoplus_{n \geq 0} S_\sigma(V)_n,\ \Lambda_\sigma(V) = \bigoplus_{n \geq 0} \Lambda_\sigma(V)_n\ ,$$

since the respective ideals in $T(V)$ are homogeneous.

Denote by $\mathcal{O}_{gr,f}$ the sub-category of $U_q(\mathfrak{g}) - Mod$ whose objects are $\mathbb{Z}_{\geq 0}$-graded:

$$V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n\ ,$$

where each V_n is an object of $\mathcal{O}_f$; and morphisms are those homomorphisms of $U_q(\mathfrak{g})$-modules which preserve the $\mathbb{Z}_{\geq 0}$-grading.

Clearly, $\mathcal{O}_{gr,f}$ is a tensor category under the natural extension of the tensor structure of $\mathcal{O}_f$. Therefore, we can speak of algebras and co-algebras in $\mathcal{O}_{gr,f}$.

By the very definition, $S_\sigma(V)$ and $\Lambda_\sigma(V)$ are algebras in $\mathcal{O}_{gr,f}$.

Proposition 2.9. *The assignments $V \mapsto S_\sigma(V)$ and $V \mapsto \Lambda_\sigma(V)$ define functors from $\mathcal{O}_f$ to the category of algebras in $\mathcal{O}_{gr,f}$.*

We conclude the section with two important features of braided symmetric exterior powers and algebras.

Proposition 2.10. [2, Prop.2.11 and Eq. 2.3] *Let V be an object of $\mathcal{O}_f$ and V^* its dual in $\mathcal{O}_f$. We have the following $U_q(\mathfrak{g})$-module isomorphisms.*

$$(S_\sigma^n V^*)^* \cong S_\sigma(V)_n,\ (\Lambda_\sigma^n V^*)^* \cong \Lambda_\sigma(V)_n\ . \tag{2.16}$$

Proposition 2.11. [2, Prop.2.13] *For any V in $\mathcal{O}_f$ each embedding $V_\lambda \hookrightarrow V$ defines embeddings $V_{n\lambda} \hookrightarrow S_\sigma^n V$ for all $n \geq 2$. In particular, the algebra $S_\sigma(V)$ is infinite-dimensional.*

2.3. **The Classical Limit of Braided Algebras.** In this section we will discuss the specialization of the braided symmetric and exterior algebras at $q = 1$, the classical limit. All of the results in this section are either well known or proved in [2]. For a more detailed discussion of the classical limit we refer the reader to [2, Section 3.2]. The explicit construction of the classical limit involves the consideration of integral lattices in $U_q(\mathfrak{g})$ and its modules, which we will omit here for brevity's sake (see e.g. [2, Section 3.2], or for a shorter version [16, Section 4.3]).

Definition 2.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an almost equivalence of $\mathcal{C}$ and $\mathcal{D}$ if:*
(a) for any objects c, c' of $\mathcal{C}$ an isomorphism $F(c) \cong F(c')$ in $\mathcal{D}$ implies that $c \cong c'$ in $\mathcal{C}$;

(b) for any object d in $\mathcal{D}$ there exists an object c in $\mathcal{C}$ such that $F(c) \cong d$ in $\mathcal{D}$.

Denote by $\overline{\mathcal{O}}_f$ the full (tensor) sub-category category of $U(\mathfrak{g}) - Mod$, whose objects $\overline{V}$ are finite-dimensional $U(\mathfrak{g})$-modules having a weight decomposition $\overline{V} = \oplus_{\mu \in P} \overline{V}(\mu)$. We have the following fact.

Proposition 2.13. [2, Cor 3.22] *The categories $\mathcal{O}_f$ and $\overline{\mathcal{O}}_f$ are almost equivalent. Under this almost equivalence a simple $U_q(\mathfrak{g})$-module V_λ is mapped to the simple $U(\mathfrak{g})$-module $\overline{V}_\lambda$.*

Let $V \cong \bigoplus_{i=1}^n V_{\lambda_i} \in \mathcal{O}_f$. We call $\overline{V} \cong \bigoplus_{i=1}^n \overline{V}_{\lambda_i} \in \overline{\mathcal{O}}_f$ the *classical limit* of V under the almost equivalence of Proposition 2.13.

The following result relates the classical limit of braided symmetric algebras and Poisson algebras.

Theorem 2.14. [2, Theorem 2.29] *Let V be an object of $\mathcal{O}_f$ and let a $\overline{V}$ in $\overline{\mathcal{O}}_f$ be be the classical limit of V. Then:*
(a)The classical limit $\overline{S_\sigma(V)}$ of the braided symmetric algebra $S_\sigma(V)$ is a quotient of the symmetric algebra $S(\overline{V})$. In particular, $\dim_{\mathbb{C}(q)} S_\sigma(V)_n = \dim_{\mathbb{C}}(\overline{S_\sigma(V)}) \leq \dim_{\mathbb{C}} \overline{S(\overline{V})}_n$.
(b)Moreover, $\overline{S_\sigma(V)}$ admits a Poisson structure defined by $\{u, v\} = r^-(u \wedge v)$, where r^- is an anti-symmetrized classical r-matrix.

For more on classical r-matrices and their relation to the R-matrices associated to quantum groups, we refer the reader to [2], [17] or [5].

