On the global solutions of the Higgs boson equation

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Abstract

In this article we study global in time (not necessarily small) solutions of the equation for the Higgs boson in the Minkowski and in the de Sitter spacetimes. We reveal some qualitative behavior of the global solutions. In particular, we formulate sufficient conditions for the existence of the zeros of global solutions in the interior of their supports, and, consequently, for the creation of the so-called bubbles, which have been studied in particle physics and inflationary cosmology. We also give some sufficient conditions for the global solution to be an oscillatory in time solution.

1 Introduction

In this article we study the global in time, not necessarily small, solutions of the Higgs boson equation in the Minkowski and in the de Sitter spacetime. The Higgs boson plays a fundamental role in unified theories of weak, strong, and electromagnetic interactions [23].

In the model of the universe proposed by de Sitter, the line element has the form

$$ds^2 = -\left(1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 + \left(1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2).$$

The constant M_{bh} may have a meaning of the "mass of the black hole". The corresponding metric with this line element is called the Schwarzschild - de Sitter metric. In the present paper we focus on the limit case; namely, we set $M_{bh} = 0$ to ignore completely any influence of the black hole. Thus, the line element in the de Sitter spacetime has the form

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) c^2\, dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

The Lamaître-Robertson transformation [15] leads to the following form for the line element [15, Sec.134]:

$$ds^2 = -c^2\, dt'^2 + e^{2ct'/R}(dx'^2 + dy'^2 + dz'^2)\,.$$

Here R is the "radius" of the universe. In the Robertson-Walker spacetime [9], one can choose coordinates so that the metric has the form

$$ds^2 = -dt^2 + S^2(t) d\sigma^2\,.$$

In particular, the metric in de Sitter spacetime in the Lamaître-Robertson coordinates [15] has this form with the cosmic scale factor $S(t) = e^t$.

The matter waves in the de Sitter spacetime are described by the function ϕ, which satisfies equations of motion. In the de Sitter universe the equation for the scalar field with potential function V is the covariant wave equation

$$\square_g \phi = V'(\phi) \quad \text{or} \quad \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|} g^{ik} \frac{\partial \phi}{\partial x^k}\right) = V'(\phi)\,,$$

with the usual summation convention. Written explicitly in coordinates in the de Sitter spacetime it, in particular, for

$$V'(\phi) = -\mu^2\phi + \lambda|\phi|^{p-1}\phi, \qquad p > 1,$$

has the form

$$\phi_{tt} + n\phi_t - e^{-2t}\Delta\phi = \mu^2\phi - \lambda|\phi|^{p-1}\phi\,, \tag{1}$$

where $\mu > 0$ and $\lambda > 0$. The equation for the Higgs real-valued scalar field in the de Sitter spacetime is a special case of (1) when $p = 3$, $n = 3$:

$$\phi_{tt} + 3\phi_t - e^{-2t}\Delta\phi = \mu^2\phi - \lambda\phi^3\,. \tag{2}$$

Scalar fields play a fundamental role in the standard model of particle physics, as well as its possible extensions. In particular, scalar fields generate spontaneous symmetry breaking and provide masses to gauge bosons and chiral fermions by the Brout-Englert-Higgs mechanism [6] using a Higgs-type potential [10].

The energy

$$E(t) = e^{nt}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\phi_t + \frac{n}{2}\phi|^2 + \frac{1}{2}e^{-2t}|\nabla_x\phi|^2 - \frac{1}{2}\left(\frac{n^2}{4} + \mu^2\right)|\phi|^2 + \frac{1}{p+1}\lambda|\phi|^{p+1}\right)dx \tag{3}$$

of the $L^q(\mathbb{R}^n)$-solution of the equation (1) is non-increasing since

$$\frac{d}{dt}E(t) = -e^{nt}\int_{\mathbb{R}^n}\left(e^{-2t}|\nabla_x\phi|^2 + \lambda\frac{n(p-1)}{2(p+1)}|\phi|^{p+1}\right)dx\,.$$

The constants $\phi = \pm\frac{\mu}{\sqrt{\lambda}}$ are non-trivial real-valued solutions of the equation (2) with the positive energy density $e^{nt}\frac{\mu^2}{8\lambda}(2n^2-1)$. The x-independent solution of (2) solves the Duffing's-type equation

$$\ddot{\phi} + 3\dot{\phi} = \mu^2\phi - \lambda\phi^3\,,$$

which describes the motion of a mechanical system in a twin-well potential field.

Unlike the equation in the Minkowski spacetime, that is, the equation

$$\phi_{tt} - \Delta\phi = \mu^2\phi - \lambda|\phi|^{p-1}\phi\,, \tag{4}$$

the equation (1) has no other time-independent solution. For the equation (4) the existence of a weak global solution in the energy space is known (see, e.g., Proposition 3.2 [8]) under certain conditions. The equation

$$\phi_{tt} - \Delta\phi = \mu^2\phi - \lambda\phi^3 \tag{5}$$

for the Higgs scalar field in the Minkowski spacetime has the time-independent flat solution

$$\phi_M(x) = \frac{\mu}{\sqrt{\lambda}}\tanh\left(\frac{\mu^2}{2}N\cdot(x-x_0)\right), \qquad N, x_0, x \in \mathbb{R}^3. \tag{6}$$

The unit vector N defines the direction of the propagation of the wave front. The solution (6), after Lorentz transformation, gives rise to a traveling solitary wave of the form

$$\phi_M(x,t) = \frac{\mu}{\sqrt{\lambda}}\tanh\left(\frac{\mu^2}{2}[N\cdot(x-x_0) \pm v(t-t_0)]\frac{1}{\sqrt{1-v^2}}\right), \qquad N, x_0, x \in \mathbb{R}^3,\ t \geq t_0, \tag{7}$$

if $0 < v < 1$, where v is the initial velocity. The set of zeros of the solitary wave $\phi = \phi_M(x,t)$, that is, the set given by $N\cdot(x-x_0) \pm v(t-t_0) = 0$, is the moving boundary of the *wall*. Existence of standing waves $\phi = \exp(i\omega t)v(x)$, which are exponentially small at infinity $|x| = \infty$, and of corresponding solitary waves for the equation (5) with $\mu^2 < 0$ and $\lambda < 0$ is known (see, e.g., [19]).

It is of considerable interest for particle physics and inflationary cosmology to study the so-called bubbles [5], [14], [22]. In [13] bubble is defined as a simply connected domain surrounded by a wall such that the field $\phi(r,t)$ approaches one of the vacuums outside of a bubble. For a spherically symmetric bubble the radius is defined at the zero of the function ϕ: $\phi(R(t),t)=0$. The creation and growth of bubbles is an interesting mathematical problem [5, Ch.7], [14] and it motivates our interest in the sign changing global solutions.

A global in time solvability of the Cauchy problem for equations (1) and (2) is not known. For the wave maps on Robertson–Walker spacetimes, whose inverse radius is integrable with respect to the cosmic time, and, in particular, on the de Sitter spacetime, a global existence is proven [4]. The local solution exists for every smooth initial data. The C^2 solution of the equation (2) is unique and obeys the finite speed of the propagation property. (See, e.g., [11].) For the Higgs scalar field in the de Sitter spacetime, that is for the equation (2), Theorem 4.1 of Section 4 gives the necessary conditions for the existence of the global solution. In particular, it gives the necessary conditions for the global in time existence of bubbles. In the forthcoming paper we prove the existence of the global in time small data solutions for the equation (1).

Since we are interested in the properties of global solutions in the de Sitter spacetime, we mention here two recent articles on linear equations on the asymptotically de Sitter spacetimes. Vasy [21] exhibited the well-posedness of the Cauchy problem and showed that on such spaces, the solution of the Klein-Gordon equation without source term and with smooth Cauchy data has an asymptotic expansion at infinity. He also showed that solutions of the wave equation exhibit scattering. Baskin [2] constructed parametrix for the forward fundamental solution of the wave and Klein-Gordon equations on asymptotically de Sitter spaces without caustics and used this parametrix to obtain asymptotic expansions for solutions of the equation with some class of source terms. (For more references on the asymptotically de Sitter spaces, see the bibliography in [2], [21].)

In order to make our results more transparent we formulate them for the function $u=e^{\frac{n}{2}t}\phi$. For this new unknown function $u=u(x,t)$, the equation (1) takes the form of the semilinear Klein-Gordon equation for u

$$u_{tt}-e^{-2t}\bigtriangleup u-M^2u=-\lambda e^{-\frac{n(p-1)}{2}t}|u|^{p-1}u, \tag{8}$$

where the "curved mass" $M\geq 0$ is defined as follows:

$$M^2:=\frac{n^2}{4}+\mu^2>0\,. \tag{9}$$

The equation (8) is the so-called equation with imaginary mass. Equations with imaginary mass appear in several physical models such as ϕ^4 field model, tachion (super-light) fields, Landau-Ginzburg equation and others.

Next, we use the fundamental solution of the corresponding linear operator in order to reduce the Cauchy problem for the semilinear equation to the integral equation and to define a weak solution. We denote by G the resolving operator of the problem

$$u_{tt}-e^{-2t}\bigtriangleup u-M^2u=f, \quad u(x,0)=0, \quad \partial_t u(x,0)=0\,. \tag{10}$$

Thus, $u=G[f]$. The equation of (10) is strictly hyperbolic. This implies the well-posedness of the Cauchy problem (10) in different functional spaces. Consequently, the operator G is well-defined in those functional spaces.

The operator G is explicitly written in [25] for the case of the real mass; that is, with the mass term $+M^2u$ in the equation (10). The analytic continuation with respect to the parameter M of this operator allows us also to use G in the case of imaginary mass. More precisely, for $M\geq 0$ we define the operator G acting on $f(x,t)\in C^\infty(\mathbb{R}\times[0,\infty))$ by

$$\begin{aligned} G[f](x,t) \quad := \quad & \int_0^t db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\, f(y,b)(4e^{-b-t})^{-M}\Big((e^{-t}+e^{-b})^2-(x-y)^2\Big)^{-\frac{1}{2}+M}\\ & \times F\Big(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-(x-y)^2}{(e^{-b}+e^{-t})^2-(x-y)^2}\Big), \end{aligned}$$

where $F\big(a,b;c;\zeta\big)$ is the hypergeometric function. (See, e.g., [3].) For analytic continuation, see e.g., [18, Sec. 1.8]. For $n\geq 2$, in both cases of even and odd n, one can write

$$G[f](x,t) \quad = \quad 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, v(x,r;b)(4e^{-b-t})^{-M}\left((e^{-t}+e^{-b})^2-r^2\right)^{-\frac{1}{2}+M}$$

$$\times F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right),$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt}-\triangle v=0\,,\quad v(x,0;b)=f(x,b)\,,\quad v_t(x,0;b)=0\,.$$

It can be proved that if $n\big(\frac{1}{q'}-\frac{1}{q}\big)\le 1$, $\frac{1}{q'}+\frac{1}{q}=1$, $1\le q'\le 2\le q\le\infty$, then for every given $T>0$ the operator G can be extended to a bounded operator:

$$G\,:\,C([0,T];L^{q'}(\mathbb{R}^n))\longrightarrow C([0,T];L^{q}(\mathbb{R}^n))\,.$$

Consequently, the operator G maps

$$G\,:\,C([0,\infty);L^{q'}(\mathbb{R}^n))\longrightarrow C([0,\infty);L^{q}(\mathbb{R}^n)),$$

in the corresponding topologies. Moreover,

$$G\,:\,C([0,\infty);L^{q'}(\mathbb{R}^n))\longrightarrow C^1([0,\infty);\mathcal{D}'(\mathbb{R}^n)).$$

Let $u_0=u_0(x,t)$ be a solution of the Cauchy problem

$$\partial_t^2u_0-e^{-2t}\triangle u_0-M^2u_0=0,\quad u_0(x,0)=\varphi_0(x),\quad \partial_tu_0(x,0)=\varphi_1(x)\,. \tag{11}$$

Then any solution $u=u(x,t)$ of the equation (8), which takes initial value $u(x,0)=\varphi_0(x)$, $\partial_tu(x,0)=\varphi_1(x)$, solves the integral equation

$$u(x,t)=u_0(x,t)-G[\lambda e^{-\frac{n(p-1)}{2}\cdot}|u|^{p-1}u](x,t)\,. \tag{12}$$

