

Vector valued Jack polynomials from scratch

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Abstract

Vector valued Jack polynomials associated to the symmetric group \mathfrak{S}_N are polynomials with multiplicities in an irreducible module of \mathfrak{S}_N and which are simultaneous eigenfunctions of the Cherednik-Dunkl operators with some additional properties concerning the leading term. These polynomials were introduced by Griffeth in the general setting of the complex reflections groups $G(r, p, N)$ and studied by one of the authors (C. Dunkl) in the specialization $r = p = 1$ (*ie.* for the symmetric group).

By adapting a construction due to Lascoux, we describe an algorithm allowing us to compute explicitly the Jack polynomials following a Yang-Baxter graph. We recover some properties already studied by C. Dunkl and restate them in terms of graphs together with additional new results. In particular, we investigate normalization, symmetrization and antisymmetrization, polynomials with minimal degree, restriction *etc.*. We give also a shifted version of the construction and we discuss about vanishing properties of the associated polynomials.

1 Introduction

The Yang-Baxter graphs introduced by Lascoux [13] can be used to study vector-valued Jack polynomials. These are associated with irreducible representations of the symmetric group \mathfrak{S}_N , that is, to partitions of N . A Yang-Baxter graph is a directed graph with no loops and a unique root, whose

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edges are labeled by generators of a certain subsemigroup of the extended affine symmetric group. In this paper the vertices are labeled by a pair consisting of a composition and the content vector of a standard tableau. The compositions are the labels of monomials which are the leading terms of polynomials, and the tableaux all have the same shape. There is a vector-valued Jack polynomial associated with each vertex. These polynomials are special cases of the polynomials introduced by Griffeth [8] for the family of complex reflection groups denoted by $G(r, p, N)$ (where $p|r$). This is the group of unitary $N \times N$ matrices such that their nonzero entries are r^{th} roots of unity, the product of the nonzero entries is a $(p/r)^{\text{th}}$ root of unity, and there is exactly one nonzero entry in each row and each column. The symmetric group is the special case $G(1, 1, N)$. The vector space in which the Jack polynomials take their values is equipped with the nonnormalized basis described by Young, namely, the simultaneous eigenvectors of the Jucys-Murphy elements.

The labels on edges denote transformations to be applied to the objects at a vertex. Vector-valued Jack polynomials are uniquely determined by their spectral vector, the vector of eigenvalues under the (pairwise commuting) Cherednik-Dunkl operators. This serves to demonstrate the claim that different paths from one vertex to another produce the same result, a situation which is linked to the braid or Yang-Baxter relations. These refer to the transformations.

Following Lascoux [13] we define the monoid $\widehat{\mathfrak{S}}_N$, a subsemigroup of the affine symmetric group, with generators $\{s_1, s_2, \dots, s_{N-1}, \Psi\}$ and relations:

$$\begin{aligned} s_i s_j &= s_j s_i, |i - j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, 1 \leq i < N - 1, \\ s_1 \Psi^2 &= \Psi^2 s_{N-1}, \\ s_i \Psi &= \Psi s_{i-1}, 2 \leq i \leq N - 1. \end{aligned}$$

The relations $s_i^2 = 1$ do not appear in this list because the graph has no loops.

The main objects of our study are polynomials in $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ with coefficients in $\mathbb{Q}(\alpha)$, where α is a transcendental (indeterminate), and with values in the \mathfrak{S}_N -module corresponding to a partition λ of N . The Yang-Baxter graph G_λ is a pictorial representation of the algorithms which produce the Jack polynomials starting with constants. The generators of $\widehat{\mathfrak{S}}_N$ correspond to transformations taking a Jack polynomial to an adjacent one. At each vertex there is such a polynomial, and a 4-tuple which identifies it.

The 4-tuple consists of a standard tableau denoting a basis element of the \mathfrak{S}_N -module, a composition (multi-index) describing the leading term of the polynomial, a spectral vector, and a permutation, essentially the rank function of the composition. The spectral vector and permutation are determined by the first two elements. For technical reasons the standard tableaux are actually reversed, that is, the entries decrease in each row and each column. This convention avoids the use of a reversing permutation, in contrast to Griffeth's paper [8] where the standard tableaux have the usual ordering.

The symmetric and antisymmetric Jack polynomials are constructed in terms of certain subgraphs of G_λ . Furthermore the graph technique leads to the definition and construction of shifted inhomogeneous vector-valued Jack polynomials.

Here is an outline of the contents of each section.

Section 2 contains the basic definitions and construction of the graph G_λ . The presentation is in terms of the 4-tuples mentioned above. It is important to note that not every possible label need appear on edges pointing away from a given vertex: if the composition at the vertex is $v \in \mathbb{N}^N$ then the transposition $(i, i + 1)$ (labeled by s_i) can be applied only when $v[i] \leq v[i + 1]$, that is, when the resulting composition is greater than or equal to v in the dominance order. The action of the affine element Ψ is given by $v \cdot \Psi = (v[2], v[3], \dots, v[N], v[1] + 1)$. Vertices of G_λ satisfying certain conditions may be mapped to vertices of a graph related to \mathfrak{S}_M , $M < N$, by a restriction map. This topic is the subject of Section 2.4.

The Murphy basis for the irreducible representation of \mathfrak{S}_N along with the definition of the action of the simple reflections $(i, i + 1)$ on the basis is presented in Section 3. Also the vector-valued polynomials, their partial ordering, and the Cherednik-Dunkl operators are introduced here.

Section 4 is the detailed development of Jack polynomials. Each edge of the graph G_λ determines a transformation that takes the Jack polynomial associated with the beginning vertex to the one at the ending vertex of the edge. There is a canonical pairing defined for the vector-valued polynomials; the pairing is nonsingular for generic α and the Cherednik-Dunkl operators are self-adjoint. The Jack polynomials are pairwise orthogonal for this pairing and the squared norm of each polynomial can be found by use of the graph. Also in this section one finds the method of producing coefficients so that the corresponding sum of Jack polynomials is symmetric or antisymmetric. The idea is explained in terms of certain subgraphs of G_λ . This section also describes the restriction map.

In Section 5 the shifted vector-valued Jack polynomials are presented. These are inhomogeneous and the parts of highest degree coincide with the homogeneous Jack polynomials of the previous section. The construction again uses the Yang-Baxter graph G_λ ; it is only necessary to change the operations associated with the edges.

Throughout the paper there are numerous figures to concretely illustrate the structure of the graphs.

2 Yang-Baxter type graph associated to a partition

2.1 Sorting a vector

Consider a vector $v \in \mathbb{N}^N$, we want to compute the unique decreasing partition v^+ , which is in the orbit of v for the action of the symmetric group \mathfrak{S}_N acting on right on the position, using the minimal number of elementary transpositions $s_i = (i \ i + 1)$.

If v is a vector we will denote by $v[i]$ its i th component. Each $\sigma \in \mathfrak{S}_N$ will be associated to the vector of its images $[\sigma(1), \dots, \sigma(N)]$. Let σ be a permutation, we will denote $\ell(\sigma) = \min\{k : \sigma = s_{i_1} \dots s_{i_k}\}$ the length of the permutation. By a straightforward induction one finds:

Proposition 2.1 *Let $v \in \mathbb{N}^N$ be a vector, there exists a unique permutation σ_v such that $v = v^+ \cdot \sigma_v$ with $\ell(\sigma_v)$ minimal.*

The permutation σ_v is obtained by a standardization process : we label with integer from 1 to N the positions in v from the largest entries to the smallest one and from left to right.

Example 2.2 Let $v = [2, 3, 3, 1, 5, 4, 6, 6, 1]$, the construction gives:

$$\begin{array}{r} \sigma_v = [\quad 7 \quad 5 \quad 6 \quad 8 \quad 3 \quad 4 \quad 1 \quad 2 \quad 9 \quad] \\ v = [\quad 2 \quad 3 \quad 3 \quad 1 \quad 5 \quad 4 \quad 6 \quad 6 \quad 1 \quad] \end{array}$$

We verify that $v \cdot \sigma_v^{-1} = [6, 6, 5, 4, 3, 3, 2, 1, 1] = v^+$.

The computation of σ_v is compatible with the action of \mathfrak{S}_N in the following sense:

Proposition 2.3 1. $\sigma_{v.s_i} = \begin{cases} \sigma_v & \text{if } v = v.s_i \\ \sigma_{v.s_i} & \text{otherwise} \end{cases}$

2. If $v[i] = v[i + 1]$ then $\sigma_{v.s_i} = s_{\sigma_v[i]}. \sigma_v$.

Proof Straightforward from the construction. \square

Define the affine operation Ψ acting on vector by

$$[v_1, \dots, v_N].\Psi = [v_2, \dots, v_N, v_1 + 1].$$

Again, one can prove easily that the computation of σ_v is compatible (in a certain sense) with the action of Ψ :

Proposition 2.4

$$\sigma_{v\Psi} = \sigma_v.[2, \dots, N, 1].$$

Example 2.5 Consider $v = [2, 3, 3, 2, 5, 4, 6, 6, 1]$, one has

$$\sigma_v = [7, 5, 6, 8, 3, 4, 1, 2, 9]$$

and

$$\sigma_v.[2, 3, 4, 5, 6, 7, 8, 9, 1] = [5, 6, 8, 3, 4, 1, 2, 9, 7].$$

But

$$v' := v.[2, 3, 4, 5, 6, 7, 8, 9, 1] = [3, 3, 2, 5, 4, 6, 6, 1, 2]$$

and $\sigma_{v'} = [5, 6, \underline{7}, 3, 4, 1, 2, 9, \underline{8}]$; here the underlined integers means that there is a difference with the same position in $\sigma_v.[2, 3, 4, 5, 6, 7, 8, 9, 1]$. This is due to the fact that $v[1]$ is the first occurrence of 2 in v whilst $v'[9]$ is the last occurrence of 2 in v' . Adding 1 to $v'[9]$ one obtains

$$v.\Psi = [3, 3, 2, 5, 4, 6, 6, 1, 3].$$

The last occurrence of 2 becomes the last occurrence of 3 (that is the number of the first occurrence of 2 minus 1). Hence,

$$\sigma_{v.\Psi} = [5, 6, 8, 3, 4, 1, 2, 9, 7] = \sigma_v.[2, 3, 4, 5, 6, 7, 8, 9, 1].$$

For convenience, we will define the operator Ψ^α by

$$[v_1, \dots, v_N].\Psi^\alpha = [v_2, \dots, v_N, v_1 + \alpha].$$

2.2 Construction and basic properties of the graph

Definition 2.6 A **tableau** of shape λ is a filling with integers weakly increasing in each row and in each column. In the sequel **row-strict** means increasing in each row and **column-strict** means increasing in each column. A **reverse standard tableau** (RST) is obtained by filling the shape λ with integers $1, \dots, N$ and with the conditions of strictly decreasing in the line and the column. We will denote by Tab_λ , the set of the RST with shape λ . Let τ be a RST, we define the **vector of contents** of τ as the vector CT_τ such that $\text{CT}_\tau[i]$ is the **content** of i in τ (that is the number of the diagonal in which i appears; the number of the main diagonal is 0, and the numbers decrease from down to up).

Example 2.7

$$\text{CT} \begin{array}{c} 2 \\ 5 \\ 6 \end{array} \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} \\ \\ 1 \end{array} = [2, -2, 1, 0, -1, 0]$$

We construct a Yang-Baxter-type graph with vertices labeled by 4-tuples (τ, ζ, v, σ) , where τ is a RST, ζ is a vector with entries in $\mathbb{Z}[\alpha]$ (ζ will be called the **spectral vector**), $v \in \mathbb{N}^N$ and $\sigma \in \mathfrak{S}_N$, as follows: First, consider a RST of shape λ and write a vertex labeled by the 4-tuple $(\tau, \text{CT}_\tau, 0^N, [1, \dots, N])$. Now, we consider the action of the elementary transposition of \mathfrak{S}_N on the 4-tuple given by

$$(\tau, \zeta, v, \sigma).s_i = \begin{cases} (\tau, \zeta.s_i, v.s_i, \sigma.s_i) & \text{if } v[i+1] \neq v[i] \\ (\tau^{(\sigma[i], \sigma[i+1])}, \zeta.s_i, v, \sigma) & \text{if } v[i] = v[i+1] \text{ and } \tau^{(\sigma[i], \sigma[i+1])} \in \text{Tab}_\lambda \\ (\tau, \zeta, v, \sigma) & \text{otherwise,} \end{cases}$$

where $\tau^{(i,j)}$ denotes the filling obtained by permuting the values i and j in τ .

Consider also the affine action given by

$$(\tau, \zeta, v, \sigma).\Psi = (\tau, \zeta.\Psi^\alpha, v.\Psi, \sigma.[2, \dots, N, 1]).$$

Example 2.8

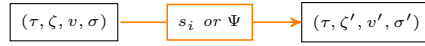
1. $\left(\begin{smallmatrix} 31 \\ 542 \end{smallmatrix}, [1, 0, 2\alpha, \alpha + 2, \alpha - 1], [0, 0, 2, 1, 1], [45123] \right) .s_4 =$
 $\left(\begin{smallmatrix} 21 \\ 543 \end{smallmatrix}, [1, 0, 2\alpha, \alpha - 1, \alpha + 2], [0, 0, 2, 1, 1], [45123] \right)$

2. $\binom{31}{542}, [1, 0, 2\alpha, \alpha + 2, \alpha - 1], [0, 0, 2, 1, 1], [45123]] .s_2 = \binom{31}{542}, [1, 2\alpha, 0, \alpha + 2, \alpha - 1], [0, 2, 0, 1, 1], [41523]]$
3. $\binom{31}{542}, [1, 0, 2\alpha, \alpha + 2, \alpha - 1], [0, 0, 2, 1, 1], [45123]] .s_1 = \binom{31}{542}, [1, 0, 2\alpha, \alpha + 2, \alpha - 1], [0, 0, 2, 1, 1], [45123]]$
4. $\binom{31}{542}, [1, 0, 2\alpha, \alpha + 2, \alpha - 1], [0, 0, 2, 1, 1], [45123]] .\Psi = \binom{31}{542}, [0, 2\alpha, \alpha + 2, \alpha - 1, \alpha + 1], [0, 2, 1, 1, 1], [51234]]$

Definition 2.9 *The graph G_λ is an infinite directed graph constructed from the 4-tuple $(\tau_\lambda, \text{CT}_{\tau_\lambda}, [0^N], [1, 2, \dots, N])$ (called the **root**) where τ_λ is obtained by filling the shape λ from bottom to top and left to right by the integers $\{1, \dots, N\}$ in the decreasing order and adding vertices and edges following the rules*

1. *We add an arrow labeled by s_i from the vertex (τ, ζ, v, σ) to $(\tau', \zeta', v', \sigma')$ if $(\tau, \zeta, v, \sigma).s_i = (\tau', \zeta', v', \sigma')$ and $v[i] < v[i+1]$ or $v[i] = v[i+1]$ and τ is obtained from τ' by interchanging the position of two integers $k < \ell$ such that k is at the south-east of ℓ (ie. $\text{CT}_\tau(k) \geq \text{CT}_\tau(\ell) + 2$).*
2. *We add an arrow labeled by Ψ from the vertex (τ, ζ, v, σ) to $(\tau', \zeta', v', \sigma')$ if $(\tau, \zeta, v, \sigma).\Psi = (\tau', \zeta', v', \sigma')$*
3. *We add an arrow s_i from the vertex (τ, ζ, v, σ) to \emptyset if $(\tau, \zeta, v, \sigma).s_i = (\tau, \zeta, v, \sigma)$.*

An arrow of the form



will be called a **step**. The other arrows will be called **jumps**, and in particular an arrow

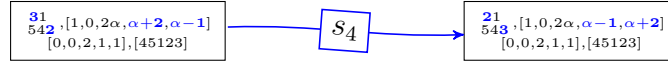


will be called a **fall**; the other jumps will be called **correct jumps**.

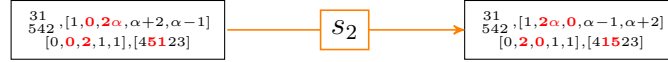
As usual a **path** is a succession of arrows in G_λ starting from the root and is denoted by the succession of the labels of its arrows. Two paths $\mathfrak{P}_1 = (a_1, \dots, a_k)$ and $\mathfrak{P}_2 = (b_1, \dots, b_\ell)$ are said to be **equivalent** (denoted by $\mathfrak{P}_1 \equiv \mathfrak{P}_2$) if they lead to the same vertex.

Remark that from proposition 2.3, in the case $v[i] = v[i+1]$, the part 1 of definition 2.9 is equivalent to the following statement: τ' is obtained from τ by interchanging $\sigma_v[i]$ and $\sigma_v[i+1] = \sigma_v[i] + 1$ where $\sigma_v[i]$ is to the south-east of $\sigma_v[i] + 1$, that is, $CT_v[\sigma_v[i]] - CT_v[\sigma_v[i] + 1] \geq 2$.

Example 2.10 The following arrow is a correct jump

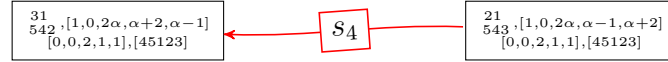


whilst

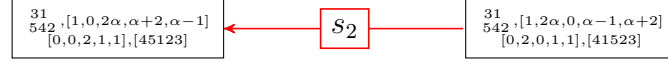


is a step.

The arrows



and



are not allowed.

Example 2.11 Consider the partition $\lambda = 21$, the graph G_{21} in Fig. 3 is obtained from the 4-tuple $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, (-1, 1, 0), (0, 0, 0), 1 \right)$ by applying the rules of def 2.9. In Fig. 3, the steps are drawn in orange, the jumps in blue and the falls have been omitted.

For a reverse standard tableau τ of shape λ , a partition of N , let

$$\text{inv}(\tau) = \#\{(i, j) : 1 \leq i < j \leq N, \text{rw}(i, \tau) > \text{rw}(j, \tau)\},$$

where $\text{rw}(i, \tau)$ is the row of τ containing i (we will denote also by $\text{cl}(i, \tau)$ the column containing i). Then a correct jump from τ to τ' implies $\text{inv}(\tau') = \text{inv}(\tau) + 1$ (the entries $\sigma[i]$ and $\sigma[i+1] = \sigma[i] + 1$ are interchanged in τ to produce τ'). Thus the number of correct jumps in a path from the root to (τ, ζ, v, σ) equals $\text{inv}(\tau) - \text{inv}(\tau_\lambda)$.

So we consider the number of steps in a path from 0^N to v ;¹ recall that one step links v to v' where either $v[i] < v[i+1]$ and $v' = v.s_i$ or $v' = v.\Psi$.

For $x \in \mathbb{Z}$ (or \mathbb{R}) let $\epsilon(x) := \frac{1}{2}(|x| + |x+1| - 1)$, then $\epsilon(x) = x$ for $x \geq 0$, $\epsilon(x) = 0$ for $-1 \leq x \leq 0$, and $\epsilon(x) = -x - 1$ for $x \leq -1$. There is a symmetry relation: $\epsilon(x) = \epsilon(-x - 1)$.

Definition 2.12 For $v \in \mathbb{N}^N$ let $|v| := \sum_{i=1}^N v[i]$ and set

$$S(v) := \sum_{1 \leq i < j \leq N} \epsilon(v[i] - v[j]).$$

The above formula can be written as

$$S(v) = \frac{1}{2} \sum_{1 \leq i < j \leq N} (|v[i] - v[j]| + |v[i] - v[j] + 1|) - \frac{N(N-1)}{4}.$$

Proposition 2.13 The number of steps in any path joining 0^N to v equals $|v| + S(v)$.

Proof The base point satisfies $|0^N| = 0$ and $S(0^N) = 0$. Consider a step of the form $v' = v.s_m$, then $|v'| = |v|$, $v[m+1] - v[m] \geq 1$ and $S(v') - S(v)$ involves only the pair $(m, m+1)$ in the sum over all pairs (i, j) , $1 \leq i < j \leq N$. Indeed

$$\begin{aligned} S(v') - S(v) &= \epsilon(v[m+1] - v[m]) - \epsilon(v[m] - v[m+1]) \\ &= (v[m+1] - v[m]) - (-v[m] + v[m+1] - 1) \\ &= 1. \end{aligned}$$

It remains to show that $S(v.\Psi) = S(v)$ (because $|v.\Psi| = |v| + 1$). Note $(v.\Psi)[N] = v[1] + 1$. Then

$$\begin{aligned} S(v) - S(v.\Psi) &= \sum_{j=2}^N \epsilon(v[1] - v[j]) - \sum_{i=2}^N \epsilon(v[i] - v[1] - 1) \\ &= \sum_{j=2}^N \epsilon(v[1] - v[j]) - \epsilon(v[j] - v[1] - 1) \\ &= 0. \end{aligned}$$

This complete the proof. \square

As a straightforward consequences, proposition 2.13 implies

¹We omitted to mention the others components of the label of the vertices.

Corollary 2.14 *All the paths relying two given vertices in G_λ have the same length.*

This suggests that some properties could be shown by induction on the length of the path relying the root to a given vertex.

For a given 4-tuple (τ, ζ, v, σ) the values of ζ and σ depend only of those of τ and v , as shown by the following proposition.

Proposition 2.15 *If (τ, ζ, v, σ) is a vertex in G_λ , then $\sigma = \sigma_v$ and $\zeta[i] = v_i\alpha + CT_\tau[\sigma[i]]$. We will set $\zeta_{v,\tau} := \zeta$.*

Proof Let us proof the result by an induction on the length k of a path (a_1, \dots, a_k) (from corollary 2.14 all the paths have the same length) from the root to (τ, ζ, v, σ) and set

$$(\tau', \zeta', v', \sigma') = (\tau_\lambda, CT_{\tau_\lambda}, 0^N, [1, \dots, N]).a_1 \dots a_{k-1}.$$

Suppose that $a_k = \Psi$ is the affine operation. More precisely, $\tau = \tau'$, $\zeta = \zeta'.\Psi^\alpha$, $v = v'.\Psi$ and $\sigma = \sigma'.[2, \dots, N, 1]$. Using the induction hypothesis one has $\sigma' = \sigma_{v'}$ and $\zeta'[i] = v'[i]\alpha + CT_{\tau'}[\sigma_{v'}[i]]$. Hence, proposition 2.4 gives $\sigma = \sigma_{v'\Psi} = \sigma_v$. Suppose that $i < N$ then

$$\zeta[i] = \zeta'[i+1] = v'[i+1]\alpha + CT_{\tau'}[\sigma_{v'}[i+1]] = v[i]\alpha + CT_\tau[\sigma_v[i]].$$

If $i = N$ then again

$$\zeta[i] = \zeta[1] + a = (v'[1] + 1)\alpha + CT_{\tau'}[\sigma_{v'}[1]] = v[N]\alpha + CT_\tau[\sigma_v[N]].$$

Suppose now that a_k is not an affine operation. Using the induction hypothesis one has $\sigma' = \sigma_{v'}$ and $\zeta'[j] = v'[j]\alpha + CT_{\tau'}[\sigma_{v'}[j]]$ for each j . If $a_k = s_i$ is a step then $\tau = \tau'$, $\zeta = \zeta'.s_i$, $v = v'.s_i$ and $\sigma = \sigma'.s_i$. Hence, proposition 2.3 gives $\sigma = \sigma_{v'.s_i} = \sigma_v$. If $j \neq i, i+1$ then one has $\zeta[j] = \zeta'[j]$, $v[j] = v'[j]$ and $\sigma[j] = \sigma'[j]$, hence $\zeta[j] = v[j]\alpha + CT_\tau[\sigma_v[j]]$. If $j = i$ then one has $\zeta[i] = \zeta'[i+1]$, $v[i] = v'[i+1]$ and $\sigma[i] = \sigma'[i+1]$, and again the result is straightforward. And similarly when $j = i+1$ one finds the correct value for $\zeta[i+1]$.