3. The Classification Theorem

Let $\mathfrak{g} = sl_2$. For convenience we denote the quantum numbers by $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$. We will employ the following notations for q-factorials $[k]_q! = [k]_q[k-1]_q \dots [1]_q$ and q-binomial coefficients $\binom{k}{\ell}_q = \frac{[k]_q!}{[k-\ell]_q![\ell]_q!}$. Recall that the irreducible $U_q(sl_2)$-modules are labeled by non-negative integers, and that $\dim(V_\ell) = \ell + 1$. The module V_ℓ has a natural weight basis $v_0, v_1, \dots, v_\ell$ such that $K(v_i) = q^{\ell - i} v_i$ and $E(v_i) = [i]_q v_{i-1}$. Moreover note that it follows from classical Lie theory and the definition of braided powers (Definition 2.4) that

$$V_\ell \otimes V_\ell = \Lambda_\sigma^2 V_\ell \oplus S_\sigma^2 V_\ell \ , \quad S_\sigma^2 V_\ell = \bigoplus_{i=0}^{\frac{\ell}{2}} V_{2\ell - 4i} \ , \quad \Lambda_\sigma^2 V_\ell = \bigoplus_{i=0}^{\frac{\ell-1}{2}} V_{2\ell - 2 - 4i} \ .$$

We will now state our main result.

Theorem 3.1. *(a) Let ℓ be odd and let V_ℓ be the $(\ell+1)$-dimensional irreducible $U_q(sl_2)$-module. The n-th braided symmetric power splits as*

$$S_\sigma^n(V_\ell) \cong \bigoplus_{i=0}^{\frac{\ell-1}{2}} V_{n\ell - 4i} \ .$$

(b) Let ℓ be even and let V_ℓ be the $\ell + 1$-dimensional irreducible $U_q(sl_2)$-module. The n-th braided symmetric power splits as

$$S_\sigma^n(V_\ell) \cong \bigoplus_{i=0}^{\frac{n\ell}{4}} V_{n\ell - 4i} \ .$$

Remark 3.2. *The above decomposition was originally conjectured in* [2, Conjecture 2.37]*, based on computer calculations.*

Proof. We first recall the main result of the paper [2] which established the Theorem in the case $n = 3$.

Proposition 3.3. [2, Theorem 2.35] *(a) Let ℓ be odd and let V_ℓ be the $\ell+1$-dimensional irreducible $U_q(sl_2)$-module. The third braided symmetric power splits as*

$$S_\sigma^3(V_\ell) \cong \bigoplus_{i=0}^{\frac{\ell-1}{2}} V_{3\ell-4i} \ .$$

(b) Let ℓ be even and let V_ℓ be the $\ell+1$-dimensional irreducible $U_q(sl_2)$-module. The third braided symmetric power splits as

$$S_\sigma^3(V_\ell) \cong \bigoplus_{i=0}^{\frac{3\ell}{4}} V_{3\ell-4i} \ .$$

Prove (a) first. Note that the assertion holds in the case $n = 2$, and in the case $n = 3$ by Proposition 3.3(a).

We first prove that one can find copies of the modules $V_{n\ell-4i}$, $i = 0, \dots, \frac{\ell-1}{2}$ in $S_\sigma(V_\ell)_n$. Note that if v is a highest weight vector in $V_\ell \otimes V_\ell$, then $v \otimes v_0^{\otimes n-2}$ is a highest weight vector in $V_\ell^{\otimes n}$. We obtain the following result.

Proposition 3.4. *(a) Let $\mathbf{v}_i$ be a highest weight vector in $S_\sigma^2 V_\ell \subset V_\ell \otimes V_\ell$ of weight $2\ell - 4i$ for $0 \le i \le \frac{\ell-1}{2}$. Then, for all $n \ge 3$*

$$\mathbf{v}_i^n = \mathbf{v}_i \otimes v_0^{\otimes n-2} \notin \langle \Lambda_\sigma^2 V_\ell \rangle_n \ .$$

(b) Moreover, $\mathbf{v}_i^n$ is a highest weight vector of weight $n\ell - 4i$.

Proof. Part (b) is obvious. The vector $\mathbf{v}_i^n$ is clearly invariant under all $\sigma_{j,j+1}$ for $j \ge 3$, hence it suffices to prove the assertion of part (a) in the case $n = 3$. To do so, we have to employ the results from [2, Section 3.4]. Using Howe duality, we identify the set $V_\ell^{\otimes 3,+}(3\ell-4i)$ of highest weight vectors of weight $3\ell - 4i$ with the weight space $V_{3\ell-2i,2i,0}(\mathbf{0})$ of the gl_3-module of highest weight $(3\ell - 2i, 2i, 0)$. Note that $V_{3\ell-2i,2i,0}(\mathbf{0})$ has odd dimension. We additionally construct two bases $\mathcal{B}_1 = \{b_1^1, \dots, b_{2m+1}^1\}$ and $\mathcal{B}_2 = \{b_1^2, \dots, b_{2m+1}^2\}$ in $V_\ell^{\otimes 3,+}(3\ell-4i)$ such that $S_\sigma^2 V_\ell \otimes V \cap V_\ell^{\otimes 3,+}(3\ell-4i)$ is spanned by $\{b_1^1, b_3^1, \dots, b_{2m+1}^1\}$ and $\Lambda_\sigma^2 V_\ell \otimes V \cap V_\ell^{\otimes 3,+}(3\ell-4i)$ is spanned by $\{b_2^1, b_4^1, \dots, b_{2m}^1\}$, whereas $V \otimes S_\sigma^2 V_\ell \cap V_\ell^{\otimes 3,+}(3\ell-4i)$ and $V \otimes \Lambda_\sigma^2 V_\ell \cap V_\ell^{\otimes 3,+}(3\ell-4i)$ are spanned by $\{b_1^2, b_3^2, \dots, b_{2m+1}^2\}$, resp. $\{b_2^2, b_4^2, \dots, b_{2m}^2\}$, for more details see [2, Theorem 3.40 and Lemma 3.55]. Moreover, we can identify $\mathbf{v}_i^3$ with the basis vector b_1^1, since it corresponds to the α_1-string of minimal length passing through $V_{3\ell-2i,2i,0}(\mathbf{0})$ (see [2, (3.19)]). Note that [2, Theorem 3.43] implies that the set $\{b_2^1, b_4^1, \dots, b_{2m}^1, b_2^2, b_4^2, \dots, b_{2m}^2\}$ spans

