

Some functional inequalities on polynomial volume growth Lie groups

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Abstract

We study in this article some Sobolev-type inequalities on polynomial volume growth Lie groups. We show in particular that improved Sobolev inequalities can be extended without the use of the Littlewood-Paley decomposition to this general framework.

Keywords: Sobolev inequalities, polynomial volume growth Lie groups.

1 Introduction

Sobolev inequalities have been studied in many different frameworks. Initially stated over \mathbb{R}^n , they were successively generalized to more general settings such as manifolds or Lie groups; see the book [15] for a detailed treatment in this last case. Since the work of P. Gérard, Y. Meyer & F. Oru [7], we know that is possible to improve the classical Sobolev inequalities introducing a well-suited Besov space, and it is worth to know if these improved inequalities can be generalized to Lie groups.

For the Heisenberg group -which is the simplest example of stratified Lie groups- this was done by H. Bahouri, P. Gérard & C-J Xu in [1] following essentially the same ideas of the original paper of P. Gérard, Y. Meyer and F. Oru; while, for general stratified Lie groups, the task was achieved in [2] using some different techniques. In this special setting we obtained a family of Sobolev-type inequalities: namely, for $G = (\mathbb{R}^n, \cdot, \delta)$ a stratified Lie group and for f a function such that $f \in \dot{W}^{s,p}(G)$ and $f \in \dot{B}_{\infty}^{-\beta,\infty}(G)$, we have:

$$\|f\|_{\dot{W}^{s,q}} \leq C \|f\|_{\dot{W}^{s_1,p}}^{\theta} \|f\|_{\dot{B}_{\infty}^{-\beta,\infty}}^{1-\theta}$$

where the parameters p, q and the indexes θ, β, s and s_1 are related in a specific manner. For precise definitions of the Sobolev space $\dot{W}^{s,p}$ and the Besov space $\dot{B}_{\infty}^{-\beta,\infty}$, see section 5 below.

These type of Lie groups are a generalization of \mathbb{R}^n when modifying dilations; and in this setting, the mathematical objects we are dealing with are constructed in such a way in order to respect the homogeneity induced by these dilations. Thus, several properties of these objects (operators, functional spaces) are very similar to the euclidean case. See [5] and [13] and the references given there for more details.

Looking for generalizations, it is a natural question to study these inequalities if we remove the dilation structure from the Lie groups we are working with. In this case we have several possibilities: for example we can treat *nilpotent Lie groups* which are a generalization of stratified Lie groups (recall that every stratified Lie group is nilpotent) but these groups are not necessarily endowed with a dilation structure. A second example is given by the *polynomial volume growth Lie groups*, where we have some interesting estimates for the Haar measure of a ball. Some other examples can be considered such as exponential growth Lie groups, see the book [15] for definitions and some related results for this last case.

In this article we will specially focus on polynomial volume growth Lie groups and we will treat improved Sobolev inequalities in this very particular setting. Here are our results:

Theorem 1 *Let G be a polynomial volume growth Lie group. If $\nabla f \in L^1(G)$ and $f \in \dot{B}_{\infty}^{-\beta, \infty}(G)$, then we have the following inequalities:*

- **[Strong inequalities $p = 1$]**

$$\|f\|_{L^q} \leq C \|\nabla f\|_{L^1}^{\theta} \|f\|_{\dot{B}_{\infty}^{-\beta, \infty}}^{1-\theta} \quad (1)$$

where $1 < q < +\infty$, $\theta = 1/q$ and $\beta = \theta/(1 - \theta)$.

- **[Strong inequalities $p > 1$]**

$$\|f\|_{\dot{W}^{s, q}} \leq C \|f\|_{\dot{W}^{s_1, p}}^{\theta} \|f\|_{\dot{B}_{\infty}^{-\beta, \infty}}^{1-\theta} \quad (2)$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s = \theta s_1 - (1 - \theta)\beta$ and $-\beta < s < s_1$.

- **[Weak inequalities]**

$$\|f\|_{\dot{W}_{\infty}^{s, q}} \leq C \|\nabla f\|_{L^1}^{\theta} \|f\|_{\dot{B}_{\infty}^{-\beta, \infty}}^{1-\theta} \quad (3)$$

where $1 < q < +\infty$, $0 < s < 1/q < 1$, $\theta = 1/q$ and $\beta = \frac{1-sq}{q-1}$.

The plan of the article is the following: the first section is devoted to a short introduction of polynomial volume growth Lie groups, section 3 gives some important estimations for the Heat kernel, section 4 expose some results concerning spectral theory, section 5 gives the precise definition of functional spaces involved in the inequalities above, while section 6 presents the proof of theorem 1.

2 Polynomial volume growth Lie groups

Let G be a connected unimodular Lie group endowed with its Haar measure dx . Denote by \mathfrak{g} the Lie algebra of G and consider a family $\mathbf{X} = \{X_1, \dots, X_k\}$ of left-invariant vector fields on G satisfying the Hörmander condition, which means that the Lie algebra generated by the X_j for $1 \leq j \leq k$ is \mathfrak{g} .

