Quantum Isometries of the finite noncommutative geometry of the Standard Model

Jyotishman Bhowmick*, Francesco D'Andrea‡, Ludwik Dąbrowski‡

* *Abdus Salam International Center for Theoretical Physics (ICTP), Strada Costiera 11, I-34151 Trieste, Italy*
‡ *Scuola Internazionale Superiore di Studi Avanzati (SISSA), via Bonomea 265, I-34136 Trieste, Italy*

Abstract

We compute the quantum isometry group of the finite noncommutative geometry F describing the internal degrees of freedom in the Standard Model of particle physics. We show that this provides genuine quantum symmetries of the spectral triple corresponding to $M \times F$ where M is a compact spin manifold. We also prove that the bosonic and fermionic part of the spectral action are preserved by these symmetries.

1 Introduction

In modern theoretical physics, symmetries play a fundamental role in determining the dynamics of a theory. In the two foremost examples, namely General Relativity and the Standard Model of elementary particles, the dynamics is dictated by invariance under diffeomorphisms and under local gauge transformations respectively. As a way to unify external (i.e. diffeomorphisms) and internal (i.e. local gauge) symmetries, Connes and Chamseddine proposed a model from Noncommutative Geometry [15] based on the product of the canonical commutative spectral triple of a compact Riemannian spin manifold M and a finite dimensional noncommutative one, describing an "internal" finite noncommutative space F [12, 13, 18, 20]. In this picture, diffeomorphisms are realized as outer automorphisms of the algebra, while inner automorphisms correspond to the gauge transformations. Inner fluctuations of the Dirac operator are divided in two classes: the 1-forms coming from commutators with the Dirac operator of M give the gauge bosons, while the 1-forms coming from the Dirac operator of F give the Higgs field. The gravitational and bosonic part S_b of the action is encoded in the spectrum of the gauged Dirac operator, which is invariant under isometries of the Hilbert space. The fermionic part S_f is also defined in terms of the spectral data. The result is an Euclidean version of the Standard Model minimally coupled to gravity (cf. [20] and references therein).

In his "Erlangen program", Klein linked the study of geometry with the analysis of its group of symmetries. Dealing with quantum geometries, it is natural to study quantum symmetries. The idea of using quantum group symmetries to understand the conceptual significance of the finite geometry F is mentioned in a final remark by Connes in [17]. Preliminary studies on the Hopf-algebra level appeared in [30, 21, 26]. Following Connes' suggestion, quantum automorphisms of finite-dimensional complex C^*-algebras were introduced by Wang in [37, 38] and later the quantum permutation groups of finite sets and graphs have been studied by a number of mathematicians, see e.g. [3, 4, 11, 34]. These are compact quantum groups in the sense of Woronowicz [41]. The notion of compact quantum symmetries for "continuous" mathematical structures, like commutative and noncommutative manifolds (spectral triples), first appeared in [28], where quantum isometry groups were defined in terms of a Laplacian, followed by the definition of "quantum groups of orientation preserving isometries" based on the theory of spectral triples in [7], and on spectral triples with a real structure in [29]. Computations of these compact quantum groups were done for several examples, including the tori, spheres, Podleś quantum spheres, and Rieffel deformations of compact Riemannian spin manifolds. For these studies we refer to [6, 7, 8, 9, 10] and references therein.

The finite noncommutative geometry $F = (A_F, H_F, D_F, \gamma_F, J_F)$ describing the internal space of the Standard Model is given by a unital real spectral triple over the finite-dimensional real C^*-algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with $\mathbb{H}$ the field of quaternions. Let $B_F \subset \mathcal{B}(\mathcal{H})$ be the smallest complex C^*-algebra containing A_F as a real C^*-subalgebra. In this article we first compute the quantum group of orientation and real structure preserving isometries of the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$; next we show that this quantum symmetry can be extended to get quantum isometries of the product of this spectral triple with the canonical spectral triple of M. Thus, we have genuine quantum symmetries of the full spectral triple of the Standard Model. Moreover these quantum symmetries preserves the spectral action in a suitable sense. Finally we compute the maximal quantum subgroup of the quantum isometry group whose coaction is a quantum automorphism of the real C^*-algebra A_F.