$$\left(\Lambda_\sigma^2 V_\ell \otimes V + V \otimes \Lambda_\sigma^2 V_\ell\right) \cap V_\ell^{\otimes 3,+}(3\ell-4i) = \langle \Lambda_\sigma^2 V_\ell \rangle_3 \cap V_\ell^{\otimes 3,+}(3\ell-4i) \ne V_\ell^{\otimes 3,+}(3\ell-4i) \, .$$

We then obtain from [2, Theorem 3.43] that the set $\{b_1^1\} \cup \{b_2^1, b_4^1, \dots, b_{2m}^1, b_2^2, b_4^2, \dots, b_{2m}^2\}$ is linearly independent, and hence that $\mathbf{v}_i^3 \notin \langle \Lambda_\sigma^2 V_\ell \rangle_3$. Proposition 3.4 is proved. □

Now we have to prove the reverse inclusion. Consider $\overline{V}_\ell$, the $(\ell+1)$-dimensional irreducible $U(sl_2)$-module. Denote by $\overline{\mathcal{J}} \subset S(\overline{V}_\ell)$ the *Jacobean ideal*, generated by

$$\bigoplus_{i=0}^{\frac{\ell-1}{2}} \overline{V}_{n\ell-4i} \subset S(\overline{V}_\ell)_3 \subset S(\overline{V}_\ell) \ .$$

Remark 3.5. *In [2] the following was proved: Consider the bracket defined by the skew-symmetrized classical r-matrix $r^- = E \wedge F$ on $S(\overline{V}_\ell)$. The Jacobean ideal is the obstruction to the bracket being a Poisson bracket, i.e. satisfying the Jacobi identity. Hence the name.*

The ideal $\mathcal{J}$ is $\mathbb{Z}$-graded and homogeneously generated in degree 3. We obtain the following fact.

Proposition 3.6. *Employ notation as above. The quotient of the n-th graded component $S(\overline{V}_\ell)_n$ by the $\mathcal{I}_n$ splits as*

$$S(\overline{V}_\ell)_n/\mathcal{I}_n \cong \bigoplus_{i=0}^{\frac{\ell-1}{2}} \overline{V}_{n\ell-4i} \ .$$

Proof. Recall that $S(\overline{V}_\ell)_n$, and analogously $S^n(\overline{V}_\ell)$ has a basis $\overline{v}_{i_1,\ldots,i_n} = \overline{v}_{i_1} \cdot \overline{v}_{i_2} \ldots \cdot \overline{v}_{i_n}$ with $0 \leq i_1 \leq \ldots \leq i_n \leq \ell+1$. We order the basis-vectors lexicographically, which means that $\overline{v}_{i_1,\ldots,i_n} < \overline{v}_{j_1,\ldots,j_n}$ if there exists k, $1 \leq k \leq n$, such that $i_h = j_h$ for all $h < k$ and $i_k < j_k$. If $\overline{w} \in S(\overline{V}_\ell)_n$, then we refer to the smallest basisvector $\overline{v}_{i_1,\ldots,i_n}$ with non-zero coefficients as the *initial monomial* of $\overline{w}$. Also note that the weight of $\overline{v}_{i_1,\ldots,i_n}$ is $n\ell - 2(i_1 + \ldots + i_n)$. The weight-space $S(\overline{V}_\ell)_n(k)$ is therefore spanned by the $\overline{v}_{i_1,\ldots,i_n}$ such that $n\ell - 2(i_1 + \ldots + i_n) = k$. We need the following fact.