We use the last equation to define a weak solution of the problem for the differential equation. Let $\Gamma\in C([0,\infty))$. For every given function $u_0\in C([0,T];L^{q'}(\mathbb{R}^n))$ we consider the integral equation

$$u(x,t)=u_0(x,t)-G\left[\Gamma(\cdot)\left|\int_{\mathbb{R}^n}|u(y,\cdot)|^{p-1}u(y,\cdot)\,dy\right|^{\beta}|u(y,\cdot)|^{p-1}u(y,\cdot)\right](x,t)\,, \tag{13}$$

for the function

$$u\in\bigcap_{i=1,p,q}C([0,T];L^{i}(\mathbb{R}^n)).$$

Here $\beta\in\mathbb{R}$, $q'\ge q>1$, $p\ge 1$. The last integral equation corresponds to a slightly more general equation than (8), namely, to the equation with *non-local nonlinearity* (*non-local self-interaction*)

$$u_{tt}-e^{-2t}\triangle u-M^2u=-\Gamma(t)\left|\int_{\mathbb{R}^n}|u(y,t)|^{p-1}u(y,t)dy\right|^{\beta}|u|^{p-1}u\,. \tag{14}$$

Definition 1.1 *If u_0 is a solution of the Cauchy problem (11), then the solution $u=u(x,t)$ of (13) is said to be a weak solution of the Cauchy problem for the equation (14) with the initial conditions $u(x,0)=\varphi_0(x)$, $\partial_tu(x,0)=\varphi_1(x)$.*

We are looking for the sufficient conditions for the continuous global solution for the changing of the sign, and, consequently, for the creation of a bubble. It turns out that, such conditions are also necessary conditions for the solution of equation (14) to exist globally in time. Equivalently, we are looking for the sufficient conditions on the weak solution of the equation that guarantee, in general, the non-existence of a global in time weak solution, namely, the blow-up phenomena. In order to prove a sign-changing property of the global solutions of the semilinear Klein-Gordon equations (1), (4) we invoke the so-called Functional Method that has been used to reveal blow-up phenomena for several equations (see, e.g., [1, Ch. 2]).

In the next definition we measure the variation of the sign of the function $\phi=\phi(x)$ by the deviation from the Hölder inequality of the inequality between the integral of the function and the self-interaction functional. Time t is regarded as a parameter.

Definition 1.2 *The function $\phi \in C([0,\infty);\ L^p(\mathbb{R}^n))$ is said to be asymptotically time-weighted L^p-non-positive (non-negative), if there is a non-negative number C_ϕ and positive non-decreasing function $\nu_\phi \in C([0,\infty))$ such that with $\sigma = 1$ ($\sigma = -1$) one has*

$$\left| \int_{\mathbb{R}^n} \phi(x,t)\,dx \right|^p \le -\sigma C_\phi \nu_\phi(t) \int_{\mathbb{R}^n} |\phi(x,t)|^{p-1}\phi(x,t)\,dx \quad \textit{for all sufficiently large} \quad t.$$

It is evident that any sign preserving function $\phi \in L^p(\mathbb{R}^n)$ with a compact support satisfies the last inequality with $\nu_\phi(t) \equiv 1$ and either $\sigma = 1$ or $\sigma = -1$, while $C_\phi^{1/(p-1)}$ is a measure of the support. Then, any smooth global non-positive (non-negative) solution $\phi = \phi(x,t)$ of the Cauchy problem for (1) or (4) with compactly supported initial data is also asymptotically time-weighted L^p-non-positive (non-negative) with the weight $\nu_\phi(t) = (1+t)^{n(p-1)}$.

Theorem 1.3 *Let $u = u(x,t) \in C([0,\infty); L^q(\mathbb{R}^n))$, $2 \le q < \infty$, be a global solution of the equation (13) with $\beta > 1/p - 1$. Suppose that the function $\Gamma \in C^1([0,\infty))$ is either non-decreasing or non-increasing. Assume that the function $u_0 \in C^1(\mathbb{R}^n \times [0,\infty))$ satisfies*

$$\sigma \left(M \int_{\mathbb{R}^n} u_0(x,0)\,dx + \int_{\mathbb{R}^n} \partial_t u_0(x,0)\,dx \right) > 0\,, \tag{15}$$

and, additionally,

$$\int_{\mathbb{R}^n} u_0(x,t)dx = \cosh(Mt) \int_{\mathbb{R}^n} u_0(x,0)\,dx + \frac{1}{M}\sinh(Mt) \int_{\mathbb{R}^n} \partial_t u_0(x,0)\,dx \quad \textit{for all} \quad t \ge 0 \tag{16}$$

if $M > 0$, while for $M = 0$

$$\int_{\mathbb{R}^n} u_0(x,t)dx = \int_{\mathbb{R}^n} u_0(x,0)\,dx + t\int_{\mathbb{R}^n} \partial_t u_0(x,0)\,dx \quad \textit{for all} \quad t \ge 0. \tag{17}$$

Assume also that the self-interaction functional satisfies

$$\sigma \int_{\mathbb{R}^n} |u(z,b)|^{p-1} u(z,b)\,dz \le 0 \tag{18}$$

for all t either outside of the sufficiently small neighborhood of zero if $M > 0$, or inside of some neighborhood of infinity if $M = 0$.

Then, the global solution $u = u(x,t)$ cannot be an asymptotically time-weighted L^p-non-positive (-non-negative) with the weight $\nu_u \in C^1([0,\infty))$ such that if $M > 0$, then

$$\Gamma(t) \ge c\nu_u(t)^{\beta+1} e^{-M(p(\beta+1)-1)t} t^{2+\varepsilon} \quad \textit{for all large} \quad t$$

with the numbers $\varepsilon > 0$ and $c > 0$, while for $M = 0$ it satisfies

$$\Gamma(t) \ge ct^{-1-p(\beta+1)} \nu_u(t)^{\beta+1} \quad \textit{for all large} \quad t\,.$$

The conditions (16), (17) are inherited from the partial differential equation (11). In fact, every smooth integrable solution of (11) satisfies (16) or (17).

Thus, the theorem shows that the continuous, asymptotically time-weighted L^p-non-positive (non-negative) global solution of the equation (13) cannot be sign preserving if it is generated by the function $u_0 = u_0(x,t)$, which obeys (15), (16), (17). Consequently, smooth asymptotically time-weighted L^p-non-positive (non-negative) global solution of the equation (8) cannot be sign preserving if its initial data φ_0, φ_1 imply (15).

An application of the last theorem to the generalized Higgs real-valued scalar field equation (1) with $\mu > 0$ results in the following corollary.

Corollary 1.4 *Let $\phi = \phi(x,t) \in C([0,\infty); L^q(\mathbb{R}^n))$, $2 \le q < \infty$, be a global weak solution of the equation (1). Assume also that the initial data of $\phi = \phi(x,t)$ satisfy*

$$\sigma \left(\left(\sqrt{\frac{n^2}{4} + \mu^2} + \frac{n}{2} \right) C_0(\phi) + C_1(\phi) \right) > 0$$

with $\sigma = 1$ ($\sigma = -1$), while

$$\sigma \int_{\mathbb{R}^3} |\phi(x,t)|^{p-1}\phi(x,t)\,dx \le 0$$

is fulfilled for all t outside of the sufficiently small neighborhood of zero.

Then, the global solution $\phi = \phi(x,t)$ cannot be an asymptotically time-weighted L^p-non-positive (-non-negative) solution with the weight $\nu_\phi(t) = e^{a_\phi t} t^{b_\phi}$, where either $a_\phi < (p-1)\left(\sqrt{\frac{n^2}{4}+\mu^2} - \frac{n}{2}\right)$, $b_\phi \in \mathbb{R}$, or $a_\phi = (p-1)\left(\sqrt{\frac{n^2}{4}+\mu^2} - \frac{n}{2}\right)$, $b_\phi < -2$.

The numbers $\pm\sqrt{\frac{n^2}{4}+\mu^2} - \frac{n}{2}$ of the last corollary are the roots of the characteristic equation of the linear ordinary differential part of (1). They also appear in the exponents $s_\pm(\lambda)$ of the suggested in [21] representation of the solution of the Cauchy problem for the linear Klein-Gordon operator on the asymptotically de Sitter-like spaces.

Consider now the case of the function $u_0 = u_0(x,t)$, which has support in the cylinder $B^n_R(0) \times [0,\infty)$, where $B^n_R(0)$ is a ball in $\mathbb{R}^n$ with radius $R > 0$. The smooth solution of the Cauchy problem for the linear equation (11) with the compactly supported initial data can exemplify such $u_0 = u_0(x,t)$. The diversity of the eigenfunctions of the Laplace operator gives a more sensitive test to find out the sign changing solutions for the equation (14) with $\beta = 0$.

Definition 1.5 *The function $\phi \in C([0,\infty); L^p(\mathbb{R}^n))$ is said to be asymptotically time-weighted $-\psi$ L^p -signed if there are a non-negative number $C_{\phi,\psi}$, and positive non-decreasing function $\nu_{\phi,\psi} \in C([0,\infty))$ such that the following inequality holds*

$$\left|\int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx\right|^p \le -C_{\phi,\psi}\nu_{\phi,\psi}(t)\int_{\mathbb{R}^n} \psi(x)|\phi(x,t)|^{p-1}\phi(x,t)\,dx \quad \textit{for all large} \quad t.$$

It is evident that if with some eigenfunction ψ the function $\psi\phi \in L^p(\mathbb{R}^n)$ has a compact support and is sign preserving, then the last inequality holds with either $\nu_{\phi,\psi}(t) \equiv 1$ or $\nu_{\phi,\psi}(t) \equiv -1$, while $C^{1/(p-1)}_{\phi,\psi}$ is a measure of the support.

Theorem 1.6 *Let $u = u(x,t) \in C([0,\infty); L^q(\mathbb{R}^n))$, $2 \le q < \infty$, be a global solution of the equation*

$$u(x,t) = u_0(x,t) - G\left[\Gamma(\cdot)|u(y,\cdot)|^{p-1}u(y,\cdot)\right](x,t) \tag{19}$$

with $p > 1$. Suppose that the function $\Gamma \in C^1([0,\infty))$ is either non-decreasing or non-increasing. Let $\psi = \psi(x)$ be an eigenfunction $\psi = \psi(x)$ of the Laplace operator in $\mathbb{R}^n$ corresponding to the eigenvalue ν. Further, suppose that $u_0 \in C^1(\mathbb{R}^n \times [0,\infty))$ with the support in the $B^n_R(0) \times [0,\infty)$ satisfies

$$(M+\nu)\int_{\mathbb{R}^n} \psi(x)u_0(x,0)\,dx + \int_{\mathbb{R}^n} \psi(x)\partial_t u_0(x,0)\,dx > 0\,, \tag{20}$$

and, additionally

$$\begin{aligned}\int_{\mathbb{R}^n} \psi(x)u_0(x,t)dx &= \cosh((M+\nu)t)\int_{\mathbb{R}^n} \psi(x)u_0(x,0)\,dx \\ &\quad + \frac{1}{M+\nu}\sinh((M+\nu)t)\int_{\mathbb{R}^n} \psi(x)\partial_t u_0(x,0)\,dx \quad \textit{for all} \quad t \ge 0,\end{aligned} \tag{21}$$

if $M+\nu > 0$, while for $M+\nu = 0$

$$\int_{\mathbb{R}^n} \psi(x)u_0(x,t)dx = \int_{\mathbb{R}^n} \psi(x)u_0(x,0)\,dx + t\int_{\mathbb{R}^n} \psi(x)\partial_t u_0(x,0)\,dx \quad \textit{for all} \quad t \ge 0\,. \tag{22}$$

Assume that the self-interaction functional satisfies

$$\int_{\mathbb{R}^n} \psi(x)|u(z,b)|^{p-1}u(z,b)\,dz \le 0\,,$$

for all t either outside of the sufficiently small neighborhood of zero if $M+\nu>0$, or inside of some neighborhood of infinity if $M+\nu=0$.

