Semigroup representations in holomorphic dynamics.

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Abstract

We use semigroup theory to describe the group of automorphisms of some semigroups of interest in holomorphic dynamical systems. We show, with some examples, that representation theory of semigroups is related to usual constructions in holomorphic dynamics. The main tool for our discussion is a theorem due to Schreier. We extend this theorem, and our results in semigroups, to the setting of correspondences and holomorphic correspondences.

1 Introduction

One of the motivations of this paper is to add a new entry to Sullivan's dictionary between holomorphic dynamics and Kleinian groups. This entry consist of the algebraic part of a holomorphic dynamical system, that corresponds to the algebraic structure of a Kleinian group. From this point of view, the natural object is a semigroup. This investigation was initiated by the following question of Étienne Ghys:

- Are there multiplicative characters, defined on the semigroup of polynomials with composition, which do not arises as a multiplicative function of the degree?

In general, multiplicative characters play an important role in representation theory, which is also the subject of this work.

We give a positive answer to Ghys' question, and suggest a general method to construct multiplicative characters on polynomials. Then, we study the automorphism groups of several semigroups of interest in holomorphic dynamical

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systems. In the second section, we show that the semigroup of polynomials and rational maps are generated by linear automorphisms and the Galois group action on $\mathbb{C}$. A result, due to Hinkkanen, states that the automorphism group of entire functions consist of the group of continuous inner automorphisms. Using Hinkkanen's Theorem, we show that, the group of automorphisms of the semigroup of meromorphic functions also consists of the group of continuous inner automorphisms. We give algebraic conditions, using sandwich semigroups, that characterize when two given polynomials, or rational maps, are conformally conjugated. The main tool is a theorem, due to Schreier, which states that any representation of a semigroup of maps S is geometric, whenever S contains constant maps. This result is useful to study the representation space of semigroups of maps and correspondences. Hence, this theorem remarks the importance of considering semigroups of maps together with constants. This point of view is also adopted in Eremenko's paper [5].

On the third section we provide several examples of semigroups representations that appear in holomorphic dynamical systems. Among these, there is a connection with the deformation space of a given rational map. The topology of the deformation space of a rational space is discussed in [2].

On the last section, we generalize the results of Section 2, to the setting of holomorphic correspondences. In particular, we prove a generalized Schreier Lemma for correspondences and holomorphic correspondences. This allow us to characterize the Galois group $Gal(\mathbb{C})$, the group of all field automorphisms of $\mathbb{C}$, as the subgroup of $Bij(\mathbb{C})$ that under conjugation functionally preserves the finite holomorphic world.

2 Semigroups.

A semigroup is a set S together with a binary operation which is associative. Given any set X, consider the semigroup $Map(X)$ of all maps $\phi : X \to X$, with composition as semigroup operation. It contains the group of bijections $Bij(X)$. A subset $I \subset S$ is called a left (respectively right) ideal, if $si \in I$ (respectively $is \in I$), for all $s \in S$ and $i \in I$.

For every element s in S, let τ_s be the left translation by s. The map $s \mapsto \tau_s$ induces a representation ϕ from S into $Map(S)$. In fact, the same map gives a representation from S into $Map(I)$ for every left ideal I in S. Note that ϕ is a faithful representation only in the case when, for every pair of elements g, h in S, there is $i \in I$ such that $gi \neq hi$.

Let X be an abstract set, then there is a canonical inclusion of X into $Map(X)$, sending every x in X to the constant map x. The image of this map is a left ideal $\mathcal{I}$ in $Map(X)$, the *ideal of constants* of X. An element x in $Map(X)$ belongs to $\mathcal{I}$ if, and only if, for every $g \in Map(X)$ we have :

i) $g \circ x \in \mathcal{I}$;

ii) $x \circ g = x$.

The ideal of constants is contained in every left ideal of $Map(X)$. In this sense, the ideal of constants is the smallest left ideal in $Map(X)$. Given any semigroup S, we can use properties (i) and (ii) to define the ideal of constants, whenever it exists.

Example. *In general, the ideal of constants in $Map(X)$ is not prime. Assume that X has at least three points x_1, x_2 and x_3, let g_1 and g_2 in $Map(X)$, such that, $Image(g_1) = \{x_1, x_2\}$, $g_2(x_1) = g_2(x_2) \neq g_2(x_3)$, then $g_2 \circ g_1$ is constant but g_2 is not. However, if we restrict to the space of continuous maps $C_0(X)$ in a topological space X with enough regularity, then the ideal of constants is prime.*

From now on, we will consider the special case where X is either the complex plane $\mathbb{C}$ or the Riemann sphere $\bar{\mathbb{C}}$. Given two subsets S_1 and S_2 in $Map(X)$, we denote by $\langle S_1, S_2 \rangle$ the semigroup generated by S_1 and S_2.

2.1 Multiplicative characters of semigroups.

Let $Pol(\mathbb{C})$ denote the semigroup of complex polynomials with composition as semigroup multiplication. Let us consider the set $Hom(Pol(\mathbb{C}), \mathbb{C})$ of all multiplicative characters, that is, the set of homomorphisms χ in satisfying

$$\chi(P_1 \circ P_2) = \chi(P_1) \cdot \chi(P_2)$$

for all $P_1, P_2 \in Pol(\mathbb{C})$.

The degree function *deg*, is a basic example of a multiplicative character in $Pol(\mathbb{C})$. Any multiplicative function of *deg* induces a multiplicative character. It was a question of É. Ghys whether there are other characters apart from these examples. We give a positive answer to this question and give a description of how to construct multiplicative characters on $Pol(\mathbb{C})$. To do so, first let us recall a theorem due to Ritt, see [13].

Definition. *A polynomial P, is called* prime*, or* indecomposable*, if whenever we have $P = Q \circ R$, where Q and R are polynomials, then either $deg(Q) = 1$ or $deg(R) = 1$. A decomposition of $P = P_1 \circ P_2 \circ ... \circ P_n$ is called a* prime decomposition *if, and only if, each P_i is a prime polynomial of degree at least* 2 *for all i.*

Given a prime decomposition of a polynomial $P = P_1 \circ ... \circ P_n$, a *Ritt transformation*, say in the j place, is the substitution of the pair $P_j \circ P_{j+1}$, in the prime decomposition of P, by the pair $Q_j \circ Q_{j+1}$. Where Q_j and Q_{j+1} are prime polynomials satisfying $P_j \circ P_{j+1} = Q_j \circ Q_{j+1}$. Now we can state Ritt's theorem.

Theorem 1 (Ritt)**.** *Let $P = P_1 \circ P_2 \circ ... \circ P_m$ and $P = Q_1 \circ ... \circ Q_n$ be two prime decompositions of P, then $n = m$. Moreover, any two given prime decompositions of P are related by a finite number of Ritt transformations.*

In [13], Ritt showed that there are three types of Ritt transformations, namely, see also [1]:

1. Substitute $P_i \circ P_{i+1}$ by $(P_i \circ A) \circ (A^{-1}) \circ P_{i+1}$, where A is an affine map.
2. Substitute $P_i \circ P_{i+1}$ by $P_{i+1} \circ P_i$, when P_i and P_{i+1} are Tchebychev polynomials.
3. If $P_i(z) = z^k$ and $P_{i+1}(z) = z^r P(z^k)$ for some polynomial P and natural numbers r and k. Define $Q_{i+1}(z) = z^r(P(z))^k$, then substitute $P_i \circ P_{i+1}$ by $Q_{i+1} \circ P_i$.

In particular, there are two invariants of a prime decomposition, the length of a prime decomposition, and the set of degrees in the prime decomposition. Hence, for every $P \in Pol(\mathbb{C})$ the length of a prime decomposition of P is a well defined additive character $l(P)$. That is, it satisfies

$$l(P_1 \circ P_2) = l(P_1) + l(P_2).$$

Now, define the function χ by $\chi(P) = e^{l(P)}$. Then, χ is a multiplicative character which is not a multiplicative function of the degree.

