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Pricing Path-Dependent Options with Jump Risk via Laplace Transforms

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We present analytical solutions for two-dimensional Laplace transforms of barrier option prices, as well as an approximation based on Laplace transforms for the prices of finite-time horizon American options, under a double exponential jump diffusion model. Our numerical results indicate that the method is fast, accurate, and easy to implement without requiring high precision calculations in Laplace inversion.

Keywords: jump diffusion, American options, barrier and lookback options
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1. Introduction

Laplace transforms have been widely used in valuing financial derivatives. For example, Carr and Madan (1999) propose Fourier transforms with respect to log-strike prices; Geman and Yor (1993), Fu, Madan, and Wang (1999) use Laplace transforms to price Asian options in the Black-Scholes setting; Laplace transforms for double-barrier and lookback options under the CEV model are given in Davydov and Linetsky (2001); Petrella and Kou (2004) use a recursion and Laplace transforms to price discretely monitored barrier and lookback options. For a survey of Laplace transforms in option pricing, see Craddock, Heath, Platen (2000).

This paper aims at using one-dimensional and two-dimensional Laplace transforms to price options under a double exponential jump diffusion model (Kou, 2002). The model is proposed to incorporate jumps into the classical Black-Scholes model, while still retaining tractability for path-dependent options, such as barrier, lookback, and American options. This is made possible mainly because the jump size in this model has a two-sided exponential distribution, which leads to an explicit calculation of the distribution of first passage times, thanks to the memoryless property of the exponential distribution; see Kou and Wang (2003, 2004).

Some identities and representations (though not explicit calculations) based on the Wiener-Hopf factorization for two-sided jump processes are given in Asmussen et al. (2004), Boyarchenko and Levendorskiĭ (2002), and Kyprianou and Pistorius (2003); see also Avram et al. (2004) and Rogers (2000) for first passage times related to one-sided jump processes. Numerical solutions based on solving partial integro-differential equations are given in Cont and Voltchkova (2005) and d'Haullin et al. (2003). For a survey of other alternative models for equity and interest rate derivatives with jumps, see Cont and Tankov (2004a), Hull (2002), Kijima (2002), and Glasserman and Kou (2004).

The current paper extends the study of option pricing under the double exponential jump diffusion model in three ways. First, we provide an approximation for finite-time horizon American options by generalizing the approximation in Ju (1998) for the classical Brownian model to the case of jump diffusions. Second, we give a simple formula for barrier options by using a two-dimensional Laplace transform, one for the space and one for the time; the new formulae after two-dimensional transforms are much simpler than the one-dimensional transform formulae in Kou and Wang (2003), and the new formulae are much easier for implementation. Third, we show ways to invert the Laplace transform via the Euler inversion, which does not require high-precision calculation and leads to fast and accurate results for a variety of options, including European call and put options, American options, barrier and lookback options.

The rest of the paper is organized as follows: In Section 2 we review the double exponential jump diffusion model and give some preliminary results for European call and put options. In Section 3 an approximation for American options are given, while we study barrier and lookback options in the last section.

2. Background and Preliminary Results

2.1. The Model

The double exponential jump diffusion model assumes the return process has two components, a continuous part modeled as Brownian motion, and a jump part with jumps having a double exponential distribution and with jump times driven by a Poisson process. It is shown (Kou, 2002) that under such a model, when using a HARA type utility function for a representative agent, the rational-expectations equilibrium price of an option is given by the expectation of the discounted option payoff under a risk-neutral measure[1] P^*. Under P^*, for the asset price $S(t)$ the return process $X(t) := \log(S(t)/S(0))$ is given by

$$X(t) = \left(r - \delta - \frac{1}{2}\sigma^2 - \lambda\zeta\right) t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad X(0) = 0, \tag{1}$$

[1] The measure P^* is called risk-neutral since $\mathsf{E}^*(e^{-(r-\delta)T} S(T)) = S(0)$.

where r is the risk-free rate, δ the continuous dividend yield, $W(t)$ standard Brownian motion, $N(t)$ a Poisson Process with rate λ and Y_i i.i.d. jumps with double exponential distribution

$$f_Y^*(y) \sim p \cdot \eta_1 e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} \mathbf{1}_{\{y<0\}}, \ \eta_1 > 1, \ \eta_2 > 0.$$

The utility function of the representative agent will affect all the risk-neutral parameters including $p, q \geq 0$, $p+q = 1$, $\lambda \geq 0$, $\eta_1 > 1$, $\eta_2 > 0$, and $\zeta := \mathsf{E}^* \left[e^Y \right] - 1 = p\eta_1/(\eta_1 - 1) + q\eta_2/(\eta_2 + 1) - 1$.

The moment generating function of $X(t)$ is

$$\mathsf{E}^* \left[e^{\theta X(t)} \right] = \exp(G(\theta)t), \tag{2}$$

where the function $G(x)$ is defined as

$$G(x) := x \left(r - \delta - \frac{1}{2}\sigma^2 - \lambda\zeta \right) + \frac{1}{2}x^2\sigma^2 + \lambda \left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1 \right), \tag{3}$$

Kou and Wang (2003) show that for $\alpha > 0$, the equation $G(x) = \alpha$ has exactly four roots $\beta_{1,\alpha}, \beta_{2,\alpha}, -\beta_{3,\alpha}, -\beta_{4,\alpha}$, where

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty, \ \ 0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty. \tag{4}$$

In Appendix B we provide the formulae for the four roots of the equation $G(x) = \alpha$, which is essentially a quartic equation. The explicit formulae of β's are crucial for the Euler algorithm in Laplace inversion.

When pricing options we often also resort to another probability measure, defined as $\widetilde{\mathsf{P}}$, under which the asset $S(t)$ is the numeraire. Kou and Wang (2004) show that, under $\widetilde{\mathsf{P}}$, $X(T)$ still has a double exponential distribution as in (1), with drift $r - \delta + \frac{1}{2}\sigma^2 - \lambda\zeta$ and jump parameters

$$\widetilde{\lambda} = \lambda(\zeta + 1), \ \ \widetilde{p} = \frac{p\eta_1}{(\zeta + 1)(\eta_1 - 1)}, \ \ \widetilde{\eta}_1 = \eta_1 - 1, \ \ \widetilde{\eta}_2 = \eta_2 + 1.$$

The moment generating function of $X(t)$ under the alternative probability measure $\widetilde{\mathsf{P}}$ is given by $\widetilde{\mathsf{E}} \left[e^{\theta X(t)} \right] = \exp(\widetilde{G}(\theta)t)$, with

$$\widetilde{G}(x) := x \left(r - \delta + \frac{1}{2}\sigma^2 - \widetilde{\lambda}\widetilde{\zeta} \right) + \frac{1}{2}x^2\sigma^2 + \widetilde{\lambda} \left(\frac{\widetilde{p}\widetilde{\eta}_1}{\widetilde{\eta}_1 - x} + \frac{\widetilde{q}\widetilde{\eta}_2}{\widetilde{\eta}_2 + x} - 1 \right).$$

2.2. Preliminary Results for European Call and Put Options

In this section we derive Laplace transforms for pricing of European call and put options. In principle, the Laplace transforms for the prices of European call and put options can be obtained by using standard results from Fourier transforms for general Lévy processes (see Cont and Tankov, 2004a, pp 361-362). For completeness, we shall include an explicit calculation for the double exponential jump model, as the proof is very simple using an idea of Carr and Madan (1999) along with a change of the order of integration; and the proof is also useful for the derivation in the later sections.

To fix the notation, the price of a European call with maturity T and strike K, is given by

$$C_T(k) = e^{-rT}\mathsf{E}^*\left[(S(T) - K)^+\right] = e^{-rT}\mathsf{E}^*\left[\left(S(0)e^{X(T)} - e^{-k}\right)^+\right], \tag{5}$$

where $k = -\log(K)$, and the price of a similar European put

$$P_T(k') = e^{-rT}\mathsf{E}^*\left[(K - S(T))^+\right] = e^{-rT}\mathsf{E}^*\left[\left(e^{k'} - S(0)e^{X(T)}\right)^+\right],$$

where $k' = \log(K)$. Alternatively, a change of numeriare argument easily yields that the price of a call/put option can be computed as

$$C_T(k) = S(0)\widetilde{\Psi}_C(k) - e^{-k}e^{-rT}\Psi_C(k), \quad P_T(k') = e^{k'}e^{-rT}\Psi_P(k') - S(0)\widetilde{\Psi}_P(k'), \tag{6}$$

where

$$\Psi_C(k) = \mathsf{P}^*(S(T) \geq e^{-k}), \quad \widetilde{\Psi}_C(k) = \widetilde{\mathsf{P}}(S(T) \geq e^{-k}),$$
$$\Psi_P(k') = \mathsf{P}^*(S(T) < e^{k'}), \quad \widetilde{\Psi}_P(k') = \widetilde{\mathsf{P}}(S(T) < e^{k'}),$$

and $\widetilde{\mathsf{P}}$ is the probability measure defined in the previous section under which the numeraire asset corresponds to $S(t)$. Therefore, we can also price a call/put option by inverting the Laplace transforms for these probabilities.