In this setting we have at our disposal the Carnot-Carathéodory metric associated with \mathbf{X} defined as follows: let $\ell : [0, 1] \rightarrow G$ be an absolutely continuous path. We say that ℓ is admissible if there exists measurable functions $\gamma_1, \dots, \gamma_k : [0, 1] \rightarrow \mathbb{C}$ such that, for almost every $t \in [0, 1]$, we have $\ell'(t) = \sum_{j=1}^k \gamma_j(t) X_j(\ell(t))$. If ℓ is admissible, define the length of ℓ by $\|\ell\| = \int_0^1 (\sum_{j=1}^k |\gamma_j(t)|^2)^{1/2} dt$. Then, for all $x, y \in G$, the distance between x and y is the infimum of the lengths of all admissible curves joining x to y . We will note $\|x\|$ the distance between the origin e and x and $\|y^{-1} \cdot x\|$ the distance between x and y .

For $r > 0$ and $x \in G$, denote by $B(x, r)$ the open ball with respect to the Carnot-Carathéodory metric centered in x and of radius r , and by $V(r)$ the Haar measure of any ball of radius r . When $0 < r < 1$, there exists $d \in \mathbb{N}^*$, c_l and $C_l > 0$ such that, for all $0 < r < 1$ we have

$$c_l r^d \leq V(r) \leq C_l r^d.$$

The integer d is the *local* dimension of (G, \mathbf{X}) . When $r \geq 1$, only two situation may occur, independently of the choice of the family \mathbf{X} : either G has polynomial volume growth and there exist $D \in \mathbb{N}^*$, c_{∞} and $C_{\infty} > 0$ such that, for all $r \geq 1$ we have

$$c_{\infty} r^D \leq V(r) \leq C_{\infty} r^D, \quad (4)$$

or G has exponential volume growth, which means that there exist $c_e, C_e, \alpha, \beta > 0$ such that, for all $r \geq 1$ we have

$$c_e e^{\alpha r} \leq V(r) \leq C_e e^{\beta r}.$$

When G has polynomial volume growth, the integer D in (4) is called the dimension at infinity of G . Recall that nilpotent groups have polynomial volume growth and that a strict subclass of the nilpotent groups consists of stratified Lie groups. For more details see the book [15].

We will assume from now on that G is a connected unimodular polynomial volume Lie group with local dimension d and dimension at infinity D .

3 Sub-Laplacian and Heat kernel

Once we have fixed the family \mathbf{X} , we define the gradient on G by $\nabla = (X_1, \dots, X_k)$ and we consider a sub-Laplacian \mathcal{J} on G defined by $\mathcal{J} = -\sum_{j=1}^k X_j^2$, which is a positive self-adjoint, hypo-elliptic operator since \mathbf{X} satisfies the Hörmander's condition. Its associated heat operator on $G \times]0, +\infty[$ is given by $\partial_t + \mathcal{J}$. We recall in the next theorem some well-known properties of the semi-group H_t obtained from the sub-Laplacian \mathcal{J} .

Theorem 2 *There exists a unique family of continuous linear operators $(H_t)_{t>0}$ defined on $L^1 + L^\infty(G)$ with the semi-group property $H_{t+s} = H_t H_s$ for all $t, s > 0$ and $H_0 = Id$, such that:*

- 1) *the sub-Laplacian \mathcal{J} is the infinitesimal generator of the semi-group $H_t = e^{-t\mathcal{J}}$;*
- 2) *H_t is a contraction operator on $L^p(G)$ for $1 \leq p \leq +\infty$ and for $t > 0$;*
- 3) *the semi-group H_t admits a convolution kernel $H_t f = f * h_t$ where h_t is the heat kernel.*
- 4) *$\|H_t f - f\|_{L^p} \rightarrow 0$ if $t \rightarrow 0$ for $f \in L^p(G)$ and $1 \leq p < +\infty$;*
- 5) *If $f \in L^p(G)$, $1 \leq p \leq +\infty$, then the function $u(x, t) = H_t f(x)$ is a solution of the heat equation.*

We obtain in particular that H_t is a symmetric diffusion semi-group as considered by Stein in [12] with infinitesimal generator \mathcal{J} .

We need to fix some terminology. To begin with, note that associated to the family \mathbf{X} , we also have a family of right-invariant vector fields $\{Y_1, \dots, Y_k\}$ with similar properties. Let $I = (j_1, \dots, j_\beta) \in \{1, \dots, k\}^\beta$ ($\beta \in \mathbb{N}$) be a multi-index, we set $|I| = \beta$ and define X^I and Y^I by the formula $X^I = X_{j_1} \cdots X_{j_\beta}$ ($Y^I = Y_{j_1} \cdots Y_{j_\beta}$ resp.) with the convention $X^I = Id$ if $\beta = 0$. The interaction of operators X^I and Y^I with convolutions is clarified by the following identities:

$$X^I(f * g) = f * (X^I g), \quad Y^I(f * g) = (Y^I f) * g, \quad (X^I f) * g = f * (Y^I g).$$

In particular we have $(\nabla f) * g = f * (\tilde{\nabla} g)$ where $\tilde{\nabla} = (Y_1, \dots, Y_k)$.

We will say now that $\varphi \in \mathcal{C}^\infty(G)$ belongs to the Schwartz's class $\mathcal{S}(G)$ if

$$N_{\alpha, I}(\varphi) = \sup_{x \in G} (1 + \|x\|)^\alpha |X^I \varphi(x)| < +\infty. \quad (\alpha \in \mathbb{N}, I \in \bigcup_{\beta \in \mathbb{N}} \{1, \dots, k\}^\beta).$$

Remark 1 To characterize the Schwartz class $\mathcal{S}(G)$ we can replace vector fields X^I in the semi-norms $N_{k, I}$ above by right-invariant vector fields Y^I .

