CONGRUENCES ON THE BELL POLYNOMIALS AND THE DERANGEMENT POLYNOMIALS

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Abstract. In this note, by the umbra calculus method, the Sun and Zagier's congruences involving the Bell numbers and the derangement numbers are generalized to the polynomial cases. Some special congruences are also provided.

Keywords: Bell polynomials; Derangement polynomials; Stirling numbers; Congruences.
2000 Mathematics Subject Classification: Primary 11B75; Secondary 05A15, 05A18, 11A07.

1. Introduction

It is well known that the first and second kind Stirling numbers $s(m, j)$ and $S(m, j)$ [8] are defined respectively by

$$x(x-1)\cdots(x-m+1) = \sum_{j=0}^{m} s(m,j)x^j, \tag{1.1}$$

$$\sum_{j=0}^{m} S(m,j)x(x-1)\cdots(x-j+1) = x^m. \tag{1.2}$$

The Bell polynomials $\{\mathcal{B}_n(x)\}_{n\geq 0}$ are defined by

$$\mathcal{B}_m(x) = \sum_{j=0}^{m} S(m,j)x^j.$$

It is clear that $\mathcal{B}_m(1)$ is the m-th Bell number, denoted by B_m, counting the number of partitions of $[m] = \{1, 2, \ldots, m\}$ (with $B_0 = 1$). The Bell polynomials $\mathcal{B}_m(x)$ satisfy the recurrence

$$\mathcal{B}_{m+1}(x) = x\sum_{j=0}^{m} \binom{m}{j}\mathcal{B}_j(x). \tag{1.3}$$

The derangement polynomials $\{\mathcal{D}_m(x)\}_{m\geq 0}$ are defined by

$$\mathcal{D}_m(x) = \sum_{j=0}^{m} \binom{m}{j} j!(x-1)^{m-j}.$$

Clearly, $\mathcal{D}_m(1) = m!$ and $\mathcal{D}_m(0)$ is the m-th derangement number, denoted by D_m, counting the number of fixed-point-free permutations on $[m]$ (with $D_0 = 1$). The derangement polynomials $\mathcal{D}_m(x)$, also called x-factorials of m, have been considerably investigated by Eriksen, Freij and Wästlund [2], Sun and Zhuang [10]. They obey the recursive relation

$$\mathcal{D}_m(x) = m\mathcal{D}_{m-1}(x) + (x-1)^m. \tag{1.4}$$

Recently, Sun [11] discovered experimentally that for a fixed positive integer m the sum $\sum_{k=0}^{p-1} B_k/(-m)^k$ modulo a prime p not dividing m is independent of the prime p, a typical case being

$$\sum_{k=0}^{p-1} \frac{B_k}{(-8)^k} \equiv -1853 \pmod p \quad \text{for all primes } p \neq 2 .$$

Later, Sun and Zagier [13] confirmed this conjecture and proved the nice result.

Theorem 1.1. *For any integer $m \geq 1$ and any prime $p \nmid m$, there hold*

$$(-x)^m \sum_{k=1}^{p-1} \frac{\mathcal{B}_k(x)}{(-m)^k} \equiv (-x)^p \sum_{k=0}^{m-1} \frac{(m-1)!}{k!}(-x)^k \pmod p.$$

Particularly, the case $x = 1$ generates

$$\sum_{k=1}^{p-1} \frac{B_k}{(-m)^k} \equiv (-1)^{m-1} D_{m-1} \pmod p. \tag{1.5}$$

Here for two polynomials $P(x), Q(x) \in \mathbb{Z}_p[x]$, by $P(x) \equiv Q(x) \pmod p$ we mean that the corresponding coefficients of $P(x)$ and $Q(x)$ are congruent modulo p.

In this note, we establish a more general result of Sun and Zagier's congruence.