The following theorem gives a method to generate multiplicative characters in $Pol(\mathbb{C})$.

Theorem 2. *Let ϕ be a complex function, defined on the set of prime polynomials, satisfying:*

(i) $\phi(c) = 0$ *for every constant c.*

(ii) If P_1, P_2, P_3, P_4 are prime polynomials with $P_1 \circ P_2 = P_3 \circ P_4$, then

$$\phi(P_1) \cdot \phi(P_2) = \phi(P_3) \cdot \phi(P_4).$$

Then, ϕ generates a multiplicative character Φ. Conversely, if Φ is a multiplicative character in $Pol(\mathbb{C})$, which is not the constant map 1, then Φ satisfies the conditions above.

Proof. Let P be a composite polynomial and $P = P_1 \circ P_2 \circ ... \circ P_n$ be a prime decomposition of P, define

$$\Phi(P) = \phi(P_1) \cdot \phi(P_2) \cdot ... \cdot \phi(P_n).$$

Let us check that Φ is well defined. By Theorem 1, it is enough to consider a step modification of $P_1 \circ P_2 \circ ... \circ P_n$. Let Q_j and Q_{j+1} be two polynomials such that $P_j \circ P_{j+1} = Q_j \circ Q_{j+1}$, then by condition (ii) we have $\phi(P_j) \cdot \phi(P_{j+1}) = \phi(Q_j) \cdot \phi(Q_{j+1})$, in consequence

$$\phi(P_1) \cdot \phi(P_2) \cdot ... \cdot \phi(P_j) \cdot \phi(P_{j+1}) \cdot ... \cdot \phi(P_n) =$$

$$\phi(P_1) \cdot \phi(P_2) \cdot ... \cdot \phi(Q_j) \cdot \phi(Q_{j+1}) \cdot ... \cdot \phi(P_n).$$

Hence Φ is invariant under step modifications and, by Theorem 1, it is independent of the prime decomposition of P. It follows, from the definition, that Φ is a multiplicative character.

Conversely, let Φ be a multiplicative character. For any pair of constants c_1 and c_2, the equations $c_1 \circ c_2 = c_1$ and $c_2 \circ c_1 = c_2$ imply

$$\Phi(c_1) = \Phi(c_1 \circ c_2) = \Phi(c_1) \cdot \Phi(c_2)$$

$$= \Phi(c_2) \cdot \Phi(c_1) = \Phi(c_2 \circ c_1) = \Phi(c_2).$$

Then, for every constant c, either we have $\Phi(c) = 1$ or $\Phi(c) = 0$. If $\Phi(c) = 1$, the equation $P(c) = P \circ c$ implies that $\Phi(P) = 1$ for all P. Hence if Φ is not constantly 1, then we have $\Phi(c) = 0$ for every constant c. The second condition follows from the fact that Φ is a multiplicative character. $\square$

Example (Affine characters.)**.** *Let H be the ideal of non injective polynomials. Any multiplicative character $\chi : \mathrm{Aff}(\mathbb{C}) \to \mathbb{C}$ admits an extension to a multiplicative character defined in $Pol(\mathbb{C})$. For instance, put $\chi(c) = 0$ for all constant c, and $\chi(h) = 0$, for all other h in H.*

In the same way, we can extend affine characters to other semigroups containing $\mathrm{Aff}(\mathbb{C})$, such as $Rat(\mathbb{C})$, $Ent(\mathbb{C})$ or the semigroup of holomorphic correspondences discussed at the end of this work.

Now let us construct non-trivial extension of the constant affine character equal to 1. In order to do so, we have to consider the bi-action, left and right, of $\mathrm{Aff}(\mathbb{C})$ on $Pol(\mathbb{C})$. The bi-orbit of a polynomial P is the set of all polynomials of the form $A \circ P \circ B$, where A, B belong to $\mathrm{Aff}(\mathbb{C})$. We say that a polynomial has no symmetries if, there are no elements A, B, in $\mathrm{Aff}(\mathbb{C})$, such that $P = A \circ P \circ B$.

Lemma 3. *Let P be a prime polynomial, and let $\mathcal{AF}(P)$ be the semigroup generated by the bi-orbit of the Affine group of the set of iterates $\{P^n\}$. Let Q and R be a pair of polynomials, of degree at least 2, such that $Q \circ R \in \mathcal{AF}(P)$, then $Q \in \mathcal{AF}(P)$ and $R \in \mathcal{AF}(P)$.*

Proof. Since $Q \circ R$ belong to $\mathcal{AF}(P)$, there is a prime decomposition of $Q \circ R$ whose elements are of the form $A \circ P \circ B$. By Ritt's Theorem, any other prime decomposition of $Q \circ R$ is obtained by a finite number of Ritt's transformations. But, Ritt's transformations are either permutations, or substitution by a pair of elements in the bi-affine orbit. Hence, all prime decompositions of $Q \circ R$ have prime elements in $\mathcal{AF}(P)$. Then the conclusion of the Lemma follows. $\square$

Example. *Let χ be the constant multiplicative character equal to 1 defined on $\mathrm{Aff}(\mathbb{C})$, and P be a prime polynomial of degree at least 2. Let us extend χ to all $Pol(\mathbb{C})$ defining $\chi(Q) = 1$ for all Q in the bi orbit by $\mathrm{Aff}(\mathbb{C})$ of the set $\{P^n\}_{n \in \mathbb{P}}$, and $\chi(Q) = 0$ for all the other polynomials Q in $Pol(\mathbb{C})$. By Lemma 3 and Theorem 2, this is a well defined character. In fact for any number a, defining $\chi(A \circ P^n \circ B) = a^n$ where A, B are elements in $\mathrm{Aff}(\mathbb{C})$ gives other extensions of χ in $Pol(\mathbb{C})$.*

To extend arbitrary multiplicative characters defined on $\text{Aff}(\mathbb{C})$*, the construction of the character is more involved. At least in the case where* P *is a prime polynomial, such that every iterate* P^n *is without symmetries, it is possible to extend any multiplicative character* χ *on* $\text{Aff}(\mathbb{C})$*.*

The ideal of constants is very useful to understand the structure of $Map(X)$. A homomorphism $\phi : Map(X) \to Map(Y)$ is called *geometric* if, there is a map $f : X \to Y$ satisfying $\phi(P) \circ f = f \circ P$ for every $P \in Map(X)$. Now we recall a result due to Schreier that describes the semigroup $Map(X)$ using the ideal of constants. For further details see [14], and also the discussions in Eremenko's paper [5] and Magill's survey [9].

Lemma 4 (Schreier's Lemma)**.** *Let* $\phi : Map(X) \to Map(Y)$ *be a homomorphism, then* ϕ *is geometric. In the case where* $\phi \in Aut(Map(X))$ *and* $\phi(P) \circ f = f \circ P$*, then* f *is a bijection of* X *and* $\phi(P) = f \circ P \circ f^{-1}$*, for all* $P \in Map(X)$*.*

Proof. Consider the restriction $f := \phi_{|X}$ to the ideal of constants. Since ϕ is a homomorphism, it maps ideals into ideals, it also preserves the properties of the ideal of constants, hence f sends constants to constants. So f is a map from X to Y. Moreover,

$$\phi(P(x)) = f(P(x))$$

since $P(x) \in X$. Also,

$$\phi(P(x)) = \phi(P \circ x) = \phi(P) \circ f(x) = \phi(P)(f(x)),$$

hence

$$\phi(P)(f(x)) = f(P(x)).$$

If $\phi \in Aut(Map(X))$, then f is a map from X to itself. Moreover, since ϕ is an automorphism, we can apply the argument to ϕ^{-1}, so we get that f is invertible. Which implies that f is a bijection and the formula $\phi(P) = f \circ P \circ f^{-1}$. □

In fact, there is no need that the homomorphism in Lemma 4 is defined in all $Map(X)$, the same proof above shows.