Then, the global solution $u=u(x,t)$ cannot be an asymptotically time-weighted $-\psi$ L^p-signed with the weight $\nu_{u,\psi}\in C^1([0,\infty))$ such that if $M+\nu>0$, then

$$\Gamma(t) \ge c\nu_{u,\psi}(t)e^{-(M+\nu)(p-1)t}t^{2+\varepsilon} \quad \textit{for all large} \quad t$$

with the numbers $\varepsilon>0$ and $c>0$, while for $M+\nu=0$ it satisfies

$$\Gamma(t) \ge ct^{-1-p}\nu_{u,\psi}(t) \quad \textit{for all large} \quad t\,.$$

For the differential equation (2) of the Higgs boson in the de Sitter spacetime, Theorem 1.6 leads to the following result (see Teeorem 4.1): The continuous global solutions obtained by prolongation of some local solutions must change a sign, and consequently, they vanish at some points. In particular, such radial global solutions have zeros and, therefore, they give rise to at least one bubble. Hence, for the global solutions Theorem 4.1 guarantees the creation of the bubble. Moreover, acccording to Corollary 4.4 the bubbles exist in any neighborhood of infinite time. Thus, the global solution is an oscillating in time solution. In particular, for the continuous global solutions we give integral conditions (see below (53) and (52)), which are sufficient conditions for the creation of the bubbles and their existence in the future. Similar conclusions are valid for the equations in the Minkowski spacetime (Section 2).

This paper is organized as follows. In Section 2 we discuss and prove properties of the Higgs boson equation in the Minkowski spacetime. First in Theorem 2.1 we give some criteria via integrals of the solution, which has unbounded support in the spatial variables. Next we use the eigenfunctions of the Laplace operator to widen those criteria by weighted integrals of the solutions, which has compact supports in the spatial variables. In Section 3 we prove Theorem 1.3 about solutions of the equation in the de Sitter spacetime with, in general, non-local self-interaction. In Section 4 we prove Theorem 1.6 and then in Theorems 4.1-4.5 we discuss in more detail the case of solutions of the Higgs boson equations in the de Sitter spacetime with $x\in\mathbb{R}^3$, which have compact supports in the spatial variables. In the Appendix, Section 5, we give some integral representations of the hyperbolic sine function and one generalization of Kato's lemma to the second order ordinary differential inequality with the exponentially decaying kernel.

2 The Higgs boson in the Minkowski spacetime

Consider the generalized Higgs boson equation in the Minkowski spacetime,

$$\phi_{tt} - \Delta\phi = \mu^2\phi - \lambda|\phi|^{p-1}\phi\,. \tag{23}$$

Here for the numbers μ and λ we assume $\mu\ge 0$ and $\lambda>0$. For the equation (23) the existence and the uniqueness of weak global solution $(\phi,\dot\phi)\in C(\mathbb{R};X_e)$ in the energy space $X_e := H^1\oplus L^2$ is known (see, Proposition 3.2 [8] and Theorems 6.2-6.3 [17]) under certain conditions on n and p, which include $p-1<4/(n-2)$. That solution satisfies the conservation of energy $E(\phi(t),\dot\phi(t)) = E(\phi(0),\dot\phi(0)) \equiv E$ and the following estimates

$$\int_{\mathbb{R}^n} |\phi(x,t)|^2\,dx \le e(E,t)\,,$$
$$\int_{\mathbb{R}^n} \left(|\dot\phi(x,t)|^2 + |\nabla\phi(x,t)|^2\right)dx \le \dot e(E,t)\,,$$

where

$$e(E,t) := \left(\int_{\mathbb{R}^n}|\phi(x,0)|^2\,dx\right)\cosh(\mu t) + \left(E+\mu^2\int_{\mathbb{R}^n}|\phi(x,0)|^2\,dx\right)^{1/2}\mu^{-1}\sinh(\mu t)\,.$$

Moreover, if $p < \frac{n+2}{n-2}$ and $n \le 9$, then for smooth data $\phi(0), \dot\phi(0) \in C^\infty$, the solution ϕ is C^∞-smooth. The equation

$$\phi_{tt} - \Delta\phi = \mu^2\phi - \lambda\phi^3 \tag{24}$$

for the Higgs scalar field in the Minkowski spacetime has the time-independent real-valued flat solution (6) as well as a traveling solitary wave.

In this paper we are looking for the qualitative properties of the global in time solutions of equation (23). More precisely, we are interested in the sign changing global solutions of equations (23) and (24). Our interest in such solutions is motivated by the problem of the creation and growth of bubbles. The theorems below give necessary conditions for the global in time existence of the bubbles.

In order to solve the Cauchy problem for the semilinear equation via the integral equation and to define a weak solution, we use the fundamental solution of the corresponding linear operator. We denote by G the resolving operator of the problem

$$u_{tt} - \triangle u - \mu^2 u = f, \quad u(x,0) = 0, \quad \partial_t u(x,0) = 0\,. \tag{25}$$

Thus, $u = G[f]$. One can write the following explicit formula (see, e.g., [27]) for the operator G, namely,

$$u(x,t) = \int_0^t db \int_0^{t-b} I_0\left(\mu\sqrt{(t-b)^2 - r^2}\right) w(x,r;b)dr,\ x \in \mathbb{R}^n, \tag{26}$$

where $w(x,t;b)$ is a solution of

$$\begin{cases} w_{tt} - \Delta w = 0 \quad \text{in} \quad \mathbb{R}^{n+1}, \\ w(x,0;\tau) = f(x,\tau), \quad w_t(x,0;\tau) = 0 \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

while τ is regarded as a parameter. The function $I_0(z)$ is the modified Bessel function of the first kind. For $w = w(x,t;b)$ there are the following representation formulas (see, e.g., [17]). If n is odd, $n = 2m+1$, $m \in \mathbb{N}$, then for $f \in C^\infty(\mathbb{R}^n \times [0,\infty))$, we have

$$w(x,t;b) \quad = \quad \frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}}\int_{S^{n-1}} f(x+ry,b)\,dS_y,$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-2)$. The constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

If n is even, $n = 2m$, $m \in \mathbb{N}$, then for $f \in C^\infty(\mathbb{R}^n \times [0,\infty))$, we have

$$w(x,t;b) \quad = \quad \frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1}c_0^{(n)}}\int_{B_1^n(0)} \frac{f(x+ry,b)}{\sqrt{1-|y|^2}}\,dV_y.$$

Here $B_1^n(0) := \{|y| \le 1\}$ is the unit ball in $\mathbb{R}^n$, while $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-1)$.

In particular,

$$\int_{\mathbb{R}^n} w(x,t;b)\,dx = \int_{\mathbb{R}^n} f(x,b)\,dx \quad \text{for all} \quad t\,. \tag{27}$$

The equation of (25) is strictly hyperbolic and the operator G is well-defined in the several functional spaces.

Let $\phi_0 = \phi_0(x,t)$ be a solution of the Cauchy problem

$$\partial_t^2\phi_0 - \triangle\phi_0 - \mu^2\phi_0 = 0, \quad \phi_0(x,0) = \varphi_0(x), \quad \partial_t\phi_0(x,0) = \varphi_1(x)\,. \tag{28}$$

Then any solution $\phi = \phi(x,t)$ of the equation (23), which takes initial value $\phi(x,0) = \varphi_0(x)$, $\partial_t\phi(x,0) = \varphi_1(x)$, solves the integral equation

$$\phi(x,t) = \phi_0(x,t) - G[\lambda|\phi|^{p-1}\phi](x,t)\,. \tag{29}$$

For every given function $\phi_0 \in C([0,T];L^{q'}(\mathbb{R}^n))$ we consider the integral equation (29) for the function

$$\phi \in \bigcap_{i=1,p,q} C([0,T];L^i(\mathbb{R}^n)).$$

Here $q' \ge q > 1$, $p \ge 1$. If ϕ_0 is generated by the Cauchy problem (28), then the solution $\phi = \phi(x,t)$ of (29) is said to be *a weak solution* of the Cauchy problem for equation (23) with the initial conditions $\phi(x,0) = \varphi_0(x)$, $\partial_t\phi(x,0) = \varphi_1(x)$.

Theorem 2.1 *Let $\phi = \phi(x,t) \in C(\mathbb{R}^n \times [0,\infty))$ be a weak global solution of the real field equation (23). Denote the integrals (functionals) of the initial values of ϕ by*

$$C_0(\phi) := \int_{\mathbb{R}^n} \phi(x,0)dx, \quad C_1(\phi) := \int_{\mathbb{R}^n} \phi_t(x,0)dx\,. \tag{30}$$

Assume that the self-interaction functional $-\lambda \int_{\mathbb{R}^n} |\phi(x,t)|^{p-1}\phi(x,t)\,dx$ satisfies

$$\sigma \int_{\mathbb{R}^n} |\phi(x,t)|^{p-1}\phi(x,t)\,dx \le 0\,, \tag{31}$$

for all t either outside of the sufficiently small neighborhood of zero if $\mu > 0$, or inside of some neighborhood of infinity if $\mu = 0$. Assume also that

$$\sigma\left(\mu C_0(\phi) + C_1(\phi)\right) > 0\,, \tag{32}$$

where either $\sigma = -1$ or $\sigma = 1$.

Then, the global solution $\phi = \phi(x,t)$ cannot be an asymptotically time-weighted L^p-non-positive (-non-negative) with the weight $\nu_\phi = e^{a_\phi t}t^{b_\phi}$, where if $\mu > 0$, then either $a_\phi < \mu(p-1)$ or $a_\phi = \mu(p-1)$ and $b_\phi < -2$, while $a_\phi = 0$ and $b_\phi \le 1+p$ if $\mu = 0$.

Proof. We consider the case of $\sigma = 1$ only, since the case of $\sigma = -1$ follows by the reflection $\phi \to -\phi$. We discuss separately two cases: with positive mass, $\mu > 0$, and vanishing mass, $\mu = 0$, respectively. We start with the case of positive mass. Let $\phi_0 \in C^1([0,\infty) \times \mathbb{R}^n)$ be a function with

$$\int_{\mathbb{R}^n} \phi_0(x,t)dx = C_0(\phi)\cosh(\mu t) + C_1(\phi)\frac{1}{\mu}\sinh(\mu t) \qquad \text{for all} \quad t \ge 0\,, \tag{33}$$

where

$$\phi_0(x,0) = \phi(x,0), \quad \partial_t\phi_0(x,0) = \partial_t\phi(x,0)\,. \tag{34}$$

The integrable in $\mathbb{R}^n_x$ solution to the problem (28) satisfies (33). Thus, $\phi \in C([0,\infty); L^q(\mathbb{R}^n))$ is a solution to (29) generated by ϕ_0. According to the definition of the solution, for every given $T > 0$ we have

$$G\left[|\phi|^{p-1}\phi\right] \in C([0,T]; L^q(\mathbb{R}^n)) \cap C^1([0,T]; \mathcal{D}'(\mathbb{R}^n))$$

and (34). Then ϕ is a continuous function of $t \in [0,\infty)$ with values in $L^1(\mathbb{R}^n)$, and we may integrate the equation (29):

$$\int_{\mathbb{R}^n} \phi(x,t)\,dx = \int_{\mathbb{R}^n} \phi_0(x,t)\,dx - \lambda \int_{\mathbb{R}^n} G\left[|\phi|^{p-1}\phi\right](x,t)\,dx. \tag{35}$$

In particular,

$$\int_{\mathbb{R}^n} \phi(x,0)\,dx = \int_{\mathbb{R}^n} \phi_0(x,0)\,dx = C_0, \quad \int_{\mathbb{R}^n} \phi_t(x,0)\,dx = \int_{\mathbb{R}^n} \partial_t\phi_0(x,0)\,dx = C_1\,. \tag{36}$$

Then, for the smooth function $\phi = \phi(x,t)$ we obtain

$$\int_{\mathbb{R}^n} G\left[|\phi|^{p-1}\phi\right](x,t)\,dx \quad = \quad \int_{\mathbb{R}^n} dx \int_0^t db \int_0^{t-b} I_0\left(\mu\sqrt{(t-b)^2 - r^2}\right) w(x,r;b)dr\,,$$

where $w(x,r;b)$ is solution of the problem

$$\begin{cases} w_{tt} - \Delta w = 0 \quad \text{in} \quad \mathbb{R}^{n+1}, \\ w(x,0;\tau) = |\phi(x,\tau)|^{p-1}\phi(x,\tau), \quad w_t(x,0;\tau) = 0 \quad \text{in} \quad \mathbb{R}^n\,, \end{cases}$$

with the parameter $\tau \in [0,\infty)$. Therefore, in view of (27),

$$\begin{aligned}
\int_{\mathbb{R}^n} G\left[|\phi|^{p-1}\phi\right](x,t)\,dx &= \int_0^t db \int_0^{t-b} I_0\left(\mu\sqrt{(t-b)^2-r^2}\right) dr \int_{\mathbb{R}^n} dx\, w(x,0;b) \\
&= \int_0^t db \int_0^{t-b} I_0\left(\mu\sqrt{(t-b)^2-r^2}\right) dr \int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\,dx \\
&= \int_0^t db \int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\,dx \int_0^{t-b} I_0\left(\mu\sqrt{(t-b)^2-r^2}\right) dr\,.
\end{aligned}$$