Lemma 1. *The Laplace transform with respect to k of $C_T(k)$ in* (5) *is given by*

$$\widehat{f_C}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k} C_T(k)dk = e^{-rT}\frac{S(0)^{\xi+1}}{\xi(\xi+1)}\exp(G(\xi+1)T), \quad \xi > 0. \tag{7}$$

and the Laplace transform with respect to k' for the put option $P_T(k')$ is

$$\widehat{f_P}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k'} P_T(k')dk' = e^{-rT}\frac{S(0)^{-(\xi-1)}}{\xi(\xi-1)}\exp(G(-(\xi-1)T), \ \xi > 1. \tag{8}$$

The Laplace transforms with respect to k and k' of $\Psi_C(k)$ and $\Psi_P(k')$ are

$$\widehat{f_{\Psi_C}}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k}\Psi_C(k)dk = \frac{S(0)^{\xi}}{\xi}\exp(G(\xi)T), \quad \xi > 0, \tag{9}$$

$$\widehat{f_{\Psi_P}}(\xi) := \int_{-\infty}^{\infty} e^{-\xi k'}\Psi_P(k')dk' = e^{-rT}\frac{S(0)^{-\xi}}{\xi}\exp(G(-\xi)T), \quad \xi > 0, \tag{10}$$

The Laplace transforms of $\widetilde{\Psi}_C(k)$ and $\widetilde{\Psi}_P(k')$ are similar except with $\widetilde{G}$ in place of G.

Proof. By (5) the Laplace transform for the call option is

$$\widehat{f_C}(\xi) = e^{-rT} \int_{-\infty}^{\infty} e^{-\xi k} \mathsf{E}^* \left[\left(S(0) e^{X(T)} - e^{-k} \right)^+ \right] dk.$$

Applying the Fubini theorem yields for every $\xi > 0$,

$$\begin{aligned}
\widehat{f_C}(\xi) &= e^{-rT} \mathsf{E}^* \left[\int_{-\infty}^{\infty} e^{-\xi k} \left(S(0) e^{X(T)} - e^{-k} \right)^+ dk \right] \\
&= e^{-rT} \mathsf{E}^* \left[\int_{-X(T)-\log S(0)}^{\infty} e^{-\xi k} \left(S(0) e^{X(T)} - e^{-k} \right) dk \right] \\
&= e^{-rT} \mathsf{E}^* \left[S(0) e^{X(T)} e^{\xi(X(T)+\log S(0))} \frac{1}{\xi} - e^{(\xi+1)(X(T)+\log S(0))} \frac{1}{\xi+1} \right] \\
&= e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} \mathsf{E}^* \left[e^{(\xi+1)X(T)} \right],
\end{aligned}$$

from which (7) follows readily from (2). The proof of (8) is similar. For (9), note that

$$\widehat{f_{\Psi_C}}(\xi) = \int_{-\infty}^{\infty} e^{-\xi k} \mathsf{E}^* 1_{\{S(T) \geq e^{-k}\}} \, dk = \int_{-\infty}^{\infty} e^{-\xi k} \mathsf{E}^* 1_{\{k \geq -\log S(T)\}} \, dk.$$

By Fubini's Theorem, we can interchange the order of integration and write

$$\widehat{f_{\Psi_C}}(\xi) = \mathsf{E}^* \left[\int_{-\log S(T)}^{\infty} e^{-\xi k} \, dk \right] = \frac{1}{\xi} \mathsf{E}^* \left[S(T)^{\xi} \right] = \frac{S(0)^{\xi}}{\xi} \mathsf{E}^* \left[e^{\xi X(T)} \right],$$

from which (9) follows. The proof of (10) is similar. □

It is also possible to compute the sensitivities of the option by inverting the derivatives of the option's Laplace transform in (7), as detailed in the following corollary.

Corollary 1. *For any maturity T and strike K, we have*

$$\Delta(C_T(k)) = \frac{\partial}{\partial S(0)} C_T(k) = \mathcal{L}_{\xi}^{-1} \left(e^{-rT} \frac{S(0)^{\xi}}{\xi} \exp(G(\xi+1)T) \right) |_{k=-\log K},$$

$$\Gamma(C_T(k)) = \frac{\partial^2}{\partial^2 S(0)} C_T(k) = \mathcal{L}_{\xi}^{-1} \left(e^{-rT} S(0)^{\xi-1} \exp(G(\xi+1)T) \right) |_{k=-\log K},$$

$$\Delta(P_T(k')) = \frac{\partial}{\partial S(0)} P_T(k') = -\mathcal{L}_{\xi}^{-1} \left(e^{-rT} \frac{S(0)^{-\xi}}{\xi} \exp(G(-(\xi-1)T) \right) |_{k'=\log K},$$

$$\Gamma(P_T(k')) = \frac{\partial^2}{\partial^2 S(0)} P_T(k') = \mathcal{L}_{\xi}^{-1} \left(e^{-rT} S(0)^{-(\xi+1)} \exp(G(-(\xi-1)T) \right) |_{k'=\log K},$$

where $\mathcal{L}_{\xi}^{-1}$ means the Laplace inversion with respect to ξ.

Proof. The results follow easily by interchanging derivatives and integrals, which is legitimate by using Theorem A. 12 on pp. 203-204 in Schiff (1999). □

We shall invert the transforms above in the complex plane, using the two-sided extension of the Euler algorithm as described and implemented in Petrella (2004). To check the accuracy of the inversion, in Table 1 we compare the inversion results with the prices obtained by using the closed-form formulae derived by Kou (2002). From the tables we see that the results from both inversion methods, LT1 and LT2, agree to the fifth decimal with the analytical solutions for European call and put options[2)].

3. American Options

For brevity we shall only consider finite-time horizon American put options. Related American calls can be priced by exploiting the symmetric relationship in Schroeder (1999)

$$C_A(S(0), K, r, \delta, \sigma, \lambda, p, \eta_1, \eta_2, T) = P_A(K, S(0), \delta, r, \sigma, \widetilde{\lambda}, \widetilde{p}, \widetilde{\eta}_1, \widetilde{\eta}_2, T).$$

To price American options we use a piecewise exponential approximation of the early exercise boundary, as suggested in Ju (1998).

Extending previous work by Carr et al. (1992), Gukhal (2001) and Pham (1997) show that under jump diffusion models the value at time t of an American put option with maturity $T > t$ on an asset with value S_t at time t ($P_A(S_t, t, T)$ from now on) is given by

$$\begin{aligned} P_A(S_t, t, T) = P_E(S_t, t, T) &+ \int_t^T e^{-r(s-t)} rK \mathsf{E}^* \left[\mathbf{1}_{\{S_s \le S_s^*\}} | S_t \right] ds \\ &- \delta \int_t^T e^{-r(s-t)} \mathsf{E}^* \left[S_s \mathbf{1}_{\{S_s \le S_s^*\}} | S_t \right] ds \qquad (11) \\ &- \lambda \int_t^T e^{-r(s-t)} \mathsf{E}^* \left[\{P_A(VS_{s^-}, s, T) - (K - VS_{s^-})\} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \mathbf{1}_{\{VS_{s^-} > S^*_{s^-}\}} | S_t \right] ds, \end{aligned}$$

where $P_E(S_t, t, T)$ is the value of the corresponding European put option, $\log(V) = Y$ with an independent double exponential distribution, and S_s^* is the early exercise boundary at time s, such that if the stocks price S_s goes below S_s^* at time s, then it is optimal to exercise immediately. Gukhal (2001) provides an interpretation of the four terms in (11): The value of an American put is given by the corresponding European put option value $P_E(X, t, T)$ to which we add the present value of

[2)] Because of the difficulty in precise calculation of the normal distribution function and the $Hh(x)$ function for very positive and negative x, it is possible that for very large values of the return variance $\sigma^2 T$ and for very high jump rate λ (though perhaps not within the typical parameter ranges seen in finance applications) the closed-form formulae may not give accurate results. In such cases, the inversion method still performs remarkably well, giving results as accurate as the ones presented herein.

Table 1 Accuracy check, the Laplace inversion methods versus closed-form (CF) soluation. In the table LT1 is obtained by inverting the Laplace transforms in (7) and (8), and LT2 by inverting seperately the Laplace transforms of the probabilities in (9) and $\widetilde{\Psi}$. The running times of LT1 and LT2 are all less than a tenth of a second for each option price on a Pentium IV, 1.8 Ghz, using a C++ implementation.