Corollary 5. *Let* S_1 *and* S_2 *be subsemigroups of* $Map(X)$ *and* $Map(Y)$*, respectively, and such that* $X_1 = S_1 \cap X$ *and* $Y_1 = S_2 \cap Y$ *are both non empty sets. If* $\phi : S_1 \to S_2$ *is a homomorphism, then there exist* $f : X_1 \to Y_1$*, such that for all* $h \in S_1$*,* $\phi(h) \circ f = f \circ h$*. Moreover,*

- *the homomorphism* ϕ *is injective, or surjective, if and only if, the map* f *is injective or surjective. In particular,* ϕ *is an isomorphism if, and only if,* f *is a bijection.*
- *When* S_1 *and* S_2 *are topological semigroups, then* Φ *is continuous if, and only if,* f *is continuous.*

Along with the ideal of constants, the affine group $\mathrm{Aff}(\mathbb{C})$ plays an important role in the description of automorphisms of polynomials. Later on, we will consider generalizations to semigroups generated by correspondences. A particular case of Lemma 4 is the following

Corollary 6. *For any set X, the group $Aut(Map(X))$ is isomorphic to $Bij(X)$.*

Let $Gal(\mathbb{C})$ denote the absolute Galois group of $\mathbb{C}$, that is, the full group of field automorphisms of $\mathbb{C}$. Remind that since every orientation preserving element in $Gal(\mathbb{C})$ must fix the complex rationals, the identity and complex conjugation are the only continuous elements in $Gal(\mathbb{C})$. The action of $Gal(\mathbb{C})$ extends to an action in $Rat(\mathbb{C})$, the semigroup of rational functions in $\mathbb{C}$. In particular, the action of $Gal(\mathbb{C})$ in $\mathbb{C}$ extends to an action in $Pol(\mathbb{C})$.

Proposition 7. *The group of automorphisms of $Pol(\mathbb{C})$ is generated by $Gal(\mathbb{C})$ and $\mathrm{Aff}(\mathbb{C})$. Moreover, $Aut(Pol(\mathbb{C})) = Aut(\mathrm{Aff}(\mathbb{C}))$.*

Proof. Let ϕ be an element of $Aut(Pol(\mathbb{C}))$. By Lemma 4, the restriction $f = \phi_{|\mathbb{C}}$ is a bijection from $\mathbb{C}$ to $\mathbb{C}$, and $\phi(P) = f \circ P \circ f^{-1}$. First, let us check that $\phi = Id$ if, and only if, $f = Id$. Note that we can realize evaluation as composition with a constant function. If $f = Id$, then we have

$$\phi(P)(z) = \phi(P) \circ f(z) = \phi(P(z)) = f(P(z)) = P(z)$$

for every polynomial P and $z \in \mathbb{C}$, that is $\phi = Id$. The converse is clear.

Since, by Lemma 4, ϕ is a conjugation, then $\phi(P)$ and P have the same degree. In particular, ϕ leaves the affine group $\mathrm{Aff}(\mathbb{C})$ invariant, so $\phi(\mathrm{Aff}(\mathbb{C})) = \mathrm{Aff}(\mathbb{C})$. This fact also follows from the characterization of $\mathrm{Aff}(\mathbb{C})$ as the set of injective polynomials. In particular, $Aut(Pol(\mathbb{C})) \subset Aut(\mathrm{Aff}(\mathbb{C}))$, the converse is also true by Lemma 4, since any conjugacy in the $\mathrm{Aff}(\mathbb{C})$ extends to a conjugacy in $Pol(\mathbb{C})$.

The group of translations T is the commutator of $\mathrm{Aff}(\mathbb{C})$, hence T is invariant under ϕ. The value of a translation at one point, determines the translation. Let τ_c denote the translation $z \mapsto z + c$, since

$$\phi(\tau_c)(f(0)) = f \circ \tau_c \circ f^{-1}(f(0)) = f(c),$$

then

$$\phi(\tau_c) = \tau_{(f(c)-f(0))}.$$

Define $g(z) = f(z) - f(0)$, then g is a bijection of $\mathbb{C}$ which is the restriction to the constants of the map $\tilde{\phi} = \tau_{-f(0)} \circ \phi$ and $g(0) = 0$. By definition, $\tilde{\phi} \in Aut(Pol(\mathbb{C}))$ and $\phi(\tau_c) = \tau_{g(c)}$, it follows that

$$\phi(\tau_{c_1+c_2}) = \phi(\tau_{c_1}) \circ \phi(\tau_{c_2}),$$

that is,

$$g(c_1 + c_2) = g(c_1) + g(c_2).$$

Let A_0 be the group of injective polynomials fixing 0, since $\tilde{\phi}(0) = 0$, then $\tilde{\phi}(A_0) = A_0$. Now we repeat the argument above, this time in multiplicative terms, to show that $h(c) = \frac{g(c)}{g(1)}$ is a bijection of $\mathbb{C}$ preserving multiplication and $h(1) = 1$. By definition, h also preserves addition with $h(0) = 0$, hence $h \in Gal(\mathbb{C})$. Note that h is the restriction to constants of the map $(g(1)^{-1}\tau_{-f(0)})\circ\phi$. This implies that $f = g(1)h + f(0)$ as we wanted to show.

□

The proof of Proposition 7 can be adapted to show

Proposition 8. *Let $Rat(\bar{\mathbb{C}})$ denote the semigroup of rational maps in the Riemann sphere, then $Aut(Rat(\bar{\mathbb{C}})) = \langle Gal(\mathbb{C}), PSL(2,\mathbb{C})\rangle$*

Proof. Since $\phi(Id) = Id$, and using the formula $R \circ R^{-1} = Id$, one can check that ϕ sends $PSL(2,\mathbb{C})$, the group of invertible rational maps, into $PSL(2,\mathbb{C})$. Post composing ϕ with an element of $PSL(2,\mathbb{C})$ we can assume that $\phi(\infty) = \infty$. In this case, it follows that $\phi(\text{Aff}(\mathbb{C})) \subset \text{Aff}(\mathbb{C})$, hence if $\phi(\infty) = \infty$ then $\phi \in Aut(\text{Aff}(\mathbb{C}))$. Since every element in $\langle Gal(\mathbb{C}), PSL(2,\mathbb{C})\rangle$ induces a conjugation in $Rat(\mathbb{C})$, we have the claim of the proposition.

□

Now we want to study the semigroup of meromorphic functions $Mer(\mathbb{C})$. This semigroup contains the semigroup of entire functions $Ent(\mathbb{C})$. We recall a theorem by Hinkkanen [6].

Theorem 9 (Hinkkanen)**.** *Let ϕ be a geometric automorphism of $Ent(\mathbb{C})$, then ϕ is affine.*

In other words, except for the identity, no element in $Gal(\mathbb{C})$ leaves the semigroup $Ent(\mathbb{C})$ invariant in the space of formal series. The following are immediate consequences of Lemma 4 and Hinkkanen's Theorem.

Proposition 10. *The group of automorphisms of $Mer(\mathbb{C})$ is isomorphic to $PSL(2,\mathbb{C})$.*

Proof. Let ϕ be an element in $Aut(Mer(\mathbb{C}))$, and γ be an element in $PSL(2,\mathbb{C})$ so that $\gamma(\phi(\infty)) = \infty$. By Lemma 4, $\gamma \circ \phi$ is a geometric automorphism in $Mer(\mathbb{C})$. Now, a meromorphic map g is entire if, and only if, g has no finite poles. Since infinity is fixed by $\gamma \circ \phi$, the map $\gamma \circ \phi$ sends entire functions into entire functions. By Theorem 9, we have $\gamma \circ \phi \in \text{Aff}(\mathbb{C})$ and $\phi \in PSL(2,\mathbb{C})$. □

Corollary 11. *Every automorphism of $Mer(\mathbb{C})$ is continuous.*

Corollary 12. *A map ϕ in $Aut(Rat(\mathbb{C}))$ is continuous if, and only if, ϕ extends to a map in $Aut(Mer(\mathbb{C}))$.*

All above gives a characterization of elements in $Bij(\mathbb{C})$ that belong to the Galois group $Gal(\mathbb{C})$.