The plan of this article is as follows. We start by recalling in Sec. 2 some basic definitions and facts about compact quantum groups and quantum isometries. In Sec. 3 we introduce the spectral triple F and state the main result. Since quantum groups, coactions, etc. are defined in the framework of complex (C^*-)algebras, we replace A_F by B_F and compute the quantum isometry group of the latter in the sense of [29]. As shown in Sec. 3.2, this is given by the free product $C(U(1)) * A_{\mathrm{aut}}(M_3(\mathbb{C}))$, where $A_{\mathrm{aut}}(M_n(\mathbb{C}))$ is Wang's quantum automorphism group of $M_n(\mathbb{C})$ [37]. In Sec. 4, we discuss the invariance of the spectral action under quantum isometries. In Sec. 5 we explain how the result changes if we work with real instead of complex algebras. The final section deals with the proof of the main result, that is, Proposition 3.3.

Throughout the paper, by the symbol $\otimes_{\mathrm{alg}}$ we will always mean the algebraic tensor product over $\mathbb{C}$, by $\otimes$ minimal tensor product of complex C^*-algebras or the completed tensor product of Hilbert modules over complex C^*-algebras. The symbol $\otimes_{\mathbb{R}}$ will denote the tensor product over the real numbers. Unless otherwise stated, all algebras are assumed to be unital complex associative involutive algebras. We denote by $\mathcal{N}^*$ the set of all bounded linear functionals $\mathcal{N} \to \mathbb{C}$ on the normed linear space $\mathcal{N}$, by $\mathcal{M}(\mathcal{A})$ the multiplier algebra of the complex C^*-algebra $\mathcal{A}$, by $\mathcal{L}(\mathcal{H})$ the adjointable operators on the Hilbert module $\mathcal{H}$ and by $\mathcal{K}(\mathcal{H})$ the compact operators on the Hilbert space $\mathcal{H}$. For a unital complex C^*-algebra $\mathcal{A}$, we will implicitly use the identification of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$ with the set of all adjointable operators on the Hilbert $\mathcal{A}$-module $\mathcal{H} \otimes \mathcal{A}$. By abelianization of $\mathcal{A}$ we will mean the quotient of $\mathcal{A}$ by its commutator C^*-ideal. Lastly, we want to attract the reader's attention to a choice of notation. The notation $\widetilde{\mathrm{QISO}}^+_J$ used in this article is the same as $\widetilde{\mathrm{QISO}}^+_{\mathrm{real}}$ of [29]. We did this to avoid confusion with the newly defined object $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}$ of Section 5 in the context of quantum isometries of real C^*-algebras.

2 Compact quantum groups and quantum isometries

2.1 Some generalities on Compact Quantum Groups

We begin by recalling the definition of compact quantum groups and their coactions from [40, 41]. We shall use most of the terminology of [36], for example Woronowicz C^*-subalgebra, Woronowicz C^*-ideal, etc., however with the exception that Woronowicz C^*-algebras will be called compact quantum groups, and we will not use the term compact quantum groups for the dual objects as done in [36].

Definition 2.1. *A* compact quantum group *(to be denoted by CQG from now on) is a pair* (Q, Δ) *given by a complex unital* C^**-algebra* Q *and a unital* C^**-algebra morphism* $\Delta : Q \to Q \otimes Q$ *such that*

i) Δ *is coassociative, i.e.*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$

as equality of maps $Q \to Q \otimes Q \otimes Q$*;*

ii) $\mathrm{Span}\{(a \otimes 1_Q)\Delta(b) \,|\, a, b \in Q\}$ *and* $\mathrm{Span}\{(1_Q \otimes a)\Delta(b) \,|\, a, b \in Q\}$ *are norm-dense in* $Q \otimes Q$.

For $Q = C(G)$, where G is a compact topological group, conditions i) and ii) correspond to the associativity and the cancellation property of the product in G, respectively.

Definition 2.2. *A* unitary corepresentation *of a compact quantum group* (Q, Δ) *on a Hilbert space* $\mathcal{H}$ *is a unitary element* $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes Q)$ *satisfying*

$$(id \otimes \Delta)U = U_{(12)}U_{(13)}$$

where we use the standard leg numbering notation (see e.g. [32]).

If $Q = C(G)$, U corresponds to a strongly continuous unitary representation of G.

For any compact quantum group Q (see [40, 41]), there always exists a canonical dense $*$-subalgebra $Q_0 \subset Q$ which is spanned by the matrix coefficients of the finite dimensional unitary corepresentations of Q and two maps $\epsilon : Q_0 \to \mathbb{C}$ (counit) and $\kappa : Q_0 \to Q_0$ (antipode) which make Q_0 a Hopf $*$-algebra.