Suppose now that $a_k = s_i$ is a jump. That is $\tau = \tau'^{(\sigma[i], \sigma[i+1])}$, $\zeta = \zeta'.s_i$, $v = v'$ and $\sigma = \sigma'$. Straightforwardly, $\sigma = \sigma_{v'} = \sigma_v$ and if $j \neq i, i+1$ then

$$\zeta[j] = \zeta'[j] = v'[j]\alpha + CT_{\tau'}[\sigma_{v'}[j]] = v[j]\alpha + CT_\tau[\sigma_v[j]].$$

Suppose that $j = i$, since a_k is a jump $v'[i] = v'[i + 1]$ and

$$\zeta[i] = \zeta'[i + 1] = v'[i + 1]\alpha + \text{CT}_{\tau'}[\sigma_{v'}[i + 1]] = v[i]\alpha + \text{CT}_{\tau}[\sigma_v[i]].$$

Similarly, when $j = i + 1$, one obtains the correct value for $\zeta[j]$. This ends the proof. \square

Example 2.16 Consider the RST $\tau = \begin{matrix} & & & 3 \\ 7 & 4 & 1 & \\ & 8 & 6 & 5 & 2 \end{matrix}$ and the vector $v = [6, 2, 4, 2, 2, 3, 1, 4]$. One has $\sigma_v = [15267483]$ and $\text{CT}_{\tau} = [1, 3, -2, 0, 2, 1, -1, 0]$ and then

$$\zeta_{v,\tau} = [6\alpha + 1, 2\alpha + 2, 4\alpha + 3, 2\alpha + 1, 2\alpha - 1, 3\alpha, \alpha, 4\alpha - 2].$$

Hence, the 4-tuple

$$\left(\begin{matrix} 3 \\ 7 & 4 & 1 \\ 8 & 6 & 5 & 2 \end{matrix}, [6\alpha + 1, 2\alpha + 2, 4\alpha + 3, 2\alpha + 1, 2\alpha - 1, 3\alpha, \alpha, 4\alpha - 2], [6, 2, 4, 2, 2, 3, 1, 4], [15267483] \right)$$

labels a vertex of G_{431} .

As a consequence,

Corollary 2.17 Let (τ, v) be a pair constituted with a RST τ of shape λ (a partition of N) and a vector $v \in \mathbb{N}^N$. Then there exists a unique vertex in G_{λ} labeled by a 4-tuple of the form (τ, ζ, v, σ) . We will denote $\mathfrak{V}_{\tau,\zeta,v,\sigma} := (\tau, v)$.

Remark also, that all the information can be retrieved from the spectral vector ζ - the coefficients of α give v , the rank function of v gives σ , and the constants in the spectral vector give the content vector which does uniquely determine the RST τ .

Definition 2.18 We define also the subgraph G_{τ} as the graph obtained from G_{λ} by erasing all the vertices labeled by RST other than τ and the associated arrows. Remark that such a graph is connected.

Note that the graph G_{λ} is the union of the graphs G_{τ} connected by jumps. Furthermore, if $G_{\tau'}$ and G_{τ} are connected by a succession of jumps then there is no step from $G_{\tau'}$ to G_{τ} .

Example 2.19 In Fig. 3, the graph G_{21} is constituted with the two graphs $G_{1_{32}}$ and $G_{2_{31}}$ connected by jumps (in blue).

2.3 Non-affine connectivity

Let us denote by H_λ the graph obtained from G_λ by removing the affine edges, all the falls and the vertex \emptyset . The purpose of this section is to investigate the connected components of H_λ .

Definition 2.20 *Let $v \in \mathbb{N}^N$ and $\tau \in \text{Tab}_\lambda$ (λ partition). We define the filling $T(\tau, v)$ obtained by replacing i by $v^+[i]$ in τ for each i .*

Proposition 2.21 *Two 4-tuples (τ, ζ, v, σ) and $(\tau', \zeta', v', \sigma')$ are in the same connected component of H_λ if and only if $T(\tau, v) = T(\tau', v')$.*

Proof Remark first that the steps and correct jumps preserve $T(\tau, v)$. Indeed steps let invariant the pairs (τ, v^+) whilst the correct jumps act on the RST by $\tau.s_{\sigma_v[i]}$ where $v[i] = v[i+1]$ (or equivalently by $\tau.s_j$ where $v^+[j] = v^+[j+1]$). Hence, we show that if (τ, ζ, v, σ) is connected to $(\tau', \zeta', v', \sigma')$ then $T(\tau, v) = T(\tau', v')$.

To prove the converse, it suffices to prove that if $T(\tau_\lambda, \mu) = T(\tau, \mu)$ then there exists a series of correct jumps from (τ, ζ', μ, Id) to $(\tau_\lambda, \zeta, \mu, Id)$ when μ is a partition. Let us prove the result by induction on the length of the permutation ω such that $\tau.\omega = \tau_\lambda$ for the weak order. The base point of the induction is straightforward. Now, choose i such that $\omega[i] > \omega[i+1]$ and i and $i+1$ are not in the same line or in the same column in τ then $\omega = s_i\omega'$ where $\ell(\omega') < \ell(\omega)$. From the construction of $T(\tau, \mu)$ this means that $\mu[i] = \mu[i+1]$ and hence, there is a correct jump from (τ, ζ', μ, Id) to $(\tau^{(i, i+1)}, \zeta'.s_i, \mu, Id)$. Applying the induction hypothesis, this shows the result. \square

This shows that the connected components of H_λ are indexed by the $T(\tau, \mu)$ where μ is a partition.

Definition 2.22 *We will denote by H_T the connected component associated to T in H_λ . The component H_T will be said **1-compatible** if T is a column-strict tableau. The component H_T will be said **(-1)-compatible** if T is a row-strict tableau.*

Example 2.23 Let $\mu = [2, 1, 1, 0, 0]$ and $\lambda = [3, 2]$. There are four connected components with vertices labeled by permutations of μ in H_λ (see Fig. 2). The possible values of $T(\tau, \mu)$ are

$$\begin{array}{ccc} 12 & 02 & 01 \\ 001 & 011 & 012 \end{array} \text{ and } \begin{array}{c} 11 \\ 002 \end{array},$$

squared in red in Fig. 2. The 1-compatible components are H_{001}^{12} and H_{002}^{11} whilst there is only one (-1) -compatible component H_{012}^{01} . The component H_{011}^{02} is neither 1-compatible neither (-1) -compatible. The component H_{001}^{12} contains vertices of G_{542}^{31} and G_{543}^{21} connected by jumps.

We use the following result in the sequel, its proof is easy and let to the reader.

Proposition 2.24 *Let (τ, ζ, v, σ) be a vertex of H_T such that $(\tau, \zeta, v, \sigma).s_i = \emptyset$. One has*

1. *If H_T is 1-compatible then $\sigma[i]$ and $\sigma[i+1](= \sigma[i] + 1)$ are in the same line in τ .*
2. *If H_T is (-1) -compatible then $\sigma[i]$ and $\sigma[i+1](= \sigma[i] + 1)$ are in the same column in τ .*

The following definition is used to find a RST corresponding to a filling of a shape.

Definition 2.25 *Let T be a filling of shape λ , the **standardization** $\text{std}(T)$ of T is the reverse standard tableau with shape λ obtained by the following process:*

1. *Denote by $|T|_i$ the number of occurrences of i in T*
2. *Read the tableau T from the left to the right and the bottom to the top and replace successively each occurrence of i by the numbers $N - |T|_0 - \dots - |T|_{i-1}$, $N - |T|_0 - \dots - |T|_{i-1} - 1$, \dots , $N - |T|_0 - \dots - |T|_i$.*

Alternatively, one has

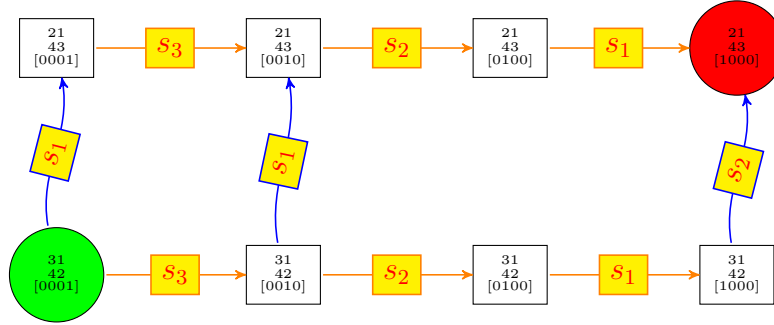
$$\begin{aligned} \text{std}(T) [i, j] := & \# \{(k, l) : T[k, l] > T[i, j]\} + \# \{(k, l) : l > j, T[k, l] = T[i, j]\} \\ & + \# \{(k, j) : k \geq i, T[k, j] = T[i, j]\}. \end{aligned}$$

We will denote by λ_T the unique partition obtained by sorting in the decreasing order all the entries of T .

Example 2.26 We have $\text{std} \begin{pmatrix} 01 \\ 002 \end{pmatrix} = \begin{pmatrix} 32 \\ 541 \end{pmatrix}$ and $\lambda_{002}^{01} = [21000]$.

Remark that each H_T has a unique sink (that is a vertex with no outward edge) and this vertex is labeled by $(\text{std}(T), \zeta_T, \lambda_T, Id)$ for a certain vector ζ_T and a unique root.

Example 2.27 Consider the tableau $T = \begin{smallmatrix} 01 \\ 21 \\ 43 \end{smallmatrix}$, its standardization is $\text{std}(T) = \begin{smallmatrix} 21 \\ 43 \\ 0001 \end{smallmatrix}$ and the graph H_T is :



The sink is denoted by a red disk and the root by a green disk.

2.4 Restrictions

Consider the operator \downarrow_M acting on the Yang-Baxter graphs G_λ by producing a new graph $G_{\lambda \downarrow_M}$ following the rules below:

1. Add all the possible edges of the form



More precisely, the action of Ψ' on the 4-tuples is given by

$$\Psi' = \Psi \cdot s_{N-1} \dots s_M.$$

2. Suppress the vertices labeled by (τ, ζ, v, σ) with $v[i] \neq 0$ for some $i > M$, with the associated inward and outward edges.
3. Relabel the remained vertices $(\tau, \zeta, v, \sigma) \downarrow_M := (\tau \downarrow_M, \zeta \downarrow_M, v \downarrow_M, \sigma \downarrow_M)$ with
 - (a) $\tau \downarrow_M$ is obtained from τ by removing the nodes labeled by $M + 1, \dots, N$. Note that the shape of $\tau \downarrow_M$ could be a skew partition.

- (b) $v \downarrow_M = [v[1], \dots, v[M]]$
- (c) $\sigma \downarrow_M = [\sigma[1], \dots, \sigma[M]]$
- (d) $\zeta \downarrow_M = [\zeta[1] - \text{CT}_\tau[M], \dots, \zeta[M] - \text{CT}_\tau[M]]$

4. Relabel by Ψ the edges labeled by Ψ' .

Example 2.28 Consider the partition $\lambda = 21$ and $M = 2$, the graph G_{21} in Fig. 3 with edges Ψ' added. We obtain the graph $G_{21} \downarrow_M$ (Fig. 4) applying the other rules.

Definition 2.29 A RST τ has the property $R(M)$ if the removal of the nodes labeled by $M + 1, \dots, N$ in τ produces a RST whose Ferrers diagram is a partition.

Example 2.30 The RST

$$\begin{array}{r} 5 \ 2 \\ 7 \ 3 \ 1 \\ 8 \ 6 \ 4 \end{array}$$

has the property $R(3)$ whilst the RST $\begin{array}{r} 2 \\ 31 \end{array}$ has not the property $R(2)$.

Denote by $\overline{G_\tau}$ the subgraph of G_λ whose root is τ . In particular, one has

Proposition 2.31 Let τ with the $R(M)$ property such $\tau \downarrow_M = \tau_{\lambda \downarrow_M}$ where $\lambda \downarrow_M$ denotes the Ferrers diagram of $\tau \downarrow_M$. The graph $G_{\lambda \downarrow_M}$ is identical to the subgraph $\overline{G_{\tau \downarrow_M}}$ of $G_{\lambda \downarrow_M}$ whose root is $\tau \downarrow_M$.

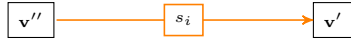
Proof Obviously, since the Ferrers diagram of $\tau \downarrow_M$ is a partition, all the spectral vectors ζ labeling the vertices of $\overline{G_{\tau \downarrow_M}}$ are obtained by subtracting the same integer (that is $\text{CT}_\tau[M]$) from the corresponding spectral vector in G_λ . It follows that the action of the s_i permutes the components of the spectral vectors in $\overline{G_{\tau \downarrow_M}}$.

Let $\mathbf{v}' = (\tau', \zeta', [v'[1], \dots, v'[M], 0 \dots, 0], \sigma')$ be a vertex of $\overline{G_\tau}$. Let us prove by induction on the length of a path from the root to \mathbf{v}' that

1. There is a vertex labeled by $\mathbf{v}' \downarrow_M := (\tau' \downarrow_M, \zeta' \downarrow_M, [v'[1], \dots, v'[M]], \sigma' \downarrow_M)$ in $G_{\lambda \downarrow_M}$.
2. If there is a non affine edge labeled by s_i with $i < M$ from $\mathbf{v}'' = (\tau'', \zeta'', [v''[1], \dots, v''[M], 0 \dots, 0], \sigma'')$ to \mathbf{v}' in $\overline{G_\tau}$ then there is the same edge from $\mathbf{v}'' \downarrow_M$ to $\mathbf{v}' \downarrow_M$ in $G_{\lambda \downarrow_M}$.
3. If there is an edge from $\mathbf{v}'' = (\tau'', \zeta'', [v''[1], \dots, v''[M], 0 \dots, 0], \sigma'')$ to \mathbf{v}' in $\overline{G_\tau}$ labeled by Ψ' then there is an edge labeled by Ψ from $\mathbf{v}'' \downarrow_M$ to $\mathbf{v}' \downarrow_M$ in $G_{\lambda \downarrow_M}$.

First, remark that if $\tau' = \tau$ and $v'[i] = 0$ for each i (*ie.* \mathbf{v}' is a component of the label of the root of $\overline{G_\tau}$) then the construction gives, straightforwardly the result.

Suppose that there is a non affine edge



in $\overline{G_\tau}$. By induction $\mathbf{v}'' \downarrow_M$ labels a vertex of $G_{\lambda \downarrow_M}$. We verify that

$$(\mathbf{v}'' \cdot s_i) \downarrow_M = (\mathbf{v}'' \downarrow_M) \cdot s_i = \mathbf{v}' \downarrow_M$$

Hence, $\mathbf{v}' \downarrow_M$ labels a vertex of $G_{\lambda \downarrow_M}$ and the assertion (2) is recovered.

Suppose now, that there is a affine edge



in $\overline{G_\tau}$. By induction $\mathbf{v}'' \downarrow_M$ labels a vertex of $G_{\lambda \downarrow_M}$. We verify that

$$(\mathbf{v}'' \cdot \Psi') \downarrow_M = (\mathbf{v}'' \Psi s_M \dots s_N) \downarrow_M = (\mathbf{v}'' \downarrow_M) \Psi = \mathbf{v}' \downarrow_M.$$

Hence, $\mathbf{v}' \downarrow_M$ labels a vertex of $G_{\lambda \downarrow_M}$ and the assertion (3) is recovered.

Conversely, if \mathbf{v}' labels a vertex of $G_{\lambda \downarrow_M}$, there exists a vertex labeled by $\mathbf{v}'^{(N)}$

in $\overline{G_\tau}$ verifying $\mathbf{v}^{(N)} \downarrow = \mathbf{v}'$. Indeed, suppose $\mathbf{v}' = (\tau', \zeta', v', \sigma')$ then $\mathbf{v}^{(N)} = (\tau'^{(N)}, \zeta'^{(N)}, v'^{(N)}, \sigma'^{(N)})$, where $\tau'^{(N)}$ is obtained from τ' by adding the nodes of τ labeled by $M + 1, \dots, N$, $v'^{(N)} = [v'[1], \dots, v'[M], 0, \dots, 0]$, $\zeta'^{(N)} = \zeta_{v'^{(N)}, \tau'^{(N)}}$ and $\sigma'^{(N)} = \sigma_{v'^{(N)}}$. Furthermore if $\mathbf{v}' \cdot s_i = \mathbf{v}''$ then $\mathbf{v}^{(N)} \cdot s_i = \mathbf{v}''^{(N)}$ and if $\mathbf{v}' \cdot \Psi = \mathbf{v}''$ then $\mathbf{v}^{(N)} \cdot \Psi' = \mathbf{v}''^{(N)}$. This concludes the proof. \square

Example 2.32 Consider in Fig. 5 the restriction problem for

$$\tau = \begin{array}{ccc} 5 & 2 & \\ 7 & 3 & 1 \\ 8 & 6 & 4 \end{array}$$

and $M = 3$. The subgraph of $\overline{G_\tau}$ obtained using only the root and

$$\begin{array}{ccc} 5 & 2 & \\ 7 & 3 & 1 \\ 8 & 6 & 4 \end{array}$$

the arrows labeled by Ψ' , s_1 and s_2 is isomorphic to the graph $\overline{G_{31}^2}$ (see Fig. 3).

3 Vector valued polynomials

3.1 About the Young seminormal representation of the symmetric group

We consider the space V_λ spanned by reverse tableaux of shape λ and the action of the symmetric group defined by Murphy in [14] by

$$\tau \cdot s_i = \begin{cases} b_\tau[i] \tau & \text{if } b_\tau[i]^2 = 1, \\ b_\tau[i] \tau + \tau^{(i, i+1)} & \text{if } 0 < b_\tau[i] \leq \frac{1}{2}, \\ b_\tau[i] \tau + (1 - b_\tau[i]^2) \tau^{(i, i+1)} & \text{otherwise,} \end{cases} \quad (1)$$

where $b_\tau[i] := \frac{1}{\text{CT}_\tau[i] - \text{CT}_\tau[i+1]}$. Note that when $|b_\tau[i]| < 1$, $\tau^{(i, i+1)}$ is always a reverse standard tableau when τ is a reverse standard tableau.

Murphy showed [14] that the RST are the simultaneous eigenfunctions of the Jucys-Murphy elements:

$$\omega_i = \sum_{j=i+1}^N s_{ij}.$$

More precisely:

Proposition 3.1

$$\tau.\omega_i = \text{CT}_\tau[i]\tau.$$

As usual, a polynomial representation for the Murphy action on the RST can be computed through the Yang-Baxter graph. We start from τ_λ and we construct the associated polynomial

$$P_{\tau_\lambda} = \prod_i \prod_{\substack{k>l \\ (i,k),(i,l) \in \lambda}} (t_{\tau_\lambda[i,k]} - t_{\tau_\lambda[i,l]}),$$

where $\tau[i, j]$ denotes the integer belonging at the column i and the row j in τ . Such a polynomial is a simultaneous eigenfunction of the Jucys-Murphy idempotents :

$$P_{\tau_\lambda}.\omega_i = \text{CT}_{\tau_\lambda}[i]P_{\tau_\lambda}.$$

Suppose that P_τ is the polynomial associated to τ . Suppose also that $0 < b_\tau[i] < 1$. Hence, the polynomial $P_{\tau(i,i+1)}$ is obtained from the polynomial P_τ by acting with $s_i - b_\tau[i]$ (with the standard action of the transposition s_i on the variables t_j).

Example 3.2

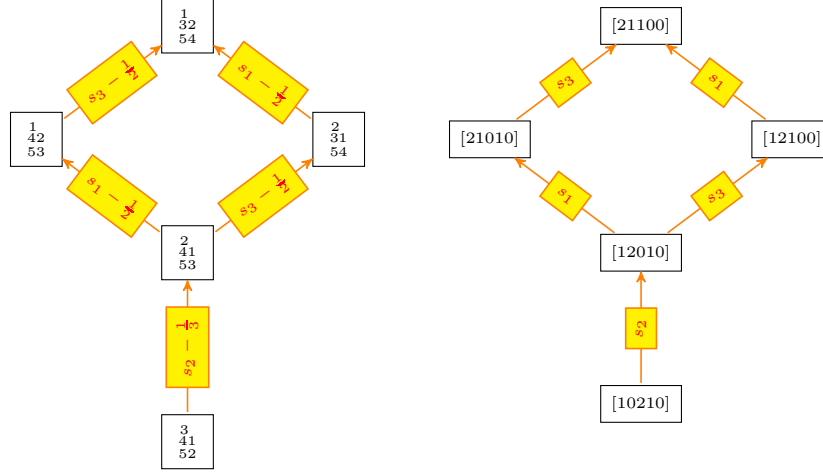
$$\begin{aligned} P_{\substack{21 \\ 43}} &= P_{\substack{31 \\ 42}}.(s_2 - \frac{1}{2}) \\ &= (t_3 - t_4)(t_1 - t_2).(s_2 - \frac{1}{2}) \\ &= t_1 t_2 - \frac{1}{2} t_4 t_1 - \frac{1}{2} t_3 t_2 + t_4 t_3 - \frac{1}{2} t_4 t_2 - \frac{1}{2} t_3 t_1 \end{aligned}$$

Let us remark that in [12], Lascoux simplified the Young construction by having recourse to the covariant algebra (of \mathfrak{S}_N) $\mathbb{C}[x_1, \dots, x_N]/\mathfrak{Sym}_+$ where \mathfrak{Sym}_+ is the ideal generated by symmetric functions without constant terms. Note that the covariant algebra is isomorphic to the regular representation. In the aim to adapt his construction to our notations, we replace each polynomial with its dominant monomial represented by the vectors of its exponents. The vector associated to the root of the graph is the vector exponent of the leading term in the product of the Vandermonde determinants associated to each column and is obtained by putting the number of the line minus 1 in the corresponding entry.