Theorem 13. *Let F be an element in $Bij(\mathbb{C})$, let us assume it fixes three points in $\mathbb{C}$, then the following are equivalent.*

i) The map F belongs to $Gal(\mathbb{C})$.

ii) The induced map in $Map(\mathbb{C})$ sends $Rat(\mathbb{C})$ into itself.

iii) The induced map in $Map(\mathbb{C})$ sends $Pol(\mathbb{C})$ into itself.

iv) The induced map in $Map(\mathbb{C})$ sends $\mathrm{Aff}(\mathbb{C})$ into itself.

2.2 Sandwich semigroups.

Here, we give an algebraic condition for when two rational maps are Möbius conjugated, for this we do not require any dynamical restrictions on the rational maps. We start with the polynomial case, where the action of $PSL(2,\mathbb{C})$ is replaced by the action of $\mathrm{Aff}(\mathbb{C})$.

Given a map $g : Y \to X$, let us define on $Map(X,Y)$ the following operation, for $f, h \in Map(X,Y)$ put $f *_g h = f \circ g \circ h$. We denote this new semigroup by $Map_g(X,Y) = (Map(X,Y), *_g)$. In particular, if S is a subsemigroup of $Map(X)$ and $g \in Map(X)$, the set $S_g := (S, *_g)$ is also a semigroup. In particular, given a polynomial P, let us consider the semigroup $Pol_P(\mathbb{C})$.

Theorem 14. *Let P_1 and P_2 be two complex polynomials. Let*

$$\Phi : Pol_{P_1}(\mathbb{C}) \to Pol_{P_2}(\mathbb{C})$$

be an isomorphism of semigroups. Then there is $f \in Bij(\mathbb{C})$, and $B \in \mathrm{Aff}(\mathbb{C})$, such that $\Phi(P) = f \circ P \circ f^{-1} \circ B^{-1}$.

Proof. We first check that $\phi(\mathrm{Aff}(\mathbb{C})) = \mathrm{Aff}(\mathbb{C})$. By definition, for every pair of polynomials P, Q, we have

$$\phi(P *_{P_1} Q) = \phi(P) *_{P_2} \phi(Q).$$

Let $f = \phi|_{\mathbb{C}}$ then, taking for Q a constant $c \in \mathbb{C}$, the equality above becomes

$$f(P \circ P_1(c)) = \phi(P) \circ P_2(f(c)), \tag{1}$$

for every polynomial $P \in Pol(\mathbb{C})$. Since ϕ is an isomorphism, f is an invertible map. Hence the equation above implies that f conjugates the polynomial $P \circ P_1$ to $\phi(P) \circ P_2$. Then $deg(P \circ P_1) = deg(\phi(P) \circ P_2)$. We obtain a similar equation for ϕ^{-1}

$$f^{-1}(P \circ P_2(c)) = \phi^{-1}(P) \circ P_1(f^{-1}(c))$$

and $deg(P \circ P_2) = deg(\phi(P) \circ P_1)$. Since deg is a multiplicative character, and takes values in $\mathbb{N}$, for every invertible polynomial A we obtain

$$deg(P_1) = deg(\phi(A)) \cdot deg(P_2)$$

and

$$deg(P_2) = deg(\phi^{-1}(A)) \cdot deg(P_1).$$

Hence $1 = deg(\phi(A)) \cdot deg(\phi^{-1}(A))$, which implies that $\phi(\text{Aff}(\mathbb{C})) = \text{Aff}(\mathbb{C})$.

Define $B = \phi(Id)$, then B is an element of $\text{Aff}(\mathbb{C})$, now consider the map $\phi_B : (Pol(\mathbb{C}), P_2) \to (Pol(\mathbb{C}), B^{-1}P_2)$, given by $\phi_B(P) = P \circ B$. The ϕ_B is an isomorphism of semigroups. Then the composition $\Phi = \phi_B \circ \phi$ is an isomorphism from $(Pol(\mathbb{C}), P_1)$ to $(Pol(\mathbb{C}), B^{-1}P_2)$, satisfying $\Phi(Id) = Id$. Last equation implies that $\Phi(P_1) = P_2$. Moreover, since $\Phi(c) = \phi(c) \circ B = \phi(c) = f(c)$, the restrictions to constants, of the maps ϕ and Φ, are equal. If $P = Id$ in (1), we obtain that $P_1 = f^{-1} \circ P_2 \circ f$, which implies from (1) that for all $c \in C$

$$f \circ P \circ P_1(f^{-1}(c)) = \Phi(P) \circ P_2(f(f^{-1}(c)) \\ = \Phi(P) \circ P_2(c),$$

then $\Phi(P) = f \circ P \circ f^{-1}$. Hence $\phi(P) = f \circ P \circ f^{-1} \circ B^{-1}$ as we wanted to show.

□

Corollary 15. *Two polynomials P_1 and P_2 are affinely conjugate if, and only if, the semigroups $Pol_{P_1}(\mathbb{C})$ and $Pol_{P_2}(\mathbb{C})$ are continuously isomorphic with an isomorphism ϕ, such that $\phi(Id) = Id$.*

By substituting $\text{Aff}(\mathbb{C})$ by $PSL(2, \mathbb{C})$, and $Pol(\mathbb{C})$ by $Rat(\mathbb{C})$ in the proof of previous theorem, we obtain the following

Theorem 16. *Let R_1 and R_2 be two complex rational maps, and consider an automorphism of semigroups $\Phi : Rat_{R_1}(\bar{\mathbb{C}}) \to Rat_{R_2}(\bar{\mathbb{C}})$. Then there is $f \in Bij(\mathbb{C})$ and $B \in PSL(2, \mathbb{C})$ such that $\Phi(R) = f \circ R \circ f^{-1} \circ B^{-1}$. In particular, if Φ is continuous with $\Phi(Id) = Id$, then Φ is conjugation by an element of $PSL(2, \mathbb{C})$.*

Which implies the following

Corollary 17. *Two rational maps R_1 and R_2 are conjugate by a map in $PSL(2, \mathbb{C})$ if, and only if, the semigroups $Rat_{R_1}(\mathbb{C})$ and $Rat_{R_2}(\mathbb{C})$ are continuously isomorphic with an isomorphism ϕ, such that $\phi(Id) = Id$.*

By Theorem 14, the condition $\phi(Id) = Id$ is equivalent to require that $\phi(R_1) = R_2$. Every automorphism of $Rat(\mathbb{C})$ induces an isomorphism of sandwich semigroups. Indeed, if $\phi \in Aut(Rat(\mathbb{C}))$ take Q, R rational maps such that $\phi(Q) = R$, then ϕ is an isomorphism between $Rat_Q(\mathbb{C})$ and $Rat_R(\mathbb{C})$. Let ψ be an isomorphism of sandwich semigroups in $Rat(\mathbb{C})$. By Theorem 14 and Lemma 4, ψ induces an automorphism of $Rat(\mathbb{C})$ if, and only if, $\psi(Id) = Id$. Let us now discuss the situation of sandwich isomorphisms for small semigroups. Let Q y R be two non-constant rational maps, and consider the semigroup $S = \langle Q, R, \mathbb{C} \rangle$. Take R_1 and R_2 in S and consider an isomorphism ϕ between

S_{R_1} and S_{R_2}. Since Q and R are the non constant elements in S with smaller degree, then we have either

$$\phi(Q) = R \text{ and } \phi(R) = Q$$

or

$$\phi(Q) = Q \text{ and } \phi(R) = R.$$

In any case, ϕ^2 fixes Q and R. Then the restriction of ϕ to constants is a non trivial bijection of $\mathbb{C}$, which commutes with Q and R.