Lemma 3.7. *Suppose that the space of highest weight vectors $S(\overline{V}_\ell)_3(k)^+$ of weight k is d-dimensional. Then, it has a basis consisting of d elements whose initial monomials are the d smallest monomials of weight d.*

Proof. Suppose that $\overline{w} = \sum_{0 \leq i_1 \leq i_2 \leq i_3 \leq \ell+1} c_{i_1,i_2,i_3} v_{i_1,i_2,i_3} \in S(\overline{V}_\ell)_3(k)^+$ where $3\ell - 2(i_1+i_2+i_3) = k$. We have

$$E(\overline{v}_{i_1,i_2,i_3}) = i_1 \overline{v}_{i_1-1,i_2,i_3} + i_2 \overline{v}_{i_1,i_2-1,i_3} + i_3 \overline{v}_{i_1,i_2,i_3-1} \ .$$

We obtain that the c_{i_1,i_2,i_3} satisfy the following system of linear equations:

$$(i_1)c_{i_1,i_2,i_3} + (i_2+1)c_{i_1-1,i_2+1,i_3} + (i_3+1)c_{i_1-1,i_2,i_3+1} = 0 \ ,$$

for all (i_1,i_2,i_3) such that $3\ell - 2(i_1+i_2+i_3) = k$. We have to argue that the resulting matrix can be transformed into row-echelon-form without column operations. But this is straightforward, since given any two rows, there is at most one column in which both rows have non-zero entries. The lemma is proved. □

We are now able to complete the proof of Proposition 3.6 by employing the following lemma.

Lemma 3.8. *Let $d \geq 0$ be an integer, and let $\mathcal{I} \subset S(\overline{V}_\ell)$ be an ideal generated in degree $S(\overline{V}_\ell)_p$, $p \geq 2$, by homogeneous elements, with respect to the weight-grading, such that there exist $\overline{w}_1, \ldots, \overline{w}_{dim S(\overline{V}_\ell)_p(k)-d} \in \mathcal{I}$ such that the initial monomial of w_i is the i-th smallest monomial in $S(\overline{V}_\ell)_p(k)$. Then, the dimension of the quotient $dim(S(\overline{V}_\ell)_j(k)/\mathcal{I}_p(k)) \leq d$ for all $j \geq p$.*

Proof. The assertion in the case $j = p$ is obvious. By induction, it now suffices to prove the assertion in the case of $j = p+1$. Now assume that $\overline{w}$ of weight k has initial monomial $\overline{v}_{i_1,\ldots,i_{p+1}}$ and suppose that $\overline{v}_{j_1,\ldots,j_{p+1}}$ is its immediate successor. Then there exist $1 \leq q \leq r \leq p+1$ such that $j_q = i_q + 1$ and $j_r = i_r - 1$ while $j_s = i_s$ for all $s \neq q, r$. Now, let $s \neq q, r$. Then, $\overline{v}_{i_1,\ldots,\hat{i}_s\ldots i_{p+1}} < \overline{v}_{j_1,\ldots,\hat{j}_s\ldots j_{p+1}}$ in $S(\overline{V}_\ell)_p(k - i_s)$. Hence, if there are d monomials greater than $\overline{v}_{i_1,\ldots,i_{p+1}}$ in $S(\overline{V}_\ell)_{p+1}(k)$, then there exists $\overline{v}_{i_1,\ldots,\hat{i}_s\ldots i_{p+1}} \in \mathcal{I}_p$, and hence $\overline{v}_{i_1,\ldots,i_{p+1}} \in \mathcal{I}$. The lemma is proved. □

Notice that the Jacobian ideal satisfies the conditions of Lemma 3.8 with $p = 3$ and $d = \frac{\ell+1}{2}$. However, we know from Proposition 3.4 and the basic properties of the classical limit (Theorem 2.14)

$$S(\overline{V}_\ell)_n/\mathcal{I}_n \supset \bigoplus_{i=0}^{\frac{\ell-1}{2}} \overline{V}_{n\ell-4i} \ .$$

The proposition is proved. □

Theorem 3.1 (a) is proved.

It remains to prove part (b). First, recall the definition and properties of the classical and quantum Veronese algebras from Appendix A, in particular the natural $U(sl_2)$, resp. $U_q(sl_2)$-module algebra structure. Notice that if ℓ is even, then we can embed V_ℓ into $S_\sigma(V_2)_{\frac{\ell}{2}}$ and we obtain a homomorphism from $S_\sigma V_\ell$ to $V_q(2, \frac{\ell}{2})$ (see Lemma A.4). Its image is the subalgebra $A(2, \frac{\ell}{2})$, introduced in Appendix A. Proposition 3.3, Lemma A.5(a) and Lemma A.3 now imply that $A(2, \frac{\ell}{2})_2 \cong (S_\sigma(V_\ell))_2$ and $A(2, \frac{\ell}{2})_3 \cong (S_\sigma(V_\ell))_3$. The kernel is an ideal generated by homogeneous elements in degrees 2 and 3, by Lemma A.5(b). We obtain therefore that $S_\sigma(V_\ell) \cong A(2, \frac{\ell}{2})$ as $U_q(sl_2)$-module algebras. Now, Theorem 3.1 (b) is a direct consequence of Lemma A.3 and Lemma A.5(a). Theorem 3.1 is proved. □

Remark 3.9. *Numerical experiments and the philosophy of* [16] *lead us to conjecture in the case of $\mathfrak{g} = sl_n$ similar description of the braided symmetric algebras of $V_{2k\omega_1}$ in terms of Veronese algebras of $V_{2\omega_1}$, where ω_1 denotes the first fundamental weight of $\mathfrak{g} = sl_n$. This is particularly interesting, as $V_{2\omega_1} = S^2(V_{\omega_1})$ is a flat $U_q(sl_n)$-module (see* [16, Theorem 1.1]*)*

3.1. **The Radical.** In this section we will consider the set of nilpotent elements in the algebras $S_\sigma(V_\ell)$. First note the following facts.