For a proof of these facts and for further details see [5], [12], [15] and the reference given there.

Theorem 3 *Let G be a polynomial volume growth Lie group, then for every $j \in \{1, \dots, k\}$, there exists $C > 0$ such that*

$$|X_j h_t(x)| \leq C t^{-1/2} V(\sqrt{t})^{-1} e^{-\frac{\|x\|^2}{ct}} \quad \text{for all } x \in G, t > 0.$$

This theorem implies the next proposition

Proposition 3.1 *For every $j \in \{1, \dots, k\}$ and for all $p \in [1, +\infty]$ there exists a constant $C > 0$ such that:*

$$\|X_j h_t(\cdot)\|_{L^p} \leq C t^{-1/2} V(\sqrt{t})^{-1/p'}, \quad t > 0; \quad (5)$$

For a proof of theorem 3 and proposition 3.1, see chapter VIII of the book [15].

4 Spectral decomposition for the sub-Laplacian

The use in this article of spectral resolution for the sub-Laplacian consists roughly in expressing this operator by the formula $\mathcal{J} = \int_0^{+\infty} \lambda \, dE_\lambda$ and, by means of this characterization, build a family of new operators $m(\mathcal{J})$ associated to a Borel function m . This kind of operators have some nice properties as shown in the next propositions.

Proposition 4.1 *If G is polynomial growth Lie group and if m is a bounded Borel function on $]0, +\infty[$ then the operator $m(\mathcal{J})$ fixed by*

$$m(\mathcal{J}) = \int_0^{+\infty} m(\lambda) \, dE_\lambda, \quad (6)$$

*is bounded on $L^2(G)$ and admits a convolution kernel M i.e.: $m(\mathcal{J})(f) = f * M \quad (\forall f \in L^2(G)).$*

Following [8] and [6] we can improve the conclusion of the above proposition. Let $k \in \mathbb{N}$ and m be a function of class $\mathcal{C}^k(\mathbb{R}^+)$, we write

$$\|m\|_{(k)} = \sup_{\substack{1 \leq r \leq k \\ \lambda > 0}} (1 + \lambda)^k |m^{(r)}(\lambda)|.$$

This formula gives us a necessary condition to obtain some properties of the operators defined by (6):

Proposition 4.2 *Let G be polynomial volume growth Lie group with local dimension d . Let $j \in \{1, \dots, k\}$ and $p \in [1, +\infty]$. There is a constant $C > 0$ and an integer k such that, for any function $m \in \mathcal{C}^k(\mathbb{R}^+)$ with $\|m\|_{(k)} < +\infty$, the kernel M_t associated to the operator $m(t\mathcal{J})$ with $t > 0$ satisfies*

$$\|X_j M_t(\cdot)\|_{L^p} \leq C t^{-(\frac{d}{2p'} + \frac{1}{2})} \|m\|_{(k)}. \quad (7)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Follow the same steps of the proof of proposition 3.2 in [6] and use inequality (5). ■

Remark 2 Notice that, when $0 < t \leq 1$, we can replace in (7) X_j by X^I for some multi-index I .

5 Functional spaces

We give in this section the precise definition of the functional spaces involved in theorem 1. In a general way, given a norm $\|\cdot\|_E$, we will define the corresponding functional space $E(G)$ by $\{f \in \mathcal{S}'(G) : \|f\|_E < +\infty\}$. For the Lebesgue spaces $L^p(G)$ with $1 \leq p < +\infty$, we will use the following characterization

$$\|f\|_{L^p}^p = \int_0^{+\infty} p \sigma^{p-1} |\{x \in G : |f(x)| > \sigma\}| \, d\sigma,$$

and for the Lorentz spaces $L^{p,\infty}(G)$ we set $\|f\|_{L^{p,\infty}} = \sup_{\sigma > 0} \{\sigma |\{x \in G : |f(x)| > \sigma\}|^{1/p}\}$.

In order to define Sobolev spaces, we need to introduce the fractional powers \mathcal{J}^s and \mathcal{J}^{-s} with $s > 0$:

$$\begin{aligned} \mathcal{J}^s f(x) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\Gamma(k-s)} \int_\varepsilon^{+\infty} t^{k-s-1} \mathcal{J}^k H_t f(x) \, dt \\ \mathcal{J}^{-s} f(x) &= \lim_{\eta \rightarrow +\infty} \frac{1}{\Gamma(s)} \int_0^\eta t^{s-1} H_t f(x) \, dt \end{aligned} \quad (8)$$

for all $f \in \mathcal{C}^\infty(G)$ with k the smallest integer greater than s . We consider then Sobolev spaces by the norms $\|f\|_{\dot{W}^{s,p}} = \|\mathcal{J}^{s/2}f\|_{L^p}$ when $1 < p < +\infty$ and when $p = s = 1$ we will note $\|f\|_{\dot{W}^{1,1}} = \|\nabla f\|_{L^1}$.

We will also need to define weak Sobolev spaces $\dot{W}_\infty^{s,p}(G)$ used in (3), and we write here

$$\|f\|_{\dot{W}_\infty^{s,p}} = \|\mathcal{J}^{s/2}f\|_{L^{p,\infty}} \quad (1 < p < +\infty) \quad (9)$$

Finally, for Besov spaces of indices $(-\beta, \infty, \infty)$ which appear in all the inequalities (1)-(3) we have:

$$\|f\|_{\dot{B}_\infty^{-\beta,\infty}} = \sup_{t>0} t^{\beta/2} \|H_t f\|_{L^\infty} \quad (10)$$

For more details on Sobolev and Besov spaces on groups see [11].