Example 3.3 The vector associated to $\frac{4}{52}$ is [010210].
631

In fact, the covariant algebra being isomorphic to the regular representation of \mathfrak{S}_N , the computation of the polynomials is completely encoded by the action of the symmetric group on the leading monomials, as shown in the following example.

Example 3.4 Consider the RST of shape 221, one has



For instance, one has

$$\begin{aligned}
P_{\begin{smallmatrix} 1 & 42 \\ 53 \end{smallmatrix}} &= -\frac{1}{2} t_4 t_2^2 t_1 + \frac{1}{2} t_4 t_2^2 t_3 + \frac{1}{2} t_2^2 t_5 t_1 - \frac{1}{2} t_2^2 t_5 t_3 + \frac{1}{2} t_4 t_3^2 t_1 \\
&+ \frac{1}{2} t_4^2 t_1 t_3 + t_1^2 t_5 t_3 - t_1^2 t_5 t_2 + \frac{1}{2} t_5^2 t_4 t_3 + \frac{1}{2} t_5 t_4^2 t_2 \\
&+ \frac{1}{2} t_5^2 t_1 t_2 + t_1^2 t_4 t_2 - \frac{1}{2} t_5^2 t_4 t_2 + \frac{1}{2} t_3^2 t_5 t_2 - \frac{1}{2} t_4^2 t_1 t_2 \\
&- t_1^2 t_4 t_3 - \frac{1}{2} t_3^2 t_5 t_1 - \frac{1}{2} t_5 t_4^2 t_3 - \frac{1}{2} t_4 t_3^2 t_2 - \frac{1}{2} t_5^2 t_1 t_3,
\end{aligned}$$

whose leading term is $t_1^2 t_2 t_4$.

Hence, the leading monomial of P_τ is the product of all the $t_i^{\text{rw}(i,\tau)-1}$. For example, the leading monomial in $P_{\begin{smallmatrix} 51 \\ 732 \\ 9864 \end{smallmatrix}}$ is $t_1^2 t_2 t_3 t_5^2 t_7$.

Now, let τ be a RST of shape $\lambda = [\lambda_1, \dots, \lambda_k]$ and $M < N$. Consider P_τ as a polynomial in $\mathbb{C}[t_1, \dots, t_M][t_{M+1}, \dots, t_N]$. Let μ the sub-partition of λ which is the shape of the RST obtained by removing the nodes labeled by $1, \dots, M$ in τ and denote by $\tau^{(M)}$ the associated RST. Consider also the skew-RST $\tau \downarrow_M$ of shape λ/μ obtained by removing the nodes labeled $M+1, \dots, N$ in τ . Let $C(\tau, M) := \{\rho : \rho^{(M)} = \tau^{(M)}\}$ and $T(\tau, M) := \{\rho \downarrow_M : \rho \in C(\rho, M)\}$.

To each skew-RST $\rho \downarrow_M$ in $T(\tau, M)$ we associate the polynomial $P'_{\rho \downarrow_M}$ which is the coefficient of the monomial $\prod_{i=M+1}^N t_i^{\lambda_{\text{cl}(i,\rho)} - \text{rw}(i,\rho)}$ in P_ρ .

Example 3.5 Consider $\tau = \begin{matrix} 2 \\ 4 & 1 \\ 5 & 3 \end{matrix}$ and $M = 3$. Then $\tau^{(3)} = \begin{matrix} 4 & 1 \\ 5 & 3 \end{matrix}$, $\tau \downarrow_3 = \begin{matrix} 2 \\ 1 \\ 3 \end{matrix}$,

$$C(\tau, 3) = \left\{ \begin{matrix} 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 2 & 3 \end{matrix}, \begin{matrix} 4 & 1 \\ 5 & 3 \end{matrix}, \begin{matrix} 4 & 2 \\ 5 & 3 \end{matrix} \right\}$$

and

$$T(\tau, 3) = \left\{ \begin{matrix} 3 & 2 & 1 \\ & 1 & 1 \\ & 2 & 3 \end{matrix}, \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \right\}.$$

The polynomial $P'_{\begin{matrix} 2 \\ 1 \\ 3 \end{matrix}}$ is the coefficient of $t_5^2 t_4$ in $P_{\begin{matrix} 2 \\ 4 & 1 \\ 5 & 3 \end{matrix}}$.

Suppose that $\gamma := \lambda/\mu$ is a partition ($\tau \downarrow_M$ is a RST). We remark that the polynomials $P'_{\rho \downarrow_M}$ for $\rho \in C(\tau, M)$ are related to each others by the same relations as those induced by the subgraph of the graph associated to λ composed uniquely of the vertices labeled by the RST of $C(\tau, M)$ and this is also the case for the polynomials $P_{\rho \downarrow_M}$ for $\rho \in C(\tau, M)$. In other words, when $i < M$, since s_i does not act on the variables s_{M+1}, \dots, s_N , the Murphy rules (eq. 1) give

$$P'_{\rho \downarrow_M} \cdot s_i = \begin{cases} b_\rho[i] P'_{\rho \downarrow_M} & \text{if } b_\rho[i]^2 = 1, \\ b_\rho[i] P'_{\rho \downarrow_M} + P'_{\rho^{(i,i+1)} \downarrow_M} & \text{if } 0 < b_\rho[i] \leq \frac{1}{2}, \\ b_\rho[i] P'_{\rho \downarrow_M} + (1 - b_\rho[i]^2) P'_{\rho^{(i,i+1)} \downarrow_M} & \text{otherwise.} \end{cases} \quad (2)$$

Since the action s_i ($i < M$) on the RST commutes with the restriction \downarrow_M , one has $P'_{\rho^{(i,i+1)} \downarrow_M} = P'_{(\rho \downarrow_M)^{(i,i+1)}}$ and proposition 3.1 and eq (2) imply that

the polynomials $P'_{\rho \downarrow M}$ are simultaneous eigenfunctions of the Jucys-Murphy operators

$$\omega_i^{(M)} = \sum_{i+1}^M s_{ij},$$

with eigenvalues $\text{CT}_{\rho \downarrow M}[i]$. Since the $P'_{\rho \downarrow M}$ for $\rho \in C(\tau, M)$ span a polynomial representation of the symmetric group \mathfrak{S}_M with minimal degree, the polynomials $P'_{\rho \downarrow M}$ are equal up to a global multiplicative coefficient to the polynomials $P_{\rho \downarrow M}$.

To summarize :

Proposition 3.6 *When $\tau \downarrow M$ is a RST, the coefficient of $\prod_{i=M+1}^N t_i^{\lambda_{\text{cl}(i,\tau)} - \text{rw}(i,\tau)}$ in $P_{\tau \downarrow M}$ is proportional to $P_{\tau \downarrow M}$.*

Example 3.7 The coefficient of $t_9^2 t_8^2 t_7 t_6$ in $P_{\frac{51}{732} \frac{9864}{32}}$ equals

$$\frac{1}{6} t_1 - \frac{1}{12} t_2 - \frac{1}{12} t_3 = \frac{1}{6} P_{\frac{1}{32}}.$$

3.2 Definition and dominance properties of vector valued polynomials

Consider the space

$$M_N = \text{span}_{\mathbb{C}}\{x_1^{v[1]} \dots x_N^{v[N]} \otimes \tau : v \in \mathbb{N}^N, \tau \in \text{Tab}_{\lambda}, \lambda \vdash N\},$$

where $\text{Tab}(N)$ denotes the set of the reverse standard tableaux on $\{1, \dots, N\}$. This space splits into a direct sum $M_N = \bigoplus_{\lambda \vdash N} M_{\lambda}$, where

$$M_{\lambda} = \text{span}_{\mathbb{C}}\{x_1^{v[1]} \dots x_N^{v[N]} \otimes \tau | v \in \mathbb{N}^N, \tau \in \text{Tab}_{\lambda}\}.$$

The algebra $\mathbb{C}[\mathfrak{S}_N] \otimes \mathbb{C}[\mathfrak{S}_N]$ acts on these spaces by commuting the vector of the powers on the variables on the left component and the action on the tableaux defined by Murphy (eq. (1)) on the right component.

Example 3.8

$$x_1^3 x_2^1 \otimes \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix} \cdot s_2 \otimes s_1 = \frac{1}{2} x_1^3 x_3^1 \otimes \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix} + x_1^3 x_3^1 \otimes \begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix}.$$

For simplicity we will denote $x^v = x_1^{v[1]} \dots x_N^{v[N]}$ and $x^{v,\tau} := x^v \otimes \tau \sigma_v$. Note that the space M_λ is spanned by the set of polynomials

$$\mathfrak{M}_\lambda := \{x^{v,\tau} : v \in \mathbb{N}^N, \tau \in \text{Tab}_\lambda\},$$

which can be naturally endowed with the strict order \triangleleft defined by

$$x^{v,\tau} \triangleleft x^{v',\tau'} \text{ iff } v \triangleleft v',$$

with $v \triangleleft v'$ means that $v^+ \prec v'^+$ or $v^+ = v'^+$ and $v \prec v'$, where \prec denotes the classical dominance order on the vectors:

$$v \preceq v' \text{ iff } \forall i, v[1] + \dots + v[i] \leq v'[1] + \dots + v'[i].$$

Example 3.9 1. $x^{031, \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}} \triangleleft x^{310, \begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix}}$ since $031 \prec 310$.

2. $x^{220, \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}} \triangleleft x^{301, \begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix}}$ since $220 \prec 310$.

3. The polynomials $x^{031, \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}}$ and $x^{031, \begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix}}$ are not comparable.

The partial order \trianglelefteq will provide us a relevant dominance notion.

Definition 3.10 A polynomial $P \in M_\lambda$ has the **dominance property** w.r.t. \trianglelefteq if and only P can be written as

$$P = \alpha_v x^{v,\tau} + \sum_{x^{v',\tau'} \triangleleft x^{v,\tau}} \alpha_{v',\tau'} x^{v',\tau'}$$

with $\alpha_v \neq 0$. The monomial $x^{v,\tau}$ is the **leading monomial** of P .

As in [13], we define $\Psi := \theta \otimes \theta \cdot x_N$, with $\theta = s_1 s_2 \dots s_{N-1}$. The dominance property is compatible with the action of main operators in the following sense.

Proposition 3.11 Suppose that P has the dominance property with leading monomial $x^{v,\tau}$ then

1. If $v[i] < v[i+1]$ then $P \cdot s_i \otimes s_i$ has the dominance property and its leading term is $x^{v \cdot s_i, \tau}$.
2. $P \cdot \Psi$ has the dominance property and its leading monomial is $x^{v \cdot \Psi, \tau}$.

3.3 Dunkl and Cherednik-Dunkl operators for vector valued polynomials

As in [4], we define the Dunkl operators

$$\mathfrak{D}_i := \frac{\partial}{\partial x_i} \otimes 1 + \frac{1}{\alpha} \sum_{i \neq j} \partial_{ij} \otimes s_{ij},$$

where s_{ij} denotes the transposition which exchanges i and j and

$$\partial_{ij} = (1 - s_{ij}) \frac{1}{x_i - x_j}$$

is the divided difference.

One has

Lemma 3.12 *If \mathfrak{D}_i denotes the Dunkl operator, one has*

$$s_i \otimes s_i \mathfrak{D}_i = \mathfrak{D}_{i+1} s_i \otimes s_i$$

Proof Straightforward from the definition of \mathfrak{D}_i and the equalities $s_i s_{ij} = s_i s_{i+1, j}$, $s_i \partial_{ij} = \partial_{i+1, j} s_i$ and $s_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}} s_i$. \square

The Cherednik-Dunkl operators are defined by [4]

$$\mathfrak{U}_i := x_i \mathfrak{D}_i - \frac{1}{\alpha} \sum_{j=1}^{i-1} s_{i, j} \otimes s_{i, j},$$

One has

Lemma 3.13 1. $s_i \otimes s_i \mathfrak{U}_i = \mathfrak{U}_{i+1} s_i \otimes s_i + \frac{1}{\alpha}$

2. $s_i \otimes s_i \mathfrak{U}_j = \mathfrak{U}_j s_i \otimes s_i$, $j \neq i, i+1$

3. $s_i \otimes s_i \mathfrak{U}_{i+1} = \mathfrak{U}_i s_i \otimes s_i - \frac{1}{\alpha}$,

Proof The three identities can be proved on the same model. We only treat the first one which follows from the equalities

$$\begin{aligned} s_i \otimes s_i \mathfrak{U}_i &= s_i \otimes s_i x_i \mathfrak{D}_i - \frac{1}{\alpha} \sum_{j=1}^{i-1} s_{i, j} \otimes s_{i, j} \\ &= \left(x_{i+1} \mathfrak{D}_{i+1} - \frac{1}{\alpha} \sum_{j=1}^{i-1} s_{i+1, j} \otimes s_{i+1, j} \right) s_i \otimes s_i \\ &= \mathfrak{U}_{i+1} s_i \otimes s_i + \frac{1}{\alpha}. \end{aligned}$$

\square

The affine operator Ψ has the following commutation properties with the Dunkl operators:

Lemma 3.14 1. $\mathfrak{D}_{i+1}\Psi = \Psi\mathfrak{D}_i + \theta \otimes \theta \cdot s_{i,N} \otimes s_{i,N}$, $i < 1$

$$2. \mathfrak{D}_1\Psi = \Psi\mathfrak{D}_N + \theta \otimes \theta \left(\sum_{j=1}^{N-1} s_{Nj} \otimes s_{Nj} - 1 \right)$$

As a consequence, one finds.

Lemma 3.15

$$\Psi\mathfrak{U}_i = \mathfrak{U}_{i+1}\Psi, \quad i \neq N$$

and

$$\Psi\mathfrak{U}_N = (\mathfrak{U}_1 + 1)\Psi.$$

The action on the RST is given by

Lemma 3.16

$$1 \otimes \tau.\mathfrak{U}_i = \left(1 + \frac{1}{\alpha} \text{CT}_\tau[i]\right) 1 \otimes \tau$$

Proof One has

$$\begin{aligned} 1 \otimes \tau.\mathfrak{U}_i &= 1 \otimes \tau.x_i\mathfrak{D}_i - \frac{1}{\alpha} \sum_{j=1}^{i-1} 1 \otimes \tau.s_{i,j} \otimes s_{i,j} \\ &= 1 \otimes \tau.(1 + \frac{1}{\alpha} 1 \otimes \omega_i) \end{aligned}$$

where $\omega_i := \sum_{j=i+1}^N (i \ j)$ denotes a Jucys-Murphy element. Since the RST are eigenfunctions of the Jucys-Murphy elements and the associated eigenvalues are given by the contents, we show the lemma. \square

For convenience, define $\tilde{\xi}_i := \alpha\mathfrak{U}_i - \alpha$. The results of this section gives the following equalities.

Proposition 3.17

$$s_i \otimes s_i \tilde{\xi}_i = \tilde{\xi}_{i+1} s_i \otimes s_i + 1, \tag{3}$$

$$s_i \otimes s_i \tilde{\xi}_{i+1} = \tilde{\xi}_i s_i \otimes s_i - 1, \tag{4}$$

$$s_i \otimes s_i \tilde{\xi}_j = \tilde{\xi}_j \cdot s_i \otimes s_i, \quad j \neq i, i+1, \tag{5}$$

$$\Psi \tilde{\xi}_i = \tilde{\xi}_{i+1} \Psi, \quad i \neq N \tag{6}$$

$$\Psi \tilde{\xi}_N = \left(\tilde{\xi}_1 + \alpha \right) \Psi. \tag{7}$$

4 Non symmetric vector valued Jack polynomials

In this section we recover the construction due to one of the authors [4] of a basis of vector valued polynomials $J_{v,\tau}$ (vector valued Jack polynomials) with the dominance property which are simultaneous eigenfunctions of the operators $\hat{\xi}_i$. This construction belongs to a huge family of vector valued Jack polynomials associated to the complex reflection groups $G(r, 1, n)$ defined by Griffeth [8]. We will denote by $\zeta_{v,\tau}$ their associated spectral vectors. We will see also that many properties of this basis can be deduced from the Yang-Baxter structure.

4.1 Yang-Baxter construction associated to G_λ

Let λ be a partition and G_λ the associated. We construct the set of the polynomials $(J_{\mathfrak{P}})_{\mathfrak{P} \text{ path in } G_\lambda}$ using the following recursive rules:

1. $J_{\emptyset} := 1 \otimes \tau_\lambda$
2. If $\mathfrak{P} = [a_1, \dots, a_{k-1}, s_i]$ then

$$J_{\mathfrak{P}} := J_{[a_1, \dots, a_{k-1}]} \left(s_i \otimes s_i + \frac{1}{\zeta[i+1] - \zeta[i]} \right),$$

where the vector ζ is defined by

$$(\tau_\lambda, \text{CT}_{\tau_\lambda}, 0^N, [1, 2, \dots, N]).a_1 \dots a_{k-1} = (\tau, \zeta, v, \sigma),$$

3. $\mathfrak{P} = [a_1, \dots, a_{k-1}, \Psi]$ then then

$$J_{\mathfrak{P}} = J_{[a_1, \dots, a_{k-1}]} \Psi.$$

One has the following theorem.

Theorem 4.1 *Let $\mathfrak{P} = [a_0, \dots, a_k]$ be a path in G_λ from the root to (τ, ζ, v, σ) . The polynomial $J_{\mathfrak{P}}$ is a simultaneous eigenfunctions of the operators $\hat{\xi}_i$ whose leading term is $x^{v,\tau}$. Furthermore, the eigenvalues of $\tilde{\xi}_i$ associated to $J_{\mathfrak{P}}$ equals $\zeta[i]$.*

Hence, for any path \mathfrak{P}' ending at (τ, ζ, v, σ) , one has $J_{v,\tau} := J_{\mathfrak{P}} = J_{\mathfrak{P}'}$ and the family $(J_{v,\tau})_{v,\tau}$ forms a basis of M_λ of simultaneous eigenfunctions of the Cherednik operators.

Furthermore, if \mathfrak{P} leads to \emptyset then $J_{\mathfrak{P}} = 0$.

Proof We will prove the result by induction on the length k . If $k = 0$ then the result follows from proposition 3.16. Suppose now that $k > 0$ and let $(\tau', \zeta', v', \sigma_{v'}) = (\tau_\lambda, \text{CT}_{\tau_\lambda}, 0^N, [1, \dots, N])_{a_1 \dots a_{k-1}}$. By induction, $J_{[a_1, \dots, a_{k-1}]}$ is a simultaneous eigenfunctions of the operators $\tilde{\xi}_i$ such that the associated vector of eigenvalues is given by

$$J_{[a_1, \dots, a_{k-1}]} \tilde{\xi}_i = \zeta'[i] J_{[a_1, \dots, a_{k-1}]}$$

and the leading term is $x^{v', \tau'}$.

If the $a_k = \Psi$ is an affine arrow, then $\tau = \tau'$, $\zeta = \zeta' \cdot \Psi^\alpha$, $v = v' \cdot \Psi$, $\sigma_v = \sigma_{v'}[2, \dots, N, 1]$ and $J_{\mathfrak{p}} = J_{[a_1, \dots, a_{k-1}]} \cdot \Psi_i$. If $i \neq N$

$$\begin{aligned} J_{\mathfrak{p}} \tilde{\xi}_i &= J_{[a_1, \dots, a_{k-1}]} \Psi \tilde{\xi}_i \\ &= J_{[a_1, \dots, a_{k-1}]} \tilde{\xi}_{i+1} \Psi \\ &= \zeta'[i+1] J_{\mathfrak{p}} \\ &= \zeta[i] J_{\mathfrak{p}}. \end{aligned}$$

If $i = N$ then,

$$\begin{aligned} J_{\mathfrak{p}} \tilde{\xi}_N &= J_{[a_1, \dots, a_{k-1}]} \Psi \tilde{\xi}_N \\ &= J_{[a_1, \dots, a_{k-1}]} (\xi_1 + \alpha) \Psi \\ &= (\zeta'[1] + \alpha) J_{\mathfrak{p}} \\ &= \zeta[N] J_{\mathfrak{p}}. \end{aligned}$$

The dominance is a consequence of proposition 3.11.

Suppose now that $a_k = s_i$ is non affine arrow, then $\zeta = \zeta' \cdot s_i$, $v = v' \cdot s_i$ and $J_{\mathfrak{p}} = J_{[a_1, \dots, a_{k-1}]} \cdot \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right)$. If $j \neq i$, $i+1$ then

$$\begin{aligned} J_{\mathfrak{p}} \tilde{\xi}_j &= J_{[a_1, \dots, a_{k-1}]} \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \tilde{\xi}_j \\ &= J_{[a_1, \dots, a_{k-1}]} \tilde{\xi}_j \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \\ &= \zeta'[j] J_{\mathfrak{p}} \\ &= \zeta[j] J_{\mathfrak{p}}. \end{aligned}$$

If $j = i$ then

$$\begin{aligned} J_{\mathfrak{p}} \tilde{\xi}_i &= J_{[a_1, \dots, a_{k-1}]} \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \tilde{\xi}_i \\ &= J_{[a_1, \dots, a_{k-1}]} \left(\tilde{\xi}_{i+1} s_i \otimes s_i + 1 + \tilde{\xi}_i \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \\ &= J_{[a_1, \dots, a_{k-1}]} \left(\zeta'[i+1] s_i \otimes s_i + 1 + \frac{\zeta'[i]}{\zeta'[i+1] - \zeta'[i]} \right) \\ &= \zeta'[i+1] J_{[a_1, \dots, a_{k-1}]} \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \\ &= \zeta[i] J_{\mathfrak{p}}. \end{aligned}$$

If $j = i + 1$ then

$$\begin{aligned}
J_{\mathfrak{P}} \tilde{\xi}_{i+1} &= J_{[a_1, \dots, a_{k-1}]} \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \tilde{\xi}_i \\
&= J_{[a_1, \dots, a_{k-1}]} \left(\tilde{\xi}_i s_i \otimes s_i - 1 + \tilde{\xi}_{i+1} \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \\
&= J_{[a_1, \dots, a_{k-1}]} \left(\zeta'[i] s_i \otimes s_i - 1 + \frac{\zeta'[i+1]}{\zeta'[i+1] - \zeta'[i]} \right) \\
&= \zeta'[i] J_{[a_1, \dots, a_{k-1}]} \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) \\
&= \zeta[i] J_{\mathfrak{P}}.
\end{aligned}$$

Let us examine the dominance properties. First, suppose that $a_k = s_i$ is a step then $\tau = \tau'$ and $\sigma_v = \sigma_{v'} s_i$. From proposition 3.11, the leading term in $J_{\mathfrak{P}}$ equals the leading term in $x^{v'\tau'} \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right)$ that is $x^{v' s_i} \otimes \tau' \sigma_{v'} s_i = x^{v, \tau}$.