Theorem 1.2. *For any integers $n \geq 0, m \geq 1$ and any prime $p \nmid m$, there hold*

$$x^m \sum_{k=1}^{p-1} \frac{\mathcal{B}_{n+k}(x)}{(-m)^k} \equiv x^p \sum_{k=0}^{n} S(n,k)(-1)^{m+k-1}\mathcal{D}_{m+k-1}(1-x) \pmod p, \tag{1.6}$$

or equivalently

$$x^m \sum_{j=0}^{n} s(n,j) \sum_{k=1}^{p-1} \frac{\mathcal{B}_{j+k}(x)}{(-m)^k} \equiv (-1)^{m+n-1} x^p \mathcal{D}_{m+n-1}(1-x) \pmod p. \tag{1.7}$$

In particular, the case $x = 1$ leads to

Corollary 1.3. *For any integers $n \geq 0, m \geq 1$ and any prime $p \nmid m$, there hold*

$$\sum_{k=1}^{p-1} \frac{B_{n+k}}{(-m)^k} \equiv \sum_{k=0}^{n} S(n,k)(-1)^{m+k-1} D_{m+k-1} \pmod p, \tag{1.8}$$

or equivalently

$$\sum_{j=0}^{n} s(n,j) \sum_{k=1}^{p-1} \frac{B_{j+k}}{(-m)^k} \equiv (-1)^{m+n-1} D_{m+n-1} \pmod p. \tag{1.9}$$

2. Proof of Theorem 1.2

Define the generalized Bell umbra $\mathbf{B_x}$, given by ${\mathbf{B_x}}^m = \mathcal{B}_m(x)$. (See [5, 6] for more information on the umbra calculus.) Then (1.3) can be rewritten as ${\mathbf{B_x}}^{m+1} = x(\mathbf{B_x} + 1)^m$. By linearity, for any polynomial $f(x)$ we have

$$\mathbf{B_x} f(\mathbf{B_x}) = x f(\mathbf{B_x} + 1),$$

which, by induction on integer $m \geq 0$, yields

$$\mathbf{B_x}(\mathbf{B_x} - 1) \cdots (\mathbf{B_x} - m + 1) f(\mathbf{B_x}) = x^m f(\mathbf{B_x} + m). \tag{2.1}$$

Lemma 2.1. *For any integers $m, n \geq 0$, there hold*

$$\mathbf{B_x}(\mathbf{B_x}-1)\cdots(\mathbf{B_x}-m+1)\mathbf{B_x}^n = x^m(\mathbf{B_x}+m)^n. \tag{2.2}$$

or equivalently

$$\sum_{j=0}^{m} s(m,j)\mathcal{B}_{j+n}(x) = x^m \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_j(x) m^{n-j}, \tag{2.3}$$

Proof. The case $f(x) = x^n$ in (2.1) produces (2.2). By setting $x = \mathbf{B_x}$ in (1.1), then (2.2) is just the umbral representation of (2.3). □

Lemma 2.2. *For any integer $m \geq 1$, there hold*

$$(\mathbf{B_x}-1)(\mathbf{B_x}-2)\cdots(\mathbf{B_x}-m+1) = (-1)^{m-1}\mathcal{D}_{m-1}(1-x), \tag{2.4}$$

or equivalently

$$\sum_{j=0}^{m} s(m,j)\mathcal{B}_{j-1}(x) = (-1)^{m-1}\mathcal{D}_{m-1}(1-x). \tag{2.5}$$

Proof. Let $\mathcal{A}_m(x)$ denote the expression on the left hand side of (2.4), by the case $n = 0$ in (2.2), we have

$$\begin{aligned}
\mathcal{A}_{m+1}(x) &= (\mathbf{B_x}-1)\cdots(\mathbf{B_x}-m+1)(\mathbf{B_x}-m) \\
&= \mathbf{B_x}(\mathbf{B_x}-1)\cdots(\mathbf{B_x}-m+1) - m(\mathbf{B_x}-1)\cdots(\mathbf{B_x}-m+1) \\
&= x^m - m\mathcal{A}_m(x).
\end{aligned}$$

By (1.4), it is routine to check that $(-1)^m\mathcal{D}_m(1-x)$ also obey the same recurrence as $\mathcal{A}_{m+1}(x)$ and $\mathcal{A}_1(x) = 1 = \mathcal{D}_0(1-x)$. Hence $\mathcal{A}_{m+1}(x) = (-1)^m\mathcal{D}_m(1-x)$, which proves (2.4).