6 Improved Sobolev Inequalities on stratified groups: the proofs

We divide the proof of the theorem 1 in two steps. First we study the inequality (2) and then we prove the strong inequality (1) and the weak inequality (3).

6.1 The general improved Sobolev inequalities ($p > 1$)

We know that the operator $\mathcal{J}^{s/2}$ carries out an isomorphism between the spaces $\dot{B}_\infty^{-\beta,\infty}(G)$ and $\dot{B}_\infty^{-\beta-s,\infty}(G)$ (see [11]); so we can rewrite the inequality (2) in the following way

$$\|\mathcal{J}^{\frac{s-s_1}{2}} f\|_{L^q} \leq C \|f\|_{L^p}^\theta \|f\|_{\dot{B}_\infty^{-\beta-s_1,\infty}}^{1-\theta}$$

where $1 < p < q < +\infty$, $\theta = p/q$, $s = \theta s_1 - (1 - \theta)\beta$ and $-\beta < s < s_1$. Using the sub-Laplacian fractionnal powers characterization (8) we have the identity

$$\mathcal{J}^{-\frac{\alpha}{2}} f(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^{+\infty} t^{\frac{\alpha}{2}-1} H_t f(x) dt = \frac{1}{\Gamma(\frac{\alpha}{2})} \left(\int_0^T t^{\frac{\alpha}{2}-1} H_t f(x) dt + \int_T^{+\infty} t^{\frac{\alpha}{2}-1} H_t f(x) dt \right) \quad (11)$$

where $\alpha = s_1 - s > 0$ and T will be fixed in the sequel.

For studying each one of these integrals we will use the estimates

- $|H_t f(x)| \leq |f(x)|$
- $|H_t f(x)| \leq C t^{\frac{-\beta-s_1}{2}} \|f\|_{\dot{B}_\infty^{-\beta-s_1,\infty}} \quad (\text{by the thermic definition of Besov spaces})$

Then, applying these inequalities in (11) we obtain

$$|\mathcal{J}^{-\frac{\alpha}{2}} f(x)| \leq \frac{c_1}{\Gamma(\frac{\alpha}{2})} T^{\frac{\alpha}{2}} |f(x)| + \frac{c_2}{\Gamma(\frac{\alpha}{2})} T^{\frac{\alpha-\beta-s}{2}} \|f\|_{\dot{B}_\infty^{-\beta-s_1,\infty}}.$$

We fix now

$$T = \left(\frac{\|f\|_{\dot{B}_\infty^{-\beta-s_1,\infty}}}{|f(x)|} \right)^{\frac{2}{\beta+s_1}}$$

and we get

$$|\mathcal{J}^{-\frac{\alpha}{2}} f(x)| \leq \frac{c_1}{\Gamma(\frac{\alpha}{2})} |f(x)|^{1-\frac{\alpha}{\beta+s_1}} + \frac{c_2}{\Gamma(\frac{\alpha}{2})} |f(x)|^{1-\frac{\alpha}{\beta+s_1}} \|f\|_{\dot{B}_\infty^{-\beta-s_1,\infty}}^{\frac{\alpha}{\beta+s_1}}.$$

Since $\frac{\alpha}{\beta+s_1} = 1 - \theta$ and $\theta = p/q$ we have

$$|\mathcal{J}^{-\frac{\alpha}{2}} f(x)| \leq \frac{c}{\Gamma(\frac{\alpha}{2})} |f(x)|^\theta \|f\|_{\dot{B}_\infty^{-\beta-s_1,\infty}}^{1-\theta}.$$

We finally obtain

$$\|\mathcal{J}^{-\frac{\alpha}{2}} f\|_{L^q} \leq c \|f\|_{L^p}^\theta \|f\|_{\dot{B}_{\infty}^{-\beta-s_1, \infty}}^{1-\theta}$$

and we are done. ■

6.2 Strong and weak inequalities ($p = 1$)

We treat now the inequalities (1) and (3). For this we will need the following result.

Theorem 4 *Let f be a function such that $\nabla f \in L^1(G)$. We have the following estimate for $0 \leq s < 1$ and for $t > 0$:*

$$\|\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f\|_{L^1} \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

Proof. To begin the proof, we observe that the following identity occurs:

$$(\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f)(x) = \left(\int_0^{+\infty} m(t\lambda) dE_\lambda \right) t^{1-s/2} \mathcal{J} f(x),$$

where we noted $m(\lambda) = \lambda^{s/2-1}(1 - e^{-\lambda})$ for $\lambda > 0$, note that m is a bounded function which tends to 0 at infinity since $s/2 - 1 < 0$. We break up this function by writing:

$$m(\lambda) = m_0(\lambda) + m_1(\lambda) = m(\lambda)\theta_0(\lambda) + m(\lambda)\theta_1(\lambda)$$

where we chose the auxiliary functions $\theta_0(\lambda), \theta_1(\lambda) \in \mathcal{C}^\infty(\mathbb{R}^+)$ defined by:

- $\theta_0(\lambda) = 1$ on $]0, 1/2]$ and 0 on $]1, +\infty[$,
- $\theta_1(\lambda) = 0$ on $]0, 1/2]$ and 1 on $]1, +\infty[$,

so that $\theta_0(\lambda) + \theta_1(\lambda) \equiv 1$. Then, we obtain the formula:

$$(\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f)(x) = \left(\int_0^{+\infty} m_0(t\lambda) dE_\lambda \right) t^{1-s/2} \mathcal{J} f(x) + \left(\int_0^{+\infty} m_1(t\lambda) dE_\lambda \right) t^{1-s/2} \mathcal{J} f(x).$$

If we note $M_t^{(i)}$ the kernel of the operator fixed by $\int_0^{+\infty} m_i(t\lambda) dE_\lambda$ for $i = 0, 1$, we have:

$$(\mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f)(x) = t^{1-s/2} \mathcal{J} f * M_t^{(0)}(x) + t^{1-s/2} \mathcal{J} f * M_t^{(1)}(x).$$

We obtain the inequality

$$\int_G \left| \mathcal{J}^{s/2} f - H_t \mathcal{J}^{s/2} f \right| dx \leq \int_G \left| t^{1-s/2} \mathcal{J} f * M_t^{(0)}(x) \right| dx + \int_G \left| t^{1-s/2} \mathcal{J} f * M_t^{(1)}(x) \right| dx. \quad (12)$$

We will now estimate the right side of the above inequality by the two following propositions:

Proposition 6.1 *For the first integral in the right-hand side of (12) we have the inequality:*

$$\int_G \left| t^{1-s/2} \mathcal{J} f * M_t^{(0)}(x) \right| dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}$$

Proof. The function m_0 is the restriction on \mathbb{R}^+ of a function belonging to the Schwartz class. This function satisfies the assumptions of the proposition 4.2 which we apply after having noticed the identity

$$I = \int_G \left| t^{1-s/2} \mathcal{J} f * M_t^{(0)}(x) \right| dx = \int_G \left| t^{1-s/2} \nabla f * \tilde{\nabla} M_t^{(0)}(x) \right| dx$$

where we noted $\tilde{\nabla}$ the gradient formed by the vectors fields $(Y_j)_{1 \leq j \leq k}$. We have then

$$I \leq \int_G \int_G t^{1-s/2} |\nabla f(y)| |\tilde{\nabla} M_t^{(0)}(y^{-1} \cdot x)| dx dy \leq t^{1-s/2} \|\nabla f\|_{L^1} \|\tilde{\nabla} M_t^{(0)}\|_{L^1}.$$

Using the inequality (7) we obtain

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(0)}(x) \right| dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

■

Proposition 6.2 *For the last integral of (12) we have the inequality*

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(1)}(x) \right| dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}$$

Proof. Here, it is necessary to make an additional step. We cut out the function m_1 in the following way:

$$m_1(\lambda) = \left(\frac{1 - e^{-\lambda}}{\lambda} \right) \theta_1(\lambda) = m_a(\lambda) - m_b(\lambda)$$

where $m_a(\lambda) = \frac{1}{\lambda} \theta_1(\lambda)$ and $m_b(\lambda) = \frac{e^{-\lambda}}{\lambda} \theta_1(\lambda)$. We will note $M_t^{(a)}$ and $M_t^{(b)}$ the associated kernels of these two operators. We obtain thus the estimate

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(1)}(x) \right| dx \leq \int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(a)}(x) \right| dx + \int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(b)}(x) \right| dx \quad (13)$$

We have the next lemma for the last integral in (13).

Lemma 6.1

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(b)}(x) \right| dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

Proof. Observe that $m_b \in \mathcal{S}(\mathbb{R}^+)$, then the proof is straightforward and follows the same steps as those of the preceding proposition 6.1.

■

We treat the other part of (13) with the following lemma:

Lemma 6.2

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(a)}(x) \right| dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1} \quad (14)$$

Proof. We consider the auxiliary function

$$\psi(\lambda) = \theta_0(\lambda/2) - \theta_0(\lambda) = \theta_1(\lambda) - \theta_1(\lambda/2)$$

in order to obtain the identity

$$\sum_{j=0}^{+\infty} \psi(2^{-j}\lambda) = \theta_1(\lambda).$$

We have then

$$m_a(t\lambda) = \frac{1}{t\lambda} \sum_{j=0}^{+\infty} \psi(2^{-j}t\lambda) = \sum_{j=0}^{+\infty} 2^{-j} \phi(2^{-j}t\lambda)$$

where $\phi(\lambda) = \frac{\psi(\lambda)}{\lambda}$ is a function in $\mathcal{C}_0^\infty(\mathbb{R}^+)$. Then, from the point of view of operators, one has:

$$M_t^{(a)} = \sum_{j=0}^{+\infty} 2^{-j} K_{j,t} \quad (15)$$

where $K_{j,t} = \phi(2^{-j}t\mathcal{J})$. With the formula (15) we return to the left side of (14):

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(a)}(x) \right| dx \leq \sum_{j=0}^{+\infty} 2^{-j} \int_G \left| t^{1-s/2} \mathcal{J}f * K_{j,t}(x) \right| dx. \quad (16)$$

Using the sub-Laplacian definition and the vector fields properties, we have

$$\sum_{j=0}^{+\infty} 2^{-j} \int_G \left| t^{1-s/2} \mathcal{J}f * K_{j,t}(x) \right| dx \leq \sum_{j=0}^{+\infty} 2^{-j} t^{1-s/2} \|\nabla f\|_{L^1} \|\tilde{\nabla} K_{j,t}\|_{L^1}.$$