Suppose that $a_k = s_i$ is not a step and set $Q := x^{v', \tau'} \left(s_i \otimes s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right)$. One has

$$J_{\mathfrak{P}} = Q + \sum_{x^{v'', \tau''} \triangleleft x^{v, \tau}} \alpha_{v'', \tau''} x^{v'', \tau''}.$$

If $a_k = s_i$ is a jump then $\tau = \tau'(\sigma_{v'}[i], \sigma_{v'}[i+1])$ and $\sigma_v = \sigma'_v$. But

$$\begin{aligned}
Q &= x^{v' s_i} \otimes \tau' \sigma_{v'} s_i + \frac{1}{\zeta'[i+1] - \zeta'[i]} x^{v'} \otimes \tau' \sigma_{v'} \\
&= x^v \otimes \tau' s_{\sigma_{v'}[i]} \sigma_{v'} + \frac{1}{\zeta'[i+1] - \zeta'[i]} x^{v'} \otimes \tau' \sigma_{v'} \\
&= x^v \otimes \tau \sigma_v + \left(b'_\tau[\sigma_{v'}[i]] + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) x^{v'} \otimes \tau' \sigma_{v'} \\
&= x^{v, \tau} + \left(b'_\tau[\sigma_{v'}[i]] + \frac{1}{\zeta'[i+1] - \zeta'[i]} \right) x^{v', \tau'}.
\end{aligned}$$

But $\zeta'[i] = \text{CT}_{\tau'}[\sigma_{v'}[i]]$ and $\zeta'[i+1] = \text{CT}_{\tau'}[\sigma_{v'}[i+1]] = \text{CT}_{\tau'}[\sigma_{v'}[i] + 1]$, hence $b_{\tau'}[\sigma_{v'}[i]] = -\frac{1}{\zeta'[i+1] - \zeta'[i]}$. And the leading term is $Q = x^{v, \tau}$ as expected.

This proves the first part of the theorem and that the family $(J_{v, \tau})_{v, \tau}$ forms a basis of M_λ of simultaneous eigenfunctions of the Cherednik operators.

Finally, if $a_k = s_i$ is a fall, Q is proportional to $x^{v'} \otimes \tau' \sigma_{v'}$ and then $J_{\mathfrak{P}}$ is proportional to $J_{[a_1, \dots, a_k]}$. But clearly, the two polynomials are eigenfunction of the Cherednik operators but with different eigenvalues. This prove that $J_{\mathfrak{P}} = 0$. \square

Example 4.2 Consider again the example $\lambda = 21$, the method allowing us to obtain the first polynomials $J_{v, \tau}$ is illustrated in Fig. 6.

Note that if $[a_1, \dots, a_{k-1}]$ leads to a vertex other than \emptyset and $[a_1, \dots, a_{k-1}, s_i]$ leads to \emptyset , the last part of theorem 4.1 implies that $J_{[a_1, \dots, a_{k-1}]}$ is symmetric or antisymmetric under the action of s_i .

The recursive rules of this section appear first in [8] Lemma 5.3 and Yang-Baxter graph is essentially what Griffeth called *calibration graph* in that paper.

4.2 Partial Yang-Baxter-type construction associated to G_τ

To compute an expression for a polynomial $J_{v,\tau}$ it suffices to find the good path in the subgraph G_τ as shown by the following examples.

Example 4.3 Consider $\tau = \begin{smallmatrix} 1 \\ 3 & 2 \end{smallmatrix}$, Fig. 7 explains how to obtain the values of $J_{v, \begin{smallmatrix} 1 \\ 32 \end{smallmatrix}}$ from the graph $G_{\begin{smallmatrix} 1 \\ 32 \end{smallmatrix}}$.

Example 4.4 For the trivial representation, remark that the Cherednik operators (in [13]) have the same eigenspaces as the Cherednik-Dunkl operators \mathfrak{U}_i (in [4]). In the notations of [13], ξ_i reads

$$\xi_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{\substack{j=1 \\ j \neq i}}^N \bar{\pi}_{ij} + (1 - i), \quad (8)$$

where

$$\bar{\pi}_{ij} = \begin{cases} x_i \partial_{ij} & \text{if } j < i \\ x_j \partial_{ij} & \text{if } i < j \end{cases},$$

where ∂_{ij} denotes the divided difference between the variables x_i and x_j . Noting that $x_i \partial_{ij} = \partial_{ij} x_i - 1$, $x_j \partial_{ij} = \partial_{ij} x_i - (ij)$ and $x_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} x_i - 1$, one finds

$$\xi_i = \alpha \mathfrak{U}_i - (\alpha + N - 1) = \tilde{\xi}_i - (N - 1).$$

Example 4.5 Consider the alternating representation given by $\tau = \begin{smallmatrix} 1 \\ \vdots \\ N \end{smallmatrix}$.

One can show that the alternated-Jack polynomials are equal to the standard one for the coefficient $-\alpha$.

Example 4.6 Let us explain the method on a bigger example: $J_{[0,0,2,1,1,0],\tau}$, for $\tau := \begin{smallmatrix} 4 & 3 \\ 6 & 5 & 2 & 1 \end{smallmatrix}$. First, we obtain the vector $[0, 0, 2, 1, 1, 0]$ from $[0, 0, 0, 0, 0, 0]$ by using only the permutations s_i and an affine operation Ψ which cyclically permutes the vector and adds 1 to the last entry $[i_1, \dots, i_N] \rightarrow [i_2, \dots, i_N, i_1 + 1]$.

$$\begin{aligned} & [0,0,0,0,0,0] \xrightarrow{\Psi} [0,0,0,0,0,1] \xrightarrow{s_5} [0,0,0,0,1,0] \xrightarrow{s_4} [0,0,0,1,0,0] \xrightarrow{s_3} [0,0,1,0,0,0] \\ \xrightarrow{s_2} & [0,1,0,0,0,0] \xrightarrow{s_1} [1,0,0,0,0,0] \xrightarrow{\Psi} [0,0,0,0,0,2] \xrightarrow{s_5} [0,0,0,0,2,0] \xrightarrow{\Psi} [0,0,0,2,0,1] \\ \xrightarrow{s_5} & [0,0,0,2,1,0] \xrightarrow{\Psi} [0,0,2,1,0,1] \xrightarrow{s_5} [0,0,2,1,1,0] \end{aligned}$$

Replace Ψ by Ψ^α in the list of the operations.

$$\Psi^\alpha s_5 s_4 s_3 s_2 s_1 \Psi^\alpha s_5 \Psi^\alpha s_5 \Psi^\alpha s_5$$

Act successively with these operators on the content vector of τ . Starting with this vector, $[3, 2, 0, -1, 1, 0]$, we obtain the vectors $\zeta_{[i_1, \dots, i_N]}$ which are the eigenvalues of the $\tilde{\xi}_i$ associated to the desired polynomials.

$$\begin{aligned} \zeta_{[0,0,0,0,0,0]} &= [3, 2, 0, -1, 1, 0] \xrightarrow{\Psi^\alpha} \zeta_{[0,0,0,0,0,1]} = [2, 0, -1, 1, 0, \alpha + 3] \xrightarrow{s_5} \\ \zeta_{[0,0,0,0,1,0]} &= [2, 0, -1, 1, \alpha + 3, 0] \xrightarrow{s_4} \zeta_{[0,0,0,1,0,0]} = [2, 0, -1, \alpha + 3, 1, 0] \xrightarrow{s_3} \\ \zeta_{[0,0,1,0,0,0]} &= [2, 0, \alpha + 3, -1, 1, 0] \xrightarrow{s_2} \zeta_{[0,1,0,0,0,0]} = [2, \alpha + 3, 0, -1, 1, 0] \xrightarrow{s_1} \\ \zeta_{[1,0,0,0,0,0]} &= [\alpha + 3, 2, 0, -1, 1, 0] \xrightarrow{\Psi^\alpha} \zeta_{[0,0,0,0,0,2]} = [2, 0, -1, 1, 0, 2\alpha + 3] \xrightarrow{s_5} \\ \zeta_{[0,0,0,0,2,0]} &= [2, 0, -1, 1, 2\alpha + 3, 0] \xrightarrow{\Psi^\alpha} \zeta_{[0,0,0,2,0,1]} = [0, -1, 1, 2\alpha + 3, 0, \alpha + 2] \xrightarrow{s_5} \\ \zeta_{[0,0,0,2,1,0]} &= [0, -1, 1, 2\alpha + 3, \alpha + 2, 0] \xrightarrow{\Psi^\alpha} \zeta_{[0,0,2,1,0,1]} = [-1, 1, 2\alpha + 3, \alpha + 2, 0, \alpha] \\ \xrightarrow{s_5} & \zeta_{[0,0,2,1,1,0]} = [-1, 1, 2\alpha + 3, \alpha + 2, \alpha, 0] \end{aligned}$$

Now, to obtain the vector valued Jack polynomial, it suffices to start from $\begin{smallmatrix} 4 & 3 \\ 6 & 5 & 2 & 1 \end{smallmatrix}$ and act successively with the affine operator Ψ (when reading Ψ^α) and with $s_i \otimes s_i + \frac{1}{\zeta_{[i+1]} - \zeta_{[i]}}$ (when reading s_i).

In conclusion, the computation of vector valued Jack for a given RST is completely independent of the computations of the vector valued Jack indexed by the other RST with the same shape.

4.3 Symmetrization and antisymmetrization

In [2], Baker and Forrester developed the idea of the coefficients and the norm calculation of the symmetric Jack polynomials obtained by symmetrizing the nonsymmetric Jack polynomials. In this section, we generalize their results and obtain symmetric and antisymmetric vector valued Jack polynomials.

For convenience, let us define :

$$(v, \tau).s_i = (v', \tau') \text{ if } (\tau, \zeta, v, \sigma).s_i = (\tau', \zeta', v', \sigma')$$

and

$$(v, \tau).s_i = \emptyset \text{ if } (\tau, \zeta, v, \sigma).s_i = \emptyset.$$

Denote also, $J_\emptyset := 0$.

Let (τ, ζ, v, σ) be a vertex of H_T , set $b_{v,\tau}[i] = \frac{1}{\zeta_{v,\tau}[i+1] - \zeta_{v,\tau}[i]}$ and $c_{v,\tau}[i] = \frac{\zeta_{v,\tau}[i] - \zeta_{v,\tau}[i+1]}{\zeta_{v,\tau}[i] - \zeta_{v,\tau}[i+1] + 1}$.

Note that

$$1 + c_{v,\tau}[i]b_{v,\tau}[i] = c_{v,\tau}[i] \quad (9)$$

and

$$c_{v,\tau}[i](1 - b_{v,\tau}[i]^2) - b_{v,\tau} = 1. \quad (10)$$

Let H_T be a 1-compatible component of G_λ . For each vertex (τ, ζ, v, σ) of H_T , we define the coefficient $\mathfrak{E}_{v,\tau}$ by the following induction:

1. $\mathfrak{E}_{v,\tau} = 1$ if there is no arrows of the form



in H_T .

2. $\mathfrak{E}_{v,\tau} = \frac{\zeta'[i+1] - \zeta'[i]}{\zeta'[i+1] - \zeta'[i] - 1} \mathfrak{E}_{v',\tau'} = \frac{\zeta[i+1] - \zeta[i]}{\zeta[i+1] - \zeta[i] + 1} \mathfrak{E}_{v',\tau'} = c_{v',\tau'} \mathfrak{E}_{v',\tau'}$ if there is an arrow



in H_T .

The symmetric group acts on the spectral vectors ζ by permuting their components. Hence the value of $\mathfrak{E}_{v,\tau}$ does not depend on the path used for its computation and the $\mathfrak{E}_{v,\tau}$ are well defined. Indeed, it suffices to check that the definition is compatible with the commutations $s_i s_j = s_j s_i$ with $|i-j| > 1$ and the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Let us first prove the compatibility with the commutation relations. Suppose

$$(\tau_0, \zeta_0, v_0, \sigma_0) s_i s_j = (\tau_1, \zeta_1, v_1, \sigma_1) s_j = (\tau_2, \zeta_2, v_2, \sigma_2),$$

with $|i-j| > 1$ and

$$(\tau_0, \zeta_0, v_0, \sigma_0) s_j s_i = (\tau'_1, \zeta'_1, v'_1, \sigma'_1) s_i = (\tau'_2, \zeta'_2, v'_2, \sigma'_2).$$

Note that $\tau'_2 = \tau_2$, $\zeta'_2 = \zeta_2$, $v'_2 = v_2$ and $\sigma'_2 = \sigma_2$. But, since the symmetric group acts on ζ by permuting its components, one has

$$\zeta_2[j+1] = \zeta'_1[j+1], \quad \zeta_2[j] = \zeta'_1[j], \quad \zeta_1[j+1] = \zeta'_2[j+1] \text{ and } \zeta_1[j] = \zeta'_2[j].$$

Hence,

$$\begin{aligned} \frac{\zeta_2[j+1]-\zeta_2[j]}{\zeta_2[j+1]-\zeta_2[j]+1} \cdot \frac{\zeta_1[i+1]-\zeta_1[i]}{\zeta_1[i+1]-\zeta_1[i]+1} &= \frac{\zeta_1[i+1]-\zeta_1[i]}{\zeta_1[i+1]-\zeta_1[i]+1} \cdot \frac{\zeta_2[j+1]-\zeta_2[j]}{\zeta_2[j+1]-\zeta_2[j]+1} \\ &= \frac{\zeta'_1[i+1]-\zeta'_1[i]}{\zeta'_1[i+1]-\zeta'_1[i]+1} \cdot \frac{\zeta'_2[j+1]-\zeta'_2[j]}{\zeta'_2[j+1]-\zeta'_2[j]+1}, \end{aligned}$$

and the definition of $\mathfrak{E}_{v,\tau}$ is compatible with the commutations.

Now, let us show that the definition is compatible with the braid relations and set

$$(\tau_0, \zeta_0, v_0, \sigma_0) s_i s_{i+1} s_i = (\tau_1, \zeta_1, v_1, \sigma_1) s_{i+1} s_i = (\tau_2, \zeta_2, v_2, \sigma_2) s_i = (\tau_3, \zeta_3, v_3, \sigma_3),$$

and

$$(\tau_0, \zeta_0, v_0, \sigma_0) s_{i+1} s_i s_{i+1} = (\tau'_1, \zeta'_1, v'_1, \sigma'_1) s_i s_{i+1} = (\tau'_2, \zeta'_2, v'_2, \sigma'_2) s_{i+1} = (\tau'_3, \zeta'_3, v'_3, \sigma'_3).$$

Note that $\tau'_3 = \tau_3$, $\zeta'_3 = \zeta_3$, $v'_3 = v_3$ and $\sigma'_3 = \sigma_3$. Since the symmetric group acts on ζ by permuting its components, one has

$$\begin{aligned} \zeta_3[i+1] &= \zeta'_1[i+1], & \zeta_3[i] &= \zeta'_1[i+1], \\ \zeta_2[i+2] &= \zeta'_2[i+1], & \zeta_2[i+1] &= \zeta'_2[i], \\ \zeta_1[i+1] &= \zeta'_2[i+2] & \text{and } \zeta_1[i] &= \zeta'_3[i+1]. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\zeta_3[i+1]-\zeta_3[i]}{\zeta_3[i+1]-\zeta_3[i]+1} \cdot \frac{\zeta_2[i+2]-\zeta_2[i+1]}{\zeta_2[i+2]-\zeta_2[i+1]+1} \cdot \frac{\zeta_1[i+1]-\zeta_1[i]}{\zeta_1[i+1]-\zeta_1[i]+1} &= \frac{\zeta_1[i+1]-\zeta_1[i]}{\zeta_1[i+1]-\zeta_1[i]+1} \cdot \frac{\zeta_2[i+2]-\zeta_2[i+1]}{\zeta_2[i+2]-\zeta_2[i+1]+1} \cdot \frac{\zeta_3[i+1]-\zeta_3[i]}{\zeta_3[i+1]-\zeta_3[i]+1} \\ &= \frac{\zeta'_3[i+2]-\zeta'_3[i+1]}{\zeta'_3[i+2]-\zeta'_3[i+1]+1} \cdot \frac{\zeta'_2[i+1]-\zeta'_2[i]}{\zeta'_2[i+1]-\zeta'_2[i]+1} \cdot \frac{\zeta'_1[i+2]-\zeta'_1[i+1]}{\zeta'_1[i+2]-\zeta'_1[i+1]+1}, \end{aligned}$$

and the definition is compatible with the braid relations.

Define the symmetrization operator

$$\mathfrak{S} := \sum_{\omega \in \mathfrak{S}_N} \omega \otimes \omega.$$

We will say that a polynomial is symmetric if it is invariant by the action of $s_i \otimes s_i$ for each $i < N$.

Theorem 4.7 1. Let H_T be a connected component of G_λ . For each vertex (τ, ζ, v, σ) of H_T , the polynomial $J_{v,\tau} \mathfrak{S}$ equals $J_{\lambda_T, \text{std}(T)} \mathfrak{S}$ up to a multiplicative constant.

2. One has $J_{\lambda_T, \text{std}(T)} \mathfrak{S} \neq 0$ if and only if H_T is 1-compatible.

3. More precisely, when H_T is 1-compatible, the polynomial

$$J_T = \sum_{(\tau, \zeta, v, \sigma) \text{ vertex of } H_T} \mathfrak{E}_{v,\tau} J_{v,\tau}$$

is symmetric.

Proof

1. Let us prove the first assertion by induction on the length of a path from (τ, ζ, v, σ) to $(\text{std}(T), \zeta_T, \lambda_T, \sigma)$ in H_T . Let $(\tau', \zeta', v', \sigma')$ such that

$$\boxed{(\tau, \zeta, v, \sigma)} \xrightarrow{s_i} \boxed{(\tau', \zeta', v', \sigma')}$$

is not a jump in H_T (hence, $-1 < b_{v,\tau}[i] < 1$). It follows that

$$\begin{aligned} J_{v,\tau} \mathfrak{S} &= \frac{1}{1-b_{v,\tau}[i]^2} (J_{v',\tau'} s_i \otimes s_i + b_{v,\tau}[i] J_{v',\tau'}) \cdot \mathfrak{S} \\ &= \frac{1}{1-b_{v,\tau}[i]^2} (1 + b_{v,\tau}[i]) J_{v',\tau'} \cdot \mathfrak{S}. \end{aligned}$$

By induction $J_{v',\tau'} \mathfrak{S}$ is proportional to $J_{\lambda_T, \text{std}(T)}$, which ends the proof.

2. If H_T is not 1-compatible, then there exists s_i such that $J_{\lambda_T, \text{std}(T)} \cdot s_i \otimes s_i = -J_{\lambda_T, \text{std}(T)}$. Hence, since $\mathfrak{S} = s_i \otimes s_i \mathfrak{S}$, one obtains $J_{\lambda_T, \text{std}(T)} \cdot \mathfrak{S} = 0$.

3. Let us prove that, when H_T is 1-compatible, $J_T.s_i \otimes s_i = J_T$ for any i . Fix i and decompose $J_T := J^+ + J_0 + J^-$ where

$$J^+ = \sum_{\tau, v}^+ \mathfrak{E}_{\tau, v} J_{v, \tau},$$

where \sum^+ means that the sum is over the pairs (τ, v) such that there exists an arrow $\boxed{(\tau', \zeta', v', \sigma')} \xrightarrow{s_i} \boxed{(\tau, \zeta, v, \sigma)}$ in H_T ,

$$J^- = \sum_{\tau, v}^- \mathfrak{E}_{\tau, v} J_{v, \tau}$$

where \sum^- means that the sum is over the pairs (τ, v) such that there exists an arrow $\boxed{(\tau, \zeta, v, \sigma)} \xrightarrow{s_i} \boxed{(\tau', \zeta', v', \sigma')}$ in H_T and

$$J_0 = \sum^0 \mathfrak{E}_{\tau, v} J_{v, \tau}.$$

where \sum^0 means that the sum is over the pairs (τ, v) such that there exists an arrow $\boxed{(\tau, \zeta, v, \sigma)} \xrightarrow{s_i} \boxed{\emptyset}$ in G_T (equivalently there is no arrow from (τ, ζ, v, σ) labeled by s_i in H_T). Suppose that $\boxed{(\tau, \zeta, v, \sigma)} \xrightarrow{s_i} \boxed{\emptyset}$ is a fall in G_T , then

$$J_{v, \tau} s_i \otimes s_i = J_{(v, \tau) s_i} - b_{v, \tau}[i] J_{v, \tau} = -b_{v, \tau}[i] J_{v, \tau}.$$

Since, H_T is 1-compatible proposition 2.24 implies that i and $i+1$ are in the same line. Hence, $b_{v, \tau}[i] = -1$ and $J_{v, \tau} s_i \otimes s_i = J_{v, \tau}$. It follows that $J_0.s_i \otimes s_i = J_0$.

Now, let $\boxed{(\tau, \zeta, v, \sigma)} \xrightarrow{s_i} \boxed{(\tau', \zeta', v', \sigma')}$ be an arrow in H_T , then

$$J_{v, \tau} s_i \otimes s_i = J_{v', \tau'} - b_{v, \tau}[i] J_{v, \tau},$$

$$J_{v', \tau'} s_i \otimes s_i = b_{v, \tau}[i] J_{v', \tau'} + (1 - b_{v, \tau}[i]^2) J_{v, \tau}$$

and

$$\mathfrak{E}_{v', \tau'} = c_{v, \tau} \mathfrak{E}_{v, \tau}.$$

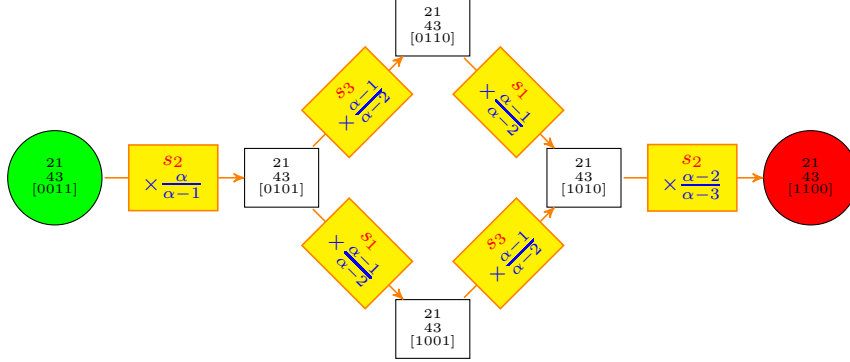
Hence, equalities (9) and (10) imply

$$\begin{aligned} (\mathfrak{E}_{v, \tau} J_{v, \tau} + \mathfrak{E}_{v', \tau'} J_{v', \tau'}) s_i \otimes s_i &= \mathfrak{E}_{v, \tau} (J_{v, \tau} + c_{v, \tau}[i] J_{v', \tau'}) . s_i \otimes s_i \\ &= \mathfrak{E}_{v, \tau} (((c_{v, \tau}[i](1 - b_{v, \tau}[i]^2) - b_{v, \tau}[i]) J_{v, \tau} \\ &\quad + (1 + c_{v, \tau}[i] b_{v, \tau}[i]) J_{v', \tau'}) \\ &= \mathfrak{E}_{v, \tau} (J_{v, \tau} + c_{v, \tau}[i] J_{v', \tau'}) \\ &= (\mathfrak{E}_{v, \tau} J_{v, \tau} + \mathfrak{E}_{v', \tau'} J_{v', \tau'}). \end{aligned}$$

This proves that $(J^+ + J^-)s_i \otimes s_i = J^+ + J^-$. Hence, $J_T \cdot s_i \otimes s_i = J_T$ for each i and J is symmetric.