		European Call - Double Exponential Jump-Diffusion Model $S_0 = 100$, $r = 0.05$, $\sigma = .3$, $T = 1.0$, $p = 0.6$					
		$\eta_1 = \eta_2 = 20.0$			$\eta_1 = \eta_2 = 40.0$		
		Price LT1	Price LT2	Price CF	Price LT1	Price LT2	Price CF
K = 90	$\lambda = 1.0$	19.9547611	19.9547611	19.9547612	19.7633112	19.7633112	19.7633113
	$\lambda = 3.0$	20.4568712	20.4568712	20.4568712	19.8941074	19.8941074	19.8941074
	$\lambda = 5.0$	20.9431418	20.9431418	20.9431418	20.0236702	20.0236702	20.0236702
K = 100	$\lambda = 1.0$	14.5393158	14.5393158	14.5393157	14.3099234	14.3099234	14.3099234
	$\lambda = 3.0$	15.1347529	15.1347529	15.1347529	14.4657297	14.4657297	14.4657297
	$\lambda = 5.0$	15.7050995	15.7050995	15.7050995	14.6195549	14.6195549	14.6195549
K = 110	$\lambda = 1.0$	10.3484566	10.3484566	10.3484566	10.1033152	10.1033153	10.1033153
	$\lambda = 3.0$	10.9816866	10.9816866	10.9816867	10.2681125	10.2681125	10.2681125
	$\lambda = 5.0$	11.5866915	11.5866915	11.5866915	10.4307424	10.4307424	10.4307424
		European Put - Double Exponential Jump-Diffusion Model $S_0 = 100$, $r = 0.05$, $\sigma = .3$, $T = 1.0$, $p = 0.3$					
		$\eta_1 = \eta_2 = 20.0$			$\eta_1 = \eta_2 = 40.0$		
		Price LT1	Price LT2	Price CF	Price LT1	Price LT2	Price CF
K = 90	$\lambda = 1.0$	5.5661158	5.5661156	5.5661156	5.3741449	5.3741447	5.3741447
	$\lambda = 3.0$	6.0666513	6.0666511	6.0666511	5.5051816	5.5051814	5.5051814
	$\lambda = 5.0$	6.5483603	6.5483601	6.5483600	5.6348256	5.6348255	5.6348254
K = 100	$\lambda = 1.0$	9.6534727	9.6534725	9.6534725	9.4317515	9.4317513	9.4317513
	$\lambda = 3.0$	10.2313887	10.2313885	10.2313885	9.5853257	9.5853255	9.5853255
	$\lambda = 5.0$	10.7844632	10.7844630	10.7844630	9.7369176	9.7369174	9.7369174
K = 110	$\lambda = 1.0$	14.9652637	14.9652635	14.9652634	14.7361097	14.7361095	14.7361094
	$\lambda = 3.0$	15.5650780	15.5650778	15.5650778	14.8961635	14.8961633	14.8961633
	$\lambda = 5.0$	16.1404471	16.1404469	16.1404469	15.0542275	15.0542272	15.0542272

interest accrued on the strike price in the exercise region (IA, from now), subtract the present value of dividends lost in the exercise region (DL, from now on), and subtract the last term in (11), to be denoted by $RCJ(t,T)$, which represents the rebalancing costs due to jumps from the early exercise region to the continuation region and is absent in the case of pure-diffusion.

The term $RCJ(t,T)$ takes into account of the possibility of an upward jump that will move the asset price from the early exercise to the continuation region. Consequently, this term diminishes when the upward jump rate λp is small. Furthermore, intuitively this term should also be very small whenever a jump from the early exercise to the continuation region only causes minimal changes in the American option value, which in particular requires that the overshoot over the exercise boundary is not too large. This happens if the overshoot jump size has small mean, which in the double exponential case is $1/\eta_1$. In other words, the term $RCJ(t,T)$ should be negligible for either small λp or large η_1. The following proposition provides a bound for $RCJ(t,T)$, which confirms our intuition.

Proposition 1. *For $T > t$, under the double exponential jump diffusion model, the following bound holds for $RCJ(t,T)$*

$$RCJ(t,T) \leq \lambda p \frac{\eta_1}{\eta_1 - 1} K \cdot U(t,T). \tag{12}$$

where $U(t,T) = \int_t^T \mathsf{E}^* \left[\left(\frac{S^*_{s^-}}{S_{s^-}} \right)^{-(\eta_1 - 1)} \mathbf{1}_{\{S_{s^-} \leq S^*_{s^-}\}} | S_t \right] ds.$

The proof is deferred to the appendix. From (12) we can conclude that the term $RCJ(t,T)$ may be neglected when we have small upside jump rate λp or when the parameter η_1 is large (in which case the integrand inside $U(t,T)$ will be small). While we refer to Cont and Tankov (2004b) for more details on parameter estimation under the double exponential model, we believe that in most practical cases $\eta_1 > 10$. Therefore, we should expect that the upper bound $U(t,T)$ in (12) is typically very small, and we can ignore the term $RCJ(t,T)$ in equation (11) for practical usage.

Observing that at the optimal exercise boundary S^*_t, $P_A(S^*_t, t, T) = K - S^*_t$, we obtain an integral equation for S^*_t

$$\begin{aligned} K - S^*_t &= P_E(S^*_t, t, T) + \int_t^T e^{-r(s-t)} r K \mathsf{E}^* \left[\mathbf{1}_{\{S_s \leq S^*_s\}} | S_t = S^*_t \right] ds \\ &\quad - \int_t^T e^{-r(s-t)} \delta \mathsf{E}^* \left[S_s \mathbf{1}_{\{S_s \leq S^*_s\}} | S_t = S^*_t \right] ds, \end{aligned}$$

ignoring the term $RCJ(t,T)$. To solve this integral equation, we shall use a piecewise exponential function representation for the early exercise boundary as in Ju (1998).

More precisely, with n intervals of size $\Delta T = T/n$ we approximate the optimal boundary S^*_t by an n-piece exponential function $\tilde{S}_t = \exp(s^*_i + \alpha_i t)$ for $t \in [(i-1)\Delta T, i\Delta T)$ with $i = 1, .., n$. In our numerical experiments, even $n = 3$ or 5 will give sufficient accuracy in most cases.

To determine the value of the constants s^*_i and α_i in each interval, we make use of the "value-matching" and "smoothing-pasting" conditions (requiring the slope at the contacting point to be -1 to make the curve smooth). Thus, starting from $i = n$ going backwards to $i = 1$ we solve recursively at $t_i = (i-1)\Delta T$ the two unknowns s^*_i and α_i in terms of the system of two equations, i.e., the value matching equation

$$K - \widetilde{S}_i = P_E(\widetilde{S}_i, t_i, T) + \sum_{j=i}^{n} IA_j(\widetilde{S}_i, t_j) - \sum_{j=i}^{n} DL_j(\widetilde{S}_i, t_j), \tag{13}$$

and the smoothing pasting equation

$$-1 = \frac{\partial}{\partial \widetilde{S}_i} P_E(\widetilde{S}_i, t_i, T) + \sum_{j=i}^{n} \frac{\partial}{\partial \widetilde{S}_i} IA_j(\widetilde{S}_i, t_j) - \sum_{j=i}^{n} \frac{\partial}{\partial \widetilde{S}_i} DL_j(\widetilde{S}_i, t_j), \tag{14}$$

where $\widetilde{S}_i \equiv \widetilde{S}_{t_i} = \exp\{s_i^* + \alpha_i t_i\}$,

$$IA_j(S_t, u) = rK \int_u^{t_{j+1}} e^{-r(s-t)} \mathsf{E}^* \left[\mathbf{1}_{\{S_s \le \tilde{S}_s\}} | S_t \right] ds, \quad t \le u, \ u \in [t_j, t_{j+1}),$$

$$DL_j(S_t, u) = \delta \int_u^{t_{j+1}} e^{-r(s-t)} \mathsf{E}^* \left[S_s \mathbf{1}_{\{S_s \le \tilde{S}_s\}} | S_t \right] ds, \ t \le u, \ u \in [t_j, t_{j+1}).$$

This system of equations can be solved numerically via an iterative procedure to be specified shortly, if the right-hand sides of (13) and (14) can be computed. To this end, we shall derive Laplace transforms for these terms in the following theorem.

Theorem 1. *Let $\xi > 0$, the Laplace transforms with respect to s_i^* of IA_j and DL_j are given by*

$$\widehat{f_{IA_j}}(\xi) = \frac{rK \cdot S_t^{-\xi}}{\xi(G(-\xi) - r + \xi\alpha_j)} e^{(G(-\xi)-r)(u-t)+\xi\alpha_j u} \left[e^{(G(-\xi)-r+\xi\alpha_j)(t_{j+1}-u)} - 1 \right], \tag{15}$$

$$\widehat{f_{DL_j}}(\xi) = \frac{\delta \cdot S_t^{-(\xi-1)}}{\xi(\tilde{G}(-\xi) - \delta + \xi\alpha_j)} e^{(\tilde{G}(-\xi)-\delta)(u-t)+\xi\alpha_j u} \left[e^{(\tilde{G}(-\xi)-\delta+\xi\alpha_j)(t_{j+1}-u)} - 1 \right], \tag{16}$$

where

$$\widehat{f_{IA_j}}(\xi) = \int_{-\infty}^{\infty} e^{-\xi s_i^*} IA_j(S_t) ds_j^*, \quad \widehat{f_{DL_j}}(\xi) = \int_{-\infty}^{\infty} e^{-\xi s_i^*} DL_j(S_t) ds_j^*.$$

Thus, for all $j = 1, ..., n$ we have

$$\frac{\partial}{\partial S_t} IA_j = -\mathcal{L}_\xi^{-1} \left(\frac{rK \cdot S_t^{-(\xi+1)}}{(G(-\xi) - r + \xi\alpha_j)} e^{(G(-\xi)-r)(u-t)+\xi\alpha_j u} \left[e^{(G(-\xi)-r+\xi\alpha_j)(t_{j+1}-u)} - 1 \right] \right), \tag{17}$$

$$\frac{\partial}{\partial S_t} DL_j = -\mathcal{L}_\xi^{-1} \left(\frac{\delta \cdot (\xi - 1) \cdot S_t^{-\xi}}{\xi(\tilde{G}(-\xi) - \delta + \xi\alpha_j)} e^{(\tilde{G}(-\xi)-\delta)(u-t)+\xi\alpha_j u} \left[e^{(\tilde{G}(-\xi)-\delta+\xi\alpha_j)(t_{j+1}-u)} - 1 \right] \right), \tag{18}$$

where $\mathcal{L}_\xi^{-1}$ means the Laplace inversion with respect to ξ.