Apply now proposition 4.2 to obtain the estimate $\|\tilde{\nabla} K_{j,t}\|_{L^1} \leq C 2^{j/2} t^{-1/2}$. We have then for (16) the following inequality

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(a)}(x) \right| dx \leq C \sum_{j=0}^{+\infty} 2^{-j/2} t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

Then, we finally get

$$\int_G \left| t^{1-s/2} \mathcal{J}f * M_t^{(a)}(x) \right| dx \leq C t^{\frac{1-s}{2}} \|\nabla f\|_{L^1}.$$

Which ends the proof of the lemma 6.2. ■

With these two last lemmas we conclude the proof of the proposition 6.2. Now, getting back to the formula (12), with propositions 6.1 and 6.2 we finally finish the proof of theorem 4. ■

6.3 Weak inequalities

To begin the proof notice that operator $\mathcal{J}^{s/2}$ carries out an isomorphism between the spaces $\dot{B}_{\infty}^{-\beta,\infty}$ and $\dot{B}_{\infty}^{-\beta-s,\infty}$, see [11] for a proof. Thus inequality (3) rewrites as:

$$\|\mathcal{J}^{s/2} f\|_{L^{q,\infty}} \leq C \|\nabla f\|_{L^1}^{\theta} \|\mathcal{J}^{s/2} f\|_{\dot{B}_{\infty}^{-\beta-s,\infty}}^{1-\theta} \quad (17)$$

By homogeneity, we can suppose that the norm $\|\mathcal{J}^{s/2} f\|_{\dot{B}_{\infty}^{-\beta-s,\infty}}$ is bounded by 1; then we have to show

$$\|\mathcal{J}^{s/2} f\|_{L^{q,\infty}} \leq C \|\nabla f\|_{L^1}^{\theta}. \quad (18)$$

We have thus to evaluate the expression $|\{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\}|$ for all $\alpha > 0$. If we use the thermic definition of the Besov space (10), we have

$$\|\mathcal{J}^{s/2} f\|_{\dot{B}_{\infty}^{-\beta-s,\infty}} \leq 1 \iff \sup_{t>0} \left\{ t^{\frac{\beta+s}{2}} \|H_t \mathcal{J}^{s/2} f\|_{L^{\infty}} \right\} \leq 1.$$

But, if one fixes $t_{\alpha} = \alpha^{-\left(\frac{2}{\beta+s}\right)}$, we obtain $\|H_{t_{\alpha}} \mathcal{J}^{s/2} f\|_{L^{\infty}} \leq \alpha$. Note also that with the definition of parameter β one has $t_{\alpha} = \alpha^{-\frac{2(q-1)}{(1-s)}}$. Therefore, since we have the following set inclusion

$$\left\{ x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha \right\} \subset \left\{ x \in G : |\mathcal{J}^{s/2} f(x) - H_{t_{\alpha}} \mathcal{J}^{s/2} f(x)| > \alpha \right\},$$

the Tchebychev inequality implies

$$\alpha^q \left| \left\{ x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha \right\} \right| \leq \alpha^{q-1} \int_G |\mathcal{J}^{s/2} f(x) - H_{t_{\alpha}} \mathcal{J}^{s/2} f(x)| dx.$$

At this point, we use the theorem 4 to estimate the right side of the preceding inequality:

$$\alpha^q \left| \{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\} \right| \leq C \alpha^{q-1} t_\alpha^{\frac{1-s}{2}} \int_G |\nabla f(x)| dx. \quad (19)$$

But, by the choice of t_α , one has $\alpha^{q-1} \alpha^{-\frac{2(q-1)}{(1-s)} \frac{(1-s)}{2}} = 1$. Then (19) implies the inequality

$$\alpha^q \left| \{x \in G : |\mathcal{J}^{s/2} f(x)| > 2\alpha\} \right| \leq C \|\nabla f\|_{L^1};$$

and, finally, using the definition (9) of weak Sobolev spaces it comes

$$\|\mathcal{J}^{s/2} f\|_{L^{q,\infty}}^q \leq C \|\nabla f\|_{L^1}$$

which is the desired result. ■

6.4 Strong inequalities

When $s = 0$ in the weak inequalities it is possible to obtain stronger estimations. To achieve this, we will need an intermediate step:

Proposition 6.3 *Let $1 < q < +\infty$, $\theta = \frac{1}{q}$ and $\beta = \theta/(1 - \theta)$. Then we have*

$$\|f\|_{L^q} \leq C \|\nabla f\|_{L^1}^\theta \|f\|_{\dot{B}_{\infty}^{-\beta,\infty}}^{1-\theta}$$

when the three norms in this inequality are bounded.

Proof. We will follow closely [9]. Just as in the preceding theorem, we will start by supposing that $\|f\|_{\dot{B}_{\infty}^{-\beta,\infty}} \leq 1$. Thus, we must show the estimate

$$\|f\|_{L^q} \leq C \|\nabla f\|_{L^1}^\theta. \quad (20)$$

Let us fix t in the following way: $t_\alpha = \alpha^{-2(q-1)/q}$ where $\alpha > 0$. We have then, by the thermic definition of Besov spaces, the estimate $\|H_t f\|_{L^\infty} \leq \alpha$. We use now the characterization of Lebesgue space given by the distribution function:

$$\frac{1}{5^q} \|f\|_{L^q}^q = \int_0^{+\infty} |\{x \in G : |f(x)| > 5\alpha\}| d(\alpha^q). \quad (21)$$

It now remains to estimate $|\{x \in G : |f(x)| > 5\alpha\}|$ and for this we introduce the following thresholding function:

$$\Theta_\alpha(t) = \begin{cases} \Theta_\alpha(-t) = -\Theta_\alpha(t) & \\ 0 & \text{if } 0 \leq T \leq \alpha \\ t - \alpha & \text{if } \alpha \leq T \leq M\alpha \\ (M-1)\alpha & \text{if } T > M\alpha \end{cases}$$

Here, M is a parameter which depends on q and which we will suppose for the moment larger than 10.