□

Example 4.8 Consider the graph $H_{11}^{11}_{00}$



The polynomial

$$J_{11}^{11}_{00} = J_{0011,43}^{21} + \frac{\alpha}{\alpha-1} J_{0101,43}^{21} + \frac{\alpha}{\alpha-2} J_{0110,43}^{21} + \frac{\alpha}{\alpha-2} J_{1001,43}^{21} + \frac{\alpha(\alpha-1)}{(\alpha-2)^2} J_{1010,43}^{21} + \frac{\alpha(\alpha-1)}{(\alpha-2)(\alpha-3)} J_{1100,43}^{21}$$

is symmetric.

Let H_T be a connected component, denote by $\text{root}(T)$ the only vertex of H_T without inward edge and by $\text{sink}(T) = (\text{std}(T), \zeta_T, \lambda_T, Id)$ the only vertex of H_T without outward edge. Denote by $\#H_T$ the number of vertices of H_T . The following proposition allows to compare the polynomial J_T to the symmetrized of $J_{\text{root}(T)}$.

Proposition 4.9 *One has*

$$J_T = \frac{\#H_T}{N!} \mathfrak{E}_{\text{sink}(T)} J_{\text{root}(T)} \cdot \mathfrak{S}.$$

Proof It suffices to compare the coefficient of $J_{\text{sink}(T)}$ in J_T and in $J_{\text{root}(T)} \cdot \mathfrak{S}$. The coefficient of $J_{\text{sink}(T)}$ in J_T equals $\mathfrak{E}_{\text{sink}(T)}$ whilst the coefficient of $J_{\text{sink}(T)}$ in $J_{\text{root}(T)} \cdot \mathfrak{S}$ equals $\frac{N!}{\#H}$. Indeed $\frac{N!}{\#H}$ is the order of the stabilizer of λ_T . The leading term of $J_{\text{sink}(T)}$ does not appear in any other $J_{v,\tau}$ so its coefficient in the symmetrization of $J_{\text{root}(T)}$ equals the order of the stabilizer. □

Let H_T be a (-1) -compatible component of G_λ . For each vertex (τ, ζ, v, σ) of H_T , we define the coefficient $\mathfrak{F}_{v,\tau}$ by the following induction:

1. $\mathfrak{F}_{v,\tau} = 1$ if there is no arrows of the form



in H_T .

2. $\mathfrak{F}_{v,\tau} = -\frac{\zeta[i]-\zeta[i+1]}{\zeta[i]-\zeta[i+1]+1}\mathfrak{F}_{v',\tau'} = -\frac{\zeta'[i+1]-\zeta[i]}{\zeta'[i+1]-\zeta[i]+1}\mathfrak{F}_{v',\tau'}$ if there is an arrow



in H_T .

Again the $\mathfrak{F}_{v,\tau}$ are well defined since the symmetric group acts on the spectral vectors by permuting their components. Define also the symmetrization operator

$$\mathfrak{A} := \sum_{\omega \in \mathfrak{S}_N} (-1)^{\ell(\omega)} \omega \otimes \omega.$$

We will say that a polynomial is antisymmetric if it vanishes through the action of $1 - s_i \otimes s_i$ for each $i < N$.

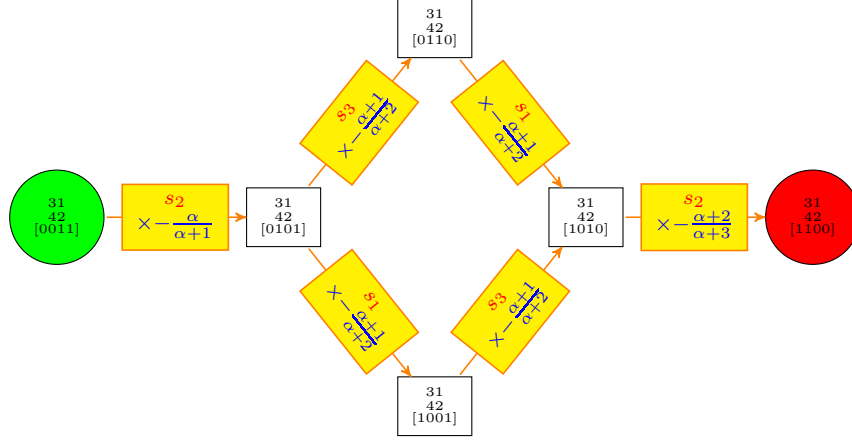
Theorem 4.10 1. Let H_T be a connected component of G_λ . For each vertex (τ, ζ, v, σ) of H_T , the polynomial $J_{v,\tau}\mathfrak{A}$ equals $J_{\lambda_T, \text{std}(T)}\mathfrak{A}$ up to a multiplicative constant.

2. One has $J_{\lambda_T, \text{std}(T)}\mathfrak{A} \neq 0$ if and only if H_T is (-1) -compatible.
 3. More precisely, when H_T is (-1) -compatible, the polynomial

$$J'_T = \sum_{(\tau, \zeta, v, \sigma) \text{ vertex of } H_T} \mathfrak{F}_{v,\tau} J_{v,\tau}$$

is antisymmetric.

Example 4.11 Consider the graph H_{01}



The polynomial

$$J'_{01} = J_{0011,42} - \frac{\alpha}{\alpha+1} J_{0101,42} + \frac{\alpha}{\alpha+2} J_{0110,42} + \frac{\alpha}{\alpha+2} J_{1001,42} - \frac{\alpha(\alpha+1)}{(\alpha+2)^2} J_{1010,42} + \frac{\alpha(\alpha+1)}{(\alpha+2)(\alpha+3)} J_{1100,42}$$

is antisymmetric.

And, as in the symmetric case, one has:

Proposition 4.12 *One has*

$$J_T = \frac{\#H_T}{N!} \mathfrak{F}_{\text{sink}(T)} J_{\text{root}(T)} \cdot \mathfrak{A}.$$

4.4 Normalization

The space V_λ spanned by the RST τ of the same shape λ is naturally endowed (up to a multiplicative constant) by \mathfrak{S}_N -invariant scalar product $\langle \cdot, \cdot \rangle_0$ for which the RST are pairwise orthogonal. As in [4], we set

$$\|\tau\|^2 = \prod_{\substack{1 \leq i < j \leq N \\ \text{CT}_\tau[i] < \text{CT}_\tau[j]-1}} \frac{(\text{CT}_\tau[i] - \text{CT}_\tau[j] - 1)(\text{CT}_\tau[i] - \text{CT}_\tau[j] + 1)}{(\text{CT}_\tau[i] - \text{CT}_\tau[j])^2}.$$

As in [4], we consider the contravariant form $\langle \cdot, \cdot \rangle$ on the space M_λ which is the symmetric \mathfrak{S}_N -invariant form extending $\langle \cdot, \cdot \rangle_0$ and such that the Dunkl operator \mathfrak{D}_i is the adjoint to the multiplication by x_i .

Since \mathfrak{U}_i is self-adjoint for the form $\langle \cdot, \cdot \rangle$, the polynomials $J_{v,\tau}$ are pairwise orthogonal.

Let us compute their squared norms $\|J_{v,\tau}\|^2$ (the bilinear form is nonsingular for generic α and positive definite for α in some subset of \mathbb{R} [6]). The method is essentially the same than in [4] and we show that the result can be read in the Yang-Baxter graph. More precisely, one has

Proposition 4.13 1. $\|J_{(v,\tau).s_i}\|^2 = \frac{(\zeta_{v,\tau}[i+1] - \zeta_{v,\tau}[i] - 1)(\zeta_{v,\tau}[i+1] - \zeta_{v,\tau}[i] + 1)}{(\zeta_{v,\tau}[i+1] - \zeta_{v,\tau}[i])^2} \|J_{v,\tau}\|^2$.

2. $\|J_{(v,\tau).\Psi}\|^2 = \left(\frac{1}{\alpha}\zeta_{v,\tau}[1] + 1\right) \|J_{v,\tau}\|^2$.

Proof

1. Since

$$J_{(v,\tau).s_i} = J_{v,\tau} \left(s_i \otimes s_i + \frac{1}{\zeta_{v,\tau}[i+1] - \zeta_{v,\tau}[i]} \right),$$

one obtains

$$\|J_{(v,\tau).s_i} \otimes s_i\|^2 = \|J_{(v,\tau).s_i}\|^2 + \frac{1}{(\zeta_{v,\tau}[i+1] - \zeta_{v,\tau}[i])^2} \|J_{v,\tau}\|^2.$$

But $\|J_{(v,\tau).s_i} \otimes s_i\|^2 = \|J_{v,\tau}\|^2$, which gives the result.

2. One has

$$\begin{aligned} \|J_{(v,\tau).\Psi}\|^2 &= \|J_{v,\tau}.\theta \otimes \theta x_N\|^2 \\ &= \langle J_{v,\tau}.\theta \otimes \theta, J_{v,\tau}.\theta \otimes \theta x_N \mathfrak{D}_N \rangle \\ &= \langle J_{v,\tau}.\theta \otimes \theta, J_{v,\tau} x_1 \mathfrak{D}_1.\theta \otimes \theta \rangle \\ &= \langle J_{v,\tau}, J_{v,\tau} x_1 \mathfrak{D}_1 \rangle; \end{aligned}$$

recall that $\theta = s_1 s_2 \dots s_{N-1}$. Since $\mathfrak{U}_1 = x_1 \mathfrak{D}_1$, one obtains the results.

□

Example 4.14 Let again $\tau = \begin{smallmatrix} 2 \\ 3 & 1 \end{smallmatrix}$, we compute the normalization following the Yang-Baxter graph (see Fig. 8)
For instance:

$$\begin{aligned} \|J_{[020],\tau}\|^2 &= \|\tau\|^2 \left(1 + \frac{1}{\alpha}\right) \left(\frac{\alpha(\alpha+2)}{(\alpha+1)^2}\right) \left(\frac{(\alpha+1)(\alpha+3)}{(\alpha+2)^2}\right) \left(2 + \frac{1}{\alpha}\right) \left(\frac{2\alpha(2\alpha+2)}{(2\alpha+1)^2}\right) \\ &= \frac{4(\alpha+3)(\alpha+1)}{(2\alpha+1)(\alpha+2)} \end{aligned}$$

As a consequence of proposition 4.13, one deduces the following result using theorems 4.7 and 4.10.

Corollary 4.15 *Let H_T be a connected component and (τ, ζ, v, σ) be a vertex of H_T . Denote by $\ell_{\tau,v}^T$ the length of a path from $\text{root}(T)$ to (τ, ζ, v, σ) . One has,*

$$\|J_{v,\tau}\|^2 = (-1)^{\ell_{\tau,v}^T} \mathfrak{E}_{v,\tau}^{-1} \mathfrak{F}_{v,\tau}^{-1} \|J_{\text{root}(T)}\|^2.$$

From theorems 4.7 and 4.10, vector valued symmetric and antisymmetric Jack polynomials are also pairwise orthogonal.

Proposition 4.16 *1. Let H_{T_1} and H_{T_2} be two 1-compatible connected components. If $T_1 \neq T_2$ then $\langle J_{T_1} | J_{T_2} \rangle = 0$.*

2. Let H_{T_1} and H_{T_2} be two (-1) -compatible connected components. If $T_1 \neq T_2$ then $\langle J'_{T_1} | J'_{T_2} \rangle = 0$.

Proof It suffices to remark that from theorem 4.7 (resp. theorem 4.10) each J_T (resp. J'_T) is a linear combination of $J_{v,\tau}$ for (τ, ζ, v, σ) vertex in the connected component H_T . \square

In the special cases when H_T is ± 1 -compatible, the value of $\|J_T\|^2$ admits a remarkable equality.

Proposition 4.17 *One has:*

1. If H_T is a 1-compatible connected component then

$$\|J_T\|^2 = \#H_T \mathfrak{E}_{\text{sink}(T)} \|J_{\text{root}(T)}\|^2.$$

2. If H_T is a (-1) -compatible connected component then

$$\|J'_T\|^2 = \#H_T \mathfrak{F}_{\text{sink}(T)} \|J_{\text{root}(T)}\|^2.$$

Proof The case being very similar, let us only prove the symmetric case. From proposition 4.9, one has:

$$\begin{aligned} \|J_T\|^2 &= \frac{\#H_T}{N!} \mathfrak{E}_{\text{sink}(T)} \langle J_T, J_{\text{root}(T)} \cdot \mathfrak{S} \rangle \\ &= \frac{\#H_T}{N!} \mathfrak{E}_{\text{sink}(T)} \sum_{\sigma \in \mathfrak{S}_N} \langle J_T, J_{\text{root}(T)} \cdot \sigma \otimes \sigma \rangle \\ &= \#H_T \mathfrak{E}_{\text{sink}(T)} \|J_{\text{root}(T)}\|^2 \end{aligned}$$

\square

From corollary 4.15 and theorem 4.17, one obtains the surprising equalities:

Corollary 4.18 *If H_T is 1-compatible, one has:*

$$\sum_{(\tau, \zeta, v, \sigma) \text{ vertex of } H_T} (-1)^{\ell_{v, \tau}^T} \frac{\mathfrak{E}_{v, \tau}}{\mathfrak{F}_{v, \tau}} = \#H_T \mathfrak{E}_{\text{sink}(T)}. \quad (11)$$

If H_T is (-1) -compatible, one has:

$$\sum_{(\tau, \zeta, v, \sigma) \text{ vertex of } H_T} (-1)^{\ell_{v, \tau}^T} \frac{\mathfrak{F}_{v, \tau}}{\mathfrak{E}_{v, \tau}} = \#H_T \mathfrak{F}_{\text{sink}(T)}. \quad (12)$$

Example 4.19 *Consider the graph H_{11}^{00} , the sum (11) gives*

$$1 + \frac{\alpha + 1}{\alpha - 1} \left(1 + \frac{\alpha}{\alpha - 2} \left(2 + \frac{\alpha}{\alpha - 2} \left(1 + \frac{\alpha - 1}{\alpha - 3} \right) \right) \right) = 6 \frac{\alpha(\alpha - 1)}{(\alpha - 2)(\alpha - 3)}$$

as expected.

4.5 Symmetric and antisymmetric polynomials with minimal degree

Since the irreducible characters of \mathfrak{S}_N are real it follows that the tensor product of an irreducible module with itself contains the trivial representation exactly once. The tensor product of the module corresponding to a partition λ with the module for ${}^t\lambda$ (the transpose) contains the determinant representation exactly once. We demonstrate these facts explicitly. Using the concepts from Section 4.1 let

$$\zeta_1 = \sum_{\tau \in \text{Tab}_\lambda} a(\tau) \tau \otimes \tau \in V_\lambda \otimes V_\lambda$$

be symmetric with (rational) coefficients $a(\tau)$ to be determined. We impose the conditions $\zeta_1 \cdot s_i \otimes s_i = \zeta_1$ for $i = 1, \dots, N - 1$. Fix some i and write

$$\begin{aligned} \zeta_1 = & \sum \{a(\tau) \tau \otimes \tau : b_\tau[i] = \pm 1\} \\ & + \sum \left\{ a(\tau) \tau \otimes \tau + a(\tau^{(i, i+1)}) \tau^{(i, i+1)} \otimes \tau^{(i, i+1)} : 0 < b_\tau[i] \leq \frac{1}{2} \right\}. \end{aligned}$$

In the first sum $(\tau \otimes \tau) \cdot (s_i \otimes s_i) = b_\tau [i]^2 \tau \otimes \tau = \tau \otimes \tau$. For the second sum, note that $\tau^{(i,i+1)} \cdot s_i = (1 - b_\tau [i]^2) \tau - b_\tau [i] \tau^{(i,i+1)}$. Simple computations show that

$$\begin{aligned} & (a(\tau) \tau \otimes \tau + a(\tau^{(i,i+1)}) \tau^{(i,i+1)} \otimes \tau^{(i,i+1)}) \cdot (s_i \otimes s_i) \\ &= a(\tau) \tau \otimes \tau + a(\tau^{(i,i+1)}) \tau^{(i,i+1)} \otimes \tau^{(i,i+1)} \end{aligned}$$

exactly when $a(\tau) = (1 - b_\tau [i]^2) a(\tau^{(i,i+1)})$. The unique (up to a constant multiple) \mathfrak{S}_N -invariant norm on V_τ satisfies $\|\tau^{(i,i+1)}\|^2 = (1 - b_\tau [i]^2) \|\tau\|^2$ (see 4.4); thus $a(\tau) = c/\|\tau\|^2$ for some constant c .

Consider the module $V_{i\lambda}$. The transpose map takes each RST τ with shape λ to the RST ${}^t\tau$ of shape ${}^t\lambda$. Thus $b_{t_\tau} [i] = -b_\tau [i]$ for $1 \leq i \leq N$. Suppose $0 < b_\tau [i] \leq \frac{1}{2}$ for some τ and i , then $-\frac{1}{2} \leq b_{t_\tau} [i] < 0$ and the following transformation rules apply:

$$\begin{aligned} {}^t\tau \cdot s_i &= b_{t_\tau} [i] {}^t\tau + (1 - b_{t_\tau} [i]^2) ({}^t\tau^{(i,i+1)}), \\ {}^t\tau^{(i,i+1)} \cdot s_i &= {}^t\tau - b_{t_\tau} [i] {}^t\tau^{(i,i+1)}. \end{aligned}$$

Let

$$\zeta_{\det} = \sum_{\tau \in \text{Tab}_\lambda} a(\tau) ({}^t\tau) \otimes \tau \in V_{i\lambda} \otimes V_\lambda$$

be antisymmetric with (rational) coefficients $a(\tau)$ to be determined. We impose the conditions $\zeta_{\det} \cdot s_i \otimes s_i = -\zeta_{\det}$ for $i = 1, \dots, N-1$. Fix some i and write

$$\begin{aligned} \zeta_{\det} &= \sum \{a(\tau) ({}^t\tau) \otimes \tau : b_\tau [i] = \pm 1\} \\ &+ \sum \left\{ a(\tau) ({}^t\tau) \otimes \tau + a(\tau^{(i,i+1)}) ({}^t\tau^{(i,i+1)}) \otimes \tau^{(i,i+1)} : 0 < b_\tau [i] \leq \frac{1}{2} \right\}. \end{aligned}$$

In the first sum $(\tau \otimes ({}^t\tau)) \cdot (s_i \otimes s_i) = b_\tau [i] b_{t_\tau} [i] ({}^t\tau) \otimes \tau = -({}^t\tau) \otimes \tau$. We find that

$$\begin{aligned} & (a(\tau) ({}^t\tau) \otimes \tau + a(\tau^{(i,i+1)}) ({}^t\tau^{(i,i+1)}) \otimes \tau^{(i,i+1)}) \cdot (s_i \otimes s_i) \\ &= - (a(\tau) ({}^t\tau) \otimes \tau + a(\tau^{(i,i+1)}) ({}^t\tau^{(i,i+1)}) \otimes \tau^{(i,i+1)}) \end{aligned}$$

exactly when $a(\tau) = -a(\tau^{(i,i+1)})$. Thus $a(\tau) = c(-1)^{\text{inv}(\tau)}$ (recall $\text{inv}(\tau) = \#\{(i, j) : 1 \leq i < j \leq N, \text{rw}(i, \tau) > \text{rw}(j, \tau)\}$, and $0 < b_\tau [i] \leq \frac{1}{2}$ implies $\text{inv}(\tau^{(i,i+1)}) = \text{inv}(\tau) + 1$).

We can now write down the symmetric and antisymmetric Jack polynomials of lowest degree, by replacing the first factors in ζ_1 and ζ_{\det} by the corresponding polynomials $P_\tau(x)$ and $P_{t\tau}(x)$ (as constructed in Section 3). Let $l = \ell(\lambda) = {}^t\lambda[1]$.

In the symmetric case let $v = \left[(l-1)^{\lambda[l]}, (l-2)^{\lambda[l-1]}, \dots, 1^{\lambda[2]}, 0^{\lambda[1]} \right]$ (using exponents to indicate the multiplicity of an entry) The corresponding tableau is

$$T_1 := \begin{array}{cccccc} l-1 & \dots & l-1 & & & (\lambda[l] \times) \\ \vdots & & & \ddots & & \vdots \\ 1 & \dots & \dots & \dots & 1 & (\lambda[2] \times) \\ 0 & \dots & \dots & \dots & \dots & 0 \quad (\lambda[1] \times) \end{array}$$

and $\text{std}(T_1)$ contains the numbers $N, N-1, \dots, 2, 1$ entered row-by-row.

Example 4.20 If $\lambda = [4, 3, 2]$ then $v = [221110000]$. The corresponding tableau is $T_1 = \begin{array}{ccc} 2 & 2 & \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}$ and $\text{std}(T_1) = \begin{array}{ccc} 5 & 4 & 3 \\ 9 & 8 & 7 \\ 6 & & \end{array}$.

In

$$\zeta_1(x) = \sum_{\tau \in \text{Tab}_\lambda} \frac{c}{\|\tau\|^2} P_\tau(x) \otimes \tau,$$

the monomial x^v occurs only when $\tau = \text{std}(T_1)$, with coefficient $c / \|\text{std}(T_1)\|^2$. This polynomial is a multiple of J_{T_1} (see Theorem 4.7).

For the antisymmetric case let

$$T_{\det} := \begin{array}{ccccccc} 0 & 1 & \dots & \lambda_l - 1 & & & \\ \vdots & \vdots & & \vdots & \ddots & & \\ 0 & 1 & \dots & \lambda_l - 1 & \dots & \lambda_2 - 1 & \\ 0 & 1 & \dots & \lambda_l - 1 & \dots & \lambda_2 - 1 & \dots & \lambda_1 - 1 \end{array}$$

Thus $\text{std}(T_{\det}) = \tau_\lambda$ and $v = \left[(\lambda[1] - 1)^{t\lambda[\lambda[1]]}, (\lambda[1] - 2)^{t\lambda[\lambda[1]-1]}, \dots, 0^{t\lambda[1]} \right]$.