One can easily check that

$$\int_0^{t-b} I_0\left(\mu\sqrt{(t-b)^2-r^2}\right) dr = \frac{1}{\mu}\sinh(\mu(t-b))\,.$$

Thus, we obtain

$$\int_{\mathbb{R}^n} G[|\phi|^{p-1}\phi](x,t)dx = \int_0^t \left(\int_{\mathbb{R}^n} |u(x,b)|^{p-1}u(x,b)\,dx\right)\frac{1}{\mu}\sinh(\mu(t-b))\,db.$$

Hence, (35) reads as follows:

$$\int_{\mathbb{R}^n} \phi(x,t)\,dx = \int_{\mathbb{R}^n} \phi_0(x,t)\,dx - \lambda\int_0^t \left(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\,dx\right)\frac{1}{\mu}\sinh(\mu(t-b))\,db\,.$$

Taking into account (36) we derive

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(x,t)\,dx &= \frac{1}{2}\left(C_0+\frac{C_1}{\mu}\right)e^{\mu t} + \frac{1}{2}\left(C_0-\frac{C_1}{\mu}\right)e^{-\mu t} \\
&\quad -\lambda\int_0^t \left(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\,dx\right)\frac{1}{\mu}\sinh(\mu(t-b))\,db\,.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
F(t) &= C_0\cosh(\mu t) + \frac{C_1}{\mu}\sinh(\mu t) \\
&\quad -\lambda\int_0^t \left(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\,dx\right)\frac{1}{\mu}\sinh(\mu(t-b))\,db\,,
\end{aligned}$$

where we have denoted

$$F(t) := \int_{\mathbb{R}^n} \phi(x,t)\,dx\,. \tag{37}$$

It follows $F \in C^2([0,\infty))$. Moreover,

$$\begin{aligned}
\dot F(t) &= C_1\cosh(\mu t) + \mu C_0\sinh(\mu t) \\
&\quad -\lambda\int_0^t \left(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\,dx\right)\cosh(\mu(t-b))\,db\,, \qquad (38)\\
\ddot F(t) &= \mu^2F(t) - \lambda\int_{\mathbb{R}^n} |\phi(x,t)|^{p-1}\phi(x,t)\,dx\,. \qquad (39)
\end{aligned}$$

In particular, since (31) and $C_\phi \ge 0$, $\nu_\phi(t) \ge 0$, there is a positive number ε, $0<\varepsilon<1$, such that

$$F(t) \ge (1-\varepsilon)\left(C_0\cosh(\mu t) + \frac{C_1}{\mu}\sinh(\mu t)\right) \qquad \text{for large } t\,. \tag{40}$$

Indeed, according to (31) there exist positive $\varepsilon < 1$ and $\delta_\phi > 0$ such that

$$\lambda \left| \int_0^{\delta_\phi} \left(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\,dx \right) \frac{1}{\mu} \sinh(\mu(t-b))\,db \right| \le \varepsilon \cosh(\mu t) \quad \text{for large } t\,. \tag{41}$$

Then, the inequality (40) is fulfilled for all $t \ge \delta_\phi$, if δ_ϕ is sufficiently large. By means of the condition $\mu C_0 + C_1 > 0$ we conclude that

$$F(t) \ge 0 \qquad \text{for large } t\,,$$

and, consequently,

$$\ddot{F}(t) \quad \ge \quad -\lambda \int_{\mathbb{R}^n} |\phi(x,t)|^{p-1}\phi(x,b)\,dx \qquad \text{for large } t. \tag{42}$$

On the other hand, using Definition 1.2 with $\nu_\phi(t) = e^{a_\phi t} t^{b_\phi}$ we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \phi(x,t)\,dx \right|^p &\le -C_\phi \nu_\phi(t) \int_{\mathbb{R}^n} |\phi(x,t)|^{p-1}\phi(x,t)\,dx \\
&\le \lambda^{-1} C_\phi \nu_\phi(t) \left(-\lambda \int_{\mathbb{R}^n} |\phi(x,t)|^{p-1}\phi(x,t)\,dx \right) \\
&= \lambda^{-1} C_\phi \nu_\phi(t) \Big(\ddot{F}(t) - \mu^2 F(t) \Big) \\
&\le \lambda^{-1} C_\phi \nu_\phi(t) \ddot{F}(t) \qquad \text{for large } t\,.
\end{aligned}$$

Thus, since $\nu_\phi(t) > 0$ we have

$$\ddot{F}(t) \ge \delta_0 \nu_u(t)^{-1} F(t)^p \qquad \text{for all large } t \quad \text{with } \delta_0 := \lambda C_\phi^{-1} > 0\,.$$

Hence, the last inequality together with (38) to (40) implies the following system of the ordinary differential inequalities

$$\left\{ \begin{array}{lclll}
F(t) & \ge & (1-\varepsilon) C_0 \cosh(\mu t) + (1-\varepsilon)\frac{C_1}{\mu} \sinh(\mu t) & \text{for all} & t \in [a,b), \\
\dot{F}(t) & \ge & C_1 \cosh(\mu t) + \mu C_0 \sinh(\mu t) & \text{for all} & t \in [a,b), \\
\ddot{F}(t) & \ge & \delta_0 \nu_\phi(t)^{-1} F(t)^p & \text{for all} & t \in [a,b),
\end{array} \right.$$

with large a. The Lemma 5.3 with $A(t) = \cosh(\mu t)$ and $\nu(t) = \cosh(p\mu t) e^{-a_\phi t} t^{-b_\phi}$ shows that if $F(t) \in C^2([0,b))$, then b must be finite.

Now consider the case of $\mu = 0$. Let C_0 and C_1 be defined in (30), while the function $\phi_0(x,t)$ satisfies

$$\int_{\mathbb{R}^n} \phi_0(x,t)\,dx = C_0 + C_1 t\,.$$

Then Corollary 5.2 implies

$$\int_{\mathbb{R}^n} G[\Gamma(\cdot)|\phi|^{p-1}\phi](x,t)\,dx \quad = \quad \int_0^t \Big(\int_{\mathbb{R}^n} |\phi(z,b)|^{p-1}\phi(z,b) dz \Big)(t-b)\,db.$$

Hence,

$$\int_{\mathbb{R}^n} \phi(x,t)\,dx \quad = \quad \int_{\mathbb{R}^n} \phi_0(x,t)\,dx - \int_0^t \Big(\int_{\mathbb{R}^n} |\phi(z,b)|^{p-1}\phi(z,b) dz \Big)(t-b)\,db\,.$$

Thus

$$F(t) \quad = \quad C_0 + C_1 t - \int_0^t \Big(\int_{\mathbb{R}^n} |\phi(z,b)|^{p-1}\phi(z,b) dz \Big)(t-b)\,db\,,$$

where $F(t)$ is defined by (37). It follows $F \in C^2([0,\infty))$. More precisely,

$$\begin{aligned}
\dot{F}(t) &= C_1 - \int_0^t \Big(\int_{\mathbb{R}^n} |\phi(z,b)|^{p-1}\phi(z,b)dz \Big)\, db\,, \\
\ddot{F}(t) &= -\Big(\int_{\mathbb{R}^n} |\phi(z,t)|^{p-1}\phi(z,t)dz \Big)\, db\,.
\end{aligned}$$

In particular, for every given positive $\varepsilon < 1$ one has

$$F(t) \quad \geq \quad C_0 + (1-\varepsilon)C_1 t \quad \text{for all large } \ t\,. \tag{43}$$

Indeed, according to (31) there exists a positive number $A_\phi > 0$ such that

$$\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\, dx \leq 0 \qquad \text{for all } \ t \geq A_\phi\,.$$

At the meantime, for every given positive ε we have

$$\begin{aligned}
\int_0^t \Big(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\, dx \Big) (t-b)\, db &= \int_0^{A_\phi} dz \int_0^z \Big(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\, dx \Big)\, db \\
&\quad + \int_{A_\phi}^t dz \int_0^z \Big(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\, dx \Big)\, db \\
&\leq \int_0^{A_\phi} dz \int_0^z \Big(\int_{\mathbb{R}^n} |\phi(x,b)|^{p-1}\phi(x,b)\, dx \Big)\, db \\
&\leq \varepsilon t \quad \text{for large } \ t\,.
\end{aligned}$$

The last inequality proves (43). Further, according to Definition 1.2 we obtain

$$\Big| \int_{\mathbb{R}^n} \phi(x,t)\, dx \Big|^p \quad \leq \quad C_\phi \nu_\phi(t) \ddot{F}(t) \quad \text{for all large } t\,,$$

where $\nu_\phi(t) := t^{b_\phi}$, $b_\phi \leq 1+p$. Thus

$$\ddot{F}(t) \geq C_\phi^{-1} \nu_\phi(t)^{-1} |F(t)|^p \quad \text{for all large } t\,.$$

By means of the condition $C_1 > 0$ we obtain $F(t) > 0$ for large t and, consequently,

$$\ddot{F}(t) \geq \delta_0 \nu_\phi(t)^{-1} F(t)^p \qquad \text{for large } t \quad \text{with } \delta_0 := C_\phi^{-1} > 0\,.$$

The last inequality together with (43) implies

$$\left\{ \begin{array}{lcll} F(t) & \geq & C_0 + (1-\varepsilon)C_1 t & \quad \text{for all} \quad t \in [a,b), \\ \ddot{F}(t) & \geq & \delta_0 \nu_u(t)^{-1} F(t)^p & \quad \text{for all} \quad t \in [a,b), \end{array} \right.$$

with a sufficiently large number a. The Kato's Lemma 2 [12] shows that if $F(t) \in C^2([0,b))$ and $\nu_\phi(t)^{-1} \geq t^{-1-p}$ with $p > 1$, then b must be finite. The theorem is proven. $\square$

Remark 2.2 *For the smooth sign preserving function with the support in the ball of radius r, in the case of $n = 3$, one has $\left|\int_{\mathbb{R}^3} \varphi(x)\, dx\right|^3 \leq Cr^6 \left|\int_{\mathbb{R}^3} \varphi^3(x)\, dx\right|$. Hence, if ϕ obeys a finite propagation speed property and its initial values have compact supports, then the inequality of Definition 1.2 is satisfied since*

$$\Big| \int_{\mathbb{R}^3} \phi(x,t)\, dx \Big|^3 \leq C_\phi (1+t)^6 \Big| \int_{\mathbb{R}^3} \phi^3(x,t)\, dx \Big| \quad \textit{for all large } \ t\,.$$

For the case of nonlinear wave equation with $\mu = 0$ condition $b_\phi \leq 1+p$ is fulfilled if $5 \leq p$. In fact, for that critical case $p = 5$ an existence of the global solution for smooth data with sufficiently small energy is known (see, e.g., Corollary 6.2 [17]).

The next conclusion from the theorem stated that the global solution has to change sign.

Corollary 2.3 *If the smooth local solution with initial data satisfying (32) with $\sigma = 1$ ($\sigma = -1$) can be prolonged to the global solution, then the global solution cannot be non-positive (non-negative) for all large time t.*

Corollary 2.4 *Let $\phi = \phi(x,t)$ be a continuous global solution of the equation (23) with the Cauchy data $\phi(x,0)$, $\phi(x,0) \in C_0^\infty$ satisfying (32) with $\sigma = 1$ ($\sigma = -1$) and such that its self-interaction functional is non-negative (non-positive) for all t outside either of the sufficiently small neighborhood of zero if $\mu > 0$, or inside of some neighborhood of infinity if $\mu = 0$. Then there exists a sequence $\{t_k\}_{k=1}^\infty$, $\lim_{k\to\infty} t_k = \infty$, such that the solution has a zero inside of the interior of its support on every hyperplane $t = t_k$, $k = 1, 2, \ldots$.*

Thus, this global solution is an oscillating in time solution. In particular, for the continuous global solutions the conditions (30),(31), and (32) the sufficient conditions for the creation of the bubbles and their existence in the future.

Remark 2.5 *The condition of the theorem about the existence of sufficiently small neighborhood of zero means that (41) is fulfilled with some ε, $0 \le \varepsilon < 1$.*

The next theorem generalizes Theorem 2.1 by embedding a proper weight provided that the time slices of the solution have compact supports. The linear Klein-Gordon equation in the Minkowski spacetime preserves a compactness property of the support of the solutions on all time slices, if it is compact on the initial hyperplane. The eigenfunctions of the Laplace operator give a wide choice for the weight functions. In the next theorem they have been used to test global solutions of the equation (23).