Definition 2.3. *A* Woronowicz C^*-ideal *of a CQG* (Q, Δ) *is a* C^**-ideal* I *of* Q *such that* $\Delta(I) \subset \ker(\pi_I \otimes \pi_I)$*, where* $\pi_I : Q \to Q/I$ *is the projection map. The quotient* Q/I *is a CQG with the induced coproduct.*

If $Q = C(G)$ are continuous functions on a compact topological group G, closed subgroups of G correspond to the quotients of Q by its Woronowicz C^*-ideals. While quotients Q/I give "compact quantum subgroups", C^*-subalgebras $Q' \subset Q$ such that $\Delta(Q') \subset Q' \otimes Q'$ describe "quotient quantum groups".

Definition 2.4. *We say that a CQG* (Q, Δ) *coacts on a unital* C^**-algebra* $\mathcal{A}$ *if there is a unital* C^**-homomorphism (called a* coaction*)* $\alpha : \mathcal{A} \to \mathcal{A} \otimes Q$ *such that:*

i) $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$*,*

ii) $\mathrm{Span}\{\alpha(a)(1_{\mathcal{A}} \otimes b) \,|\, a \in \mathcal{A},\, b \in Q\}$ *is norm-dense in* $\mathcal{A} \otimes Q$.

The coaction is faithful *if any CQG $Q' \subset Q$ coacting on $\mathcal{A}$ coincides with Q.*

It is well known (cf. [33, 37]) that condition (ii) in Def. 2.4 is equivalent to the existence of a norm-dense unital $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ such that $\alpha(\mathcal{A}_0) \subset \mathcal{A}_0 \otimes_{\mathrm{alg}} Q_0$ and $(id \otimes \epsilon)\alpha = id$ on $\mathcal{A}_0$. For later use, let us now recall the concept of universal CQGs $A_u(R)$ as defined in [35, 38] and references therein.

Definition 2.5. *For a fixed $n \times n$ positive invertible matrix R, $A_u(R)$ is the universal C^*-algebra generated by $\{u_{ij},\, i,j = 1,\dots,n\}$ such that*

$$uu^* = u^*u = \mathbb{I}_n \;, \qquad u^t(R\overline{u}R^{-1}) = (R\overline{u}R^{-1})u^t = \mathbb{I}_n$$

where $u := ((u_{ij}))$, $u^* := ((u^*_{ji}))$ and $\overline{u} := (u^*)^t$. The coproduct Δ is given by

$$\Delta(u_{ij}) = \sum\nolimits_k u_{ik} \otimes u_{kj} \;.$$

Note that u is a unitary corepresentation of $A_u(R)$ on $\mathbb{C}^n$.

The $A_u(R)$'s are universal in the sense that every compact *matrix* quantum group (i.e. every CQG generated by the matrix entries of a finite-dimensional unitary corepresentation) is a quantum subgroup of $A_u(R)$ for some $R > 0$ [38]. It may also be noted that $A_u(R)$ is the universal object in the category of CQGs which admit a unitary corepresentation on $\mathbb{C}^n$ such that the adjoint coaction on the finite-dimensional C^*-algebra $M_n(\mathbb{C})$ preserves the functional $M_n(\mathbb{C}) \ni m \mapsto \mathrm{Tr}(R^t m)$ (see [39]).

We observe the following elementary fact which is going to be used in the sequel.

Lemma 2.6. *Let $\mathcal{H} = \mathbb{C}^n$, $n \in \mathbb{N}$ and $B \in M_n(\mathcal{B})$ be a matrix with entries in a unital $*$-algebra $\mathcal{B}$. Then*

$$(\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\, B(L \otimes 1)B^* = \mathrm{Tr}_{\mathcal{H}}(L) \cdot 1_{\mathcal{B}}$$

for any linear operator L on $\mathcal{H}$ if and only if B^t is unitary.

A matrix B (with entries in a unital $*$-algebra $\mathcal{B}$) such that both B and B^t are unitary is called a *biunitary* [5]. We remark that the CQG $A_u(n) := A_u(\mathbb{I}_n)$, called the *free quantum unitary group*, is generated by the biunitary matrix u given in Def. 2.5. We refer to [38] for a detailed discussion on the structure and classification of such quantum groups.

The analogue of projective unitary groups was introduced in [2] (see also Sec. 3 of [5]). Let us recall the definition.

Definition 2.7. *We denote by $PA_u(n)$ the C^*-subalgebra of $A_u(n)$ generated by $\{u_{ij}(u_{kl})^* : i,j,k,l = 1,\dots,n\}$. This is a CQG with the coproduct induced from $A_u(n)$.*

Remark 2.8. *The projective version of any quantum subgroup of $A_u(n)$ can be defined similarly.*

In [37], Wang defines the quantum automorphism group of $M_n(\mathbb{C})$, denoted by $A_{\mathrm{aut}}(M_n(\mathbb{C}))$ to be the universal object in the category of CQGs with a coaction on $M_n(\mathbb{C})$ preserving the trace (and with morphisms given by CQGs homomorphisms intertwining the coactions). The explicit definition is in Theorem 4.1 of [37].