For (2.5), by setting $x = \mathbf{B_x}$ in (1.1) after dividing an x on the two sides of (1.1), we can represent (2.5) umbrally as (2.4) and vice versa. □

Remark 2.3. *The case $x = 1$ in (2.5) produces*

$$\sum_{j=0}^{m} s(m,j)B_{j-1} = (-1)^{m-1}D_{m-1}.$$

It is curious that such a simple and interesting identity did not appear in the literature.

Remark 2.4. *By the orthogonal relationship between the two types of Stirling numbers,*

$$\sum_{j=k}^{m} s(m,j)S(j,k) = \delta_{m,k}, \tag{2.6}$$

where $\delta_{m,k}$ is the Kronecker symbol defined by $\delta_{m,k} = 1$ if $m = k$ and $\delta_{m,k} = 0$ otherwise, one can obtain another equivalent form of (2.3) and (2.5)

$$\begin{aligned}
\mathcal{B}_{m+n}(x) &= \sum_{k=0}^{m} S(m,k)x^k \sum_{j=0}^{n} \binom{n}{j}\mathcal{B}_j(x)k^{n-j}, \\
\mathcal{B}_{m-1}(x) &= \sum_{k=0}^{m} S(m,k)(-1)^{k-1}\mathcal{D}_{k-1}(1-x).
\end{aligned} \tag{2.7}$$

It should be noticed that (2.7) has been obtained by Spivey [7] in the case $x = 1$, Gould and Quaintance [4], Belbachir and Mihoubi [1] using different methods. By $S(p,1) = S(p,p) = 1$

and $p|S(p,k)$ for a prime p and $1<k<p$, we have immediately the following congruence relations

$$\mathcal{B}_{p+n}(x) \equiv x^p\mathcal{B}_n(x)+\mathcal{B}_{n+1}(x) \pmod p, \tag{2.8}$$
$$\mathcal{B}_{p-1}(x) \equiv 1+\mathcal{D}_{p-1}(1-x) \pmod p. \tag{2.9}$$

Note that (2.8) has been obtained by Gertsch and Robert [3]*, and the case $x=1$ in (2.8) reduces to the well-known Touchard's congruence $B_{p+n}\equiv B_n+B_{n+1}$ (mod p)* [14]*. The case $x=1$ in (2.9) yields a new congruence*

$$B_{p-1}\equiv 1+D_{p-1} \pmod p. \tag{2.10}$$

Later, Sun [12] *informed us that they also independently obtained (2.10) as a corollary of (1.5).*

Proof of Theorem 1.2. It suffices to prove (1.7), for (1.6) can be obtained from (1.7) by using the orthogonality in (2.6). Setting $x=\mathbf{B_x}$ in the Lagrange congruence

$$x(x-1)\cdots(x-p+1)\equiv x^p-x \pmod p,$$

by (2.2) in the case $n=0$, we have

$$\mathbf{B_x}^p-\mathbf{B_x}\equiv x^p \pmod p.$$

Using the congruence $\binom{p-1}{k}\equiv(-1)^{p-k-1}(\text{mod }p)$ and the Fermat's congruence $m^{p-1}\equiv 1$ (mod p), where m is any integer not divided by the prime p, we get

$$\begin{aligned}
&x^m\sum_{j=0}^{n}s(n,j)\sum_{k=1}^{p-1}\frac{\mathcal{B}_{j+k}(x)}{(-m)^k}\\
&= x^m\sum_{k=1}^{p-1}\frac{1}{(-m)^k}\sum_{j=0}^{n}s(n,j)\mathcal{B}_{j+k}(x)\\
&\equiv x^m\sum_{k=1}^{p-1}\binom{p-1}{k}m^{p-k-1}x^n(\mathbf{B_x}+n)^k \pmod p\\
&= x^{m+n}((\mathbf{B_x}+n+m)^{p-1}-m^{p-1})\\
&= \mathbf{B_x}(\mathbf{B_x}-1)\cdots(\mathbf{B_x}-(m+n)+1)(\mathbf{B_x}^{p-1}-m^{p-1})\\
&\equiv \mathbf{B_x}(\mathbf{B_x}-1)\cdots(\mathbf{B_x}-(m+n)+1)(\mathbf{B_x}^{p-1}-1) \pmod p\\
&= (\mathbf{B_x}-1)\cdots(\mathbf{B_x}-(m+n)+1)(\mathbf{B_x}^{p}-\mathbf{B_x})\\
&\equiv x^p(\mathbf{B_x}-1)\cdots(\mathbf{B_x}-(m+n)+1) \pmod p\\
&= (-1)^{m+n-1}x^p\mathcal{D}_{m+n-1}(1-x),
\end{aligned}$$