Proposition 3.10. *(a) Let ℓ be even. Then* 0 *is the only nilpotent element.*
(b) Let ℓ be odd. The set of nilpotent elements $\mathcal{N}$ is a graded $U_q(sl_2)$-module ideal of $S_\sigma(V_\ell)$. More precisely,

$$\mathcal{N} \cong \bigoplus_{n=2}^{\infty} \mathcal{N}_n \ , \text{where} \quad \mathcal{N}_n \cong \bigoplus_{i=1}^{\frac{\ell-1}{2}} V_{n\ell-4i} \ .$$

Proof. Part (a) follows from the description of $S_\sigma(V_\ell)$ as a subalgebra of the quantum Veronese algebra $\mathcal{V}(2, \frac{\ell}{2})$ which clearly has no zero divisors. In order to prove part(b), we first show that the submodules $\mathcal{N}_n$ are contained in the radical. Consider $\mathbf{v} \in \mathcal{N}_n$. If $\mathbf{v}^\ell \neq 0$, then $\mathbf{v}^\ell \in S_\sigma(V_\ell)_{n\ell}$ lies in a submodule which contains no highest weight vector of weight greater than $\ell n\ell - 4\ell$. But Theorem 3.1 implies that there exists no such highest weight vector in $S_\sigma(V_\ell)_{n\ell}$, hence $\mathbf{v}^\ell = 0$. Now suppose that $\mathbf{v}^\ell \neq \mathcal{N}$. There exists $n \geq 0$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in V_{n\ell} \subset S_\sigma(V_\ell)_n$, and $\mathbf{w} \in \mathcal{N}_n \bigoplus_{i<n} S_\sigma(V_\ell)_i$. Suppose that $E^k(\mathbf{u}) \neq 0$ while $E^{k+1}(\mathbf{u}) = 0$. We easily obtain from the action of $U_q(sl_2)$ on $S_\sigma(V_\ell)$ that $E^{mk}(\mathbf{u}^m) \neq 0$ for all $m \geq 1$, hence $\mathbf{v}$ is not nilpotent. It is easy to verify that $\mathcal{N}$ is indeed a $U_q(sl_2)$-module ideal. Part(b) is proved. The proposition is proved. $\square$

4. Applications

4.1. **The Poisson Closure.** In this section, we will connect the results on braided symmetric and exterior powers to the semiclassical limits, and the notion of the Poisson closure, as developed in [2]. Recall from Theorem 2.14 (b) that the semiclassical limit $\overline{S_\sigma(V)}$ of $S_\sigma(V)$ admits a Poisson bracket defined as $\{u, v\} = r^-(u \wedge v)$. In [2] and [16] one considers the Jacobian map:

$$J : S(\overline{V}) \cdot S(\overline{V}) \cdot S(\overline{V}) \to S(\overline{V}),$$

$$u \otimes v \otimes w \mapsto \{u, \{v, w\}\} + \{w, \{u, v\}\} + \{v, \{w, u\}\} .$$

Indeed, we showed that the image of J is a graded $U(\mathfrak{g})$-module ideal and that $S(\overline{V})/J$ together with the bracket induced by r^- is a Poisson algebra, which we refer to as the *Poisson closure* of $(S(\overline{V}), \{\cdot,\cdot\})$ (Theorem 2.14 (b) and [2, Corollary 2.20]).

Using the construction of the classical limit developed in Section 2.3 we now derive the following corollary from Theorem 3.1.

Corollary 4.1. *Let V be a simple $U_q(sl_2)$-module. The classical limit of the braided symmetric algebra $\overline{S_\sigma(V)}$ is isomorphic, as an algebra, to the Poisson closure, of $S(\overline{V})$.*

Employing the notion of bracketed superalgebras (see [2, Section 2.2]), an analogous assertion for the braided exterior algebras of simple $U_q(sl_2)$-module was proved in [2]. Thus, our result gives further evidence for the following conjecture.

Conjecture 4.2. *Let V be an object in $\mathcal{O}_f$. The classical limit of the braided symmetric, resp. exterior algebra $\overline{S_\sigma(V)}$, resp. $\overline{\Lambda_\sigma(V)}$ is isomorphic, as an algebra, to the Poisson closure, of $S(\overline{V})$, resp. $\Lambda(\overline{V})$.*

4.2. **Non-Commutative Geometry.** In this section we will associate the braided symmetric algebras with non-commutative geometry in the sense of Artin, Tate and Van den Bergh [1]. First we will recall some of their definitions and results. Let V be a n-dimensional vectorspace. Every element of $f \in (V^*)^{\otimes k}$ defines a multilinear form on $V^{\otimes k}$. It induces a form on $(\mathbb{P}V)^{\otimes k}$, where $\mathbb{P}V$ is the projective space of lines in V. Since the form is multi-homogeneous, it defines a zero locus in $(\mathbb{P}V)^{\otimes k}$. Let $\mathcal{I}$ be a graded ideal in $T(V^*)$. Then the d-th graded component $\mathcal{I}_d$ of $\mathcal{I}$ defines a scheme Γ_d, where Γ_d is the scheme of zeros of $\mathcal{I}_d \subset (\mathbb{P}V)^{\otimes d}$. We have the following fact.