Proof. By the Fubini theorem, the Laplace transform for IA_j is

$$\begin{aligned}
\widehat{f_{IA_j}}(\xi) &= rK\int_{-\infty}^{\infty} e^{-\xi s_j^*}\int_u^{t_{j+1}} e^{-r(s-t)}\mathsf{E}^*\left[1_{\{S_s\le \exp(s_j^*+\alpha_j s)\}}\,|S_t\right] ds\,ds_j^* \\
&= rK\int_u^{t_{j+1}} e^{-r(s-t)}\mathsf{E}^*\left[\int_{-\infty}^{\infty} e^{-\xi s_j^*}1_{\{s_j^*\ge \log S_s-\alpha_j s\}}ds_j^*\,|S_t\right] ds \\
&= \frac{rK}{\xi}\int_u^{t_{j+1}} e^{-r(s-t)}\mathsf{E}^*\left[e^{-\xi(\log S_s-\alpha_j s)}\,|S_t\right] ds \\
&= \frac{rK\cdot S_t^{-\xi}}{\xi}\int_u^{t_{j+1}} e^{-r(s-t)+\xi\alpha_j s}\mathsf{E}^*\left[\left(\frac{S_s}{S_t}\right)^{-\xi}\,|S_t\right] ds \\
&= \frac{rK\cdot S_t^{-\xi}}{\xi}\int_u^{t_{j+1}} e^{-r(s-t)+\xi\alpha_j s+G(-\xi)(s-t)}\, ds,
\end{aligned}$$

from which (15) follows readily. Equation (16) can be derived in the same way by using the measure $\widetilde{\mathsf{P}}$, and is thus omitted. Theorem A. 12 on pp. 203-204 in Schiff (1999) justifies interchanging derivatives and integrals, which leads to (17) and (18). □

In summary, we have the following algorithm.

The Algorithm:

1. Compute the approximation exercise boundary $\tilde{S}$ by letting i going backwards from n to 1 while, at each time point t_i one solves the system of two equations in (13) and (14) to get s_i^* and α_i, with the right hand side of (13) and (14) being obtained by inverting Laplace transforms in Theorem 1. The system of two equations can be solved, for example, by using the multi-dimensional secant method by Broydn (as implemented in Press et al., 1993).

2. After the boundary $\tilde{S}$ is obtained, at any time $t \in [t_i, t_{i+1})$, the value of the American put option is given by

$$P_E(S_t,t,T) + IA_i(S_t,t) + \sum_{j=i+1}^{n} IA_j(S_t,t_j) - DL_i(S_t,t) - \sum_{j=i+1}^{n} DL_j(S_t,t_j).$$

Note that IA_j and DL_j involve both S_t and $\tilde{S}$.

In our numerical implementation, we use the two-sided Euler algorithm in Petrella (2004) to do inversion in Step 1. The initial values for the secant method is obtained by setting $\alpha_i = 0$ and using the critical value in the approximation given by Kou and Wang (2004) as an initial value of S_i^*.

In Tables 2 and 3 we report the prices using a 3 and 5-piece exponential function approximation of the boundary (3EXP and 5EXP respectively, from now on and in the tables). We compare our results with the "true" values computed using the tree method as in Amin (1993) and the prices obtained by the analytic approximation in Kou and Wang (2004) (KW from now on and in the tables). In Amin's tree method we use 1600 steps and the two-point Richardson extrapolation for the square-root convergence rate, ensuring an accuracy of about a penny. The running time of the new algorithm is less than 2 seconds for 3EXP and 4 seconds for 5EXP, compared to more than an hour required by the Amin's tree method. In most cases 3EXP provides an estimate of the option price more accurate than KW, and, as we would expect, 5EXP has even better accuracy. We also find that adding additional segments to the piecewise function does not significantly increase the accuracy of the results beyond what we get using 5EXP.

While here we focus on jump diffusion processes, one may speculate that the method might work for general processes, so long as the overshoot from upward jumps is not too large so that the rebalance cost term can still be ignored, since the calculation in Theorem 1 can be easily extended to more general models. This will be on our future research agenda.

4. Barrier and Lookback Options

Barrier and lookback options are among the most popular path-dependent derivatives traded in exchanges and over-the-counter markets worldwide. The payoffs of these options depend on the extrema of the underlying asset. For a complete description of these and other related contracts we refer the reader to Hull (2002). To study barrier and lookback options, it is crucial to understand the first passage times τ_b defined by

$$\tau_b := \inf\{t \geq 0;\ X(t) \geq b\}, \quad b > 0,$$

where $X(\tau_b) := \limsup_{t\to\infty} X(t)$, on the set $\{\tau_b = \infty\}$. In the standard Black-Scholes setting, closed-form solutions for barrier and lookback options have been derived by Merton (1973) and Goldman et al. (1979). For the double exponential jump diffusion model, Kou and Wang (2003) shows that the memoryless property of the exponential distribution leads to (1) the conditional memoryless property of the jump overshoot; (2) the conditional independence of the overshoot, $X(\tau_b) - b$, and the first passage time τ_b, given that the overshoot is bigger than 0; (3) and analytical solutions for the Laplace transforms of τ_b.

4.1. Pricing Barrier Options

We will focus on the pricing of an up-and-in call option (UIC, from now on); other types of barrier options can be priced similarly and using the symmetries described in the Appendix of Petrella and Kou (2004) and Haug (1999). The price of an UIC is given by

$$UIC(k,T) = \mathsf{E}^*\left[e^{-rT}\left(S(T) - e^{-k}\right)^+ \mathbf{1}_{\{\tau_b<T\}}\right], \tag{19}$$

where $H > S(0)$ is the barrier level, $k = -\log(K)$ the transformed strike and $b = \log(H/S(0))$. Using a change of numeraire argument, Kou and Wang (2004) obtain

$$UIC(k,T) = S(0)\widetilde{\Psi}_{UI}(k,T) - Ke^{-rT}\Psi_{UI}(k,T), \tag{20}$$

Table 2 Comparison of the new approximation with the approximation in Kou and Wang (2004) with $S(0) = 100$ and $T = 0.25$ years. The "true price" is caculated by Amin's tree method. The CPU time for Amin's method is more than one hour, while the CPU times for 3EXP and 5EXP are about 2 and 4 seconds, respectively.