as desired. □

3. Special Consequences

The cases $n=1$ and $n=2$ in (1.8) produce

Corollary 3.1. *For any integer $m\ge 1$ and any prime $p\nmid m$, there hold*

$$\sum_{k=1}^{p-1}\frac{B_{k+1}}{(-m)^k} \equiv (-1)^m D_m \pmod p, \tag{3.1}$$
$$\sum_{k=1}^{p-1}\frac{B_{k+2}}{(-m)^k} \equiv (-1)^m (D_m-D_{m+1}) \pmod p.$$

Corollary 3.2. *For any integers $n, m \geq 0$ and any prime p, there hold*

$$D_{pn+m} \equiv (-1)^n D_m \pmod{p}, \tag{3.2}$$

$$\sum_{k=1}^{p-1}(-1)^k B_{n+k} \equiv V_n \pmod{p}, \tag{3.3}$$

$$\sum_{k=1}^{p-1} B_{n+k} - \sum_{k=1}^{n-1} B_k \equiv D_{p-1} \pmod{p}. \tag{3.4}$$

where $V_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_k$ is the number of partitions of $[n]$ without singletons (i.e., one-element subsets) [9].

Proof. Setting $m := pn + m$ in (3.1), by $(-1)^{pn} \equiv (-1)^n \pmod{p}$, one can get (3.2). Setting $m = 1$ in (1.8), we have

$$\begin{aligned}
\sum_{k=1}^{p-1}(-1)^k B_{n+k} &\equiv \sum_{k=0}^{n} S(n,k)(-1)^k D_k \pmod{p} \\
&= \sum_{k=0}^{n} S(n,k)(\mathbf{B}-1)(\mathbf{B}-2)\cdots(\mathbf{B}-k) \\
&= (\mathbf{B}-1)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathbf{B}^k \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_k = V_n,
\end{aligned}$$

where $\mathbf{B} := \mathbf{B_x}|_{x=1}$ is the Bell umbra. Thus (3.3) follows.

By the Touchard's congruence, we have

$$\begin{aligned}
&\sum_{k=1}^{p-1} B_{n+k} - \sum_{k=1}^{n-1} B_k \\
&\equiv \sum_{k=1}^{p-1} B_{n+k} - \sum_{j=1}^{n-1}(B_{p+j} - B_{j+1}) \pmod{p} \\
&= \sum_{k=1}^{p-1} B_{n+k} - \sum_{j=1}^{n-1}\sum_{k=1}^{p-1}(B_{j+k+1} - B_{j+k}) \\
&= \sum_{k=1}^{p-1} B_{n+k} - \sum_{k=1}^{p-1}\sum_{j=1}^{n-1}(B_{j+k+1} - B_{j+k}) \\
&= \sum_{k=1}^{p-1} B_{n+k} - \sum_{k=1}^{p-1}(B_{n+k} - B_{k+1}) \\
&= \sum_{k=1}^{p-1} B_{k+1} \equiv D_{p-1} \pmod{p},
\end{aligned}$$

where the last step is obtained by setting $m := p - 1$ in (3.1). □

Acknowledgements

The authors are grateful to Prof. Zhi-Wei Sun for the helpful suggestions and comments. The work was partially supported by The National Science Foundation of China (Grant No. 10801020 and 11001035) and by the Fundamental Research Funds for the Central Universities (Grant No. 2009QN070 and 2009QN071).

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