3 Semigroup representations.

In this section, we give examples of how the theory of semigroup representations applies to holomorphic dynamics. For every X, let us consider the decomposition of $Map(X)$ into the ideal of constants, $\mathcal{I}(X)$, the group of bijections $Bij(X)$ and the rest $H(X)$. That is $Map(X) = \mathcal{I}(X) \cup Bij(X) \cup H(X)$, as a consequence of Corollary 5, it follows that every homomorphism of $Map(X)$ into $\mathcal{I}$ is constant. Similarly, the only homomorphism from $Map(X)$ to $Bij(X)$ is the constant map with value Id.

In the spirit of Lemma 4, we consider semigroups together with the ideal of constants. Let A be any set in X and S a subset of $Map(X)$, then we denote by $\langle S, A\rangle$ the semigroup generated by S and the constants in A regarded as semigroups of $Map(\mathcal{O}_S^+(A))$, where $\mathcal{O}^+{}_S(A)$ denotes the forward S-orbit of A. With this construction, the ideal of constants of $\langle S, A\rangle$ is precisely $\mathcal{O}_S^+(A) \cup A$.

Example. *Let $f_0 = z^2$, then $\langle f_0, 1\rangle = 1$, since $f_0 = Id = 1$ in $Map(\{1\})$. Analogously, if a is a periodic orbit of f_0, then $\langle f_0, a\rangle$ consists of the orbit of a and the cyclic permutations of this orbit.*

We can generalize the previous example to rational functions $R : \mathbb{C} \to \mathbb{C}$. In this case, we obtain a family of semigroups $\langle R, a\rangle$ parametrized by a point a in the plane $\mathbb{C}$. In this way, the set $\mathcal{D}_R = \{\langle R, a\rangle : a \in \mathbb{C}\}$ inherits the usual topology from $\mathbb{C}$. Let $\mathcal{X}_R \subset \mathcal{D}_R$ be the set of finite semigroups, we call the set $\mathcal{J}_R = \overline{\mathcal{X}_R} \setminus \{\text{isolated points}\}$, the *algebraic Julia set* of R. The complement $\mathcal{F}_R = \{\langle R, a\rangle : a \in \mathbb{C}\} \setminus \mathcal{J}_R$ will be called the *algebraic Fatou set* of R in $\mathcal{D}_R$. In this setting, the algebraic Fatou set is the interior of the set of free semigroups in $\mathcal{D}_R$. These definitions reflect the dynamical Julia set $J(R)$, which is the closure of the repelling periodic points in $\mathbb{C}$ and, the dynamical Fatou set $F(R)$ which is the complement of $J(R)$ in $\mathbb{C}$.

3.1 Representations of semigroups of polynomials.

Let $\mathcal{P}$ be a partition of $Pol(\mathbb{C})$, we say that $\mathcal{P}$ is a *compatible partition* if for $A, B \in \mathcal{P}$, and a pair of points $a \in A$, $b \in B$, the composition $a \circ b$ belongs to a component C in $\mathcal{P}$ which do not depend on the representatives a and b. A *graduation* is a partition of $Pol(\mathbb{C})$ which is compatible with composition.

As we discussed earlier in the paper, $Pol(\mathbb{C})$ has a non empty set of multiplicative characters. Each multiplicative character in $Pol(\mathbb{C})$ induces a graduation in $Pol(\mathbb{C})$. The fibers of multiplicative characters induce compatible partitions. In particular, the degree of a polynomial induces a compatible partition of $Pol(\mathbb{C})$. In this case, the classes of this partition are $Pol_d(\mathbb{C})$, the set of polynomials of given degree d. We will describe now some examples of representations of semigroups of the form $\langle P, A\rangle$ into $Pol_d(\mathbb{C})$.

Note that since we are including an ideal of constants A in the domain, then we have to include the constants in $Pol_d(\mathbb{C})$ as well. Otherwise, there is no representation from $\langle P, A\rangle$ into $Pol_d(\mathbb{C})$. Nevertheless, including constants, in both domain and range, is consistent with the philosophy of Lemma 4. In this setting, every representation of S in $Pol(\mathbb{C})$ is geometric, and realized by a map defined in the complex plane. Let S be a subsemigroup in $Pol(\mathbb{C})$ containing Id, and let us consider representations of S into $Pol_0(\mathbb{C})$, the semigroup of constant polynomials. Let $\phi : S \to Pol_0(\mathbb{C})$ be a homomorphism, since $\phi(Id)$ is constant, then $\phi(R) = \phi(Id) \circ \phi(R) = \phi(Id)$. Hence, any representation of S into the constant polynomials is a constant map.

The theory of representation of semigroups of the form $\langle P, J(P)\rangle$ into $Pol(\mathbb{C})$ is widely discussed in holomorphic dynamics in other terms. For example, the theory of the continuous representations of $\langle P, J(P)\rangle$ into $Pol(\mathbb{C})$ is parameterized by the J-stable components of P. For example, see [2].

Another important situation is representations of semigroups $\langle P, \mathcal{P}(P)\rangle$ into $Pol(\mathbb{C})$, here $\mathcal{P}(P)$ is the postcritical set of P. Interior components of the representation space can be parameterized by combinatorially equivalent polynomials. Uniformization of these components by suitable geometric objects (like suitable Teichmüller spaces), shed light on many problems in holomorphic dynamics. In this direction, important advances were made by Douady, Hubbard, Lyubich, McMullen, Sullivan and Thurston, among many others. See for example [4] and [10].

Now, let us consider the space of representations of affine semigroups into the space of polynomials of degree d. This space includes all linearizations around periodic orbits. Here, we review the repelling case. A complete treatment of linearization theory in holomorphic dynamics can be found in Milnor's book [12].

Let A_λ in Aff$(\mathbb{C})$ of the form $z \mapsto \lambda z$. Let P be a polynomial such that there exist a repelling cycle $\mathcal{O} = \{z_0, z_1, ..., z_n\}$ with multiplier λ. The Poincaré function associated to z_0, is a map $\phi : \mathbb{C} \to \mathbb{C}$ sending 0 to z_0 which locally conjugates A_λ to P^n around z_0. This construction induces a representation of $\langle A_\lambda, \mathbb{C}\rangle$ into $\langle P, U_0\rangle$ for a suitable neighborhood U_0 of z_0. Moreover, since Poincaré functions turn out to be meromorphic functions, it also induces a representation of affine semigroups into the semigroups of meromorphic functions. Similar constructions apply to other kind of linearizations. In the attracting case, the inverse of the Poincaré function, defined on a neighborhood U_0 of z_0, is known as König's coordinate and gives a representation of $\langle P, U_0\rangle$ into $\langle A_\lambda, \mathbb{D}_r\rangle$, where $\mathbb{D}_r$ denotes the disk of radius r and $r < 1$. This construction can also be applied to the parabolic case.

The process of renormalization, in holomorphic dynamics, gives examples of semigroups of the form $\langle P, U\rangle$ that admit representations into themselves.

Let P be a polynomial $P(z)$, of degree n, with connected and locally connected Julia set. Then, ∞ is a superattracting fixed point of P. If $A_0(\infty)$ denotes the basin of ∞ of P, by Böttcher's theorem, there is a homeomorphism $\phi : \bar{\mathbb{C}} \setminus \bar{\mathbb{D}} \to A_0(\infty)$, that conjugates $z \mapsto z^n$ in $\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}$ with P in $A_0(\infty)$. Since $J(P)$ is connected and locally connected, the map ϕ extends to the boundaries by Caratheodory's theorem. The map on the boundaries induces a representation of $\langle z^n, \mathbb{S}^1\rangle$ into $\langle P, J(P)\rangle$.