This cut-off function enables us to define a new function posing $f_\alpha = \Theta_\alpha(f)$. We write in the next lemma some significant properties of this function f_α :

Lemma 6.3

1. the set defined by $\{x \in G : |f(x)| > 5\alpha\}$ is included in the set $\{x \in G : |f_\alpha(x)| > 4\alpha\}$.

2. On the set $\{x \in G : |f(x)| \leq M\alpha\}$ one has the estimate $|f - f_\alpha| \leq \alpha$.

3. If $f \in \mathcal{C}^1(G)$, one has the equality $\nabla f_\alpha = (\nabla f)\mathbb{1}_{\{\alpha \leq |f| \leq M\alpha\}}$ almost everywhere.

We leave the verification of this lemma to the reader.

Let us return now to (21). By the first point of the lemma above we have

$$\int_0^{+\infty} |\{x \in G : |f(x)| > 5\alpha\}| d(\alpha^q) \leq \int_0^{+\infty} |\{x \in G : |f_\alpha(x)| > 4\alpha\}| d(\alpha^q) = I. \quad (22)$$

We note $A_\alpha = \{x \in G : |f_\alpha(x)| > 4\alpha\}$, $B_\alpha = \{x \in G : |f_\alpha(x) - H_{t_\alpha}(f_\alpha)(x)| > \alpha\}$ and $C_\alpha = \{x \in G : |H_{t_\alpha}(f_\alpha - f)(x)| > 2\alpha\}$. Now, by linearity of H_t we can write: $f_\alpha = f_\alpha - h_{t_\alpha}(f_\alpha) + h_{t_\alpha}(f_\alpha - f) + h_{t_\alpha}(f)$. Then, holding in account the fact $\|H_t f\|_{L^\infty} \leq \alpha$, we obtain $A_\alpha \subset B_\alpha \cup C_\alpha$. Returning to (22), this set inclusion gives us the following inequality

$$I \leq \int_0^{+\infty} |B_\alpha| d(\alpha^q) + \int_0^{+\infty} |C_\alpha| d(\alpha^q) \quad (23)$$

We will study and estimate these two integrals, which we will call I_1 and I_2 respectively, by the two following lemmas:

Lemma 6.4 *For the first integral of (23) we have the estimate:*

$$I_1 = \int_0^{+\infty} |B_\alpha| d(\alpha^q) \leq C q \log(M) \|\nabla f\|_{L^1} \quad (24)$$

Proof. The Tchebychev's inequality implies

$$|B_\alpha| \leq \alpha^{-1} \int_G |f_\alpha(x) - H_{t_\alpha}(f_\alpha)(x)| dx.$$

Using the theorem 4 with $s = 0$ in the above integral we obtain:

$$|B_\alpha| \leq C \alpha^{-1} t_\alpha^{1/2} \int_G |\nabla f_\alpha(x)| dx$$

Remark that the choice of t_α fixed before gives $t_\alpha^{1/2} = \alpha^{1-q}$, then we have

$$|B_\alpha| \leq C \alpha^{-q} \int_{\{\alpha \leq |f| \leq M\alpha\}} |\nabla f(x)| dx.$$

We integrate now the preceding expression with respect to $d(\alpha^q)$:

$$I_1 \leq C \int_0^{+\infty} \alpha^{-q} \left(\int_{\{\alpha \leq |f| \leq M\alpha\}} |\nabla f(x)| dx \right) d(\alpha^q) = C q \int_G |\nabla f(x)| \left(\int_{\frac{|f|}{M}}^{|f|} \frac{d\alpha}{\alpha} \right) dx$$

It follows then $I_1 \leq C q \log(M) \|\nabla f\|_{L^1}$ and one obtains the estimation needed for the first integral. ■

Lemma 6.5 *For the second integral of (23) one has the following result:*

$$I_2 = \int_0^{+\infty} |C_\alpha| d(\alpha^q) \leq \frac{q}{q-1} \frac{1}{M^{q-1}} \|f\|_{L^q}^q$$

Proof. For the proof of this lemma, we write:

$$|f - f_\alpha| = |f - f_\alpha| \mathbb{1}_{\{|f| \leq M\alpha\}} + |f - f_\alpha| \mathbb{1}_{\{|f| > M\alpha\}}.$$

As the distance between f and f_α is lower than α on the set $\{x \in G : |f(x)| \leq M\alpha\}$, one has the inequality

$$|f - f_\alpha| \leq \alpha + |f| \mathbb{1}_{\{|f| > M\alpha\}}$$

By applying the heat semi-group to both sides of this inequality we obtain $H_{t_\alpha}(|f - f_\alpha|) \leq \alpha + H_{t_\alpha}(|f| \mathbb{1}_{\{|f| > M\alpha\}})$ and we have then the following set inclusion $C_\alpha \subset \{x \in G : H_{t_\alpha}(|f| \mathbb{1}_{\{|f| > M\alpha\}}) > \alpha\}$. Thus, considering the measure of these sets and integrating with respect to $d(\alpha^q)$, it comes

$$I_2 = \int_0^{+\infty} |C_\alpha| d(\alpha^q) \leq \int_0^{+\infty} \left| \{H_{t_\alpha}(|f| \mathbb{1}_{\{|f| > M\alpha\}}) > \alpha\} \right| d(\alpha^q)$$