Example 4.21 If $\lambda = [4, 3, 2]$ then ${}^t\lambda = [3, 3, 2, 1]$ and $v = [322111000]$. The

corresponding tableau is $T_{\det} = \begin{array}{cccc} & 0 & 1 & \\ & 0 & 1 & 2 \\ & 0 & 1 & 2 & 3 \end{array}$ and

$$\text{std}(T_{\det}) = \begin{array}{cccc} & 7 & 4 & \\ & 8 & 5 & 2 \\ & 9 & 6 & 3 & 1 \end{array} = \tau_{[4,3,2]}.$$

Let

$$\zeta_{\det}(x) = \sum_{\tau \in \text{Tab}_{\lambda}} (-1)^{\text{inv}(\tau)} P_{t_{\tau}}(x) \otimes \tau.$$

The monomial x^v occurs only in the term $\tau = \tau_{\lambda}$ (see Definition 2.9). This polynomial is a constant multiple of $J'_{T_{\det}}$ (see Theorem 4.10).

We summarize the results of this section in the following theorem.

Theorem 4.22 *The subspace of M_{λ} of the symmetric (resp. antisymmetric) polynomials with minimal degree is spanned by only one generator: the symmetric (resp. antisymmetric) Jack polynomial J_{T_1} (resp. $J_{T_{\det}}$).*

As a consequence one observes a surprising property.

Corollary 4.23 *The Jack polynomial J_{T_1} (resp. $J_{T_{\det}}$) equals a polynomial which does not depend on the parameter α multiplied by the global multiplicative constant $\mathfrak{C}_{\text{sink}(T_1)}$ (resp. $\mathfrak{F}_{\text{sink}(T_{\det})}$).*

Proof The first part of the sentence is a consequence of theorem 4.22 since the dimension of the space is 1. The values of the multiplicative constants follow from Theorems 4.7 and 4.10 together with the fact that the coefficient of the leading terms in an Jack polynomials $J_{v,\tau}$ is 1 (see Theorem 4.1). \square

Remark also that T_1 (resp. T_{\det}) is not the only tableau for which the corresponding symmetric (resp. antisymmetric) Jack does not depend on α (up to a global multiplicative constant).

Example 4.24 Consider the partition $\lambda = [221]$ together with the vector $v = [2, 1, 1, 0, 0]$. The corresponding symmetric Jack $\frac{1}{\mathfrak{C}_{21100, \frac{1}{32}} J_2 \frac{11}{00}}$ has no term with α in its expansion on the monomials.

There are two symmetric Jack polynomials in degree 5: $J_{\frac{12}{00}}^2$ and $J_{\frac{11}{00}}^3$. Remark that the (non minimal) polynomial $\frac{1}{\mathfrak{e}_{22100, \frac{1}{32}, \frac{12}{00}}} J_{\frac{12}{00}}^2$ does not depend on α whilst the parameter α appears in $\frac{1}{\mathfrak{e}_{31100, \frac{1}{54}, \frac{11}{00}}} J_{\frac{11}{00}}^3$ even after simplifying the expression.

4.6 Restrictions

Consider the linear map \downarrow_M which consists to send each x_i to 0 when $i > M$ and each τ to $\tau \downarrow_M$.

Theorem 4.25 *Let τ having the $R(M)$ property. Then*

$$J_{[v[1], \dots, v[M], 0, \dots, 0], \tau \cdot \downarrow_M} = J_{[v[1], \dots, v[M]], \tau \downarrow_M}$$

Proof From proposition 2.31 the graph $\overline{G_{\tau \downarrow_M}}$ and $G_{\text{shape}(\tau \downarrow_M)}$ are the same.

Remark that:

1. The result is correct for the roots:

$$J_{([0, \dots, 0], \tau_{root}) \downarrow_M} = J_{[0, \dots, 0], \tau_{root} \downarrow_M}$$

where τ_{root} denotes the RST obtained from τ by replacing subtableau constituted with the nodes labeled $1, \dots, M$ by $\tau_{\text{shape}(\tau \downarrow_M)}$.

2. The action of the edges are compatible with the restriction:

- (a) The non-affine edges: We use only the difference between the content of two cases. Since, the values of the spectral vector ζ in $\overline{G_{\tau_{root} \downarrow_M}}$ are obtained from the values of the spectral vector in $G_{\tau_{root}}$ by adding to each component the same integer. Hence, the differences are the same and then the action of the non-affine edges are the same.

- (b) The affine edges: One has to verify that

$$J_{[v[1], \dots, v[M], 0, \dots, 0], \tau \cdot \Psi_N(s_{N-1} \otimes s_{N-1} + (*)) \dots (s_M \otimes s_M + (*)) \downarrow_M} = J_{[v[1], \dots, v[M], 0, \dots, 0], \tau \cdot \downarrow_M} \Psi_M,$$

where $(*)$ denote the correct rational numbers corresponding to the edges of the Yang-Baxter graph and Ψ_M means that one applies the operator Ψ for an alphabet of size M . This identity is easy to obtain from the construction: since Ψ gives a polynomial whose a factor is x_N , the only non vanishing part of

$$J_{[v[1], \dots, v[M], 0, \dots, 0], \tau} \cdot \Psi_N (s_{N-1} \otimes s_{N-1} + (*)) \dots (s_M \otimes s_M + (*)) \downarrow_M$$

is

$$J_{[v[1], \dots, v[M], 0, \dots, 0], \tau} \cdot \Psi_N s_{N-1} \dots s_M \otimes s_{N-1} \dots s_M \downarrow_M = J_{[v[1], \dots, v[M], 0, \dots, 0], \tau} \cdot \Psi_M \downarrow_M,$$

the last part of the proof follows from the commutation $\Psi_M \downarrow_M = \downarrow_M \Psi_M$.

This shows that the polynomials $J_{[v[1], \dots, v[M], 0, \dots, 0], \tau} \downarrow_M$ are inductively generated following the same Yang-Baxter graph as the polynomials $J_{[v[1], \dots, v[M]], \tau} \downarrow_M$ with the same initial conditions. Hence $J_{[v[1], \dots, v[M], 0, \dots, 0], \tau} \downarrow_M = J_{[v[1], \dots, v[M]], \tau} \downarrow_M$ as expected. \square

5 Shifted vector valued Jack polynomials

5.1 Knop and Sahi operators for vector valued polynomials

Let us define the following operators which are the vector valued versions of the operators defined in [9]:

1. $\varsigma_i := \partial_i \otimes 1 + s_i \otimes s_i$, where $\partial_i := \partial_{i, i+1} = (1 - s_i) \cdot \frac{1}{x_i - x_{i+1}}$ is a divided difference.
2. Denote by Φ the operator sending each x_i to x_{i-1} for $i > 1$ and x_1 to $x_N - \alpha$ and $T := \Phi \otimes s_1 s_2 \dots s_{N-1}$,
3. $\hat{\Psi} := T \cdot (x_N + N - 1)$.

Proposition 5.1 *The operators ς_i satisfy the braid relations*

$$\varsigma_i \varsigma_{i+1} \varsigma_i = \varsigma_{i+1} \varsigma_i \varsigma_{i+1}, \quad \varsigma_i \varsigma_j = \varsigma_j \varsigma_i, \quad |i - j| > 1,$$

and the relations between the ς_i and the multiplication by the indeterminates are given by the Leibniz rules:

$$x_i \varsigma_i = \varsigma_{i+1} x_i + 1, \quad x_{i+1} \varsigma_i = \varsigma_i x_i - 1, \quad x_j \varsigma_i = \varsigma_i x_j, \quad j \neq i, i + 1.$$

Proof The Leibniz rules are straightforward from the definition whilst the braid relations are a direct consequence of

1. The braid relations on the s_i and the braid relations on the ∂_i ,
2. $\partial_{i+1} \partial_i s_i + s_{i+1} \partial_i \partial_{i+1} = \partial_i s_{i+1} \partial_i$,
3. $\partial_i \partial_{i+1} s_i + s_i \partial_{i+1} \partial_i = \partial_{i+1} s_i \partial_{i+1}$,
4. $s_i \partial_{i+1} s_i = s_{i+1} \partial_i s_{i+1}$,
5. $\partial_i s_{i+1} s_i = s_{i+1} s_i \partial_{i+1}$,
6. $s_i s_{i+1} \partial_i = \partial_{i+1} s_i s_{i+1}$.

□

Since the ς_i verify the braid relations, they represent the braid group : given a permutation $\omega \in \mathfrak{S}_N$ and a reduced decomposition $\omega = s_{i_1} \dots s_{i_k}$, the product $\varsigma_{i_1} \dots \varsigma_{i_k}$ is independent of the choice of the reduced decomposition. We will denote $\varsigma_\omega := \varsigma_{i_1} \dots \varsigma_{i_k}$.

Furthermore, the algebra generated by the ς_i and the x_j is isomorphic to the degenerate Hecke affine algebra.

Our goal is to find a basis of simultaneous eigenvectors of the following operators

$$\hat{\xi}_i := x_i + N - 1 - \varsigma_i \dots \varsigma_{N-1} \hat{\Psi} \varsigma_1 \dots \varsigma_{i-1}.$$

These operators commute and play the role of Cherednik elements for our representation of the degenerate Hecke affine algebra. As a consequence, one has the following relations:

Proposition 5.2 1. $\varsigma_i \hat{\xi}_{i+1} = \hat{\xi}_i \varsigma_i - 1$,

$$2. \quad \varsigma_i \hat{\xi}_i = \hat{\xi}_{i+1} \varsigma_i + 1,$$

3. $\varsigma_i \hat{\xi}_j = \hat{\xi}_j \varsigma_i$ for $j \neq i, i + 1$,
4. $\hat{\Psi} \hat{\xi}_i = \hat{\xi}_{i-1} \hat{\Psi}$ for $i \neq 1$,
5. $\hat{\Psi} \hat{\xi}_1 = (\hat{\xi}_N - \alpha) \hat{\Psi}$.

Furthermore, the RST are simultaneous eigenfunctions of the operators $\hat{\xi}_i$. More precisely

Proposition 5.3

$$\tau.\hat{\xi}_i = \text{CT}_\tau[i]\tau$$

Proof The action of $\hat{\xi}_i$ on polynomials with degree 0 in the x_i equals the action of the operators $\tilde{\xi}_i$. Hence, the result follows from the non-shifted version of the equality (prop 3.16). \square

Straightforwardly, the operators ς_i and $\hat{\Psi}$ are compatible with the dominance properties in the following sense:

Proposition 5.4 *Suppose that P a polynomial such that its highest degree component has the dominance property with the leading monomial $x^{v,\tau}$ then*

1. *If $v[i] < v[i + 1]$ then the highest degree component of $P.\varsigma_i$ has the dominance property and its leading term is $x^{v.s_i,\tau}$.*
2. *The highest degree component of $P.\hat{\Psi}$ has the dominance property and its leading monomial is $x^{v.\Psi,\tau}$.*

5.2 The Yang-Baxter Graph

Let λ be a partition and G_λ the associated graph. We construct the set of the polynomials $\left(\hat{J}_{\mathfrak{P}}\right)_{\mathfrak{P} \text{ path in } G_\lambda}$ using the following recursive rules:

1. $\hat{J}_{\square} := 1 \otimes \tau_\lambda$
2. If $\mathfrak{P} = [a_1, \dots, a_{k-1}, s_i]$ then

$$\hat{J}_{\mathfrak{P}} := \hat{J}_{[a_1, \dots, a_{k-1}]} \left(\varsigma_i + \frac{1}{\zeta[i + 1] - \zeta[i]} \right),$$

where the vector ζ is defined by

$$(\tau_\lambda, \text{CT}_{\tau_\lambda}, 0^N, [1, 2, \dots, N]).a_1 \dots a_{k-1} = (\tau, \zeta, v, \sigma),$$

3. $\mathfrak{P} = [a_1, \dots, a_{k-1}, \Psi]$ then then

$$\hat{J}_{\mathfrak{P}} = \hat{J}_{[a_1, \dots, a_{k-1}]} \hat{\Psi}.$$

As expected one obtains

Theorem 5.5 *Let $\mathfrak{P} = [a_0, \dots, a_k]$ be a path in G_λ from the root to (τ, ζ, v, σ) . The polynomial $\hat{J}_{\mathfrak{P}}$ is a simultaneous eigenfunction of the operators $\hat{\xi}_i$ whose leading term in the highest degree component is $x^{v, \tau}$. Furthermore, the eigenvalue of $\hat{\xi}_i$ associated to $\hat{J}_{\mathfrak{P}}$ equals $\zeta[i]$.*

Hence, for any path \mathfrak{P}' ending to (τ, ζ, v, σ) , one has $\hat{J}_{v, \tau} := \hat{J}_{\mathfrak{P}} = \hat{J}_{\mathfrak{P}'}$ and the family $(\hat{J}_{v, \tau})_{v, \tau}$ forms a basis of M_λ of simultaneous eigenfunctions of the Cherednik operators.

Furthermore, if \mathfrak{P} leads to \emptyset then $\hat{J}_{\mathfrak{P}} = 0$.

Proof The proof goes as in Theorem 4.1 using respectively propositions 5.2, 5.3 and 5.4 instead of propositions 3.17, 3.16 and 3.11. \square

Example 5.6 Let again $\tau = \begin{smallmatrix} 2 \\ 3 & 1 \end{smallmatrix}$ and consider the Yang-Baxter Graph $G_{\begin{smallmatrix} 2 \\ 31 \end{smallmatrix}}$ (see fig. 9).

The eigenvalues of τ are $[1, -1, 0]$. Let $\tau' = \begin{smallmatrix} 1 \\ 3 & 2 \end{smallmatrix}$, one has

$$\begin{aligned} \hat{J}_{[001], \tau} &= \tau \cdot \hat{\Psi} \\ &= (x_3 + 2) \otimes \left(-\frac{1}{2} \tau + \tau'\right) \end{aligned}$$

with associated eigenvalues $[-1, 0, \alpha + 1]$ and

$$\begin{aligned} \hat{J}_{[010], \tau} &= \hat{J}_{[001], \tau} \left((s_2 + \partial_{2,3}) \otimes s_2 + \frac{1}{\alpha+1} \right) \\ &= \frac{1}{2} \frac{(3\alpha+1+x_2+\alpha x_2-x_3)}{\alpha+1} \tau + \frac{(x_3+3+x_2+\alpha x_2+\alpha)}{\alpha+1} \tau' \end{aligned}$$

with eigenvalues $[-1, \alpha + 1, 0]$.

The non-shifted vector valued Jack polynomials can be recovered easily from the shifted one:

Proposition 5.7 *The restriction of $\hat{J}_{v, \tau}$ to its component of top degree equals $J_{v, \tau}$.*

Proof It suffices to remark that $\varsigma_i = s_i \otimes s_i + \varsigma_i^-$ and $\hat{\Psi} = \Psi + \Psi^-$ where ς^- and Ψ^- are operators which decrease the degree in the x_j . Hence, one computes the component of the top degree of $\hat{J}_{v, \tau}$ following the Yang-Baxter graph replacing ς_i by $s_i \otimes s_i$ and $\hat{\Psi}$ by Ψ , that is the Yang-Baxter graph which was used to obtain the polynomials $J_{v, \tau}$. This ends the proof. \square

5.3 Symmetrization, Antisymmetrization

We will say that a polynomial is symmetric if it is invariant under the action of ς_i for each $i < N - 1$. Denote also $\hat{\mathfrak{S}} = \sum_{\omega \in \mathfrak{S}_N} \varsigma_\omega$. As for non shifted Jack, one has:

Theorem 5.8 1. Let H_T be a connected component of G_λ . For each vertex (τ, ζ, v, σ) of H_T , the polynomial $\hat{J}_{v,\tau} \hat{\mathfrak{S}}$ equals $\hat{J}_{\lambda_T, \text{std}(T)} \hat{\mathfrak{S}}$ up to a multiplicative constant.

2. One has $\hat{J}_{\lambda_T, \text{std}(T)} \hat{\mathfrak{S}} \neq 0$ if and only if H_T is 1-compatible.

3. More precisely, when H_T is 1-compatible, the polynomial

$$\hat{J}_T = \sum_{(\tau, \zeta, v, \sigma) \text{ vertex of } H_T} \mathfrak{E}_{v,\tau} \hat{J}_{v,\tau}$$

is symmetric.

Proof The proof is identical to the non-shifted case. \square

In the same way, for the antisymmetrization, denoting $\hat{\mathfrak{A}} := \sum_{\omega \in \mathfrak{S}_N} (-1)^{\ell(\omega)} \varsigma_\omega$, we have:

Theorem 5.9 1. Let H_T be a connected component of G_λ . For each vertex (τ, ζ, v, σ) of H_T , the polynomial $\hat{J}_{v,\tau} \hat{\mathfrak{A}}$ equals $\hat{J}_{\lambda_T, \text{std}(T)} \hat{\mathfrak{A}}$ up to a multiplicative constant.

2. One has $\hat{J}_{\lambda_T, \text{std}(T)} \hat{\mathfrak{A}} \neq 0$ if and only if H_T is (-1) -compatible.

3. More precisely, when H_T is (-1) -compatible the polynomial

$$\hat{J}'_T = \sum_{(\tau, \zeta, v, \sigma) \text{ vertex of } H_T} \mathfrak{F}_{v,\tau} \hat{J}_{v,\tau}$$

is antisymmetric.

5.4 Propagation of vanishing properties

Some phenomena of propagation of vanishing properties could be deduced from the classical case (see [13]).

Lemma 5.10 (*Lascoux*) *Let $f(x, y)$ be a function of two variables. Suppose that $f(b, a) = 0$ with $a \neq b$ then*

$$f.(s_{x,y} + \gamma.\partial_{x,y} + \frac{\gamma}{b-a})(a, b) = 0$$

Indeed, similar properties occur for vector valued polynomials:

Lemma 5.11 *Let $f \in \mathbb{C}[x, y]V_\lambda$ then*

$$f.(s_{x,y} \otimes s_{x,y} + \gamma\partial_{x,y} \otimes 1 + \frac{\gamma}{b-a})(a, b) = 0$$

when $f(b, a) = 0$.

Proof Write $f(x, y) = \sum_\tau f^\tau(x, y) \otimes \tau$ and remark that $f(b, a) = 0$ implies $f^\tau(a, b) = 0$ for each τ . But $f.s_{x,y} = \sum_\tau f^\tau s_{x,y} \otimes \tau s_{x,y}$ and by lemma 5.10 $f^\tau s_{x,y}(a, b) = 0$. Hence $f.s_{x,y}(a, b) = 0$. In the same way, lemma 5.10 implies $f^\tau(\partial_{x,y} + \frac{1}{b-a})(a, b) = 0$ and then $f(\partial_{x,y} \otimes 1 + \frac{1}{b-a})(a, b) = 0$. This proves the result. \square

Example 5.12 Consider the polynomial $\hat{J}_{001, \frac{2}{31}}$. This polynomial vanishes for $x_3 = -2$: $\hat{J}_{001, \frac{2}{31}}(x_1, x_2, -2) = 0$. Since

$$\hat{J}_{010, \frac{2}{31}} = \hat{J}_{001, \frac{2}{31}}(\varsigma_2 + \frac{1}{\alpha + 1})$$

one has $\hat{J}_{010, \frac{2}{31}}(x_1, -2, \alpha - 1) = 0$. Finally,

$$\hat{J}_{100, \frac{2}{31}} = \hat{J}_{010, \frac{2}{31}}(\varsigma_1 + \frac{1}{\alpha + 2})$$

implies $\hat{J}_{100, \frac{2}{31}}(-2, \alpha, \alpha - 1) = 0$.

Denote by Φ^+ the operator defined by $[x_1, \dots, x_N].\Phi^+ = [x_2, x_3, \dots, x_N, x_1 + \alpha]$. The action of the affine operator $\hat{\Psi}$ propagates information about vanishing properties :

Lemma 5.13 *One has*

$$f.\hat{\Psi}([a_1, \dots, a_N]\Phi^+) = 0$$

when $f(a_1, \dots, a_N) = 0$.

Hence, for each pair (v, τ) one can compute at least one N -tuple $(a_1^{v,\tau}, \dots, a_N^{v,\tau})$ such that $J_{v,\tau}(a_1^{v,\tau}, \dots, a_N^{v,\tau}) = 0$.

Proposition 5.14 *Denote by $\mathbf{V}_{v,\tau}$ the vector whose i th components is*

$$\mathbf{V}_{v,\tau}[i] := \alpha(v^+[1] - v[i]) + (\text{CT}_\tau[1] - \text{CT}_\tau[\sigma_v[i]]) - N + 1.$$

Let m be the smallest integer such that $v[m] = \max\{v[i] : 1 \leq i \leq N\}$.