It would be interesting to have results, analogous to Theorem 13 or Theorem 14, that characterizes the action of quasiconformal maps in $\mathbb{C}$. This would allow us to determine quasiconformal conjugation in terms of semigroup representations.

3.2 When Julia set is homeomorphic to a Cantor set.

Now consider the special case where $J(R)$ is homeomorphic to a Cantor set. For simplicity in the arguments, let us assume that $deg(R) = 2$.

Let us consider a Jordan curve γ containing in its interior the Julia set and a critical point; while the other critical point and all critical values lie outside γ. Choosing a suitable γ, we assume that $R^{-1}(\gamma)$ is contained in the interior of γ and consists of two Jordan curves γ_1 and γ_2. We get an scheme similar to the one sketched in Figure 1. Let us call D_1 and D_2 the interiors of γ_1 and γ_2, respectively.

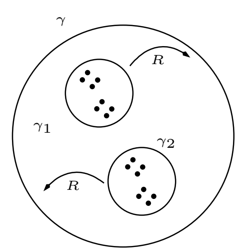

Figure 1: Cantor scheme.

With this scheme, we obtain representations of $\langle R, D\rangle$ into other semigroups. To do so, let us modify topologically the Cantor scheme, and instead of the restrictions of R on D_i, consider affine maps A_i sending the modified γ_i to γ. This induces a representation of $\langle R, J(R)\rangle$ into $\mathrm{Aff}(\mathbb{C})$.

If we modify the curves γ, γ_1 and γ_2 to circles and considering Möbius transformations g_i, instead of the maps that send γ_i to γ. We get a representation Φ of $\langle R, J(R)\rangle$ into a "half" classical Schottky group Γ with two generators. This is an example of a representation of non cyclic Kleinian groups in rational semigroups. The conjugating map of Φ may be taken quasiconformal, hence the Hausdorff dimension of the limit set of the Schottky group can be estimated in terms of the Hausdorff dimension of $J(R)$. In particular, let $R(z)$ be a quadratic polynomial of the form $z^2 + c$, such that $J(R)$ is a Cantor set. In this case, the parameter c belongs to the complement of the Mandelbrot set. A theorem of Shishikura shows that there are sequences of quadratic polynomials $R_{c_n}(z) = z^2 + c_n$, with parameters c_n tending to the boundary of the Mandelbrot set, and such that the Hausdorff dimension of the Julia sets tends to 2. With this result, Shishikura showed that the Hausdorff dimension of the boundary of the Mandelbrot set is 2. Perhaps, using the representation above is possible to get a result analogous of Shishikura's theorem for the boundary of the Classical Schottky space.

It is interesting to solve the extremal problem between these two objects from holomorphic dynamics. In case there exist an extremal map from $\langle R, J(R)\rangle$ into the Classical Schottky space, there would be a sort of estimate from above of the distance between this two pieces of Sullivan's dictionary.

The problem to describe the set of representations of $\langle P, A\rangle$, for an invariant set A, into $\langle \text{Aff}(\mathbb{C}), \mathbb{C}\rangle$ is difficult, still remain many questions. In the case where $S \in Rat(\mathbb{C})$, it is interesting to understand the space of representations of $\langle S, A\rangle$ into $PSL(2, \mathbb{C})$.

3.3 Binding semigroups of maps with constants.

Let us consider two semigroups of the form $S_1 = \langle g_1, A_1\rangle$ and $S_2 = \langle g_2, A_2\rangle$, in this case the categorical sum, or coproduct, $S_1 \coprod S_2$, is defined as $\langle g_1 \coprod g_2, A_1 \times \{1\} \sqcup A_2 \times \{2\}\rangle$, where $g_1 \coprod g_2$ is a map defined on the disjoint union $A_1 \times \{1\} \sqcup A_2 \times \{2\}$ by

$$g_1 \coprod g_2(x) = \begin{cases} g_1(x) & \text{if } x \in A_1 \\ g_2(x) & \text{if } x \in A_2 \end{cases}$$

Analogously, we define the binding of a countable family of semigroups of the form $\langle g_i, A_i\rangle$. A classical example of such construction in dynamics is the process of mating of quadratic polynomials, first described by Douady in [3]. We start with two mateable polynomials $S_1 = \langle P_1, \mathbb{C}\rangle$ and $S_2 = \langle P_1, \mathbb{C}\rangle$. Using a topological construction, the mating $P_1 \coprod P_2$ is a quadratic rational map $\langle R, \mathbb{C}\rangle$. Thus we have a representation of $S_1 \coprod S_2$ into the space of rational maps of degree 2.

3.3.1 Simultaneous linearizations and deformation spaces.

Let us now discuss a more elaborated example, associated to a fixed rational map R_0 of degree d. Let $\{a_0, a_1, ..., a_{n-1}\}$ be a periodic cycle of R_0, of period n and multiplier λ, with $|\lambda| > 1$. Let us denote by A_λ the map $z \mapsto \lambda z$, and ϕ the Poincaré function associated to R_0^n and a_0. As we discussed above ϕ induces a representation of $\langle A_\lambda, \mathbb{C}\rangle$ into $\langle R_0, \mathbb{C}\rangle$. The same is true for $R_0^i \circ \phi$, for each $i = 0, ..., n-1$, all together, induce a representation of the binding $\langle \coprod_{i=0}^{n-1} A_\lambda, \sqcup_{i=0}^{n-1} \mathbb{C} \times \{i\}\rangle$ into $\langle R_0, \mathbb{C}\rangle$, here we put a component $\langle A_\lambda, \mathbb{C}\rangle$ for each periodic point in the cycle $\{a_0, a_1, ..., a_{n-1}\}$. Let us carry this construction further considering all repelling periodic cycles of R_0, we obtain a countable binding of semigroups of the form $\langle A_\lambda, \mathbb{C}\rangle$ associated to all Poincaré functions of R_0. Let us call $\mathcal{A}(R_0)$ this countable binding, so we have a representation $\Psi : \mathcal{A}(R_0) \to \langle R_0, \mathbb{C}\rangle$. Taking instead of $\langle R_0, \mathbb{C}\rangle$, the corresponding Poincaré functions, we obtain a representation $\tilde{\Psi}$ from $\mathcal{A}(R_0)$ into $Mer(\mathbb{C})$. The image $\phi(\mathcal{A}(R_0))$ has a compactification which is related to Lyubich-Minsky laminations discussed in Section 7 of [8].

Let us assume that R_0 is hyperbolic of degree d. Since $\langle R_0, \mathbb{C}\rangle$ is a subsemigroup of $Rat(\mathbb{C})$, let us now regard Ψ as a homomorphism from $\mathcal{A}(R_0)$ into $Rat(\mathbb{C})$. Let $\mathcal{X}(R_0)$ be the space of representations from $\mathcal{A}(R_0)$ into $Rat(\mathbb{C})$, whose image is of the form $\langle R, \mathbb{C}\rangle$ for some R of degree d. In other, words we are considering all graduated representations that arise by deformations of the semigroup $\langle R_0, \mathbb{C}\rangle$. Let us define the map $P : \mathcal{X}(R_0) \to Rat_d(\mathbb{C})$, such that for every $\Phi \in \mathcal{X}(R_0)$, let $P(\Phi) = R$ where R is the non constant rational map generating $P(\Phi)$.

Let $Par_d(\mathbb{C})$ be the set of all rational maps, of degree d, that admit a parabolic periodic point. Then $P(\mathcal{X}(R_0))$, in $Rat_d(\mathbb{C})$, is equal to $Rat_d(\mathbb{C}) \setminus \overline{Par_d(\mathbb{C})}$. By a result of Lyubich, see [7], the space $Rat_d(\mathbb{C}) \setminus \overline{Par_d(\mathbb{C})}$ consists of the union of J-stable components in $Rat_d(\mathbb{C})$.