We obtain now, by applying the Tchebychev inequality, the estimate

$$I_2 \leq \int_0^{+\infty} \alpha^{-1} \left(\int_G H_{t_\alpha}(|f| \mathbb{1}_{\{|f| > M\alpha\}}) dx \right) d(\alpha^q),$$

then by Fubini's theorem we have

$$I_2 \leq q \int_G |f(x)| \left(\int_0^{+\infty} \mathbb{1}_{\{|f| > M\alpha\}} \alpha^{q-2} d\alpha \right) dx = \frac{q}{q-1} \int_G |f(x)| \frac{|f(x)|^{q-1}}{M^{q-1}} dx = \frac{q}{q-1} \frac{1}{M^{q-1}} \|f\|_{L^q}^q.$$

And this concludes the proof of this lemma. ■

We finish the proof of proposition 6.3 by connecting together these two lemmas *i.e.*:

$$\frac{1}{5^q} \|f\|_{L^q}^q \leq Cq \log(M) \|\nabla f\|_{L^1} + \frac{q}{q-1} \frac{1}{M^{q-1}} \|f\|_{L^q}^q$$

Since we supposed all the norms bounded and $M \gg 1$, we finally have

$$\left(\frac{1}{5^q} - \frac{q}{q-1} \frac{1}{M^{q-1}} \right) \|f\|_{L^q}^q \leq Cq \log(M) \|\nabla f\|_{L^1}$$
■

The proof of the theorem 1 is not yet completely finished. The last step is provided by the

Proposition 6.4 *In the proposition 6.3 it is possible to consider only the two assumptions $\nabla f \in L^1(G)$ and $f \in \dot{B}_{\infty}^{-\beta, \infty}(G)$.*

Proof. For the proof of this proposition we will build an approximation of f writing:

$$f_j = \left(\int_0^{+\infty} (\varphi(2^{-2j}\lambda) - \varphi(2^{2j}\lambda)) dE_\lambda \right) (f)$$

where φ is a $C^\infty(\mathbb{R}^+)$ function such that $\varphi = 1$ on $]0, 1/4[$ and $\varphi = 0$ on $[1, +\infty[$.

Lemma 6.6 *If $q > 1$, if $\nabla f \in L^1(G)$ and if $f \in \dot{B}_{\infty}^{-\beta, \infty}(G)$ then $\nabla f_j \in L^1(G)$, $f_j \in \dot{B}_{\infty}^{-\beta, \infty}(G)$ and $f_j \in L^q(G)$.*

Proof. The fact that $\nabla f_j \in L^1(G)$ and $f_j \in \dot{B}_{\infty}^{-\beta, \infty}(G)$ is an easy consequence of the definition of f_j . For $f_j \in L^q(G)$ the starting point is given by the relation:

$$f_j = \left(\int_0^{+\infty} m(2^{-2j}\lambda) dE_{\lambda} \right) 2^{-2j} \mathcal{J}(f),$$

where we noted

$$m(2^{-2j}\lambda) = \frac{\varphi(2^{-2j}\lambda) - \varphi(2^{2j}\lambda)}{2^{-2j}\lambda}.$$

Observe that the function m vanishes near of the origin and satisfies the assumptions of proposition 4.2. We obtain then the following identity where M_j is the kernel of the operator $m(2^{-2j}\mathcal{J})$:

$$f_j = 2^{-2j} \mathcal{J} f * M_j = 2^{-2j} \nabla f * \tilde{\nabla} M_j,$$

Using inequality (7), we estimate the norm $L^q(G)$ in the preceding identity:

$$\|f_j\|_{L^q} = \|2^{-2j} \nabla f * \tilde{\nabla} M_j\|_{L^q} \leq 2^{-2j} \|\nabla f\|_{L^1} \|\tilde{\nabla} M_j\|_{L^q}.$$

Finally, we obtain:

$$\|f_j\|_{L^q} \leq C 2^{j(d(1-\frac{1}{q})-1)} \|\nabla f\|_{L^1} < +\infty$$

■

Thanks to this estimate, we can apply the proposition 6.3 to f_j whose $L^q(G)$ norm is bounded, and we obtain:

$$\|f_j\|_{L^q} \leq C \|\nabla f_j\|_{L^1}^{\theta} \|f_j\|_{\dot{B}_{\infty}^{-\beta, \infty}}^{1-\theta}.$$

Now, since $f \in \dot{B}_{\infty}^{-\beta, \infty}(G)$, we have $f_j \rightharpoonup f$ in the sense of distributions. It follows

$$\|f\|_{L^q} \leq \liminf_{j \rightarrow +\infty} \|f_j\|_{L^q} \leq C \|\nabla f\|_{L^1}^{\theta} \|f\|_{\dot{B}_{\infty}^{-\beta, \infty}}^{1-\theta}.$$

We restricted ourselves to the two initial assumptions, namely $\nabla f \in L^1(G)$ and $f \in \dot{B}_{\infty}^{-\beta, \infty}(G)$. The strong inequalities (1) are now completely proved for stratified groups.

■

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