We conclude this section by quoting Théorème 1(iv) of [2] (cf. also Prop. 3.1(3) of [5]).

Proposition 2.9 ([2, 5])**.** *We have $PA_u(n) \simeq A_{\mathrm{aut}}(M_n(\mathbb{C}))$.*

2.2 Noncommutative Geometry and quantum isometries

In noncommutative geometry, compact Riemannian spin manifolds are replaced by real spectral triples. Recall that a *unital spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is the datum of: a complex Hilbert space $\mathcal{H}$, a complex unital associative involutive algebra $\mathcal{A}$ with a faithful unital $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ (the representation symbol is usually omitted), a (possibly unbounded) self-adjoint operator D on $\mathcal{H}$ with compact resolvent and having bounded commutators with all $a \in \mathcal{A}$. The canonical commutative example is given by $(C^\infty(M), L^2(M,S), \not{D})$, where $C^\infty(M)$ are complex-valued smooth functions on a compact Riemannian spin manifold with no boundary, $L^2(M,S)$ is the Hilbert space of square integrable spinors and $\not{D}$ is the Dirac operator.

A spectral triple is *even* if there is a $\mathbb{Z}_2$-grading γ on $\mathcal{H}$ commuting with $\mathcal{A}$ and anticommuting with D. We will set $\gamma = 1$ when the spectral triple is odd.

A spectral triple is *real* if there is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, called the *real structure*, such that

$$J^2 = \epsilon 1 \ , \qquad JD = \epsilon' DJ \ , \qquad J\gamma = \epsilon'' \gamma J \ , \tag{2.1}$$

and

$$[a, JbJ^{-1}] = 0 \ , \qquad [[D,a], JbJ^{-1}] = 0 \ , \tag{2.2}$$

for all $a, b \in \mathcal{A}$ [1]. ϵ, ϵ' and ϵ'' are signs and determine the KO-dimension of the space [16].

For the finite part of the Standard Model $\epsilon = +1$, $\epsilon' = +1$, $\epsilon'' = -1$ and the KO-dimension is 6 [14]. Imposing a few additional conditions, it is possible to reconstruct a compact Riemannian spin manifold from any commutative real spectral triple [19].

In the example $(C^\infty(M), L^2(M,S), \not{D}, J, \gamma)$ of the spectral triple associated to a compact Riemannian spin manifold M with no boundary, there exists a covering group $\widetilde{G}$ of the group of orientation preserving isometries G of M having a unitary representation U on the Hilbert space of spinors $L^2(M,S)$ commuting with $\not{D}, J, \gamma$ whose adjoint action $\mathrm{Ad_U}$ on $\mathcal{B}(L^2(M,S))$ preserves the subalgebra $C^\infty(M)$. This picture is used to generalize the notion of isometries as in [29].

Definition 2.10. *A CQG (Q, Δ) coacts by "orientation and real structure preserving isometries" on the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ if there is a unitary corepresentation $U \in \mathcal{M}(\mathcal{K}(H) \otimes Q)$ such that*

$$U \text{ commutes with } D \otimes 1 \text{ and } \gamma \otimes 1; \tag{2.3a}$$

$$(J \otimes *)U(\xi \otimes 1_Q) = U(J\xi \otimes 1_Q) \text{ for all } \xi \in \mathcal{H}; \tag{2.3b}$$

$$(id \otimes \varphi)\mathrm{Ad_U}(a) \in \mathcal{A}'' \text{ for all } a \in \mathcal{A} \text{ and every state } \varphi \text{ on } Q, \tag{2.3c}$$

where $\mathrm{Ad_U} = U(\,.\, \otimes 1_Q)U^*$ is the adjoint coaction and $\mathcal{A}''$ is the double commutant of $\mathcal{A}$.

Note that in Definition 4 of [29] two antilinear operators J and $\tilde{J}$ appear. $\tilde{J}$ is a generalized real structure (it is not assumed to be an isometry) and J is its antiunitary part. As in the case of this article the real structure is an antilinear isometry J and $\tilde{J}$ coincide and hence our definition is a particular instance of Definition 4 of [29].