In [2], the authors construct a dynamical Teichmüller space $T_2(R_0)$, which uniformize the J-stable components of R_0. It turns out that the space $T_2(R_0)$ is isomorphic to $\mathcal{X}(R_0)$.

4 Correspondences.

Let A and B be two sets, let G be a subset of $A \times B$. A *correspondence* is a triple (G, A, B). If $(a, b) \in G$ we say that b corresponds to a under G. The notion of correspondences generalizes, in a way, the notion of functions. Indeed, for every map $f : X \to Y$, the graph of f induces a correspondence in $X \times Y$. Borrowing notation from Function Theory, we define the set

$$Im(G) = \{b \in B : \exists a \in A \text{ such that } (a, b) \in G\}$$

is called the *image* of G, analogously the *domain* of G is defined by

$$Dom(G) = \{a \in A : \exists b \in B \text{ such that } (a, b) \in G\}.$$

For every $b \in Im(G)$ we call $G^{-1}(b) = \{a \in A : (a, b) \in G\}$ the preimage of b under G. Similarly, the image of an element $a \in A$ is the set $G(a) = \{b \in B : (a, b) \in G\}$. Given a set $G \subset A \times B$, the set $G^{-1} = \{(b, a) \in B \times A : (a, b) \in G\}$ is called the inverse of G. Let $G_1 \subset A \times B$ and $G_2 \subset B \times C$ be two correspondences, the composition $G_1 \circ G_2$ of G_1 and G_2 is the correspondence induced by the set

$$G_2 \circ G_1 = \{(a, c) \in A \times C : \exists b \in B \text{ such that } (a, b) \in G_1 \text{ and } (b, c) \in G_2\}.$$

Let X be a set, a correspondence K in X is a correspondence of the form (K, X, X), additionally we require that $Dom(K) = X$. A correspondence K in

X is called *surjective* if $Im(K) = X$ and, *finite* if every image is a finite set. In particular, constant maps are finite correspondences.

If G is a finite correspondence, the degree of the image of G is the maximum of the cardinalities of its images.

4.1 Schreier's Lemma for correspondences.

With composition, the set of correspondences $Corr(X)$ in X is a semigroup. Since functions are special cases of correspondences, the semigroup of correspondences of X contains $Map(X)$. The proof of the following lemma is immediate by contradiction.

Lemma 18. *Let K_1 and K_2 be two correspondences in $Corr(X)$ such that $g = K_1 \circ K_2$ is a map and K_2 is surjective, then K_1 is a map.*

We will start by generalizing Schreier's lemma restricted to correspondences generated by maps. Let us start with some definitions,

Definition. *A correspondence K in a set X is called a* block *if K has the form $R_1 \circ R_2^{-1}$, where R_1 and R_2 belong to $Map(X)$ and R_2 is surjective.*

We denote by $\mathcal{BL}(X)$ the subsemigroup of $Corr(X)$, generated by all block correspondences.

Theorem 19 (Schreier Lemma for blocks)**.** *Let $\phi : \mathcal{BL}(X) \to \mathcal{BL}(Y)$ be an homomorphism then, there exist $f \in Map(X, Y)$ such that for every $K \in \mathcal{BL}(X)$ we have $\phi(K) = f \circ K \circ f^{-1}$.*

Proof. The identity Id is characterized among $Corr(X)$ by the properties $Id \circ Id = Id$ and that for every $C \in Corr(X)$ we have $Id \circ C = C \circ Id = C$. Since this properties are preserved by homomorphisms we have $\Phi(Id) = Id$. Let $R \in Corr(X)$ be any map, then $R \circ R^{-1} = Id$. But then $\Phi(R \circ R^{-1}) = \Phi(R) \circ \phi(R^{-1}) = Id$ is a map, by Lemma 18 then $\Phi(R)$ is a map. Hence Φ sends maps into maps, so ϕ restricted to $Map(X)$ is a homomorphism of semigroups. By Lemma 4, there exist $f \in Map(X, Y)$ such that, for every map R, $\Phi(R) \circ f = f \circ R$. Then $\Phi(R) = f \circ R \circ f^{-1}$ for all maps R.

Since blocks generate $\mathcal{BL}(X)$, it is enough to check that the theorem holds for every correspondence of the form $K = R_1^{-1}$, where R_1 is a map. Since $R_1 \circ R_1^{-1} = Id$ we have

$$\Phi(R_1 \circ R_1^{-1}) = \Phi(R_1) \circ \Phi(R_1^{-1}) = Id$$

on the other hand,

$$\Phi(R_1) = f \circ R_1^{-1}$$

then

$$f \circ R_1 \circ f^{-1} \circ \Phi(R_1^{-1}) = Id$$

it follows that

$$\Phi(R_1^{-1}) = f \circ R_1^{-1} \circ f^{-1}$$

and then for every block K, $\Phi(K) = f \circ K \circ f^{-1}$ as we wanted to prove. ∎

We now include in the discussion the constant maps in $Corr(X)$, these are no longer an ideal, but we can consider the unique *minimal left ideal* $\mathcal{I}$ in $Corr(X)$, which is generated by all constant maps. The semigroup of correspondences acts on $\mathcal{I}$. That is, there is a map $\alpha : Corr(X) \to Map(\mathcal{I})$ that sends every correspondence $K \in Corr(X)$ to the left translation by K in $Map(\mathcal{I})$.

Lemma 20. *The map $\alpha : Corr(X) \to Map(\mathcal{I})$ is a one-to-one map. Moreover, for every $c \in \mathcal{I}$, we have $\alpha(c) = c$.*

Proof. Suppose that K_1 and K_2 are correspondences in $Corr(X)$ such that $\alpha(K_1) = \alpha(K_2)$. In particular, for every constant $c \in \mathcal{I}$, we have $K_1 \circ c = K_2 \circ c$. However, a correspondence is characterized by the set of images, then $K_1 = K_2$. The second part of the Lemma follows from the equation $c \circ K = c$ for all $c \in \mathcal{I}$. ☐

Now we are set to prove:

Theorem 21. *[Schreier's Lemma for correspondences] Let*

$$\Phi : Corr(X) \to Corr(Y)$$

be a homomorphism of semigroups. Then, there is a map $f \in Map(X, Y)$, such that, for every $K \in Corr(X)$ we have $\Phi(K) = f \circ K \circ f^{-1}$.

Proof. By the same argument in the proof of Theorem 19, the map Φ sends maps to maps. Moreover, the restriction of Φ to $\mathcal{BL}(X)$, is an homomorphism from $\mathcal{BL}(X)$ to $\mathcal{BL}(Y)$, by Theorem 19 there is $f \in Map(X, Y)$ such that for every $K \in \mathcal{BL}(X)$, we have that $\phi(K) = f \circ K \circ f^{-1}$.

Let $\mathcal{I}$ and $\mathcal{J}$ denote the minimal ideals in $Corr(X)$ and $Corr(Y)$, respectively. Let us consider the maps $\alpha_X : Corr(X) \to Map(\mathcal{I})$, and $\alpha_Y : Corr(Y) \to Map(\mathcal{J})$ as in Lemma 20, and define $S_X = \alpha(Corr(X))$ and $S_Y = \alpha(Corr(Y))$. By Lemma 20, the maps α_X and α_Y are bijections to their images. Moreover, α_X and α_Y send constants to constants. Hence the map $\alpha_Y \circ \phi \circ \alpha_X^{-1}$ is a homomorphism between the semigroups G_X and G_Y, sending constants to constants. By Corollary 5, there exist $F \in Map(\mathcal{I}, \mathcal{J})$ such that for every $g \in G$, we have

$$\alpha_Y \circ \phi \circ \alpha_X^{-1}(g) = F \circ g \circ F^{-1}.$$

Since α, restricted to minimal ideal is the identity, then for every c in $\mathcal{I}$ and every correspondence $K \in Corr(X)$ we have $F(c) = f(c)$, also

$$\alpha_Y(K) \circ c = K \circ c,$$

and

$$\alpha_X^{-1}(c) = c$$

evaluating in c the equation above, we get

$$(\alpha_Y \circ \phi(K) \circ \alpha_X^{-1}) \circ c = \phi(K) \circ c$$

$$= F \circ K \circ F^{-1} \circ c.$$

But then $\phi(K) = f \circ K \circ f^{-1}$.