Proposition 4.3. [1, Proposition 3.5 ii] *Let $\mathbb{P}V_i$ denote the i-th factor of $(\mathbb{P}V)^{\otimes d}$ and for $1 \leq i \leq j \leq d$ denote by pr_{ij} the projection of $(\mathbb{P}V)^{\otimes d}$ onto $\mathbb{P}V_i \times \ldots \times \mathbb{P}V_j$. Then $pr_{ij}(\Gamma_d)$ is a closed subscheme of Γ_{j-i+1}.*

Notice that by applying projections $pr_{1(d-1)}$ to $(\mathbb{P}V)^{\otimes d}$, therefore mapping Γ_d to Γ_{d-1}, we can now consider the inverse limit Γ of the sets Γ_d. We need it in order to classify the point modules which we introduce in the following. Denote by A the algebra $A = T(V^*)/\mathcal{I}$.

Definition 4.4. [1, Definition 3.8] *A graded right A-module M is called a point module if it satisfies the following conditions:*

- *M is generated in degree zero,*
- *$M_0 = k$, and*
- *$\dim M_i = 1$ for all $i \geq 0$.*

Artin, Tate and Van Den Bergh classified the point modules in the following terms.

Proposition 4.5. [1, Corollary 3.13] *The point modules of A are in one-to-one correspondence with the points of Γ.*

Using our explicit description of the braided symmetric algebras of simple $U_q(sl_2)$-modules we obtain the following result.

Theorem 4.6. *(a) Let ℓ be odd. Then the point modules of $S_\sigma(V_\ell)$ are parametrized by the points of the curve, defined by the Veronese algebra $V_q(1,\ell)$ in $\mathbb{P}V_\ell$.*
(b) Let ℓ be even. Then the point modules of $S_\sigma(V_\ell)$ are parametrized by the points of the surface, defined by the Veronese algebra $V_q(2,\frac{\ell}{2})$ in $\mathbb{P}V_\ell$ which we embed $\mathbb{P}V_\ell \subset \mathbb{P}^{\binom{2+\frac{\ell}{2}}{\frac{\ell}{2}}}$ as the generating set of $S_\sigma(V_\ell)$, as in the proof of Theorem 3.1 (b).

Proof. Part (b) follows directly from Proposition 4.5 and the proof of Theorem 3.1 (b).

In order to prove (a) we have to observe that the quotient of $S_\sigma(V_\ell)$ by the ideal $\mathcal{N}$ is isomorphic to the Veronese algebra $V_q(1,\ell)$. The assertion follows. The theorem is proved. $\square$

Remark 4.7. *The assertion of Theorem 4.6 was proved in the special case $\ell = 3$ by Vancliff in* [14].

Appendix A. Veronese Algebras–Classical and Quantum

In this section we shall review the definitions of the quantum and classical Veronese algebras and varieties. For more details and proofs of the classical facts see e.g. [7]. Recall the definition of the *skew polynomials* $\mathbb{C}_q[x_0,\ldots,x_n]$ which are defined as the free algebra generated by $x_0,\ldots x_n$ and subject to the relations

$$x_j x_i = q^{-1} x_i x_j$$

for all $0 < i < j < n$. Note that it is a flat deformation of the polynomial algebra $\mathbb{C}[x_0,\ldots,x_n]$. If $x_I = x_{i_1}\ldots x_{i_d}$ and $x_J = x_{j_1}\ldots x_{j_\ell}$ with $i_1 \leq \ldots i_d$ and $j_1 \leq \ldots j_d$, then denote by $\Lambda(I,J)$ the number

$$\Lambda(I,J) = \sum_{m=1}^{\ell} \max\{k : i_k < j_m\} \ .$$

Clearly $x_I x_J = q^{\Lambda(I,J)} x_K$, where $K = x_{k_1}\ldots x_{k_{d+\ell}}$ with $k_1 \leq k_2 \leq \ldots \leq k_{d+\ell}$.

Definition A.1. *(a)The Veronese algebra* $V(n,d)$ *is the subalgebra of* $\mathbb{C}[x_0,\dots,x_n]$ *generated by the homogeneous elements of degree* d.
(b)The quantum Veronese algebra $V_q(n,d)$ *is the subalgebra of* $\mathbb{C}_q[x_0,\dots,x_n]$ *generated by the homogeneous elements of degree* d.

The Veronese algebra induces a map from $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$. Its image is called the *Veronese variety* $\mathcal{V}(n,d)$. The relations defining the Veronese variety can be described easily. Notice that we can describe the label coordinate functions x_I on $\mathbb{C}^{\binom{n+d}{d}}$ by the d-element partitions $I=(0\le i_1\le i_2\le\ldots\le i_d\le n)$, or rather $x_I=x_{i_1}\ldots x_{i_d}$. We have the following fact.

Lemma A.2. *(a)* [7, example 2.4] *The Veronese variety* $\mathcal{V}(n,d)$ *is the zero locus of all relations* $x_Ix_J=x_Kx_L\in\mathbb{C}[x_0,\dots,x_{\binom{n+d}{d}-1}]$ *such that* $x_Ix_J=x_Kx_L\in\mathbb{C}[x_0,\dots,x_n]$. *In particular, all the relations are quadratic.*
(b) The quantum Veronese algebra is the quotient of $\mathbb{C}_q[x_0,\dots,x_{\binom{n+d}{d}-1}]$ *by the ideal generated by the relations*

$$x_Ix_J=q^{\Lambda(I,J;K,L)}x_Kx_L\in\mathbb{C}_q[x_0,\dots,x_n]\ ,$$

where $\Lambda(I,J;K,L)=\Lambda(I,J)-\Lambda(K,L)$.