Theorem 2.6 *Let $\phi = \phi(x,t) \in C(\mathbb{R}^n \times [0,\infty))$ be a weak global solution of the real field equation (23), which for every given time $t > 0$ has a compact support in x. Let $\psi = \psi(x)$ be a solution of the equation $\Delta\psi = \nu\psi$ in $\mathbb{R}^n$, with some number ν such that $\mu^2 + \nu \ge 0$. Denote the integrals (functionals) of the initial values of ϕ by*

$$C_{0\psi}(\phi) := \int_{\mathbb{R}^n} \psi(x)\phi(x,0)dx, \quad C_{1\psi}(\phi) := \int_{\mathbb{R}^n} \psi(x)\phi_t(x,0)dx\,.$$

Assume that the ψ-weighted self-interaction functional $-\lambda \int_{\mathbb{R}^3} \psi(x)|\phi(x)|^{p-1}\phi(x,t)\,dx$ satisfies

$$\int_{\mathbb{R}^n} \psi(x)|\phi(x,t)|^{p-1}\phi(x,t)\,dx \le 0\,, \tag{44}$$

for all t either outside of the sufficiently small neighborhood of zero if $\mu^2 + \nu > 0$, or inside of some neighborhood of infinity if $\mu^2 + \nu = 0$. Assume also that

$$\mu_1 C_{0\psi}(\phi) + C_{1\psi}(\phi) > 0\,. \tag{45}$$

Then, the global solution $\phi = \phi(x,t)$ cannot be an asymptotically time-weighted $-\psi$ L^p-signed with the weight $\nu_\phi = e^{a_{\phi,\psi}t}t^{b_{\phi,\psi}}$, where if $\mu_1 > 0$ then either $a_{\phi,\psi} < \mu_1(p-1)$, or $a_{\phi,\psi} = \mu_1(p-1)$ and $b_{\phi,\psi} < -2$, where $\mu_1 := \sqrt{\mu^2+\nu}$, while $a_{\phi,\psi} = 0$ and $b_{\phi,\psi} \le 1 + p$ if $\mu_1 = 0$.

Thus, according to the theorem, *for the global solutions for any eigenfunction $\psi = \psi(x)$ of the Laplace operator in $\mathbb{R}^n$ the conditions (44), (45), and the inequality of Definition 1.5 cannot hold simultaneously.*

To check solution for the subject of a zero, one can choose a positive eigenfunction ψ of the Laplace operator in $\mathbb{R}^n$ constructed, for example, in Lemma 3.1 [28]. Moreover, that eigenfunction ψ has an exponential growth and, consequently, it allows to generalize the last theorem to the solutions decaying exponentially at infinity. Then, for an arbitrary positive number ν existence of the positive or negative eigenfunction ψ of the Laplace operator can be proved by the scaling arguments.

The proof of Theorem 2.6 is very similar to the one of Theorem 2.1 with some modifications based on the following lemma.

Lemma 2.7 *Assume that the smooth function $f = f(x,t)$ for every given time $t > 0$ has a compact support. Let $\psi = \psi(x)$ be a solution of the equation $\Delta\psi = \nu\psi$ in $\mathbb{R}^n$ with some number $\nu \in \mathbb{R}$. Then,* (i) *if $\mu^2+\nu > 0$, then for all $t > 0$ we have*

$$\int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx = \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)f(x,b)\,dx\right) \frac{1}{\sqrt{\mu^2+\nu}} \sinh(\sqrt{\mu^2+\nu}(t-b))\,db\,;$$

(ii) if $\mu^2 + \nu = 0$, then for all $t > 0$ we have

$$\int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx = \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)f(x,b)\,dx\right)(t-b)\,db\,;$$

(iii) if $\mu^2 + \nu < 0$, then for all $t > 0$ we have

$$\int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx = \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)f(x,b)\,dx\right) \frac{1}{\sqrt{|\mu^2+\nu|}} \sin(\sqrt{|\mu^2+\nu|}(t-b))\,db\,.$$

Proof of lemma. Let $\phi = \phi(x,t)$ be a function defined as follows:

$$\phi(x,t) := G[f](x,t)\,.$$

For every given time $t > 0$ it has a compact support. Let $\psi = \psi(x)$ be a solution of the equation $\Delta\psi = \nu\psi$ in $\mathbb{R}^n$. We integrate the identity $\psi(x)\phi(x,t) = \psi(x)G[f](x,t)$ and obtain

$$\int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx = \int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx\,.$$

Thus, for $F_\psi(t) := \int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx$ we have

$$F_\psi(t) = \int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx\,.$$

The function $\phi = \phi(x,t)$ solves the Cauchy problem

$$\phi_{tt} - \triangle\phi - \mu^2\phi = f, \quad \phi(x,0) = 0, \quad \partial_t\phi(x,0) = 0\,.$$

From the last equation we derive

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx - \int_{\mathbb{R}^n} \psi(x)\Delta\phi(x,t)\,dx - \mu^2\int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx = \int_{\mathbb{R}^n} \psi(x)f(x,t)\,dx$$

and, consequently,

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx - (\mu^2+\nu)\int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx = \int_{\mathbb{R}^n} \psi(x)f(x,t)\,dx\,.$$

In the case of $\mu^2 + \nu > 0$ it follows then that the function $F_\psi(t)$ is

$$F_\psi(t) = \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)f(x,b)\,dx\right) \frac{1}{\sqrt{\mu^2+\nu}} \sinh(\sqrt{\mu^2+\nu}(t-b))\,db\,.$$

The remaining cases also follow in a similar manner. The lemma is proven. □

Corollary 2.8 *Assume that the smooth function $f = f(x,t)$ for every given time $t > 0$ has a compact support. Let $\psi = \psi(x)$ be a harmonic function in $\mathbb{R}^n$. Then*

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx &= \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)f(x,b)\,dx\right) \frac{1}{\mu}\sinh(\mu(t-b))\,db, \quad \mu > 0\,,\\
\int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx &= \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)f(x,b)\,dx\right)(t-b)\,db, \quad \mu = 0,\\
\int_{\mathbb{R}^n} \psi(x)G[f](x,t)\,dx &= \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)f(x,b)\,dx\right) \frac{1}{|\mu|}\sin(|\mu|(t-b))\,db, \quad \mu^2 < 0\,.
\end{aligned}$$

In particular,

$$
\begin{aligned}
\int_{\mathbb{R}^n} G[f](x,t)\,dx &= \int_0^t \left(\int_{\mathbb{R}^n} f(x,b)\,dx \right) \frac{1}{\mu} \sinh(\mu(t-b))\,db, \quad \mu > 0\,, \\
\int_{\mathbb{R}^n} G[f](x,t)\,dx &= \int_0^t \left(\int_{\mathbb{R}^n} f(x,b)\,dx \right) (t-b)\,db, \quad \mu = 0\,, \\
\int_{\mathbb{R}^n} G[f](x,t)\,dx &= \int_0^t \left(\int_{\mathbb{R}^n} f(x,b)\,dx \right) \frac{1}{|\mu|} \sin(|\mu|(t-b))\,db, \quad \mu^2 < 0\,.
\end{aligned}
$$

Proof of Theorem 2.6. Let $\phi_0 = \phi_0(x,t)$ be a solution of the Cauchy problem (28). Then the weak solution $\phi = \phi(x,t)$ of the equation (23) that takes initial values $\phi(x,0) = \varphi_0(x)$, $\partial_t \phi(x,0) = \varphi_1(x)$, solves the integral equation (29). It follows

$$
\psi(x)\phi(x,t) = \psi(x)\phi_0(x,t) - \psi(x) G[\lambda|\phi|^{p-1}\phi](x,t)\,. \tag{46}
$$

We have

$$
\begin{aligned}
C_{0\psi} &:= \int_{\mathbb{R}^n} \psi(x)\phi(x,0)\,dx = \int_{\mathbb{R}^n} \psi(x)\varphi_0(x)\,dx, \\
C_{1\psi} &:= \int_{\mathbb{R}^n} \psi(x)\partial_t\phi(x,0)\,dx = \int_{\mathbb{R}^n} \psi(x)\varphi_1(x)\,dx\,.
\end{aligned}
$$

If $\mu_1 > 0$, then it is easily seen that

$$
\int_{\mathbb{R}^n} \psi(x)\phi_0(x,t)dx = C_{0\psi}\cosh(\mu_1 t) + C_{1\psi}\frac{1}{\mu_1}\sinh(\mu_1 t) \qquad \text{for all} \quad t \geq 0\,.
$$

We integrate (46) and obtain

$$
\int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx = \int_{\mathbb{R}^n} \psi(x)\phi_0(x,t)\,dx - \int_{\mathbb{R}^n} \psi(x) G[\lambda|\phi|^{p-1}\phi](x,t)\,dx\,.
$$

Finally, for the function $F_\psi(t) := \int_{\mathbb{R}^n} \psi(x)\phi(x,t)\,dx$ we obtain

$$
F_\psi(t) = C_{0\psi}\cosh(\mu_1 t) + C_{1\psi}\frac{1}{\mu_1}\sinh(\mu_1 t) - \int_{\mathbb{R}^n} \psi(x) G[\lambda|\phi|^{p-1}\phi](x,t)\,dx\,.
$$

On the other hand, according to Corollary 2.8, we have

$$
\int_{\mathbb{R}^n} \psi(x) G[|\phi|^{p-1}\phi](x,t)\,dx = \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)|\phi(x,b)|^{p-1}\phi(x,b)\,dx \right) \frac{1}{\mu_1}\sinh(\mu_1(t-b))\,db\,.
$$

Thus,

$$
\begin{aligned}
F_\psi(t) &= C_{0\psi}\cosh(\mu_1 t) + C_{1\psi}\frac{1}{\mu_1}\sinh(\mu_1 t) \\
&\quad - \int_0^t \left(\int_{\mathbb{R}^n} \psi(x)|\phi(x,b)|^{p-1}\phi(x,b)\,dx \right) \frac{1}{\mu_1}\sinh(\mu_1(t-b))\,db\,.
\end{aligned}
$$

The remaining part of the proof is similar to the proof of Theorem 2.1 and we skip it. □

We can similarly consider the following equation

$$
\phi_{tt} - \triangle\phi = \mu^2\phi - \Gamma(t) \left| \int_{\mathbb{R}^n} |\phi(y,t)|^{p-1}\phi(y,t)dy \right|^{\beta} |\phi|^{p-1}\phi\,,
$$

which contains the *non-local nonlinearity* (*non-local self-interaction*).