One has

$$\hat{J}_{v,\tau}(x_1, \dots, x_{m-1}, \mathbf{V}_{v,\tau}[m], \dots, \mathbf{V}_{v,\tau}[N]) = 0$$

Proof Denote by $\tilde{\Phi}$ the operator defined by

$$[x_1, \dots, x_N] \cdot \tilde{\Phi} = \begin{cases} [x_1, \dots, x_N] \cdot \Phi & \text{if } x_1 = a\alpha + b \text{ with } a > 0 \\ [x_2 + \alpha, \dots, x_N + \alpha, x_1] & \text{otherwise.} \end{cases}$$

Remark first that the vectors $\mathbf{V}_{v,\tau}$ are obtained by substituting Ψ by $\tilde{\Phi}$ in G_τ :

Lemma 5.15 *Recall the notation $[v_1, \dots, v_N] \cdot \Psi = [v_2, \dots, v_N, v_1 + 1]$. One has*

$$\mathbf{V}_{v\Psi,\tau} = \mathbf{V}_{v,\tau} \tilde{\Phi}.$$

Proof Let m be the smallest integer such that $v[m] = \max\{v[i] : 1 \leq i \leq N\}$. If $m > 1$ then $v^+[1] = v\Psi^+[1]$. It follows that $\mathbf{V}_{v\Psi,\tau}[i] = \mathbf{V}_{v,\tau}[i + 1]$ if $i < N$ and, since $v\Psi[N] = v[1] + 1$ one has $\mathbf{V}_{v\Psi,\tau}[N] = \mathbf{V}_{v,\tau}[1] - \alpha$. That is $\mathbf{V}_{v\Psi,\tau} = \mathbf{V}_{v,\tau} \tilde{\Phi}$. If $m = 1$ then $v^+[1] = (v\Psi)^+[1] - 1$. Hence, $\mathbf{V}_{v\Psi,\tau}[i] = \mathbf{V}_{v,\tau}[i + 1] + \alpha$ for $i < N$ and $\mathbf{V}_{v\Psi,\tau}[N] = \mathbf{V}_{v,\tau}[1]$. This ends the proof. \square

Hence, using lemma 5.15, a straightforward induction shows that

$$\mathbf{V}_{v,\tau}[m] = 1 - N. \tag{13}$$

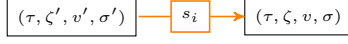
Let us prove the proposition by an induction on the length of the path from $\hat{J}_{0\dots 01,\tau}$ to $\hat{J}_{v,\tau}$. First remark that

$$\mathbf{V}_{0\dots 01,\tau}[N] = -N + 1,$$

implies $\hat{J}_{0\dots 01,\tau}(x_1, \dots, x_{N-1}, \mathbf{V}_{0\dots 01,\tau}[N]) = 0$, since

$$\hat{J}_{0\dots 01,\tau} = \hat{J}_{0\dots 0,\tau} \hat{\Psi} = \hat{J}_{0\dots 0,\tau} \cdot T(x_N - N + 1).$$

Suppose that there is an arrow



in G_τ . Let m' be the smallest integer such that $v'[m'] = \max\{v'[i] : 1 \leq i \leq N\}$. We have by induction,

$$\hat{J}_{v',\tau}(x_1, \dots, x_{m'-1}, \mathbf{V}_{v',\tau}[m'], \dots, \mathbf{V}_{v',\tau}[N]) = 0. \quad (14)$$

Remark that by hypothesis, $i \neq m'$ (otherwise $v'[i] \geq v'[i+1]$ and the arrow is not in G_τ). Furthermore we have:

$$\begin{aligned} \hat{J}_{v,\tau} &= \hat{J}_{v',\tau} \cdot (s_i \otimes s_i + \partial_i \otimes 1 + \frac{1}{(v'[i+1]-v'[i])\alpha + (\text{CT}_\tau[\sigma_{v'}[i+1]] - \text{CT}_\tau[\sigma_{v'}[i]])}) \\ &= \hat{J}_{v',\tau} \cdot (s_i \otimes s_i + \partial_i \otimes 1 + \frac{1}{(v[i]-v[i+1])\alpha + (\text{CT}_\tau[\sigma_v[i]] - \text{CT}_\tau[\sigma_v[i+1]])}). \end{aligned} \quad (15)$$

Consider three cases

1. If $i < m' - 1$ then lemma 5.11 implies

$$\hat{J}_{v,\tau}(x_1, \dots, x_{m-1}, \mathbf{V}_{v',\tau}[m'], \dots, \mathbf{V}_{v',\tau}[N]) = 0$$

But $\mathbf{V}_{v',\tau}[m', m' + 1, \dots, N] = \mathbf{V}_{v,\tau}[m', m' + 1, \dots, N]$ and $m' = m$. This proves the result.

2. If $i = m' - 1$ then $m = m' - 1$ and, as a special case of (14), one has

$$\hat{J}_{v',\tau}(x_1, \dots, x_{m'-2}, \mathbf{V}_{v,\tau}[m], \mathbf{V}_{v',\tau}[m'], \dots, \mathbf{V}_{v',\tau}[N]) = 0.$$

But

$$\begin{aligned} \mathbf{V}_{v',\tau}[m'] - \mathbf{V}_{v,\tau}[m] &= \mathbf{V}_{v,\tau}[m+1] - \mathbf{V}_{v,\tau}[m] \\ &= (v[m] - v[m+1])\alpha + (\text{CT}_\tau[\sigma_v[m]] - \text{CT}_\tau[\sigma_v[m+1]]). \end{aligned}$$

The result is, now, a direct consequence of lemma 5.11.

3. if $i > m'$ then $\mathbf{V}_{v',\tau}[m', m' + 1, \dots, N] = \mathbf{V}_{v,\tau}[m', m' + 1, \dots, N].s_i$ and $m' = m$. Remarking that,

$$\begin{aligned} \mathbf{V}_{v',\tau}[i] - \mathbf{V}_{v,\tau}[i+1] &= \mathbf{V}_{v,\tau}[i+1] - \mathbf{V}_{v,\tau}[i] \\ &= (v[i] - v[i+1])\alpha + (\text{CT}_\tau[\sigma_v[i]] - \text{CT}_\tau[\sigma_v[i+1]]), \end{aligned}$$

Lemma 5.11 gives the result.

Suppose that there is an arrow



in G_τ . Let m' be the smallest integer such that $v'[m'] = \max\{v'[i] : 1 \leq i \leq N\}$. We have by induction,

$$\hat{J}_{v',\tau}(x_1, \dots, x_{m'-1}, \mathbf{V}_{v',\tau}[m'], \dots, \mathbf{V}_{v',\tau}[N]) = 0. \quad (16)$$

We need to consider two cases:

1. If $m' > 1$ then, since, $\hat{J}_{v,\tau} = \hat{J}_{v',\tau} \cdot \hat{\Psi}$ lemma 5.13 and eq. (16) imply

$$\begin{aligned} \hat{J}_{v,\tau}(x_2, \dots, x_m, \mathbf{V}_{v,\tau}[m], \dots, \mathbf{V}_{v,\tau}[N-1], x_1) &= \hat{J}_{v,\tau}(x_2, \dots, x_{m'-1}, \mathbf{V}_{v',\tau}[m'], \dots, \mathbf{V}_{v',\tau}[N], x_1) \\ &= 0 \end{aligned}$$

In particular,

$$\hat{J}_{v,\tau}(x_1, \dots, x_{m-1}, \mathbf{V}_{v,\tau}[m], \dots, \mathbf{V}_{v,\tau}[N]) = 0.)$$

2. If $m' = 1$ then, from (13), one has $\mathbf{V}_{v',\tau}[1] = \mathbf{V}_{v,\tau}[N] = 1 - N$. But, since $\hat{J}_{v,\tau} = \hat{J}_{v',\tau} \cdot T(x_N - N + 1)$, one has

$$\hat{J}_{v,\tau}(x_1, \dots, x_N - 1, \mathbf{V}_{v,\tau}[N]) = 0,$$

and the result is just a special case obtained from this equality by specializing the values of the x_i .

□

Example 5.16 One has :

$$\mathbf{V}_{[0,2,2,1,0,3,5,1], \frac{1}{75}} = [5\alpha - 9, 3\alpha - 8, 3\alpha - 12, 4\alpha - 10, 5\alpha - 10, 2\alpha - 13, -7, 4\alpha - 11].$$

Indeed,

$$\sigma_{[0,2,2,1,0,3,5,1]} = [7, 3, 4, 5, 8, 2, 1, 6] \text{ and } [0, 2, 2, 1, 0, 3, 5, 1]^+ = [5, 3, 2, 2, 1, 1, 0, 0]$$

and the values of $v^+[1] - v[i]$ and $(\text{CT}_\tau[1] - \text{CT}_\tau[\sigma_v[i]]) - N + 1$ are computed by taking the corresponding values in the RST

$$\begin{array}{cccc} 5 & & & -9 \\ 3 & & & -12 \\ 0 & 5 & \text{and} & -7 \quad -10 \\ 4 & 2 & 4 & 3 & -11 & -13 & -10 & -8 \end{array}$$

Hence,

$$\hat{J}_{[0,2,2,1,0,3,5,1], \frac{1}{75}}(x_1, x_2, x_3, x_4, x_5, x_6, -7, 4\alpha - 11) = 0$$

The vanishing properties described in proposition 5.14 are obtained by combining the actions of the s_i and $\tilde{\Phi}$ on the initial vectors $\mathbf{V}_{0^N, \tau}$.

Example 5.17 Consider the propagation of vanishing properties described in Fig.10 .

Lemma 5.13 and proposition 5.14 suggest that one can compute other vanishing properties by combining the actions of the s_i and Φ^+ . A general closed formula remained to be found and, unfortunately, the vanishing properties obtained by propagation from $\mathbf{V}_{0^N, \tau}$ are not sufficient to characterize the shifted Jack polynomials.

Example 5.18 One has

$$\hat{J}_{[002], 31}^2(\mathbf{V}_{[001], 31}^2 s_2 s_1 \Phi^+) = \hat{J}_{[002], 31}^2(\alpha, \alpha - 1, \alpha - 2) = 0.$$

Note that the only vanishing property obtained by propagation for $\hat{J}_{[100], 31}^2$ is

$$\hat{J}_{[100], 31}^2(-2, \alpha, \alpha - 1) = 0,$$

and this is not sufficient to characterize the polynomial $\hat{J}_{[100], 31}^2$.

6 Conclusion

In this paper we used the Yang-Baxter graph technique to produce a structure describing the nonsymmetric Jack polynomials whose values lie in an irreducible \mathfrak{S}_N -module. The graph is directed with no loops and has exactly one root or base point. Any path joining the root to a vertex is essentially an algorithm for constructing the Jack polynomial at that vertex, and the edges making up the path are the steps of the algorithm. The edges are labeled by the generators of the braid group or by an affine operation.

These techniques are used to analyze restriction to a subgroup \mathfrak{S}_M and also to construct symmetric and antisymmetric Jack polynomials. These are associated with certain subgraphs.

Finally the graph technique is used to construct shifted, or inhomogeneous, vector-valued Jack polynomials.

The theory is independent of the numerical value of the parameter α provided that the eigenspaces of $\tilde{\xi}_i$ all have multiplicity one, that is, that no two vertices of the graph have the same spectral vector. Future work is needed

to analyze situations where this condition is violated, in particular when α has a singular value, $\{\frac{n}{m} : 2 \leq n \leq N, m \in \mathbb{Z}, \frac{m}{n} \notin \mathbb{Z}\}$. There may not be a basis of Jack polynomials for the space of all polynomials. For particular choices of λ there may exist symmetric Jack polynomials of highest weight, that is, those annihilated by $\sum_{i=1}^N \mathfrak{D}_i$. It seems plausible that any graph describing such a special case would be significantly different from G_λ . As a final remark, note that in the case of the trivial representation, some families of highest weight Jack polynomials have been found (see *e.g.* [3, 10, 1]) and related to the theory of the fractional quantum Hall effect [11].

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References

- [1] W. Baratta and P. Forrester, *Jack polynomials fractional quantum Hall states and their generalizations*, arXiv:1007.2692 (2010).
- [2] T. Baker and P. Forrester, *Symmetric Jack polynomials from non-symmetric theory*, Ann. Comb. **3** (1999) 159-170.
- [3] B.A. Bervenig and F.D.M. Haldane, *Generalized clustering conditions of Jack polynomials at negative Jack parameter α .*, Phys. Rev. Lett. **101** (2009) 066802.
- [4] C. Dunkl, *Symmetric and antisymmetric Vector valued Jack polynomials*, Sém. Lothar. Combin. B64a (2010), 31 pp..
- [5] C. Dunkl and S. Griffeth, *Generalized Jack polynomials and the representation theory of rational Cherednik algebras*, Selecta Math. (N.S.) **16** (2010), 791-818 .
- [6] P. Etingof and E. Stoica, appendix by S. Griffeth, *Unitary representation of rational Cherednik algebras*, arXiv: 0901.4595v3 (2009).
- [7] S. Griffeth, *Jack polynomials and the coinvariant ring of $G(r, p, n)$* , Proceedings of the American Mathematical Society vol. **137** 5 (2009) pp. 1621-1629.

- [8] S. Griffeth, *Orthogonal functions generalizing Jack polynomials*, Trans. Amer. Math. Soc. **362** (2010), 6131-6157.
- [9] F. Knop and S. Sahi, *A recursion and a combinatorial formula for Jack polynomials*, Invent. Math. **128** (1997) 9-22.
- [10] T. Jolicoeur and J.-G. Luque, *Highest weight Macdonald and Jack polynomials*, arXiv:1003.4858 (2010).
- [11] R. B. Laughlin, *Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations*, Phys. Rev. Lett. **50** (1983) 1395- 1398.
- [12] A. Lascoux, *Young's Representation of the symmetric group*, <http://www-igm.univ-mlv.fr/~al/ARTICLES/ProcCrac.ps.gz>
- [13] A. Lascoux, *Yang-Baxter graphs, Jack and Macdonald polynomials*, Ann. Comb. **5** 3-4 (2001) 397-424.
- [14] G. Murphy, *A new construction of Young seminormal representation of symmetric groups*, J. Algebra **69** (1981) 287-297.

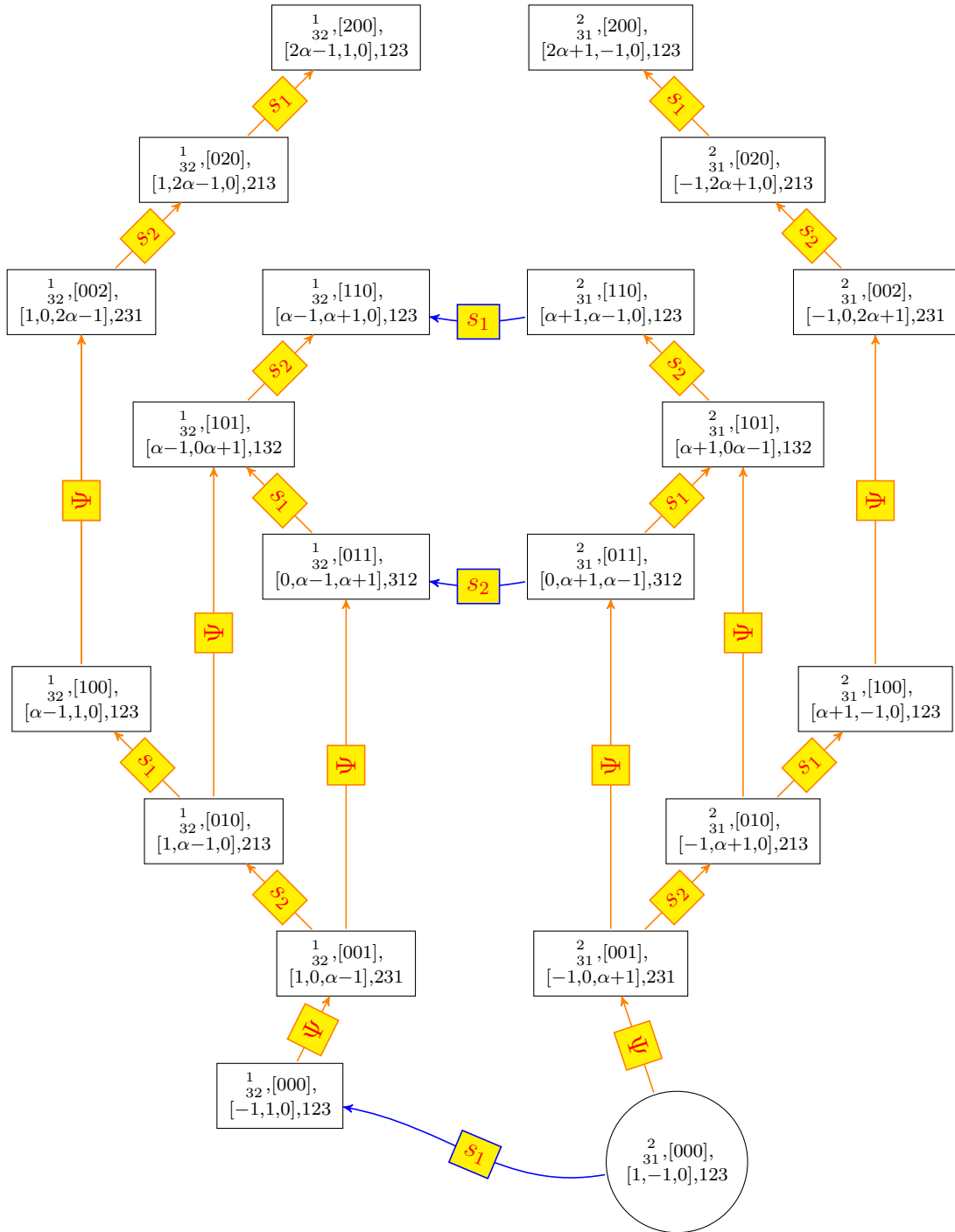


Figure 1: The first vertices of the graph G_{21} where we omit to write the vertex \emptyset and the associated arrows.

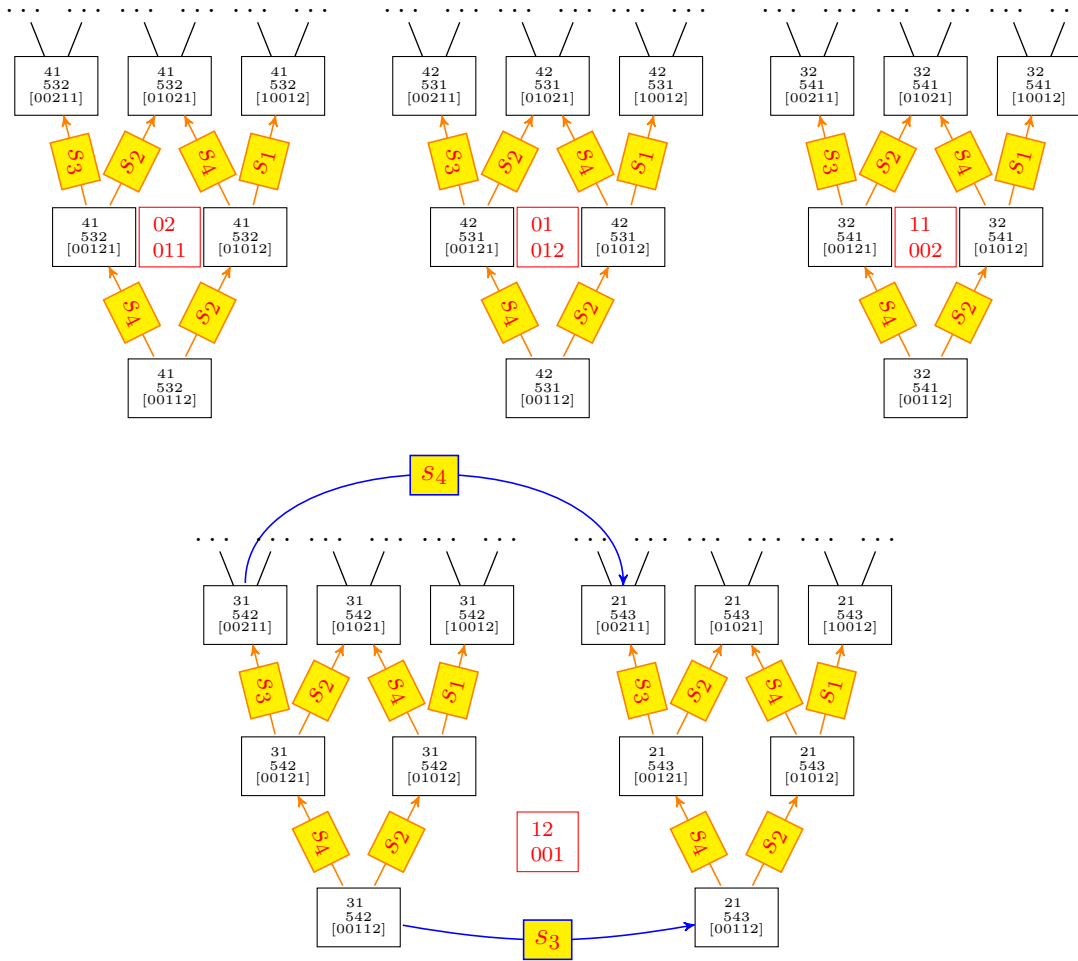


Figure 2: Some connected components of H_{32}

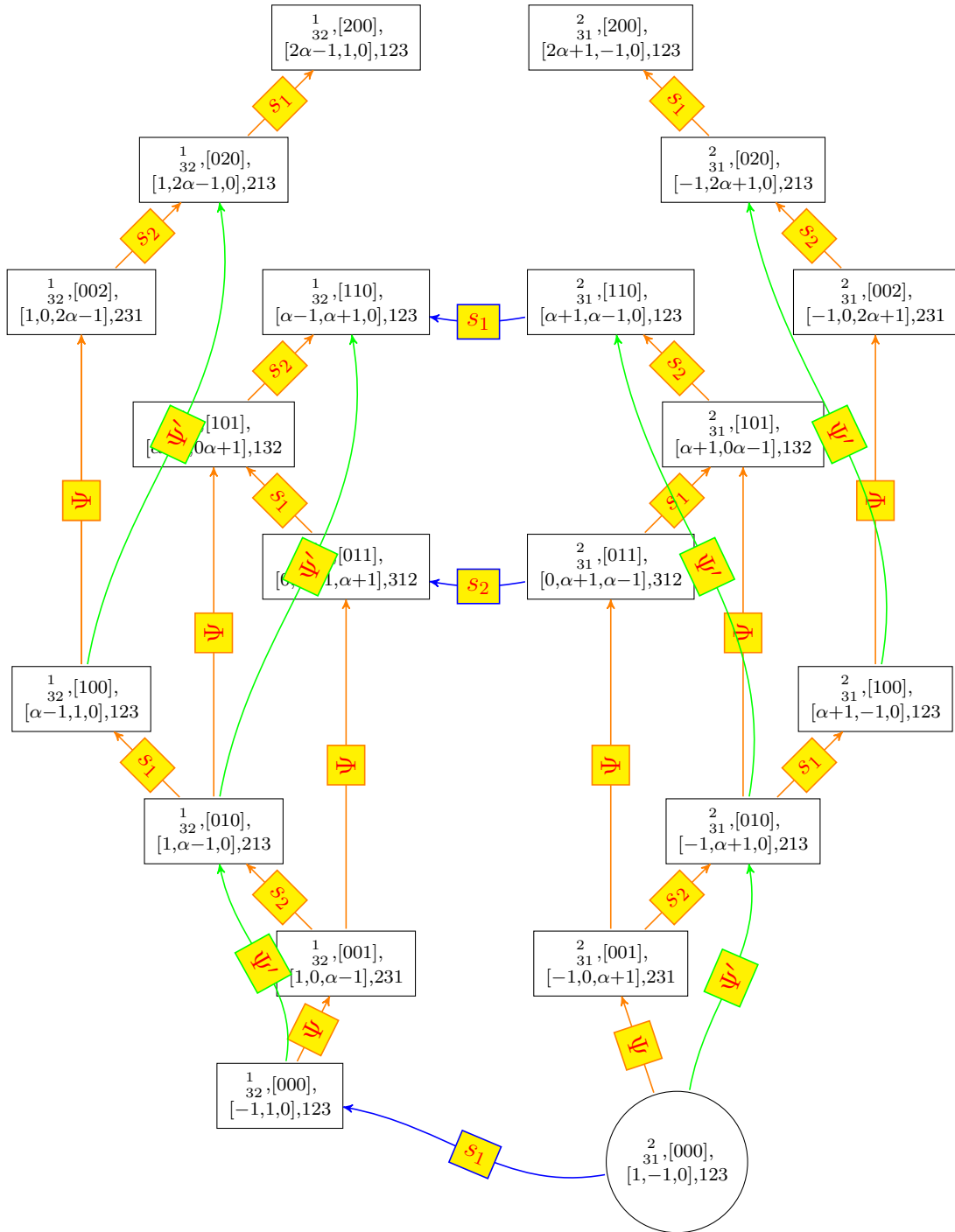


Figure 3: The first vertices of the graph G_{21} with edges Ψ' for $M = 2$.