American Put - Double Exponential Jump-Diffusion Model													
Parameter Values					True	3EXP			5EXP			KW	
K	σ	λ	η_1	η_2	Value	Value	Time	Rel. Err.	Value	Time	Rel. Err.	Value	Rel. Err.
110	0.2	3	25	25	10.48	10.45	1.23	-0.3%	10.46	3.05	-0.2%	10.43	-0.5%
110	0.2	3	25	50	10.42	10.40	1.29	-0.2%	10.41	3.27	-0.1%	10.38	-0.4%
110	0.2	3	50	25	10.36	10.36	1.36	0.0%	10.36	2.83	0.0%	10.31	-0.5%
110	0.2	3	50	50	10.31	10.31	1.41	0.0%	10.31	3.44	0.0%	10.26	-0.5%
110	0.2	7	25	25	10.81	10.78	1.52	-0.3%	10.80	3.65	-0.1%	10.79	-0.2%
110	0.2	7	25	50	10.68	10.65	1.43	-0.3%	10.66	3.66	-0.2%	10.64	-0.4%
110	0.2	7	50	25	10.51	10.51	1.50	0.0%	10.51	2.26	0.0%	10.47	-0.4%
110	0.2	7	50	50	10.39	10.39	1.37	0.0%	10.39	2.32	0.0%	10.34	-0.5%
110	0.3	3	25	25	11.90	11.89	1.35	-0.1%	11.90	2.41	0.0%	11.86	-0.3%
110	0.3	3	25	50	11.84	11.83	1.32	-0.1%	11.83	2.41	-0.1%	11.79	-0.4%
110	0.3	3	50	25	11.78	11.78	1.71	0.0%	11.78	2.65	0.0%	11.73	-0.4%
110	0.3	3	50	50	11.72	11.72	1.82	0.0%	11.72	2.69	0.0%	11.67	-0.4%
110	0.3	7	25	25	12.23	12.21	1.22	-0.2%	12.22	2.95	-0.1%	12.19	-0.3%
110	0.3	7	25	50	12.09	12.07	1.35	-0.2%	12.08	3.00	-0.1%	12.05	-0.3%
110	0.3	7	50	25	11.94	11.95	1.45	0.1%	11.95	2.73	0.1%	11.90	-0.3%
110	0.3	7	50	50	11.80	11.80	1.29	0.0%	11.81	2.59	0.1%	11.75	-0.4%
100	0.2	3	25	25	3.78	3.76	1.25	-0.5%	3.77	3.08	-0.3%	3.78	0.0%
100	0.2	3	25	50	3.66	3.65	1.27	-0.3%	3.65	3.29	-0.3%	3.66	0.0%
100	0.2	3	50	25	3.62	3.62	1.31	0.0%	3.62	2.88	0.0%	3.62	0.0%
100	0.2	3	50	50	3.50	3.50	1.42	0.0%	3.50	3.00	0.0%	3.50	0.0%
100	0.2	7	25	25	4.26	4.25	1.51	-0.2%	4.26	3.48	0.0%	4.27	0.2%
100	0.2	7	25	50	4.01	4.00	1.43	-0.2%	4.00	3.69	-0.2%	4.02	0.2%
100	0.2	7	50	25	3.91	3.91	1.71	0.0%	3.91	2.29	0.0%	3.91	0.0%
100	0.2	7	50	50	3.64	3.64	1.36	0.0%	3.64	2.34	0.0%	3.64	0.0%
100	0.3	3	25	25	5.63	5.62	1.40	-0.2%	5.63	2.44	0.0%	5.62	-0.2%
100	0.3	3	25	50	5.55	5.54	1.27	-0.2%	5.54	2.40	-0.2%	5.54	-0.2%
100	0.3	3	50	25	5.50	5.50	1.75	0.0%	5.51	2.64	0.2%	5.50	0.0%
100	0.3	3	50	50	5.42	5.42	1.88	0.0%	5.42	2.59	0.0%	5.41	-0.2%
100	0.3	7	25	25	5.99	5.98	1.19	-0.2%	5.99	2.95	0.0%	5.99	0.0%
100	0.3	7	25	50	5.81	5.80	1.32	-0.2%	5.80	2.95	-0.2%	5.81	0.0%
100	0.3	7	50	25	5.71	5.71	1.49	0.0%	5.71	2.64	0.0%	5.71	0.0%
100	0.3	7	50	50	5.52	5.52	1.29	0.0%	5.52	2.55	0.0%	5.51	-0.2%
90	0.2	3	25	25	0.75	0.74	1.28	1.3%	0.74	3.21	-1.3%	0.76	1.3%
90	0.2	3	25	50	0.65	0.65	1.30	0.0%	0.65	3.25	0.0%	0.66	1.5%
90	0.2	3	50	25	0.68	0.68	1.23	0.0%	0.68	2.97	0.0%	0.69	1.5%
90	0.2	3	50	50	0.59	0.59	1.44	0.0%	0.59	2.89	0.0%	0.60	1.7%
90	0.2	7	25	25	1.03	1.02	1.51	-1.0%	1.02	3.18	-1.0%	1.04	1.0%
90	0.2	7	25	50	0.82	0.82	1.43	0.0%	0.82	3.65	0.0%	0.83	1.2%
90	0.2	7	50	25	0.87	0.87	1.68	0.0%	0.87	2.26	0.0%	0.88	1.1%
90	0.2	7	50	50	0.66	0.66	1.45	0.0%	0.66	2.47	0.0%	0.67	1.5%
90	0.3	3	25	25	1.92	1.91	1.34	-0.5%	1.92	2.40	0.0%	1.93	0.5%
90	0.3	3	25	50	1.85	1.84	1.23	-0.5%	1.84	2.39	-0.5%	1.86	0.5%
90	0.3	3	50	25	1.84	1.84	1.69	0.0%	1.84	2.60	0.0%	1.85	0.5%
90	0.3	3	50	50	1.77	1.77	1.65	0.0%	1.77	2.56	0.0%	1.78	0.6%
90	0.3	7	25	25	2.19	2.18	1.38	-0.5%	2.18	2.97	-0.5%	2.20	0.5%
90	0.3	7	25	50	2.03	2.02	1.28	-0.5%	2.02	2.96	-0.5%	2.03	0.0%
90	0.3	7	50	25	2.01	2.00	1.39	-0.5%	2.01	2.61	0.0%	2.02	0.5%
90	0.3	7	50	50	1.84	1.84	1.43	0.0%	1.84	2.54	0.0%	1.85	0.5%

Table 3 Comparison of the new approximation with the true price and the approximation in Kou and Wang (2003) with $S(0) = 100$ and $T = 1.0$ years.

American Put - Double Exponential Jump-Diffusion Model													
Parameter Values					True	3EXP			5EXP			KW	
K	σ	λ	η_1	η_2	Value	Value	Time	Rel. Err.	Value	Time	Rel. Err.	Value	Rel. Err.
110	0.2	3	25	25	12.37	12.33	1.06	-0.3%	12.35	2.48	-0.2%	12.32	-0.4%
110	0.2	3	25	50	12.17	12.12	1.06	-0.4%	12.14	2.48	-0.2%	12.11	-0.5%
110	0.2	3	50	25	12.04	12.05	1.08	0.1%	12.06	2.55	0.2%	12.00	-0.3%
110	0.2	3	50	50	11.84	11.84	1.10	0.0%	11.84	2.44	0.0%	11.78	-0.5%
110	0.2	7	25	25	13.29	13.23	1.02	-0.5%	13.26	3.06	-0.2%	13.27	-0.2%
110	0.2	7	25	50	12.85	12.76	1.21	-0.7%	12.78	3.25	-0.5%	12.79	-0.5%
110	0.2	7	50	25	12.54	12.57	1.10	0.2%	12.58	2.30	0.3%	12.54	0.0%
110	0.2	7	50	50	12.08	12.08	1.08	0.0%	12.09	2.57	0.1%	12.03	-0.4%
110	0.3	3	25	25	15.79	15.77	1.12	-0.1%	15.78	2.58	-0.1%	15.76	-0.2%
110	0.3	3	25	50	15.63	15.61	1.11	-0.1%	15.62	2.55	-0.1%	15.59	-0.3%
110	0.3	3	50	25	15.51	15.53	1.39	0.1%	15.53	2.84	0.1%	15.49	-0.1%
110	0.3	3	50	50	15.36	15.36	1.34	0.0%	15.36	2.80	0.0%	15.32	-0.3%
110	0.3	7	25	25	16.51	16.50	1.03	-0.1%	16.51	2.96	0.0%	16.51	0.0%
110	0.3	7	25	50	16.17	16.13	1.05	-0.2%	16.14	3.15	-0.2%	16.14	-0.2%
110	0.3	7	50	25	15.89	15.94	1.15	0.3%	15.94	3.13	0.3%	15.91	0.1%
110	0.3	7	50	50	15.53	15.55	1.14	0.1%	15.56	2.90	0.2%	15.52	-0.1%
100	0.2	3	25	25	6.60	6.57	1.04	-0.5%	6.58	2.40	-0.3%	6.62	0.3%
100	0.2	3	25	50	6.36	6.32	1.04	-0.6%	6.33	2.42	-0.5%	6.37	0.2%
100	0.2	3	50	25	6.26	6.26	1.08	0.0%	6.27	2.46	0.2%	6.29	0.5%
100	0.2	3	50	50	6.01	6.00	1.09	-0.2%	6.01	2.32	0.0%	6.03	0.3%
100	0.2	7	25	25	7.57	7.53	1.03	-0.5%	7.55	3.01	-0.3%	7.62	0.7%
100	0.2	7	25	50	7.07	7.01	1.22	-0.8%	7.02	3.50	-0.7%	7.09	0.3%
100	0.2	7	50	25	6.83	6.85	1.09	0.3%	6.85	2.29	0.3%	6.88	0.7%
100	0.2	7	50	50	6.28	6.28	1.05	0.0%	6.28	2.37	0.0%	6.31	0.5%
100	0.3	3	25	25	10.10	10.09	1.11	-0.1%	10.09	2.40	-0.1%	10.13	0.3%
100	0.3	3	25	50	9.94	9.92	1.12	-0.2%	9.92	2.46	-0.2%	9.96	0.2%
100	0.3	3	50	25	9.83	9.84	1.36	0.1%	9.85	2.83	0.2%	9.87	0.4%
100	0.3	3	50	50	9.67	9.67	1.34	0.0%	9.68	2.83	0.1%	9.70	0.3%
100	0.3	7	25	25	10.81	10.80	1.03	-0.1%	10.81	2.96	0.0%	10.86	0.5%
100	0.3	7	25	50	10.46	10.43	1.04	-0.3%	10.44	3.16	-0.2%	10.49	0.3%
100	0.3	7	50	25	10.22	10.26	1.15	0.4%	10.26	3.18	0.4%	10.29	0.7%
100	0.3	7	50	50	9.85	9.86	1.13	0.1%	9.87	3.66	0.2%	9.89	0.4%
90	0.2	3	25	25	2.91	2.89	1.05	-0.7%	2.90	2.43	-0.3%	2.96	1.7%
90	0.2	3	25	50	2.70	2.68	1.04	-0.7%	2.69	2.38	-0.4%	2.75	1.9%
90	0.2	3	50	25	2.66	2.67	1.08	0.4%	2.67	2.55	0.4%	2.72	2.3%
90	0.2	3	50	50	2.46	2.45	1.08	-0.4%	2.45	2.30	-0.4%	2.51	2.0%
90	0.2	7	25	25	3.68	3.66	1.03	-0.5%	3.67	2.96	-0.3%	3.75	1.9%
90	0.2	7	25	50	3.24	3.20	1.21	-1.2%	3.21	3.23	-0.9%	3.29	1.5%
90	0.2	7	50	25	3.12	3.14	1.09	0.6%	3.14	2.28	0.6%	3.20	2.6%
90	0.2	7	50	50	2.66	2.66	1.08	0.0%	2.66	2.43	0.0%	2.72	2.3%
90	0.3	3	25	25	5.79	5.78	1.15	-0.2%	5.79	2.48	0.0%	5.85	1.0%
90	0.3	3	25	50	5.65	5.63	1.11	-0.4%	5.63	2.46	-0.4%	5.70	0.9%
90	0.3	3	50	25	5.58	5.58	1.37	0.0%	5.58	2.83	0.0%	5.64	1.1%
90	0.3	3	50	50	5.43	5.42	1.34	-0.2%	5.43	2.78	0.0%	5.49	1.1%
90	0.3	7	25	25	6.42	6.40	1.03	-0.3%	6.41	2.95	-0.2%	6.49	1.1%
90	0.3	7	25	50	6.09	6.07	1.03	-0.3%	6.07	3.11	-0.3%	6.15	1.0%
90	0.3	7	50	25	5.92	5.94	1.15	0.3%	5.94	3.92	0.3%	6.00	1.4%
90	0.3	7	50	50	5.59	5.59	1.14	0.0%	5.59	3.55	0.0%	5.65	1.1%