We end this section by recalling Theorem 1 of [29]. Let $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ be a real spectral triple with $\epsilon' = 1$ and $\mathfrak{C}_J$ be the category with objects (Q, U) as in Definition 2.10 and morphisms given by CQG morphisms intertwining the corresponding corepresentations. Then

[1] Notice that in some examples, although not in the present case, the condition (2.2) has to be slightly relaxed, cf. [22, 23, 24, 25].

Theorem 2.11. *The category* $\mathfrak{C}_J$ *has a universal object denoted by* $\widetilde{\mathrm{QISO}}^+(\mathcal{A},\mathcal{H},D,\gamma,J)$ *(or simply* $\widetilde{\mathrm{QISO}}^+_J(D)$*) whose unitary corepresentation, say* U_0*, is faithful. The* quantum isometry group*, denoted by* $\mathrm{QISO}^+(\mathcal{A},\mathcal{H},D,\gamma,J)$*, (or simply* $\mathrm{QISO}^+_J(D)$*) is given by the quantum subgroup of* $\widetilde{\mathrm{QISO}}^+_J(D)$ *generated by the elements* $\{(\varphi\otimes\mathrm{id})\mathrm{Ad}_{\mathrm{U}_0}(a) : a\in\mathcal{A},\ \varphi\in\mathcal{A}^*\}$.

$\widetilde{\mathrm{QISO}}^+_J(D)$ is the quantum analogue of the covering $\widetilde{G}$ of the classical group G of orientation preserving isometries of a spin manifold M. It's projective version (in the sense of Sec. 3 of [5]) is the quantum group $\mathrm{QISO}^+_J(D)$, which is the quantum analogue of G.

3 Quantum isometries of the internal non-commutative space of the Standard Model

3.1 The finite non-commutative space F

The spectral triple $(A_F,H_F,D_F,\gamma_F,J_F)$ describing the internal space F of the Standard Model is defined as follows (cf. [20] and references therein). The algebra A_F is

$$A_F := \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})\ , \tag{3.1}$$

where we identify $\mathbb{H}$ with the real subalgebra of $M_2(\mathbb{C})$ with elements

$$q=\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} \tag{3.2}$$

for $\alpha,\beta\in\mathbb{C}$ (cf. Cayley-Dickson construction).

Let us denote by $\mathbb{C}[v_1,\dots,v_k]\simeq\mathbb{C}^k$ the vector space with basis $v_1,\dots,v_k$. For our convenience, we adopt the following notation for the Hilbert space H_F. It can be written as a tensor product

$$H_F := \mathbb{C}^2\otimes\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^n$$

where, in the notations of [20], we have

i) the first two factors $\mathbb{C}^2\otimes\mathbb{C}^4$ with

$$\mathbb{C}^2=\mathbb{C}[\uparrow,\downarrow]\ ,\qquad \mathbb{C}^4=\mathbb{C}[\ell,\{q_c\}_{c=1,2,3}]\ ,$$

where $\uparrow$ and $\downarrow$ stand for weak isospin up and down, ℓ and q_c stand for lepton and quark of color c respectively. These may be combined into

$$\mathbb{C}^8=\mathbb{C}[\nu,e,\{u_c,d_c\}_{c=1,2,3}]\ ,$$

where ν stands for "neutrino", e for "electron", u_c and d_c for quarks with weak isospin $+1/2$ and $-1/2$ respectively and of color c. Explicitly, the isomorphism $\mathbb{C}^2\otimes\mathbb{C}^4\to\mathbb{C}^8$ is the map

$$\uparrow\otimes\,\ell\mapsto\nu\ ,\quad \downarrow\otimes\,\ell\mapsto e\ ,\quad \uparrow\otimes\, q_c\mapsto u_c\ ,\quad \downarrow\otimes\, q_c\mapsto d_c\ .$$

ii) a factor

$$\mathbb{C}^4=\mathbb{C}[p_L,\overline{p}_R,\overline{p}_L,p_R]\ ,$$

where L,R stand for the two chiralities, p for "particle" and $\overline{p}$ for "antiparticle";

iii) a factor $\mathbb{C}^n$ since each particle comes in n generations. Presently only 3 generations have been observed, but for the sake of generality we will work with an arbitrary $n \geq 3$.