$\square$

Note that in the proof of Theorem 21, we need the theorem on block correspondences to get the existence of the map f. Once we have Schreier's lemma for the whole semigroup of correspondences, we can generalize it for subsemigroups of correspondences, as long as they contain the minimal ideal of constants.

Corollary 22. *Let S_1 and S_2 be subsemigroups of $Corr(X)$ and $Corr(Y)$, respectively, such that $X_1 = S_1 \cap X$ and $Y_1 = S_2 \cap Y$ are both non empty. If $\phi : S_1 \to S_2$ is a homomorphism of semigroups, then there is $f : X_1 \to Y_1$, such that for all $K \in S_1$, $\phi(K) = f \circ K \circ f^{-1}$. Moreover,*

- *the homomorphism ϕ is injective, or surjective, if and only if, the map f is injective or surjective. In particular, ϕ is an isomorphism if, and only if, f is a bijection.*
- *When S_1 and S_2 are topological semigroups, then Φ is continuous if, and only if, f is continuous.*

4.2 Holomorphic correspondences.

A correspondence K in $\mathbb{C}$ is *holomorphic* if, as a set of $\mathbb{C} \times \mathbb{C}$, K can be decomposed as a countable union of analytic varieties, see McMullen's book [11]. However recall that, in our setting, we require that $Dom(K) = \mathbb{C}$. Moreover, we assume that the preimage of every point admits an analytic extension to the whole Riemann sphere, with the exception of finitely many points. Let us denote by $HCorr(\mathbb{C})$ the semigroup of holomorphic correspondences, which includes the semigroup of entire maps and constants. We denote by $FHCorr(\bar{\mathbb{C}})$, the semigroup of finite correspondences on the Riemann sphere. By definition $FHCorr(\bar{\mathbb{C}})$ contains the semigroup of rational maps $Rat(\mathbb{C})$ together with all constant maps. Hence, there exist a minimal left ideal of finite holomorphic correspondences. Since rational maps are onto the Riemann sphere, if R_1 and R_2 are rational maps, the block $R_1^{-1} \circ R_2$ belongs to $FHCorr(\mathbb{C})$.

Let $K \in FHCorr(\bar{\mathbb{C}})$, a holomorphic correspondence with degree d. That is there is z such that $K(z)$ consists of d points. Let $S_1, S_2, ..., S_d$ denote all the symmetric polynomials with d variables. For every i, $S_i(K)$ induces a holomorphic map from $\bar{\mathbb{C}}$ to $\bar{\mathbb{C}}$, it follows that $S_i(K)$ is a rational map. Moreover, for every z the image $K(z)$ are the roots of the polynomial

$$S_1(K(z)) + S_2(K(z))Z + ... + S_d(K(z))Z^{d-1} + Z^d.$$

Reciprocally any polynomial in Z, whose coefficients are rational maps in z, defines a finite holomorphic correspondence in $\bar{\mathbb{C}}$. From this discussion, we have the following known fact.

Proposition 23. *The space $FHCorr(\bar{\mathbb{C}})$ is equivalent to the space of monic polynomials with coefficients in $Rat(\mathbb{C})$.*

The proof of Theorem 21, can be repeated in the setting of holomorphic correspondences. In this case, every automorphism of $HCorr(\mathbb{C})$ or $FHCorr(\bar{\mathbb{C}})$ is induced by conjugation of some function in $Bij(\mathbb{C})$ or $Bij(\bar{\mathbb{C}})$. Nevertheless, the holomorphic structure imposes holomorphic conditions in such bijections.

Theorem 24. *The following statements are true*

- *Every automorphism of $HCorr(\mathbb{C})$ is continuous. Moreover,*

$$Aut(HCorr(\mathbb{C})) \simeq \mathrm{Aff}(\mathbb{C}).$$

- *The action of $Gal(\mathbb{C})$ extends to an action in $FHCorr(\bar{\mathbb{C}})$. In fact,*

$$Aut(FHCorr(\bar{\mathbb{C}})) \simeq \langle PSL(2,\mathbb{C}), Gal(\mathbb{C})\rangle.$$

Proof. The semigroup of maps in $HCorr(\mathbb{C})$ coincides with the semigroup of entire maps. The first part of the theorem is a consequence of Corollary 11. Since the semigroup of maps in $FHCorr(\bar{\mathbb{C}})$ is equal to $Rat(\bar{\mathbb{C}})$. By restriction, any automorphism of $FHCorr(\bar{\mathbb{C}})$ induces an automorphism of $Rat(\bar{\mathbb{C}})$. But every automorphism of $Rat(\bar{\mathbb{C}})$ is generated by $PSL(2,\mathbb{C})$ and $Gal(\mathbb{C})$. Now let us see that, in fact, $Gal(\mathbb{C})$ also acts on $FHCorr(\bar{\mathbb{C}})$. Let γ an element in $Gal(\mathbb{C})$, and let $K \in FHCorr(\bar{\mathbb{C}})$, then $\gamma \circ K \circ \gamma^{-1}$ is a finite correspondence in $Corr(\bar{\mathbb{C}})$. Let d be the maximum cardinality of a fiber of K. Remind that K is holomorphic in the Riemann sphere if, and only if, there is a symmetric polynomial S_d in d variables, such that $S_d(K)$ is a rational map in $\bar{\mathbb{C}}$. Since γ acts on symmetric polynomials, there is a symmetric polynomial $\tilde{S}_d$ such that

$$\tilde{S}_d(K) = \gamma \circ S_d(K) \circ \gamma^{-1} = S_d(\gamma \circ K \circ \gamma^{-1})$$

But the second equality is the conjugation of a rational map by a Galois map, hence is rational. This implies that $\gamma \circ K \circ \gamma^{-1}$ is a holomorphic correspondence. It follows that the group of automorphisms of $FHCorr(\bar{C})$ is isomorphic to the group of automorphisms of $Rat(\mathbb{C})$, which by Proposition 8 is isomorphic to $\langle PSL(2,\mathbb{C}), Gal(\mathbb{C})\rangle$. □

The central argument for Theorem 14 is Schreier's Lemma, with some modifications we can prove the corresponding theorem for holomorphic correspondences.

Theorem 25. *Let K_1 and K_2 be two holomorphic correspondences. Let*

$$\Phi : Corr_{K_1}(\mathbb{C}) \to Corr_{K_2}(\mathbb{C})$$

be an isomorphism of sandwich semigroups. Then there is $f \in Bij(\mathbb{C})$, and $B \in \mathrm{Aff}(\mathbb{C})$, such that $\Phi(P) = f \circ P \circ f^{-1} \circ B^{-1}$.

It is not clear whether the Galois group action acts on holomorphic correspondences. Perhaps there is a generalization to Hinkkanen's argument in this setting. Now, we can state an analogous statement to Theorem 13 for $FHCorr(\bar{\mathbb{C}})$.

Corollary 26. *Let F be an element in $Bij(\bar{\mathbb{C}})$, that fixes $0, 1$ and ∞. Then F belongs to $Gal(\mathbb{C})$ if, and only if, F induces an automorphism of $FHCorr(\bar{\mathbb{C}})$.*

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