where

$$\Psi_{UI}(k,T) = \mathsf{P}^*(S(T) \geq e^{-k},\ \tau_b < T), \qquad \widetilde{\Psi}_{UI}(k,T) = \widetilde{\mathsf{P}}(S(T) \geq e^{-k},\ \tau_b < T), \tag{21}$$

and show how to price an UIC option by inverting the one-dimensional Laplace transforms for the joint distributions in (20) as in Kou and Wang (2003).

Here we present an alternative approach that relies on a two-dimensional Laplace transform for both the option price in (19) and the probabilities in (20). The formulae after doing two-dimensional transforms become much simpler than the one-dimensional formulae in Kou and Wang (2003), which involve many special functions.

Theorem 2. *For ξ and α such that $0 < \xi < \eta_1 - 1$ and $\alpha > \max(G(\xi+1) - r, 0)$ (such a choice of ξ and α is possible for all small enough ξ as $G(1) - r = -\delta < 0$), The Laplace transform with respect to k and T of $UIC(k,T)$ is given by*

$$\begin{aligned} \widehat{f}_{UIC}(\xi,\alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k - \alpha T} UIC(k,T) dk dT \\ &= \frac{H^{\xi+1}}{\xi(\xi+1)} \frac{1}{r+\alpha-G(\xi+1)} \left(A(r+\alpha) \frac{\eta_1}{\eta_1 - (\xi+1)} + B(r+\alpha) \right), \end{aligned} \tag{22}$$

where

$$A(h) := \mathsf{E}^*\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b) > b\}}\right] = \frac{(\eta_1 - \beta_{1,h})(\beta_{2,h} - \eta_1)}{\eta_1 (\beta_{2,h} - \beta_{1,h})} \left[e^{-b\beta_{1,h}} - e^{-b\beta_{2,h}}\right], \tag{23}$$

$$B(h) := \mathsf{E}^*\left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b) = b\}}\right] = \frac{\eta_1 - \beta_{1,h}}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{1,h}} + \frac{\beta_{2,h} - \eta_1}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{2,h}}, \tag{24}$$

with $b = \log(H/S(0))$. If $0 < \xi < \eta_1$ and $\alpha > \max(G(\xi), 0)$ (again this choice of ξ and α is possible for all ξ small enough as $G(0) = 0$), then the Laplace transform with respect to k and T of $\Psi_{UI}(k,T)$ in (21) is

$$\widehat{f}_{\Psi_{UI}}(\xi,\alpha) = \int_{-\infty}^\infty \left(\int_0^\infty e^{-\xi k - \alpha T} \Psi_{UI}(k,T) dT \right) dk = \frac{H^\xi}{\xi} \frac{1}{\alpha - G(\xi)} \left(A(\alpha) \frac{\eta_1}{\eta_1 - \xi} + B(\alpha) \right). \tag{25}$$

The Laplace transforms with respect to k and T of $\widetilde{\Psi}_{UI}(k,T)$ is given similarly with $\widetilde{G}$ replacing G and the functions $\tilde{A}$ and $\tilde{B}$ defined similarly.

Proof. It follows from (19) and the Fubini theorem that

$$
\begin{aligned}
\widehat{f}_{UIC}(\xi,\alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k-(r+\alpha)T}\mathsf{E}^*\left[\left(S(T)-e^{-k}\right)^+ \mathbf{1}_{\{\tau_b<T\}}\right]dkdT \\
&= \mathsf{E}^*\left[\int_0^\infty e^{-(r+\alpha)T}\mathbf{1}_{\{\tau_b<T\}}\left(\int_{-\log S(T)}^\infty e^{-\xi k}\left(S(T)-e^{-k}\right)\right)dkdT\right] \\
&= \frac{1}{\xi(\xi+1)}\mathsf{E}^*\left[\int_0^\infty e^{-(r+\alpha)T}\mathbf{1}_{\{\tau_b<T\}}S(T)^{\xi+1}dT\right] \\
&= \frac{1}{\xi(\xi+1)}\mathsf{E}^*\left[e^{-(r+\alpha)\tau_b}\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_b)^{\xi+1}dt\right].
\end{aligned}
$$

However, the strong Markov property of X implies that

$$
\begin{aligned}
\mathsf{E}^*\left[\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_b)^{\xi+1}dt\,|\mathcal{F}_{\tau_b}\right] &= S(\tau_b)^{\xi+1}\int_0^\infty e^{-(r+\alpha)t}e^{G(\xi+1)t}\,dt \\
&= \frac{S(\tau_b)^{\xi+1}}{r+\alpha-G(\xi+1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\widehat{f}_{UIC}(\xi,\alpha) \\
=\;& \frac{1}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\mathsf{E}^*\left[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\xi+1}\right] \\
=\;& \frac{1}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\left\{\mathsf{E}^*\left[e^{-(r+\alpha)\tau_b}H^{\xi+1}\mathbf{1}_{\{X(\tau_b)>b\}}\right]\mathsf{E}^*[e^{(\xi+1)\chi^+}]\right. \\
&\left.+\mathsf{E}^*\left[e^{-(r+\alpha)\tau_b}H^{\xi+1}\mathbf{1}_{\{X(\tau_b)=b\}}\right]\right\} \\
=\;& \frac{H^{\xi+1}}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\left\{A(r+\alpha)\frac{\eta_1}{\eta_1-(\xi+1)}+B(r+\alpha)\right\},
\end{aligned}
$$

where χ^+ is an exponential random variable with rate η_1. Here the second equality follows from a conditional independent and memoryless property shown in Kou and Wang (2003). The calculations of $A(h)$ and $B(h)$ are also from Kou and Wang (2003).

For the Laplace transform of the probability Ψ_{UI}, we have

$$
\begin{aligned}
\widehat{f}_{\Psi_{UI}}(\xi,\alpha) &= \int_0^\infty\left[\int_{-\infty}^\infty e^{-\xi k-\alpha T}\cdot\mathsf{E}^*\left\{\mathbf{1}_{\{k>-\log(S(T)),\,\tau_b<T\}}\right\}dk\right]dT \\
&= \mathsf{E}^*\left\{\int_{\tau_b}^\infty\left[\int_{-\log S(T)}^\infty e^{-\xi k-\alpha T}dk\right]dT\right\} \\
&= \frac{1}{\xi}\mathsf{E}^*\left\{\int_{\tau_b}^\infty S(T)^\xi e^{-\alpha T}dT\right\} \\
&= \frac{1}{\xi}\mathsf{E}^*\left\{e^{-\alpha\tau_b}\int_0^\infty\{S(t+\tau_b)\}^\xi e^{-\alpha t}dt\right\}.
\end{aligned}
$$

The strong Markov property implies that

$$\mathsf{E}^*\left\{\int_0^\infty \{S(t+\tau_b)\}^\xi e^{-\alpha t}dt|\mathcal{F}_{\tau_b}\right\} = \{S(\tau_b)\}^\xi \int_0^\infty e^{tG(\xi)}e^{-\alpha t}dt = \frac{\{S(\tau_b)\}^\xi}{\alpha - G(\xi)}.$$

Therefore,

$$\begin{aligned}
&\widehat{f}_{\Psi_{UI}}(\xi,\alpha)\\
= \;& \frac{1}{\xi}\frac{1}{\alpha - G(\xi)}\mathsf{E}^*\left\{e^{-\alpha\tau_b}\{S(\tau_b)\}^\xi\right\}\\
= \;& \frac{1}{\xi}\frac{1}{\alpha - G(\xi)}\left\{\mathsf{E}^*\left[e^{-\alpha\tau_b}H^\xi\mathbf{1}_{\{X(\tau_b)>b\}}\right]\mathsf{E}^*[e^{\xi X^+}] + \mathsf{E}^*\left[e^{-\alpha\tau_b}H^\xi\mathbf{1}_{\{X(\tau_b)=b\}}\right]\right\}\\
= \;& \frac{H^\xi}{\xi}\frac{1}{\alpha - G(\xi)}\left\{A(\alpha)\frac{\eta_1}{\eta_1 - \xi} + B(\alpha)\right\},
\end{aligned}$$

by the conditional memoryless property, from which (25) follows. □

In Table 4 we price up-and-in calls using the two-dimensional transform herein and compare the results with the one-dimensional transform in Kou and Wang (2003) (KW from now on and in the table) and Monte Carlo simulation (MC). LT1 indicates the price obtained by inverting (22), LT2 uses (20), in which the probabilities are obtained by inverting the transforms in (25). To perform the inversion, we use the two-sided Euler method as in Petrella (2004). Our results from LT1 match to the fourth digit the ones obtained by KW in which a one-dimensional transform is inverted via the Gaver-Stehfest (GS) algorithm. From the tables we see that three inversion methods provide values which are all within the 95% confidence interval obtained via Monte Carlo simulation. Furthermore, the results obtained either inverting directly (22) or inverting (25) and then using (20) differ for less than 2×10^{-4}, confirming the accuracy of the transform approach.