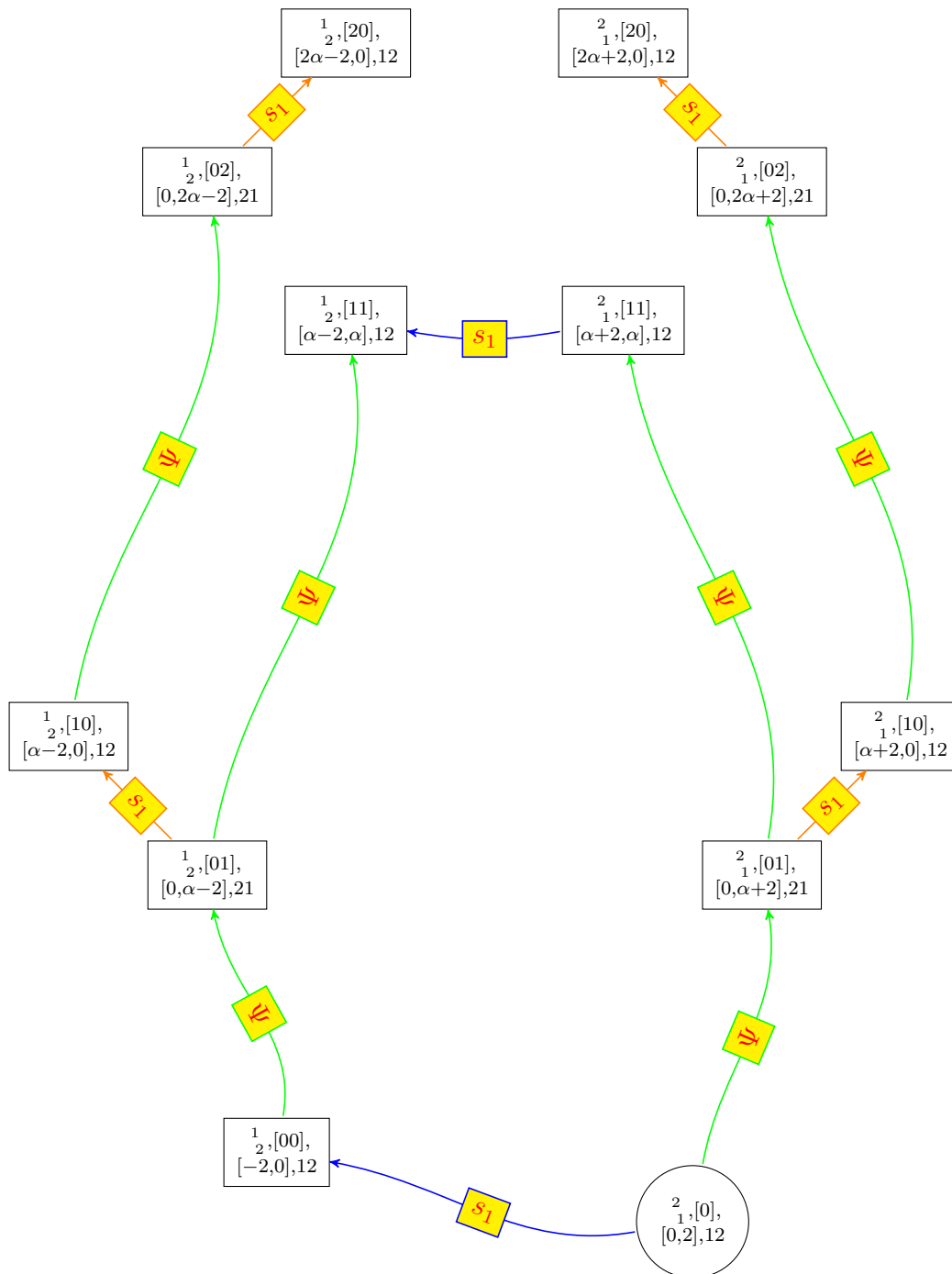


Figure 4: The first vertices of the graph $G_{21} \downarrow_2$.

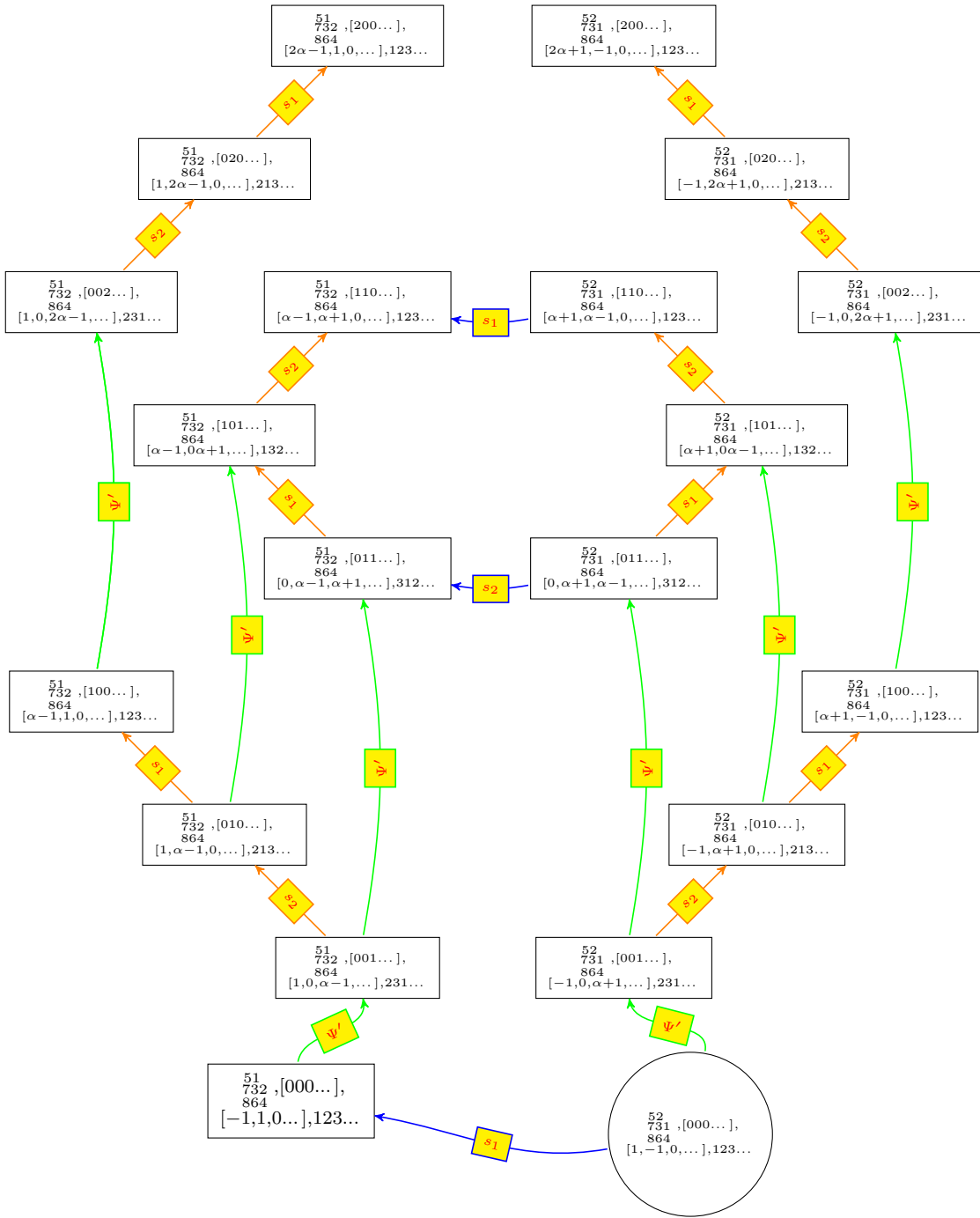


Figure 5: The first vertices of the subgraph of G_{332} associated to the restriction for $M = 3$ whose root is $\begin{smallmatrix} 52 \\ 731 \\ 864 \end{smallmatrix}$ with the edges Ψ' added.

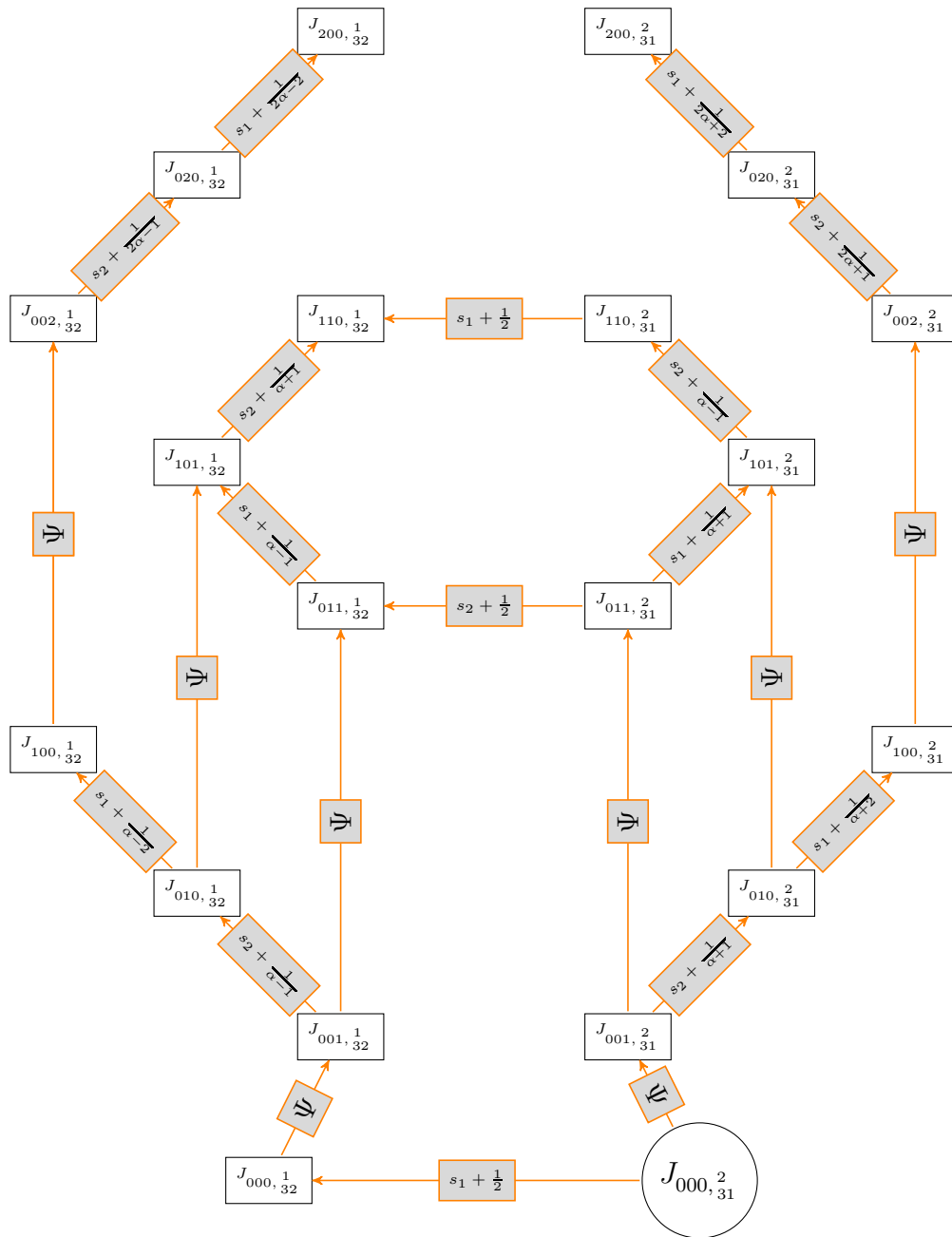


Figure 6: First values of the polynomials $J_{v,\tau}$ for $\lambda = 21$ (s_i means $s_i \otimes s_i$).

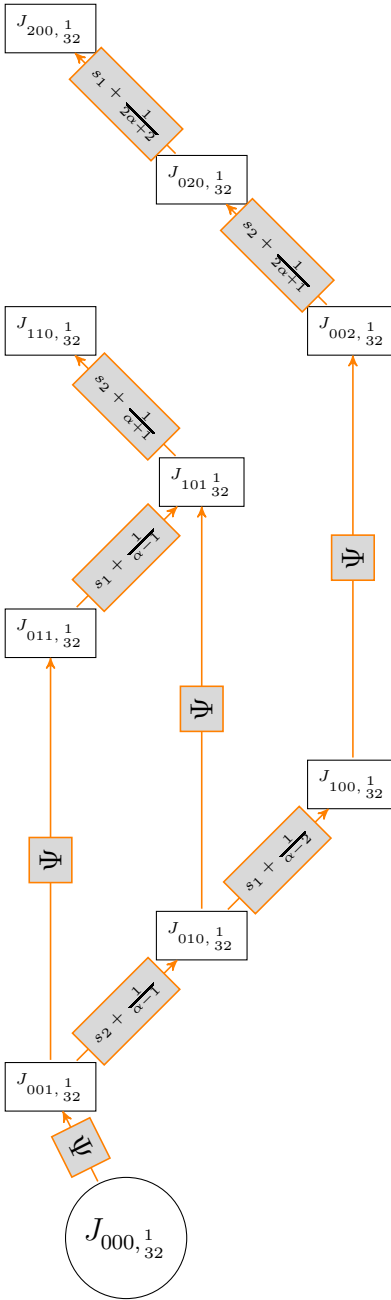


Figure 7: First values of the polynomials $J_{v,1}$

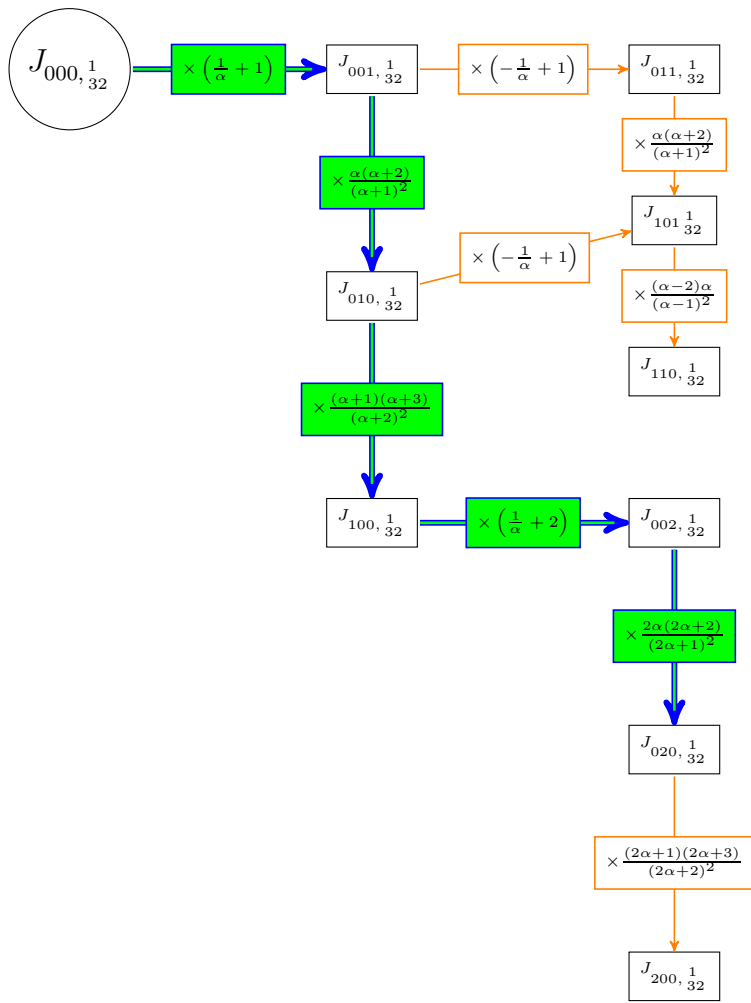


Figure 8: Computation of $\|J_{020, \frac{1}{32}}\|^2$ using the graph $G_{\frac{1}{32}}$.

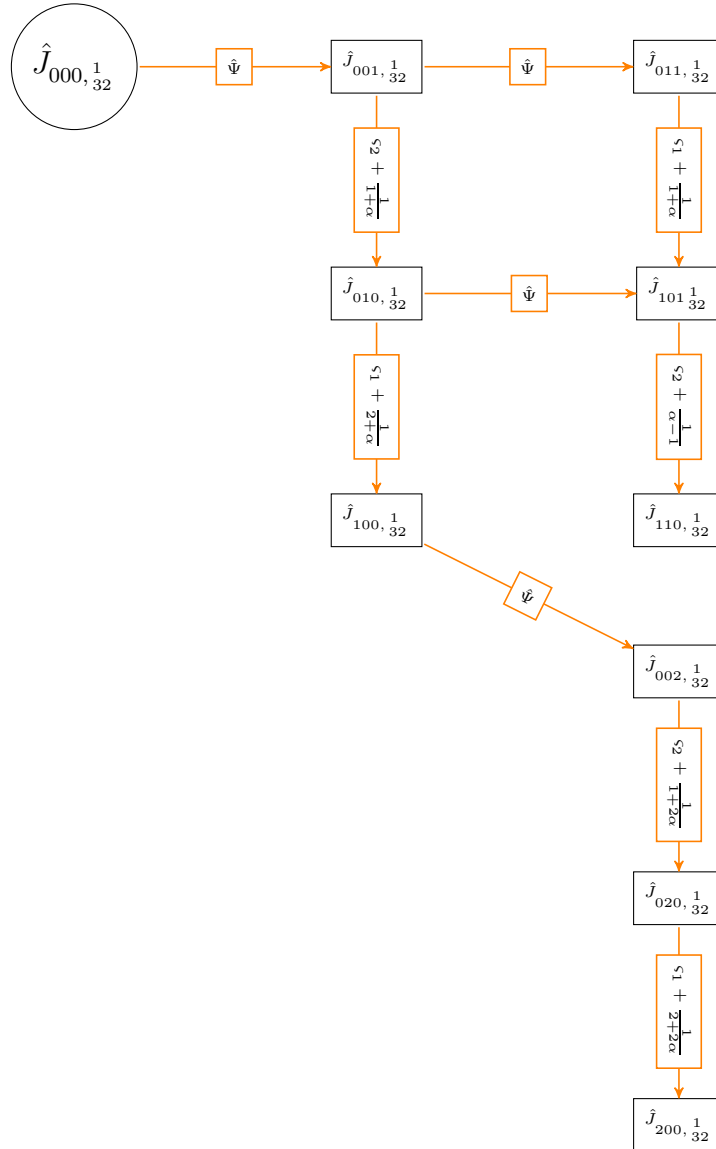


Figure 9: First values of $\hat{J}_{v, 31}^2$.

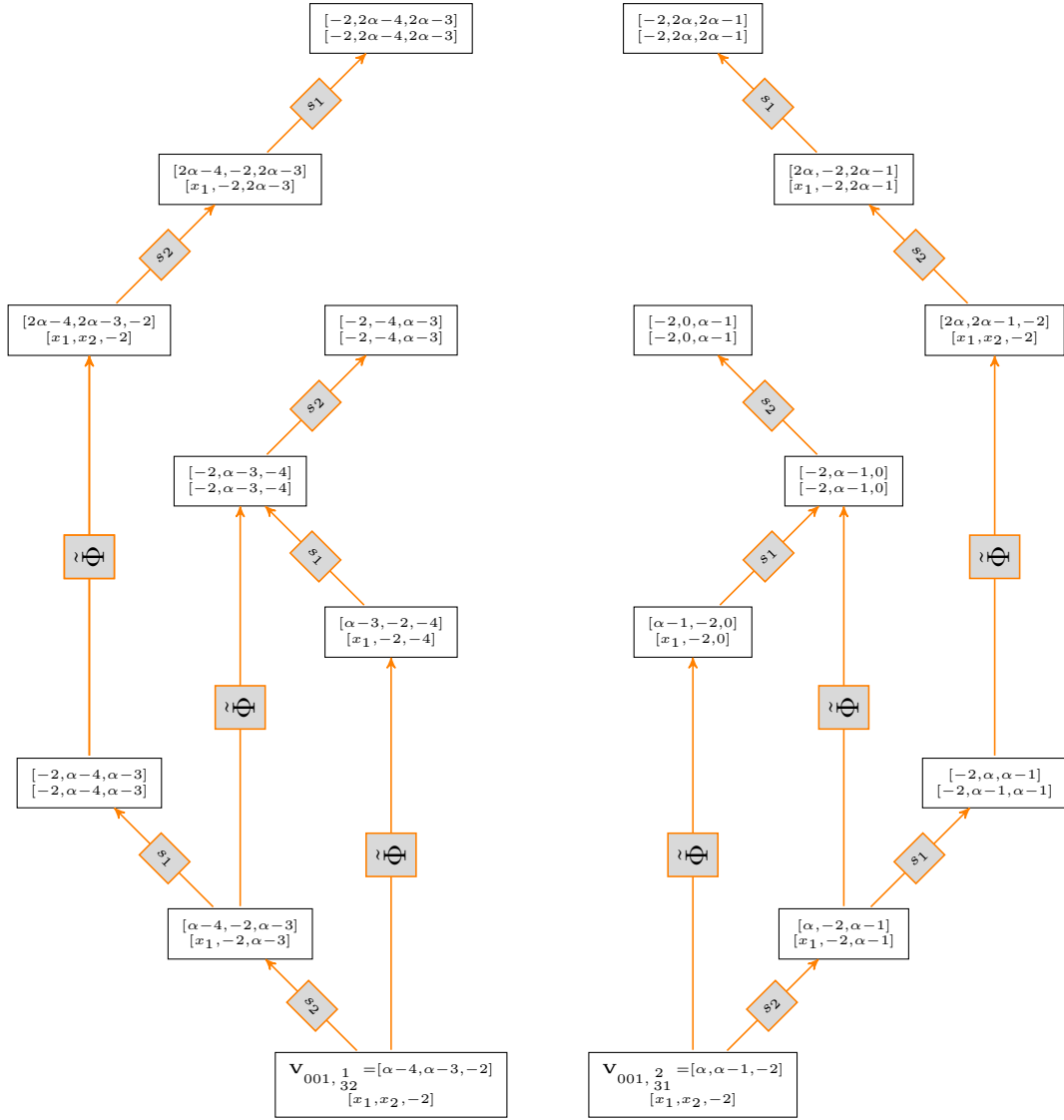


Figure 10: Propagation of some vanishing properties for $\lambda = 21$.