3 Equation in the de Sitter spacetime. Proof of Theorem 1.3

We consider the case of $\sigma = 1$ only, since case of $\sigma = -1$ follows by reflection $\phi \to -\phi$. Let $u_0 \in C^1([0,\infty)\times\mathbb{R}^n)$ be a function with

$$\int_{\mathbb{R}^n} u_0(x,t)dx = C_0\cosh(Mt) + C_1\frac{1}{M}\sinh(Mt) \qquad \text{for all} \quad t \ge 0\,,$$

where

$$C_0 := \int_{\mathbb{R}^n} u_0(x,0)\,dx, \quad C_1 := \int_{\mathbb{R}^n} \partial_t u_0(x,0)\,dx\,. \tag{47}$$

Suppose that $u \in C([0,\infty); L^q(\mathbb{R}^n))$ is a solution to (13) generated by u_0. According to the definition of the solution, for every given $T > 0$ we have

$$G\left[\Gamma(\cdot)\left|\int_{\mathbb{R}^n}|u(y,\cdot)|^{p-1}u(y,\cdot)\,dy\right|^{\beta}|u|^{p-1}u\right] \in C([0,T]; L^q(\mathbb{R}^n)) \cap C^1([0,T]; \mathcal{D}'(\mathbb{R}^n))\,.$$

Then $u \in C([0,\infty); L^1(\mathbb{R}^n))$ and we may integrate the equation (13):

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x,t)\,dx - \int_{\mathbb{R}^n} G\left[\Gamma(\cdot)\left|\int_{\mathbb{R}^n}|u(y,\cdot)|^{p-1}u(y,\cdot)\,dy\right|^{\beta}|u|^{p-1}u\right](x,t)\,dx. \tag{48}$$

In particular,

$$\int_{\mathbb{R}^n} u(x,0)\,dx = \int_{\mathbb{R}^n} u_0(x,0)\,dx = C_0, \quad \int_{\mathbb{R}^n} u_t(x,0)\,dx = \int_{\mathbb{R}^n} \partial_t u_0(x,0)\,dx = C_1\,.$$

Consider the case of odd $n \ge 3$. The case of even n can be discussed similarly. Then, for the smooth function $u = u(x,t)$ we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} G\left[\Gamma(\cdot)\left|\int_{\mathbb{R}^n}|u(y,\cdot)|^{p-1}u(y,\cdot)\,dy\right|^{\beta}|u|^{p-1}u\right](x,t)\,dx \\
= & \int_{\mathbb{R}^n} dx\, 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr_1 \Bigg(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}}\frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}} \\
& \times \int_{S^{n-1}}\left[\Gamma(b)\left|\int_{\mathbb{R}^n}|u(z,b)|^{p-1}u(z,b)\,dz\right|^{\beta}|u(x+ry,b)|^{p-1}u(x+ry,b)\right]dS_y\Bigg)_{r=r_1} \\
& \times(4e^{-b-t})^{-M}\left((e^{-t}+e^{-b})^2-r_1^2\right)^{-\frac{1}{2}+M} F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_{\mathbb{R}^n} G\left[\Gamma(\cdot)\left|\int_{\mathbb{R}^n}|u(y,\cdot)|^{p-1}u(y,\cdot)dy\right|^{\beta}|u|^{p-1}u\right](x,t)\,dx \\
= & \; 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr_1 \Bigg\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}}\frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}} \\
& \times \int_{S^{n-1}}\left[\Gamma(b)\left|\int_{\mathbb{R}^n}|u(z,b)|^{p-1}u(z,b)\,dz\right|^{\beta}\left(\int_{\mathbb{R}^n}|u(x+ry,b)|^{p-1}u(x+ry,b)\,dx\right)\right]dS_y\Bigg\}_{r=r_1} \\
& \times(4e^{-b-t})^{-M}\left((e^{-t}+e^{-b})^2-r_1^2\right)^{-\frac{1}{2}+M} F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right)
\end{aligned}$$

implies,

$$
\begin{aligned}
& \int_{\mathbb{R}^n} G\left[\Gamma(\cdot)\left|\int_{\mathbb{R}^n} |u(z,\cdot)|^{p-1}u(z,\cdot)\,dz\right|^{\beta} |u|^{p-1}u\right](x,t)\,dx \\
= \; & 2\int_0^t db\Big[\Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\Big|^{\beta}\left(\int_{\mathbb{R}^n} |u(x,b)|^{p-1}u(x,b)\,dx\right)\Big] \\
& \times \int_0^{e^{-b}-e^{-t}} dr_1 \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}} \times \int_{S^{n-1}} dS_y\right)_{r=r_1} \\
& \times (4e^{-b-t})^{-M}\left((e^{-t}+e^{-b})^2-r_1^2\right)^{-\frac{1}{2}+M} F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right).
\end{aligned}
$$

We discuss the following two cases separately: with positive curved mass, $M>0$, and vanishing curved mass, $M=0$, respectively. In the case of $M>0$ we apply Proposition 5.1 to evaluate the last term and obtain

$$
\begin{aligned}
& \int_{\mathbb{R}^n} G[\Gamma(\cdot)\Big|\int_{\mathbb{R}^n} |u(z,\cdot)|^{p-1}u(z,\cdot)\,dz\Big|^{\beta}|u|^{p-1}u](x,t)dx \\
= \; & \int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\Big|^{\beta}\left(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\right)\frac{1}{M}\sinh(M(t-b))\,db.
\end{aligned}
$$

Hence, (48) reads as follows:

$$
\begin{aligned}
\int_{\mathbb{R}^n} u(x,t)\,dx \; = \; & \int_{\mathbb{R}^n} u_0(x,t)\,dx - \int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\Big|^{\beta} \\
& \times\left(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\right)\frac{1}{M}\sinh(M(t-b))\,db\,.
\end{aligned}
$$

Taking into account (16) and (47) we derive

$$
\begin{aligned}
\int_{\mathbb{R}^n} u(x,t)\,dx \; = \; & \frac{1}{2}\left(C_0+\frac{C_1}{M}\right)e^{Mt}+\frac{1}{2}\left(C_0-\frac{C_1}{M}\right)e^{-Mt} \\
& -\int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\Big|^{\beta} \\
& \times\left(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\right)\frac{1}{M}\sinh(M(t-b))\,db\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
F(t) \; = \; & C_0\cosh(Mt)+\frac{C_1}{M}\sinh(Mt) \\
& -\int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\Big|^{\beta}\left(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\right)\frac{1}{M}\sinh(M(t-b))\,db\,,
\end{aligned}
$$

where $F(t):=\int_{\mathbb{R}^n} u(x,t)\,dx$. It follows $F\in C^2([0,\infty))$. More precisely,

$$
\begin{aligned}
\dot{F}(t) \; = \; & C_1\cosh(Mt)+MC_0\sinh(Mt) \qquad (49)\\
& -\int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\Big|^{\beta}\left(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\right)\cosh(M(t-b))\,db\,, \\
\ddot{F}(t) \; = \; & M^2F(t)-\Gamma(t)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)\,dz\Big|^{\beta}\int_{\mathbb{R}^n} |u(z,t)|^{p-1}u(z,t)\,dz\,. \qquad (50)
\end{aligned}
$$

In particular, due to (18) and to $\Gamma(t)\ge 0$, there is a positive number ε, $\varepsilon<1$, such that

$$
F(t) \;\ge\; (1-\varepsilon)\left(C_0\cosh(Mt)+\frac{C_1}{M}\sinh(Mt)\right) \quad \text{for large} \quad t\,.
$$

Indeed, due to (18) there exist positive $\varepsilon<1$ and $\delta_\phi>0$ such that

$$\left|\int_0^{\delta_\phi}\Gamma(b)\Big|\int_{\mathbb{R}^n}|u(z,b)|^{p-1}u(z,b)\,dz\Big|^\beta\left(\int_{\mathbb{R}^n}|u(z,b)|^{p-1}u(z,b)\,dz\right)\frac{1}{M}\sinh(M(t-b))\,db\right|$$
$$\le\varepsilon\cosh(Mt)\quad\text{for large }\ t\,.$$

According to the conditions of the theorem, we have $MC_0+C_1>0$. By means of the last inequality we conclude that $F(t)\ge0$ for large t and, consequently,

$$\ddot F(t)\quad\ge\quad-\Gamma(t)\Big|\int_{\mathbb{R}^n}|u(z,b)|^{p-1}u(z,b)\,dz\Big|^\beta\int_{\mathbb{R}^n}|u(z,t)|^{p-1}u(z,b)\,dz\qquad\text{for large }\ t.$$

On the other hand, using the condition of the theorem we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n}u(x,t)\,dx\Big|^p\\
\le\;& C_u\nu_u(t)\int_{\mathbb{R}^n}|u(x,t)|^{p-1}u(x,t)\,dx\\
\le\;& C_u\nu_u(t)\Gamma(t)^{-1/(\beta+1)}\left(-\Gamma(t)\left|\int_{\mathbb{R}^n}|u(x,t)|^{p-1}u(x,t)dx\right|^\beta\int_{\mathbb{R}^n}|u(x,t)|^{p-1}u(x,t)dx\right)^{1/(\beta+1)}\\
=\;& C_u\nu_u(t)\Gamma(t)^{-1/(\beta+1)}\left(\ddot F(t)-M^2F(t)\right)^{1/(\beta+1)}\\
\le\;& C_u\nu_u(t)\Gamma(t)^{-1/(\beta+1)}\ddot F(t)^{1/(\beta+1)}\qquad\text{for large }\ t\,.
\end{aligned}$$

Here we have used the inequality $\Gamma(t)>0$. Thus, since $\nu_u(t)>0$ we obtain

$$\ddot F(t)\ge C_u^{-(\beta+1)}\nu_u(t)^{-(\beta+1)}\Gamma(t)|F(t)|^{p(\beta+1)}\quad\text{for all large}\quad t\,.$$

The inequality $F(t)\ge0$ allows us to rewrite this estimate as follows

$$\ddot F(t)\ge\delta_0\nu_u(t)^{-\beta-1}\Gamma(t)F(t)^{p(\beta+1)}\qquad\text{for all large }\ t\quad\text{with }\ \delta_0:=C_u^{-(\beta+1)}>0\,.$$

Hence, taking into account the last inequality we arrive at the following system of the ordinary differential inequalities

$$\left\{\begin{array}{lcll}
F(t)&\ge&(1-\varepsilon)C_0\cosh(Mt)+(1-\varepsilon)\frac{C_1}{M}\sinh(Mt)&\text{for all}\quad t\in[a,b),\\
\dot F(t)&\ge&C_1\cosh(Mt)+MC_0\sinh(Mt)&\text{for all}\quad t\in[a,b),\\
\ddot F(t)&\ge&\delta_0\nu_u(t)^{-\beta-1}\Gamma(t)F(t)^{p(\beta+1)}&\text{for all}\quad t\in[a,b),
\end{array}\right.$$

with large a. Lemma 5.3 shows that if $F(t)\in C^2([0,b))$, then b must be finite.

Indeed, we apply Lemma 5.3 with $A(t)=e^{Mt}$ and p replaced with $p(\beta+1)$. More precisely, if we set

$$A(t)=e^{Mt},\qquad\gamma(t)=\nu_u(t)^{-\beta-1}\Gamma(t)e^{Mp(\beta+1)t},$$

then the conditions of Lemma 5.3 read as follows:

$$p(\beta+1)>1\quad\text{and}\quad\Gamma_t(t)\le0\quad\text{for all}\quad t\in[0,\infty).$$

The last inequality follows from the monotonicity of $\Gamma(t)$. For the increasing function $\Gamma(t)$ in order to apply Lemma 5.3 we replace it with the positive constant, which does not affect the above written system of the ordinary differential inequalities. By the condition of the theorem, if the global solution $u=u(x,t)$ is an asymptotically time-weighted L^p-non-positive (-non-negative) with the weight ν_u , then there exist $\varepsilon>0$ and $c>0$ such that

$$\Gamma(t)\ge c\nu_u(t)^{\beta+1}e^{-M(p(\beta+1)-1)t}t^{2+\varepsilon}\quad\text{for all}\quad t\in[a,\infty),$$

that coincides with (56). The case of $M > 0$ is proved.

Now consider the case of $M = 0$. Let

$$\int_{\mathbb{R}^n} u_0(x,t)dx = C_0 + C_1 t\,.$$

Then Corollary 5.2 allows us to write

$$\begin{aligned}
&\int_{\mathbb{R}^n} G[\Gamma(\cdot)\Big|\int_{\mathbb{R}^n} |u(z,\cdot)|^{p-1}u(z,\cdot)dz\Big|^\beta |u|^{p-1}u](x,t)\,dx \\
= &\int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big|^\beta \Big(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big)(t-b)\,db.
\end{aligned}$$

Hence, (48) reads as follows:

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x,t)\,dx - \int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big|^\beta \Big(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big)(t-b)\,db.$$

Thus,

$$F(t) = C_0 + C_1 t - \int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big|^\beta \Big(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big)(t-b)\,db,$$

where $F(t) := \int_{\mathbb{R}^n} u(x,t)\,dx$. It follows $F \in C^2([0,\infty))$. More precisely,

$$\begin{aligned}
\dot F(t) &= C_1 - \int_0^t \Gamma(b)\Big|\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big|^\beta \Big(\int_{\mathbb{R}^n} |u(z,b)|^{p-1}u(z,b)dz\Big)\,db \\
\ddot F(t) &= -\Gamma(t)\Big|\int_{\mathbb{R}^n} |u(z,t)|^{p-1}u(z,t)dz\Big|^\beta \Big(\int_{\mathbb{R}^n} |u(z,t)|^{p-1}u(z,t)dz\Big)\,db\,.
\end{aligned}$$

In particular, with some positive $\varepsilon < 1$ we have

$$F(t) \quad \geq \quad C_0 + (1-\varepsilon)C_1 t \quad \text{for large} \quad t\,. \tag{51}$$

On the other hand, according to the conditions of the theorem, we obtain

$$\Big|\int_{\mathbb{R}^n} u(x,t)\,dx\Big|^p \quad \leq \quad C_u \nu_u(t)\Gamma(t)^{-1/(\beta+1)}\ddot F(t)^{1/(\beta+1)}\,,$$

where $\nu_u(t) = t^{b_u}$, $b_u \leq 1+p$. Thus

$$\ddot F(t) \geq C_u^{-(\beta+1)}\nu_u(t)^{-(\beta+1)}\Gamma(t)|F(t)|^{p(\beta+1)}$$

for all large t. By means of the condition $C_1 > 0$ we conclude

$$\ddot F(t) \geq \delta_0 \nu_u(t)^{-(\beta+1)}\Gamma(t)F(t)^{p(\beta+1)} \qquad \text{for large } t$$

with $\delta_0 := C_u^{-(\beta+1)} > 0$. The last inequality together with (51) implies

$$\begin{cases} F(t) \geq C_0 + (1-\varepsilon)C_1 t & \text{for all} \quad t \in [a,b), \\ \ddot F(t) \geq \delta_0 \nu_u(t)^{-(\beta+1)}\Gamma(t)F(t)^{p(\beta+1)} & \text{for all} \quad t \in [a,b), \end{cases}$$

with some a. The Kato's Lemma 2 [12] shows that if $F(t) \in C^2([0,b))$ and $\nu_u(t)^{-(\beta+1)}\Gamma(t) \geq t^{-1-p(\beta+1)}$ with $p(\beta+1) > 1$, then b must be finite. The theorem is proven. □

Remark 3.1 *In fact, we have proved that any solution $u = u(x,t)$ with permanently bounded support blows up if $MC_0 + C_1 > 0$ and $M \geq 0$.*

4 The Higgs boson in the de Sitter spacetime. Proof of Theorem 1.6

To prove Theorem 1.6 we have to apply Lemma 2.7 with $\mu^2+\nu$ replaced with $M^2+\nu$, and follow the outline of the proof of Theorem 2.6. We leave details of the proof to the reader.