The LT1 and LT2 algorithms have three advantages compared to KW: (1) The formulae for the two-dimensional transforms are much easier to compute, simplifying the implementation of the methods. (2) Although we are inverting two-dimensional transforms, the LT methods are significantly faster, mainly because of the simplicity in the Laplace transform formulae. (3) High-precision calculation (with about 80 digit accuracy) as required by the GS inversion is no longer needed in the EUL inversion. The EUL inversion is made possible mainly because of the simplicity of the two-dimensional inversion formulae in Theorem 2, as no special functions are involved and all the roots of $G(x)$ are given in analytical forms.

4.2. Pricing Lookback Options via Euler Inversion

For simplicity, we shall focus on a standard lookback put option, while the derivation for a standard lookback call is similar. The price of a standard lookback

Table 4 The two dimensional Laplace inversion using the Euler Method (EUL) vs. Monte Carlo (MC) and the one-dimensional inversion in Kou and Wang (2004) via the Gaver-Stehfest inversion method (GS). The MC (along with its standard error reported in the brackets) is obtained by using 16,000 time steps and 20,000 simulation paths. Note that the MC underestimates the option's price due to the systematic discretization bias in simulation. On a Pentium IV 1.8 GHz, the EUL requires about 6 and 11 seconds for LT1 and LT2, respectively, in a C++ implementation, while the GS takes about 70 seconds running in Mathematica. A precision of 12 digits will suffice for the EUL, compared with the 80 digits accuracy required by the GS.

Up-and-In Call - Double Exponential Jump-Diffusion Model, Varying H
$S_0 = 100$, $K = 102$, $r = 0.05$, $\sigma = .2$, $T = 1.0$, $p = 0.5$

		$\eta_1 = \eta_2 = 30.0$				$\eta_1 = \eta_2 = 40.0$			
H	λ	Price LT1	Price LT2	Price KW	Price MC	Price LT1	Price LT2	Price KW	Price MC
	0.5	9.52560	9.52560	9.52565	9.50518 (0.10102)	9.48082	9.48082	9.48082	9.45745 (0.10038)
105	1.0	9.62850	9.62850	9.62850	9.63707 (0.10311)	9.53974	9.53974	9.53975	9.54124 (0.10162)
	2.0	9.83076	9.83076	9.83073	9.84272 (0.10602)	9.65637	9.65636	9.65634	9.65847 (0.10329)
	0.5	9.46795	9.46787	9.46795	9.44522 (0.10124)	9.42263	9.42256	9.42264	9.39553 (0.10061)
110	1.0	9.57222	9.57214	9.57222	9.58343 (0.10330)	9.48243	9.48236	9.48244	9.48706 (0.10182)
	2.0	9.77707	9.77699	9.77706	9.78397 (0.10623)	9.60074	9.60067	9.60074	9.60157 (0.10350)
	0.5	9.21925	9.21913	9.21924	9.19926 (0.10195)	9.17155	9.17143	9.17156	9.15492 (0.10131)
115	1.0	9.32954	9.32942	9.32955	9.33598 (0.10402)	9.23512	9.23501	9.23514	9.23907 (0.10253)
	2.0	9.54573	9.54562	9.54574	9.54535 (0.10693)	9.36072	9.36060	9.36072	9.35698 (0.10421)

Up-and-In Call - Double Exponential Jump-Diffusion Model, Varying K
$S_0 = 100$, $H = 115$, $r = 0.05$, $\sigma = .2$, $T = 1.0$, $p = 0.5$

		$\eta_1 = \eta_2 = 30.0$				$\eta_1 = \eta_2 = 40.0$			
K	λ	Price LT1	Price LT2	Price KW	Price MC	Price LT1	Price LT2	Price KW	Price MC
	0.5	9.64680	9.64686	9.64682	9.62480 (0.10435)	9.59898	9.59903	9.5990	9.58070 (0.10371)
101	1.0	9.75755	9.75760	9.75757	9.76223 (0.10640)	9.66291	9.66297	9.66293	9.66519 (0.10493)
	2.0	9.97456	9.97461	9.97457	9.97351 (0.10929)	9.78917	9.78923	9.78919	9.78551 (0.10658)
	0.5	7.98683	7.98689	7.98685	7.97048 (0.09488)	7.93950	7.93956	7.93952	7.92508 (0.09423)
105	1.0	8.09581	8.09586	8.09582	8.10582 (0.09698)	8.00209	8.00215	8.00211	8.00779 (0.09548)
	2.0	8.30966	8.30971	8.30967	8.30908 (0.09995)	8.12582	8.12588	8.12584	8.12085 (0.09720)
	0.5	6.47897	6.47905	6.47897	6.46558 (0.08571)	6.43239	6.43247	6.43241	6.41893 (0.08506)
109	1.0	6.58586	6.58593	6.58588	6.59610 (0.08788)	6.49355	6.49363	6.49357	6.49715 (0.08635)
	2.0	6.79588	6.79595	6.79590	6.79550 (0.09092)	6.61457	6.61465	6.61459	6.60972 (0.08811)

put is given by

$$
\begin{aligned}
LP(T) &= \mathsf{E}^*\left[e^{-rT}\left\{\max\left\{M, \max_{0\le t\le T} S(t)\right\} - S(t)\right\}\right] \\
&= \mathsf{E}^*\left[e^{-rT}\max\left\{M, \max_{0\le t\le T} S(t)\right\}\right] - S(0),
\end{aligned}
$$

where $M \geq S(0)$ is the prefixed maximum at time 0. For any $\xi > 0$, the Laplace transform of the lookback put with respect to the time to maturity T is given by (see Kou and Wang, 2004)

$$
\int_0^\infty e^{-\alpha T} LP(T) dT = \frac{S(0)A_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{1,\alpha+r}-1} + \frac{S(0)B_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha}, \tag{26}
$$

where

$$
A_\alpha = \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1}, \quad B_\alpha = \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1}, \quad C_\alpha = (\alpha+r)\eta_1\left(\beta_{2,\alpha+r} - \beta_{1,\alpha+r}\right),
$$

and $\beta_{1,\alpha+r}, \beta_{2,\alpha+r}$ are the two positive roots of the equation $G(x) = \alpha + r$, as in (4).

We shall invert the transform in (26) in the complex domain by using the Euler inversion algorithm (EUL from now on) developed by Abate and Whitt (1995), rather than in the real domain by the Gaver-Stehfest algorithm (GS) as in Kou and Wang (2004). The main reason for this is that the EUL inversion (which is carried out in the complex-domain) does not require the high numerical precision of the GS: A precision of 12 digits will suffice for the EUL, compared with the 80 digits accuracy required by the GS. The EUL algorithm is made possible partly due to an explicit formula for the roots of $G(x)$ given in Appendix B.

In Table 5 the results of a standard lookback put from both the EUL and GS are compared to Monte Carlo simulation. The difference between the EUL and GS results are small, always less than 3×10^{-5}. Ultimately, the EUL implementation is preferable, since it's simple to implement, and it converges fast without requiring high numerical precision as in the GS.

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Table 5 The one-dimensional Laplace inversion using the Euler Method (EUL) vs. Monte Carlo (MC) and the Gaver-Stehfest inversion (GS). The MC (along with the standard errors reported in the brackets) is obtained by using 16,000 time steps and 20,000 simulation paths. Note that the MC underestimates the option's price due to the systematic discretization bias in simulation. On a Pentium IV 1.8 GHz, the EUL requires less than a tenth of a second in C++, while the GS takes about a second running in Mathematica. However, a precision of 12 digits will suffice for the EUL, compared with the 80 digits accuracy required by the GS.