From a physical point of view, rays (lines through the origin) of H_F are states describing the internal degrees of freedom of the elementary fermions. The *charge conjugation* J_F changes a particle into its antiparticle, and is the composition of the componentwise complex conjugation on H_F with the linear operator

$$J_0 := 1 \otimes 1 \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes 1 \,. \tag{3.3}$$

The grading is

$$\gamma_F := 1 \otimes 1 \otimes \mathrm{diag}(1,1,-1,-1) \otimes 1 \,.$$

The element $a = (\lambda, q, m) \in A_F$ (with $\lambda \in \mathbb{C}$, $q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$) is represented by

$$\begin{aligned} \pi(a) &= q \otimes 1 \otimes e_{11} \otimes 1 + \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \otimes 1 \otimes e_{44} \otimes 1 \\ &\quad + 1 \otimes \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & & & \\ 0 & & m & \\ 0 & & & \end{pmatrix} \otimes (e_{22} + e_{33}) \otimes 1 \,, \end{aligned} \tag{3.4}$$

where m is a 3×3 block and $\{e_{ij}\}_{i,j=1,\ldots,k}$ is the canonical basis of $M_k(\mathbb{C})$ (e_{ij} is the matrix with 1 in the (i,j)-th position and 0 everywhere else). In particular, in (3.4) e_{11} projects on the space $\mathbb{C}[p_L]$ of particles with left chirality, e_{22} on $\mathbb{C}[\overline{p}_R]$, e_{33} on $\mathbb{C}[\overline{p}_L]$ and e_{44} on $\mathbb{C}[p_R]$.

The Dirac operator is

$$\begin{aligned} D_F &:= e_{11} \otimes e_{11} \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_\nu \\ 0 & 0 & \Upsilon_\nu^t & \Upsilon_R \\ 0 & \overline{\Upsilon}_\nu & 0 & 0 \\ \Upsilon_\nu^* & \Upsilon_R^* & 0 & 0 \end{pmatrix} + e_{11} \otimes (1 - e_{11}) \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_u \\ 0 & 0 & \Upsilon_u^t & 0 \\ 0 & \overline{\Upsilon}_u & 0 & 0 \\ \Upsilon_u^* & 0 & 0 & 0 \end{pmatrix} \\ &\quad + e_{22} \otimes e_{11} \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_e \\ 0 & 0 & \Upsilon_e^t & 0 \\ 0 & \overline{\Upsilon}_e & 0 & 0 \\ \Upsilon_e^* & 0 & 0 & 0 \end{pmatrix} + e_{22} \otimes (1 - e_{11}) \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_d \\ 0 & 0 & \Upsilon_d^t & 0 \\ 0 & \overline{\Upsilon}_d & 0 & 0 \\ \Upsilon_d^* & 0 & 0 & 0 \end{pmatrix} \,, \end{aligned} \tag{3.5}$$

where each of the Υ matrices are in $M_n(\mathbb{C})$, $\overline{m} := (m^*)^t$ is the matrix obtained from m by conjugating each entry, and we identify $\mathcal{B}(H_F) = M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes \big(M_4(\mathbb{C}) \otimes M_n(\mathbb{C})\big)$ with $M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes M_{4n}(\mathbb{C})$ by writing $M_{4n}(\mathbb{C})$ as a 4×4 matrix with entries in $M_n(\mathbb{C})$; in particular $e_{ij} \otimes m \in M_4(\mathbb{C}) \otimes M_n(\mathbb{C})$ will be the matrix with the $n \times n$ block m in position (i,j).

The physical meaning of the Υ matrices is explained in section 17.4 of [20]: for $x = e, u, d$ the eigenvalues of $\Upsilon_x^* \Upsilon_x$ give the square of the masses of the n generations of the particle x; the eigenvalues of $\Upsilon_\nu^* \Upsilon_\nu$ give the Dirac masses of neutrinos; the eigenvalues of $\Upsilon_R^* \Upsilon_R$ give the Majorana masses of neutrinos.

If we replace a spectral triple with one that is unitary equivalent we do not change the symmetries. Lemma 1.190 in [20] tells us that modulo an unitary equivalence, we can take Υ_e and Υ_d diagonal with non-negative eigenvalues, and we can take $\Upsilon_\nu = C\delta_\uparrow C^*$ and $\Upsilon_u = C'\delta'_\uparrow C'^*$

where $\delta_\uparrow$ resp. $\delta'_\uparrow$ are diagonal with non-negative eigenvalues and $C, C' \in SU(n)$. In view of their physical meaning — the masses of the n generations of the electron and of the $2n$ quarks — we can assume that the eigenvalues of $\Upsilon_e, \Upsilon_d, \Upsilon_u$ are all distinct and non-zero (this is true for the three generations that we know).