For the differential equation (2) of the Higgs boson in the de Sitter spacetime, Theorem 1.6 leads to the following result, which will be discussed in the remaining part of this paper.

Theorem 4.1 *Let $\phi=\phi(x,t)\in C(\mathbb{R}^3\times[0,\infty))$ be a weak global solution of the real field equation (2). Let $\psi=\psi(x)$ be an eigenfunction of the Laplace operator in $\mathbb{R}^3$ corresponding to the eigenvalue ν. Denote by*

$$C_0(\phi,\psi):=\int_{\mathbb{R}^3}\psi(x)\phi(x,0)dx,\quad C_1(\phi,\psi):=\int_{\mathbb{R}^3}\psi(x)\phi_t(x,0)dx\,,$$

the integrals (functionals) of its ψ-weighted initial values and assume that

$$\left(\sqrt{9+4(\mu^2+\nu)}+3\right)C_0(\phi,\psi)+2C_1(\phi,\psi)>0. \tag{52}$$

Assume also that the ψ-weighted self-interaction functional $-\lambda\int_{\mathbb{R}^3}\psi(x)\phi^3(x,t)\,dx$ satisfies

$$\int_{\mathbb{R}^3}\psi(x)\phi^3(x,t)\,dx\le 0 \tag{53}$$

for all t either outside of the sufficiently small neighborhood of zero if $\mu^2+\nu>0$, or inside of some neighborhood of infinity if $\mu^2+\nu=0$.

Then, the global solution $\phi=\phi(x,t)$ cannot be an asymptotically time-weighted $-\psi$ L^3-signed with the weight $\nu_\phi=e^{a_{\phi,\psi}t}t^{b_{\phi,\psi}}$, where if $\mu^2+\nu>0$, then either $a_{\phi,\psi}<\sqrt{9+4(\mu^2+\nu)}-3$ or $a_{\phi,\psi}=\sqrt{9+4(\mu^2+\nu)}-3$ and $b_{\phi,\psi}<-2$, while $a_{\phi,\psi}=0$ and $b_{\phi,\psi}\le 4$ if $\mu^2+\nu=0$.

Proof. In this case $\Gamma(t)=e^{-3t}$, $M=\sqrt{9/4+\mu^2}$, and one can apply Theorem 1.3. $\square$

Note, in (52) the constants $C_0(\phi,\psi)$ and $C_1(\phi,\psi)$ can be arbitrarily small. Then, the set of functions ϕ and ψ with the properties (52), (53) and the inequality of Definition 1.5, is invariant under action of the multiplicative group of positive numbers, and, consequently, is a conic set. In particular, for every given positive number ε the action ψ to $\varepsilon\psi$, shows that the constant $C_{\phi,\psi}$ can be made arbitrarily small. Then, the transform ψ to $-\psi$, changes signs in all inequalities with the opposite signs.

Corollary 4.2 *For the global solutions (52), (53) and the inequality of Definition 1.5 cannot hold simultaneously.*

It must be noted that the range of the number $b_{\phi,\psi}$ jumps when $\nu\to-\mu^2$. Moreover, in (53) the *sufficiently small* neighborhood is changed with *some* neighborhood as $\nu\to-\mu^2$. That reveals some kind of resonance phenomena.

According to the theorem there is no global in time non-positive solution $\phi=\phi(x,t)$ to the equation (2) such that $(\sqrt{9+4\mu^2}+3)C_0(\phi)+2C_1(\phi)>0$. Indeed, in that case, to verify the last statement, we set $\psi(x)\equiv 1$. Analogously, there is no global in time non-negative solution $\phi=\phi(x,t)$ to the equation (2) such that $(\sqrt{9+4\mu^2}+3)C_0(\phi)+2C_1(\phi)<0$. Hence, we have proved the following result.

Corollary 4.3 *Let $\phi=\phi(x,t)$ be a non-trivial local in time solution of the equation (2) with the Cauchy data $\phi(x,0)$, $\phi_t(x,0)\in C_0^\infty$ satisfying (52) with $\psi(x)\equiv 1$ ($\psi(x)\equiv -1$) and (53) for all t outside of the sufficiently small neighborhood of zero. Then that local solution cannot be prolonged to the global solution, which is non-positive (non-negative) for all large t.*

Thus, the continuous global solution obtained by prolongation of such local solution must change a sign and, consequently, it vanishes at some points. In particular, such radial global solution has zeros and therefore it gives rise to at least one bubble. Hence, for the global solutions, Theorem 4.1 guarantees the creation of the bubble. Moreover, the next corollary states that the bubbles exist in any neighborhood of infinite time.

Corollary 4.4 *Let $\phi = \phi(x,t)$ be a continuous global solution of the equation (2) with the Cauchy data $\phi(x,0)$, $\phi_t(x,0) \in C_0^\infty$ satisfying (52) with $\psi(x) \equiv 1$ ($\psi(x) \equiv -1$) and such that its self-interaction functional is non-negative (non-positive) for all t outside of the sufficiently small neighborhood of zero. Then there exists a sequence $\{t_k\}_{k=1}^\infty$, $\lim_{k\to\infty} t_k = \infty$, such that the solution has a zero inside of the interior of its support on every hyperplane $t = t_k$, $k = 1, 2, \ldots$.*

Thus, the global solution is an oscillating in time solution. In particular, for the continuous global solutions, the conditions (53) and (52) are the sufficient conditions for the creation of the bubbles and their existence in the future.

Furthermore, all statements of the above corollaries are also true if the eigenfunction $\psi = \psi(x)$ is non-constant.

If initial data have compact support, then the support of solution is contained in some cylinder $B_R(0) \times [0,\infty)$. Therefore, if $\psi(x)\phi(x,t)$ does not change sign, then the inequality of Definition 1.5 is satisfied with $a_{\phi,\psi} = b_{\phi,\psi} = 0$ for $\varepsilon\psi(x)\phi(x,t)$, provided that $\varepsilon > 0$ is sufficiently small.

The last case deserves special consideration. Assume that the Cauchy data $\phi(x,0)$, $\phi_t(x,0) \in C_0^\infty(B_R(0))$. Then by the finite speed of propagation property for the solution we have $\operatorname{supp}\phi \subseteq B_{R+1}(0) \times [0,\infty)$. Now we choose the function $\psi = \psi(x)$, in particular, as an eigenfunction of the Laplace operator in $B_{\widetilde{R}}(0)$, $\widetilde{R} \geq R+1$, with the Dirichlet data $\psi(x)|_{|x|=\widetilde{R}} = 0$. The eigenvalues of such a problem are well-known (see, e.g., [20]):

$$\nu_{n,k} = -\left(\frac{\rho_k^{(n)}}{\widetilde{R}}\right)^2 \qquad n = 0,1,2,\ldots; \quad k = 1,2,3,\ldots ,$$

where $\rho_k^{(n)}$ are the positive zeros of the Bessel function $J_{n+\frac{1}{2}}$, that is the positive roots of the equation $J_{n+\frac{1}{2}}(\rho) = 0$. There are $2n+1$ eigenfunctions belonging to each eigenvalue $\nu_{n,k}$. In fact, $J_{n+\frac{1}{2}}$ can be written via elementary functions (see, e.g., [3]). The corresponding eigenfunctions in the spherical coordinates are

$$\begin{aligned}
\psi_{njk}^1 &= \sqrt{\frac{\pi\widetilde{R}}{2\rho_k^{(n)} r}}\, J_{n+\frac{1}{2}}\left(\rho_k^{(n)}\frac{r}{\widetilde{R}}\right) P_n^{(j)}(\cos\vartheta)\cos(j\varphi)\,, \quad j = 0,1,2,\ldots,n;\\
\psi_{njk}^2 &= \sqrt{\frac{\pi\widetilde{R}}{2\rho_k^{(n)} r}}\, J_{n+\frac{1}{2}}\left(\rho_k^{(n)}\frac{r}{\widetilde{R}}\right) P_n^{(j)}(\cos\vartheta)\sin(j\varphi)\,, \quad j = 1,2,3,\ldots,n\,.
\end{aligned}$$

Here $P_n^{(j)}(\xi)$ are the associated Legendre polynomials. Thus, we arrive at the following theorem.

Theorem 4.5 *Let $\phi = \phi(x,t) \in C(\mathbb{R}^3 \times [0,\infty))$, $\operatorname{supp}\phi \subseteq B_{\widetilde{R}}(0) \times [0,\infty)$, be a weak global solution of the real field equation (2). Let $\psi_{njk}^i(x)$ be an eigenfunction of the Laplace operator with the vanishing Dirichlet data corresponding to the eigenvalue $\nu_{n,k}$. Denote by*

$$C_0{}_{njk}^{\,i}(\phi) := \int_{\mathbb{R}^3} \psi_{njk}^i(x)\phi(x,0)dx, \quad C_1{}_{njk}^{\,i}(\phi) := \int_{\mathbb{R}^3} \psi_{njk}^i(x)\phi_t(x,0)dx\,,$$

the integrals (functionals) of its ψ-weighted initial values and assume that

$$\left(\sqrt{9+4(\mu^2+\nu_{n,k})}+3\right) C_0{}_{njk}^{\,i}(\phi,\psi) + 2C_1{}_{njk}^{\,i}(\phi,\psi) > 0.$$

Assume also that the ψ_{njk}^i-weighted self-interaction functional $-\lambda\int_{\mathbb{R}^3}\psi_{njk}^i(x)\phi^3(x,t)\,dx$ satisfies

$$\int_{\mathbb{R}^3} \psi_{njk}^i(x)\phi^3(x,t)\,dx \leq 0 \tag{54}$$

for all t either outside of the sufficiently small neighborhood of zero if $\mu^2+\nu_{n,k} > 0$, or in some neighborhood of infinity if $\mu^2+\nu_{n,k} = 0$.

Then, the global solution $\phi = \phi(x,t)$ cannot be an asymptotically time-weighted $-\psi_{njk}^i$ L^3-signed with the weight $\nu_\phi = e^{a_{\phi,\psi}t}t^{b_{\phi,\psi}}$, where if $\mu^2+\nu_{n,k} > 0$, then either $a_{\phi,\psi} < \sqrt{9+4(\mu^2+\nu_{n,k})}-3$, or $a_{\phi,\psi} = \sqrt{9+4(\mu^2+\nu_{n,k})}-3$ and $b_{\phi,\psi} < -2$, while $a_{\phi,\psi} = 0$ and $b_{\phi,\psi} \leq 4$ if $\mu^2+\nu_{n,k} = 0$.

The next corollary describes a resonance case, when $-\mu^2$ coincides with some eigenvalue of the Laplace operator with the Dirichlet condition in some ball with a diameter no less than the diameter of the spatial trace of the support of the solution. Although one can always find such eigenvalues, it does not mean that the corresponding conditions of the theorem are satisfied.