		Lookback Put - Double Exponential Jump-Diffusion Model $S_0 = 100$, $r = 0.05$, $\sigma = .3$, $T = 1.0$, $p = 0.6$					
		$\eta_1 = \eta_2 = 20.0$			$\eta_1 = \eta_2 = 40.0$		
		Price EU	Price GS	Price MC	Price EU	Price GS	Price MC
M = 105	$\lambda = 1.0$	24.23879	24.23882	24.16946 (0.10869)	23.77979	23.77982	23.70445 (0.10677)
	$\lambda = 3.0$	25.48160	25.48163	25.37234 (0.11432)	24.12663	24.12666	24.03585 (0.10842)
	$\lambda = 5.0$	26.69433	26.69436	26.56639 (0.11851)	24.47053	24.47056	24.38635 (0.10919)
M = 107	$\lambda = 1.0$	24.52690	24.52693	24.47471 (0.10962)	24.06711	24.06713	24.00876 (0.10772)
	$\lambda = 3.0$	25.77029	25.77032	25.68097 (0.11522)	24.41321	24.41324	24.33702 (0.10936)
	$\lambda = 5.0$	26.98331	26.98334	26.86858 (0.11933)	24.75636	24.75639	24.68697 (0.11010)
M = 109	$\lambda = 1.0$	24.90497	24.90499	24.87471 (0.11087)	24.44494	24.44497	24.40841 (0.10902)
	$\lambda = 3.0$	26.14708	26.14711	26.08224 (0.11643)	24.78954	24.78957	24.73374 (0.11064)
	$\lambda = 5.0$	27.35861	27.35864	27.26071 (0.12045)	25.13121	25.13124	25.08016 (0.11136)
M = 111	$\lambda = 1.0$	25.36984	25.36987	25.36425 (0.11245)	24.91037	24.91040	24.89819 (0.11065)
	$\lambda = 3.0$	26.60843	26.60846	26.56792 (0.11794)	25.25268	25.25271	25.22194 (0.11226)
	$\lambda = 5.0$	27.81635	27.81638	27.73967 (0.12187)	25.59210	25.59212	25.56427 (0.11296)
M = 113	$\lambda = 1.0$	25.91832	25.91835	25.93659 (0.11434)	25.46031	25.46035	25.47421 (0.11258)
	$\lambda = 3.0$	27.15091	27.15094	27.13539 (0.11974)	25.79953	25.79955	25.79597 (0.11416)
	$\lambda = 5.0$	28.35295	28.35298	28.29797 (0.12358)	26.13593	26.13595	26.13531 (0.11485)
M = 115	$\lambda = 1.0$	26.54712	26.54715	26.58930 (0.11650)	26.09154	26.09157	26.13197 (0.11480)
	$\lambda = 3.0$	27.77118	27.77121	27.78305 (0.12181)	26.42690	26.42693	26.45186 (0.11637)
	$\lambda = 5.0$	28.96505	28.96508	28.93415 (0.12556)	26.75954	26.75957	26.78472 (0.11703)

A. Appendix: Proof of Proposition 1

We assume for notational simplicity that $t = 0$. We want to bound the quantity

$$\lambda \int_0^T e^{-rs} \mathsf{E}^* \left[\{ P_A(VS_{s^-}, s, T) - (K - VS_{s^-}) \} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \mathbf{1}_{\{VS_{s^-} > S^*_{s^-}\}} \right] ds. \tag{27}$$

We assume $r \ge \delta$, since, if $r < \delta$, it is never optimal to exercise the American put option early and its price is given by the equivalent European put option price. When $r \ge \delta$, Chen and Yeh (2002) provide the upper bound $P_A(S_s, s, T) < \mathsf{E}^* \left[\max \left(K - e^{(\delta - r)(T-s)} S_T, 0 \right) | S_s \right]$. We also know that

$$\begin{aligned} & \mathsf{E}^* \left[\max \left(K - e^{(\delta - r)(T-s)} S_T, 0 \right) | S_s = VS_{s^-} \right] \\ = \; & \mathsf{E}^* \left[K - e^{(\delta - r)(T-s)} S_T | S_s = VS_{s^-} \right] + \mathsf{E}^* \left[\max \left(e^{(\delta - r)(T-s)} S_T - K, 0 \right) | S_s = VS_{s^-} \right] \\ = \; & K - VS_{s^-} + \mathsf{E}^* \left[\max \left(e^{(\delta - r)(T-s)} S_T - K, 0 \right) | S_s = VS_{s^-} \right]. \end{aligned} \tag{28}$$

Combining (28) and the upper bound above we have

$$P_A(VS_{s^-}, s, T) - (K - VS_{s^-}) \le \mathsf{E}^* \left[\max \left(e^{(\delta - r)(T-s)} S_T - K, 0 \right) | S_s = VS_{s^-} \right] \le VS_{s^-}.$$

Therefore,

$$\begin{aligned} & \lambda \int_0^T e^{-rs} \mathsf{E}^* \left[\{ P_A(VS_{s^-}, s, T) - (K - VS_{s^-}) \} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \mathbf{1}_{\{VS_{s^-} > S^*_{s^-}\}} \right] ds \\ \le \; & \lambda \int_0^T e^{-rs} \mathsf{E}^* \left[VS_{s^-} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \mathbf{1}_{\{VS_{s^-} > S^*_{s^-}\}} \right] ds \\ = \; & \lambda \int_0^T e^{-rs} \mathsf{E}^* \left[S_{s^-} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \mathsf{E}^* \left[V \mathbf{1}_{\{VS_{s^-} > S^*_{s^-}, S_{s^-} \le S^*_{s^-}\}} | S_{s^-} \right] \right] ds. \end{aligned} \tag{29}$$

But by the memoryless property and the conditional independence,

$$\mathsf{E}^* \left[V \mathbf{1}_{\{VS_{s^-} > S^*_{s^-}, S_{s^-} \le S^*_{s^-}\}} | S_{s^-} \right] = \mathsf{E}^* \left[e^Y \mathbf{1}_{\{Y > \ln(S^*_{s^-}/S_{s^-}) > 0\}} \right] = \frac{p\eta_1}{\eta_1 - 1} \left(\frac{S^*_{s^-}}{S_{s^-}} \right)^{-(\eta_1 - 1)}. \tag{30}$$

Plugging (30) back in (29) we obtain

$$\begin{aligned} & \lambda \int_0^T e^{-rs} \mathsf{E}^* \left[\{ P_A(VS_{s^-}, s, T) - (K - VS_{s^-}) \} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \mathbf{1}_{\{VS_{s^-} > S^*_{s^-}\}} \right] ds \\ \le \; & \lambda \frac{p\eta_1}{\eta_1 - 1} \int_0^T \mathsf{E}^* \left[S_{s^-} \left(\frac{S^*_{s^-}}{S_{s^-}} \right)^{-(\eta_1 - 1)} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \right] ds \\ \le \; & \lambda K \frac{p\eta_1}{\eta_1 - 1} \int_0^T \mathsf{E}^* \left[\left(\frac{S^*_{s^-}}{S_{s^-}} \right)^{-(\eta_1 - 1)} \mathbf{1}_{\{S_{s^-} \le S^*_{s^-}\}} \right] ds, \end{aligned}$$

since $S_{s^-} \le S^*_{s^-} \le K$. □

B. Appendix: Roots of the equation $G(x) = \alpha$

The equation $G(x) = \alpha$, with $G(x)$ defined in (3), is essentially a quartic equation. Rearranging the terms, it can be shown that the roots of the equation satisfy

$$ax^4 + bx^3 + cx^2 + dx_1 + e = 0,$$

where

$$\begin{aligned} a &= \sigma^2, \;\; b = 2\mu - \sigma^2(\eta_1 - \eta_2), \;\; c = -\sigma^2\eta_1\eta_2 - 2\mu(\eta_1 - \eta_2) - 2\lambda - 2\alpha, \\ d &= -2\mu\eta_1\eta_2 - 2\lambda p(\eta_1 + \eta_2) + 2\lambda\eta_1 + 2\alpha(\eta_1 - \eta_2), \;\; e = 2\alpha\eta_1\eta_2, \end{aligned}$$

with $\mu = r - \delta - \frac{1}{2}\sigma^2 - \lambda\zeta$. The technique to solve the quartic equation was first developed by Ferrari (we refer the interested reader to Boyer and Merzbach, 1991, and Borwein and Erdélyi, 1995). It can be shown that the roots in (4) are given by

$$\beta_1 = -\frac{b}{4a} + \frac{p_1 - \widetilde{p}_2}{2}, \; \beta_2 = -\frac{b}{4a} + \frac{p_1 + \widetilde{p}_2}{2}, \;\; \beta_3 = \frac{b}{4a} + \frac{p_1 - p_2}{2}, \; \beta_4 = \frac{b}{4a} + \frac{p_1 + p_2}{2},$$

where

$$\begin{aligned} p_1 &= \sqrt{B_3 + C_0 + C_1}, \;\; p_2 = \sqrt{B_4 - C_0 - C_1 - \frac{B_5}{4p_1}}, \;\; \widetilde{p}_2 = \sqrt{B_4 - C_0 - C_1 + \frac{B_5}{4p_1}} \\ B_0 &= c^2 - 3bd + 12ae, \;\; B_1 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace, \\ B_2 &= \sqrt{B_1^2 - 4B_0^3}, \;\; B_3 = \frac{b^2}{4a^2} - \frac{2c}{3a}, \;\; B_4 = \frac{b^2}{2a^2} - \frac{4c}{3a}, \;\; B_5 = 4\frac{bc}{a^2} - 8\frac{d}{a} - \left(\frac{b}{a}\right)^3, \\ \widetilde{B} &= \sqrt[3]{B_1 + B_2}, \;\; C_0 = \frac{\sqrt[3]{2}B_0}{3a\widetilde{B}}, \; C_1 = \frac{\widetilde{B}}{3\sqrt[3]{2}a}. \end{aligned}$$

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