Lemma 3.1. *Up to a unitary transformation commuting with A_F, J_F and γ_F, we can assume that $\Upsilon_e, \Upsilon_d, \Upsilon_u$ and Υ_ν are diagonal (positive) matrices.*

Proof. The first two matrices Υ_e and Υ_d are already diagonal. The change of basis of H_F given by the (unitary) matrix

$$e_{11} \otimes (1 - e_{11}) \otimes \mathrm{diag}(C', \overline{C}', \overline{C}', C') + e_{11} \otimes e_{11} \otimes 1 \otimes 1 + e_{22} \otimes 1 \otimes 1 \otimes 1$$

commutes with J_F, γ_F and $\pi(a)$, for any $a \in A_F$, and its effect on D_F is to diagonalize Υ_u. Similarly the unitary matrix

$$e_{11} \otimes e_{11} \otimes \mathrm{diag}(C, \overline{C}, \overline{C}, C) + e_{11} \otimes (1 - e_{11}) \otimes 1 \otimes 1 + e_{22} \otimes 1 \otimes 1 \otimes 1$$

has the only effect of diagonalizing Υ_ν and transforming Υ_R into the matrix $\Upsilon'_R = C^t \Upsilon_R C$. The new matrix Υ'_R is still a complex symmetric matrix, and will be denoted by the same symbol Υ_R in the sequel. □

In view of previous lemma, $\Upsilon_\nu = \Upsilon_\nu^t = \Upsilon_\nu^* = \overline{\Upsilon}_\nu$ and similar for $\Upsilon_e, \Upsilon_u, \Upsilon_d$. Therefore

$$\begin{aligned} D_F = e_{11} \otimes e_{11} \otimes (X \otimes \Upsilon_\nu + e_{24} \otimes \Upsilon_R + e_{42} \otimes \Upsilon_R^*) + e_{11} \otimes (1 - e_{11}) \otimes X \otimes \Upsilon_u \\ + e_{22} \otimes e_{11} \otimes X \otimes \Upsilon_e + e_{22} \otimes (1 - e_{11}) \otimes X \otimes \Upsilon_d \,, \end{aligned} \tag{3.6}$$

with $X := e_{14} + e_{23} + e_{32} + e_{41}$.

3.2 Quantum isometries of F

Since the definition of quantum isometry group is given for spectral triples over complex $*$-algebras, we first need to explain how to canonically associate one to any spectral triple over a real $*$-algebra.

Lemma 3.2. *To any real spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ over a real $*$-algebra $\mathcal{A}$ we can associate a real spectral triple $(\mathcal{B}, \mathcal{H}, D, \gamma, J)$ over the complex $*$-algebra $\mathcal{B} \simeq \mathcal{A}_\mathbb{C} / \ker \pi_\mathbb{C}$, where $\mathcal{A}_\mathbb{C} \simeq \mathcal{A} \otimes_\mathbb{R} \mathbb{C}$ is the complexification of $\mathcal{A}$, with conjugation defined by $(a \otimes_\mathbb{R} z)^* = a^* \otimes_\mathbb{R} \overline{z}$ for $a \in \mathcal{A}$ and $z \in \mathbb{C}$, and $\pi_\mathbb{C} : \mathcal{A}_\mathbb{C} \to \mathcal{B}(\mathcal{H})$ is the $*$-representation*

$$\pi_\mathbb{C}(a \otimes_\mathbb{R} z) = z\pi(a) \,, \qquad a \in \mathcal{A} \,,\ z \in \mathbb{C} \,. \tag{3.7}$$

Notice that $\ker \pi_\mathbb{C}$ may be nontrivial since the representation $\pi_\mathbb{C}$ is not always faithful. For example, if $\mathcal{A}$ is itself a complex $*$-algebra (every complex $*$-algebra is also a real $*$-algebra) and π is complex linear, then for any $a \in \mathcal{A}$ the element $a \otimes_\mathbb{R} 1 + ia \otimes_\mathbb{R} i$ of $\mathcal{A}_\mathbb{C}$ is in the kernel of $\pi_\mathbb{C}$. This happens in the Standard Model case, where the complexification of $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is the algebra $(A_F)_\mathbb{C} := \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$, where we have used the complex $*$-algebra isomorphism $M_n(\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} \to M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ given by

$$m \otimes_\mathbb{R} z \mapsto (mz, \overline{m}z)$$

having inverse

$$(m, m') \mapsto \tfrac{m+\overline{m}'}{2} \otimes_{\mathbb{R}} 1 + \tfrac{m-\overline{m}'}{2i} \otimes_{\mathbb{R}} i \tag{3.8}$$

for all $m, m' \in M_n(\mathbb{C})$, $z \in \mathbb{C}$.