We will now consider the Veronese subalgebras in the case of $n=1$ and $n=2$. Recall that $\mathbb{C}[x_0,x_1]$ and $\mathbb{C}[x_0,x_1,x_2]$, resp. $\mathbb{C}_q[x_0,x_1]$and $\mathbb{C}_q[x_0,x_1,x_2]$ can be interpreted as symmetric algebras of the simple $U(sl_2)$-modules $S(\overline{V}_1)$ and $S(\overline{V}_2)$, resp. the simple $U_q(sl_2)$-modules $S_\sigma(V_1)$ and $S_\sigma(V_2)$. We can therefore think about the classical and quantum Veronese algebras as graded $U(sl_2)$, resp. $U_q(sl_2)$-module algebras. Using the well known decomposition of the symmetric powers of V_1 and V_2, we obtain the following fact.

Lemma A.3. *The graded components of the Veronese algebras* $V(1,d)$, $V_q(1,d)$, $V(2,d)$, $V_q(2,d)$ *split as follows:*

$$V(1,d)_k\cong\overline{V}_{kd}\ ,\quad V_q(1,d)_k\cong V_{kd}\ ,$$
$$V(2,d)_k\cong\bigoplus_{i=0}^{\frac{kd}{2}}\overline{V}_{2kd-4i}\ ,\quad V_q(2,d)_k\cong\bigoplus_{i=0}^{\frac{kd}{2}}V_{2kd-4i}\ .$$

Additionally, we have the following fact, relating Veronese algebras with classical and braided symmetric algebras.

Lemma A.4. *(a) Let* $\overline{V}$ *be a* $U(sl_2)$*-submodule of* $V(n,d)_1$. *Then there exists a surjective* $U(sl_2)$*-module homomorphism from the symmetric algebra* $S(\overline{V})$ *onto the subalgebra of* $V_{n,d}$ *generated by* $\overline{V}$.
(b) Let V *be a* $U_q(sl_2)$*-submodule of* $V_q(n,d)_1$, $n=1,2$. *Then there exists a surjective* $U_q(sl_2)$*-module homomorphism from the braided symmetric algebra* $S_\sigma(V)$ *onto the subalgebra of* $V_q(n,d)$ *generated by* V.

Proof. We first prove (b), and the proof of (a) will be analogous. Notice that if $V\subset V_q(n,d)_1$, then we can embed V in $V_n^{\otimes kd}$. Recall the alternate defiition of $S_\sigma(V)$ as in Remark 2.6. Recall that we can write $\mathcal{R}_{12\ldots kd,(kd+1)\ldots 2kd}$ acting on $V_n^{\otimes kd}\otimes V_n^{\otimes kd}$ as a product of $\mathcal{R}_{i,i+1}$. Because $S_\sigma(V_n)$ is flat when $n=1,2$, we see that if $\mathbf{v}\in V_n^{\otimes kd}\otimes V_n^{\otimes kd}$ is in the subspace spanned by the "positive" eigenvectors for each $\mathcal{R}_{i,i+1}$ then it can be expressed as a sum of eigenvectors with positive eigenvalue for $\mathcal{R}_{12\ldots kd,(kd+1)\ldots 2kd}$. The homomorphism $T(V) \to V_q(n,d)$ therefore factors through $S_\sigma(V)$. Part (b) is proved. Part(a) follows by an analogous argument, as the symmetric algebras $S(\overline{V}_n)$ are trivially flat. □

Notice that $\overline{V}_{2d} \subset V(2,d)_1$ and $V_{2d} \subset V_q(2,d)_1$. We have the following fact about the subalgebras $A(2,d)$, resp. $A_q(2,d)$ generated by $\overline{V}_{2d}$ and V_{2d}.

Lemma A.5. *(a) The subalgebra $A(2,d) \subset V(2,d)$ is a graded $U(sl_2)$-module algebra and $A_q(2,d) \subset V_q(2,d)$ is a graded $U_q(sl_2)$-module algebra. If $k \geq 2$ then*

$$A(2,d)_k = V(2,d)_k \ , \quad \text{and} \quad A_q(2,d)_k = V_q(2,d)_k \ .$$

(b) The ideal of relations of $A(2,d)$, resp. $A_q(2,d)$ is generated by quadratic and cubic elements.

Proof. Prove (a) first. We prove the assertion by induction on k. First, consider the submodule $\overline{V}_{2d} \subset V_q(2,d)_1$ of resp. $V_{2d} \subset V_q(2,d)_1$. It is generated by $\overline{x}_0^d$, resp. x_0^d. We have the following fact which can be proved by an easy induction proof using the definitions of Section 2.1.

Claim A.6. *(a) For each $m \geq 0$, we have*

$$F^m(\overline{x}_0^d) = \sum_{i,j\geq 0} \overline{c}_{ij}\overline{x}_0^i\overline{x}_1^{d-i-j}\overline{x}_2^j \ ,$$

$$F^m(x_0^d) = \sum_{i,j} c_{ij}x_0^ix_1^{d-i-j}x_2^j \ ,$$

where $\overline{c}_{ij} \in \mathbb{Z}_{>0}$ resp. the classical limit of c_{ij} is a positive integer, if and only if $2i-2j = 2d-2m$ (i.e. the monomial lies in the correct weight-space). Otherwise $\overline{c}_{ij} = c_{ij} = 0$.
(b)There exists a unique solution $(i,j) \in \mathbb{Z}^2_{\geq 0}$ of the system of equations $i+j=d$ and $i-j=d-m$ if and only if m is even. That means that if m is odd, then $F^m(x_0^d)$ is divisible by x_1.