Corollary 4.6 *Let $\phi = \phi(x,t) \in C(\mathbb{R}^3 \times [0,\infty))$, $\operatorname{supp}\phi \subseteq B_{\widetilde{R}}(0) \times [0,\infty)$, be a weak global solution of the real field equation (2). Let $\psi^i_{njk}(x)$ be an eigenfunction of the Laplace operator with the vanishing Dirichlet data corresponding to the eigenvalue $\nu_{n,k}$. Assume that (resonance)*

$$\mu^2 = -\nu_{n,k}\,.$$

and

$$3C_{0\,injk}(\phi) + C_{1\,injk}(\phi) > 0.$$

Assume also that the ψ^i_{njk}-weighted self-interaction functional $-\lambda\int_{\mathbb{R}^3}\psi^i_{njk}(x)\phi^3(x,t)\,dx$ satisfies (54) for all t in some neighborhood of infinity.

Then, the global solution $\phi = \phi(x,t)$ cannot be an asymptotically time-weighted $-\psi^i_{njk}$ L^3-signed with the weight $\nu_\phi = t^{b_{\phi,\psi}}$, where $b_{\phi,\psi} \le 4$.

We note here that, for functions with compact support, the Hölder inequality allows us to verify that all conditions of Theorem 4.5 and Corollary 4.6 are fulfilled as long as the function $\psi^i_{njk}(x)\phi(x,t)$ preserves its sign.

5 Appendix

5.1 Integral representations for the hyperbolic sine function

In [3, Sec. 2.4] one can find one-dimensional integrals involving hypergeometric function. In this section we present one more example of such an integral as well as examples of multidimensional integrals appearing in the fundamental solutions for the Klein-Gordon equation in the de Sitter spacetime. One can find more examples related to the Tricomi and Gellerstedt equations in [24], [27].

Proposition 5.1 [26] *The function $M^{-1}\sinh(M(t-b))$, $M > 0$, with $t \ge b \ge 0$, can be represented as follows:*
(i) *In the form of a one-dimensional integral*

$$\begin{aligned}\frac{1}{M}\sinh(M(t-b)) \quad = \quad & \int_{-(e^{-b}-e^{-t})}^{e^{-b}-e^{-t}} (4e^{-b-t})^{-M}\Big((e^{-t}+e^{-b})^2 - z^2\Big)^{-\frac{1}{2}+M} \\ & \times F\Big(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-z^2}{(e^{-b}+e^{-t})^2-z^2}\Big)\,dz.\end{aligned}$$

(ii) *If n is odd, $n = 2m+1$, $m \in \mathbb{N}$, then with $c_0^{(n)} = 1\cdot 3\cdot\ldots\cdot(n-2)$,*

$$\begin{aligned}& \frac{1}{M}\sinh(M(t-b)) \\ = \quad & 2\int_0^{e^{-b}-e^{-t}} dr_1\,\left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-3}{2}}\frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}}\int_{S^{n-1}} dS_y\right)_{r=r_1}(4e^{-b-t})^{-M} \\ & \times\Big((e^{-t}+e^{-b})^2 - r_1^2\Big)^{-\frac{1}{2}+M} F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r_1^2}{(e^{-b}+e^{-t})^2-r_1^2}\right).\end{aligned}$$

(iii) *If n is even, $n = 2m$, $m \in \mathbb{N}$, then with $c_0^{(n)} = 1\cdot 3\cdot\ldots\cdot(n-1)$,*

$$\frac{1}{M}\sinh(M(t-b))$$

$$= \quad 2\int_0^{e^{-b}-e^{-t}} dr_1 \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1}c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}} dV_y \right)_{r=r_1}$$

$$\times (4e^{-b-t})^{-M}\left((e^{-t}+e^{-b})^2 - r_1^2\right)^{-\frac{1}{2}+M}$$

$$\times F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^{-b}-e^{-t})^2 - r_1^2}{(e^{-b}+e^{-t})^2 - r_1^2}\right).$$

Here the constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

If we set $b = 0$ in the above integrals, then we get integral representations of the function $\sinh(Mt)$ depending on the parameter $M > 0$. By passing to the limit as $M \to 0$ we arrive at the following corollary.

Corollary 5.2 [26] *The function $t - b$ with $t \geq b \geq 0$, can be represented as follows:*
(i) *In the form of a one-dimensional integral*

$$t-b \quad = \quad \int_{-(e^{-b}-e^{-t})}^{e^{-b}-e^{-t}} \left((e^{-t}+e^{-b})^2 - z^2\right)^{-\frac{1}{2}} F\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2 - z^2}{(e^{-b}+e^{-t})^2 - z^2}\Big)\, dz.$$

(ii) *If n is odd, $n = 2m+1$, $m \in \mathbb{N}$, then with $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-2)$,*

$$t-b \quad = \quad 2\int_0^{e^{-b}-e^{-t}} dr_1 \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}} \int_{S^{n-1}} dS_y \right)_{r=r_1}$$

$$\times \left((e^{-t}+e^{-b})^2 - r_1^2\right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2 - r_1^2}{(e^{-b}+e^{-t})^2 - r_1^2}\right).$$

(iii) *If n is even, $n = 2m$, $m \in \mathbb{N}$, then with $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-1)$,*

$$t-b \quad = \quad 2\int_0^{e^{-b}-e^{-t}} dr_1 \left(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1}c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}}\, dV_y \right)_{r=r_1}$$

$$\times \left((e^{-t}+e^{-b})^2 - r_1^2\right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2 - r_1^2}{(e^{-b}+e^{-t})^2 - r_1^2}\right).$$

5.2 Second order differential inequalities

The second order differential inequalities with power decreasing kernel play a key role in proving blow-up of the solutions to semilinear equations. Kato's lemma [12] allows us to derive from the inequality

$$\ddot{w} \geq bt^{-1-p}w^p, \qquad p > 1,\ b > 0, \quad t \text{ large},$$

the boundedness of the life-span of a solution with the property $\dot{w} \geq a > 0$. For the equation in the de Sitter spacetime the kernel e^{-Mt} of the corresponding ordinary differential inequality decreases exponentially:

$$\ddot{w} \geq be^{-Mt}w^p, \qquad p > 1,\ b > 0,\ M > 0, \quad t \text{ large}.$$

There is a non-trivial global solution to the last differential inequality. Hence, to generalize Kato's lemma we need proper supplementary conditions on the involved functions.

Lemma 5.3 [26] *Suppose $F(t) \in C^2([a,b))$, and*

$$F(t) \geq c_0 A(t), \quad \dot{F}(t) \geq 0, \quad \ddot{F}(t) \geq \gamma(t)A(t)^{-p}F(t)^p \quad \text{for all } t \in [a,b), \tag{55}$$

where $A, \gamma \in C^1([a,\infty))$ are non-negative functions and $p > 1$, $c_0 > 0$. Assume that

$$\lim_{t\to\infty} A(t) = \infty\,,$$

and that

$$\frac{d}{dt}\left(\gamma(t)A(t)^{-p}\right) \le 0 \quad \text{for all} \quad t \in [a,b)\,.$$

If there exist $\varepsilon > 0$ and $c > 0$ such that

$$\gamma(t) \ge cA(t)(\ln A(t))^{2+\varepsilon} \quad \text{for all} \quad t \in [a,b), \tag{56}$$

then b must be finite.

We note here that the equation

$$\ddot{F}(t) = e^{-dt}F(t)^p\,, \quad d > 0,$$

has a global solution $F(t) = c_F e^{\frac{d}{p-1}t}$, where $c_F = (d/(p-1))^{2/(p-1)}$, while the corresponding $A(t) = c_A e^{at}$, $a > 0$, and $\gamma(t) = c_\gamma e^{(pa-d)t}$. The condition (56) implies $a > d/(p-1)$. On the other hand, the first inequality of (55) holds only if $a \le d/(p-1)$.

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References

[1] Alinhac, S. (1995). *Blowup for nonlinear hyperbolic equations.* Progress in Nonlinear Differential Equations and their Applications, 17: Birkhäuser Boston, Inc., Boston, MA.

[2] Baskin, D. (2010). A parametrix for the fundamental solution of the KleinGordon equation on asymptotically de Sitter spaces. *Journal of Functional Analysis* 259:1673–1719.

[3] Bateman, H., Erdelyi, A. (1953). *Higher Transcendental Functions.* vol. 1,2, New York: McGraw-Hill.

[4] Choquet-Bruhat, Y. (2000). Global wave maps on curved space times. Mathematical and quantum aspects of relativity and cosmology (Pythagoreon, 1998), 1–29, Lecture Notes in Phys., 537, Berlin: Springer.

[5] Coleman, S. (1985). *Aspects of Symmetry: Selected Erice Lectures.* Cambridge University Press.

[6] Englert, F., Brout, R. (1964). Broken Symmetry and the Mass of Gauge Vector Mesons. *Phys. Rev. Lett.* 13, no. 9: 321–323.

[7] Ginibre, J., Velo, G. (1985). The global Cauchy problem for the nonlinear Klein-Gordon equation. *Math. Z.* 189, no. 4: 487–505.

[8] Ginibre, J., Velo, G. (1989). The global Cauchy problem for the nonlinear Klein-Gordon equation. II. *Ann. Inst. H. Poincar Anal. Non Linaire.* 6, no. 1: 15–35.

[9] Hawking, S. W., Ellis, G. F. R. (1973). *The large scale structure of space-time.* Cambridge Monographs on Mathematical Physics, No. 1. London-New York: Cambridge University Press.

[10] Higgs, P.W. (1964). Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.* 13, no. 16: 508–509.

[11] Hörmander, L. (1997). *Lectures on nonlinear hyperbolic differential equations.* Berlin: Springer-Verlag.

[12] Kato, T. (1980). Blow-up of solutions of some nonlinear hyperbolic equations. *Comm. Pure Appl. Math.* 33: 501–505.

[13] Lee, T.D, Wick, G.C. (1974). Vacuum stability and Vacuum Excitation in Spin-0 Field. *Phys. Rev. D* 9 (8): 2291–2316.

[14] Linde, A. (1990). *Particle Physics and Inflationary Cosmology*. Harwood, Chur, Switzerland.

[15] Møller, C. (1952). *The theory of relativity*. Oxford: Clarendon Press.

[16] Rendall, A. (2008). *Partial differential equations in general relativity*. Oxford Graduate Texts in Mathematics, 16, Oxford: Oxford University Press.

[17] Shatah J., Struwe, M. (1998). *Geometric wave equations*. Courant Lect. Notes Math., 2. New York Univ., New York: Courant Inst. Math. Sci.

[18] Slater, L. J. (1966). *Generalized hypergeometric functions*. Cambridge: Cambridge University Press.

[19] Strauss, W. A. (1977). Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* 55, no. 2: 149–162.

[20] Tychonov, A. N., Samarski, A. A. (1977). *Partial differential equations of mathematical physics*. Moscow.

[21] Vasy, A. (2010). The wave equation on asymptotically de Sitter-like spaces. *Adv. Math.* 223, no. 1: 49–97.

[22] Voronov, N. A., Dyshko, A. L., Konyukhova, N. B. (2005). On the Stability of a Self-Similar Spherical Bubble of a Scalar Higgs Field in de Sitter Space. *Physics of Atomic Nuclei* 68, no. 7: 1218–1226.

[23] Weinberg, S. (1996). *The quantum theory of fields*. vol. 2. New York: Cambridge University Press.

[24] Yagdjian, K. (2006). Global existence for the n-dimensional semilinear Tricomi-type equations. *Comm. Partial Diff. Equations* 31: 907–944.

[25] Yagdjian, K., Galstian, A. (2009). Fundamental Solutions for the Klein-Gordon Equation in de Sitter Spacetime. *Comm. Math. Phys.* 285: 293–344.

[26] Yagdjian, K. (2009). The semilinear Klein-Gordon equation in de Sitter spacetime. *Discrete Contin. Dyn. Syst. Ser. S* 2, no. 3: 679–696.

[27] Yagdjian, K. (2010). Fundamental Solutions for Hyperbolic Operators with Variable Coeffcients. *Rend. Istit. Mat. Univ. Trieste* 42 Suppl.: 221–243.

[28] Yordanov, B., Zhang, Qi S. (2005): Finite-time blow up for wave equations with a potential. *SIAM J. Math. Anal.* 36, no. 5: 1426–1433.