Secondly, we also need the following fact which is easy to prove.

Claim A.7. *The highest weight vectors $\overline{\mathbf{v}}_\mathbf{m}$ in $S(\overline{V}_2)_k$, resp. $\mathbf{v}_\mathbf{m}$ in $S_\sigma(V_2)_k$ of weight $2d-2m$ are of the form*

$$\overline{\mathbf{v}}_\mathbf{m} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}\overline{x}_0^{d-m-i}\ \overline{x}_1^{2i}\ \overline{x}_2^{m+i} \ ,$$

$$\mathbf{v}_\mathbf{m} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}_q x_0^{d-m-i}x_1^{2i}x_2^{m+i} \ .$$

Now consider the products $\overline{V}_{2d}\cdot\overline{V}_{2d} \subset V(2,d)_2$ and $V_{2d}\cdot V_{2d} \subset V_q(2,d)_2$. It is easy to see that
(A.1)

$$E(\overline{\mathbf{v}}_\mathbf{i}) = E\left(\sum_{j=0}^{i}(-1)^j\binom{i}{j}\overline{v}_j\overline{v}_{2i-j}\right) = 0\ ,\ E(\mathbf{v}_\mathbf{i}) = E\left(\sum_{j=0}^{i}(-1)^j\binom{i}{j}_q v_jv_{2i-j}\right) = 0$$

we have to verify, however, that $\overline{\mathbf{v}}_{\mathbf{i}} \neq 0 \in V(2,d)$, resp. $\mathbf{v}_{\mathbf{i}} \neq 0 \in V_q(2,d)$. Claim A.6 (a) and (b) and (A.1) allow us to determine that

$$\overline{\mathbf{v}}_{\mathbf{i}} = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \overline{v}_j \overline{v}_{2i-j} = c\overline{x}_0^{2d-i}\overline{x}_2^{i} + \overline{x}_1(\dots) \ ,$$

$$\mathbf{v}_{\mathbf{i}} = \sum_{j=0}^{i} (-1)^j \binom{i}{j}_q v_j v_{2i-j} = c x_0^{2d-i} x_2^{i} + x_1(\dots)$$

with non-zero c, because of the following argument. Terms of the form $\overline{x}_0^{d-k}\overline{x}_2^{k}$, resp. $x_0^{d-k}x_2^k$ appear only in the weight vectors $\overline{v}_{2k}$, resp. v_{2k} in V_{2d}. But these terms appear only in summands with positive sign in (A.1). This implies that $c \neq 0$. Hence the $\overline{\mathbf{v}}_{\mathbf{i}}$, resp. $\mathbf{v}_{\mathbf{i}}$ are highest weight vectors of the desired weight and $\overline{V}_{2d} \cdot \overline{V}_{2d}$, therefore, generates $V(2,d)_2$, while $V_{2d} \cdot V_{2d}$ generates $V_q(2,d)_2$.

In the case of $V(2,d)_k$ and $V_q(2,d)_k$, $k \geq 3$, assume by induction that the assertion has been proven for $k-1$. We observe that we can express any monomial $\overline{x}_0^i\ \overline{x}_1^j\ \overline{x}_2^{kd-i-j}$, resp. $x_0^i x_1^j x_2^{kd-i-j}$ as

$$\overline{x}_0^i\ \overline{x}_1^j\ \overline{x}_2^{kd-i-j} = \sum_{m=0}^{2d} a_m \cdot \overline{v}_m \ ,$$

$$x_0^i x_1^j x_2^{kd-i-j} = \sum_{m=0}^{2d} a_m \cdot v_m \ ,$$

where $a_m \in V(2,d)_{k-1}$ and $v_m \in V_{2d}$ as defined above. In the cases when $i \geq 2d$ or $kd - i - j \geq 2d$, then it is obvious as there is only one nonzero summand v_0 or v_{2d}. In the remaining cases, we will set up a system of equations as follows. We write

$$\overline{x}_0^i\ \overline{x}_1^j x_2^{kd-i-j} = \sum_{m=0}^{d-i} a_m \cdot \overline{v}_m \ ,$$

$$x_0^i x_1^j x_2^{kd-i-j} = \sum_{m=0}^{d-i} a_m \cdot v_m \ ,$$

and it is easy to see that we can solve the resulting system of equations for the a_m. Lemma A.5(a) follows.

Now consider (b). Recall that $A(2,d)$ is a quotient of the tensor algebra by an ideal generated by homogeneous relations. Since $A(2,d)_k = V(2,d)_k$, for $k \geq 2$ and $A(2,d)_k \cdot A(2,d)_\ell = V(2,d)_m$ for all $k + \ell = m$, for $k, \ell \geq 2$ and since there are only quadratic relations in $V(2,d)$, the ideal is generated by homogeneous elements of degree higher ≤ 3. One argues analogously in the case of $A_q(2,d)$ and $V_q(2,d)$. Part(b) and, therefore, Lemma A.5 are proved. □

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