Using (3.7), (3.8) and (3.4) we get $\pi_{\mathbb{C}}(\lambda, \lambda', q, m, m') = \langle \lambda, \lambda', q, m \rangle$, where

$$\begin{aligned} \langle \lambda, \lambda', q, m \rangle := q \otimes 1 \otimes e_{11} \otimes 1 + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \otimes 1 \otimes e_{44} \otimes 1 \\ + 1 \otimes \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & & & \\ 0 & & m & \\ 0 & & & \end{pmatrix} \otimes (e_{22} + e_{33}) \otimes 1 \,. \end{aligned} \tag{3.9}$$

The complex $*$-algebra $B_F := (A_F)_{\mathbb{C}} / \ker \pi_{\mathbb{C}}$ is simply the algebra $B_F \simeq \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ with elements $\langle \lambda, \lambda', q, m \rangle$. With A_F replaced by B_F, we can now study quantum isometries.

We notice that in the case of the spectral triple of the internal part of the Standard Model, the conditions (2.3b-2.3c) are equivalent to

$$(J_0 \otimes 1)\overline{U} = U(J_0 \otimes 1) \,; \tag{3.10a}$$

$$\mathrm{Ad}_{\mathrm{U}}(B_F) \subset B_F \otimes_{\mathrm{alg}} Q \,; \tag{3.10b}$$

with J_0 given by (3.3). The equivalence between (2.3b) and (3.10a) is an immediate consequence of the definition of J_F. The equivalence between (2.3c) and (3.10b) follows from the equality of B_F'' and B_F, the latter being a finite-dimensional C^*-algebra.

We state here the main proposition (whose proof can be found in Sec. 6).

Proposition 3.3. $\widetilde{\mathrm{QISO}}_J^+(D_F)$ *is generated by unitaries* x_k, *(*$k = 0, \ldots, n$*), elements of* n 3×3 *biunitaries* $\{T_m : m = 1, 2, ..., n\}$, *an* $n \times n$ *biunitary* V *such that*

$$V\Upsilon_\nu = V^t\Upsilon_\nu = \mathrm{diag}(x_1^* x_0^*, ..., x_n^* x_0^*)\Upsilon_\nu \,, \qquad V\Upsilon_R = \Upsilon_R \overline{V} \,, \tag{3.11a}$$

$$((T_m)_{ki})^*(T_m)_{lj} = ((T_{m'})_{ki})^*(T_{m'})_{lj} \qquad \forall\, m, m' \,. \tag{3.11b}$$

The coproduct is given by

$$\Delta(x_k) = x_k \otimes x_k, \quad \Delta(V_{ij}) = \sum_{k=1}^{n} V_{ik} \otimes V_{kj}, \quad \Delta((T_m)_{ij}) = \sum_{k=1}^{3} (T_m)_{ik} \otimes (T_m)_{kj}.$$

The corepresentation on H_F *is given by*

$$\begin{aligned} U \;=\;\; & e_{11} \otimes e_{11} \otimes e_{11} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_0 x_k + e_{22} \otimes e_{11} \otimes e_{11} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k \\ & + e_{11} \otimes e_{11} \otimes e_{22} \otimes \sum_{j,k=1}^{n} e_{jk} \otimes V_{jk} + e_{22} \otimes e_{11} \otimes e_{22} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k^* \\ & + e_{11} \otimes e_{11} \otimes e_{33} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k^* x_0^* + e_{22} \otimes e_{11} \otimes e_{33} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k^* \\ & + e_{11} \otimes e_{11} \otimes e_{44} \otimes \sum_{j,k=1}^{n} e_{jk} \otimes \overline{V}_{jk} + e_{22} \otimes e_{11} \otimes e_{44} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k \end{aligned}$$

$$
\begin{aligned}
&+ e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{11} \otimes \sum_{m=1}^{n} e_{mm} \otimes (T_m)_{j,k} \\
&+ e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{11} \otimes \sum_{m=1}^{n} e_{mm} \otimes x_0^*(T_m)_{j,k} \\
&+ e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{22} \otimes \sum_{m=1}^{n} e_{mm} \otimes ((T_m)_{j,k})^* \\
&+ e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{22} \otimes \sum_{m=1}^{n} e_{mm} \otimes ((T_m)_{j,k})^* x_0 \\
&+ e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{33} \otimes \sum_{m=1}^{n} e_{mm} \otimes ((T_m)_{j,k})^* \\
&+ e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{33} \otimes \sum_{m=1}^{n} e_{mm} \otimes ((T_m)_{j,k})^* x_0 \\
&+ e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{44} \otimes \sum_{m=1}^{n} e_{mm} \otimes (T_m)_{j,k} \\
&+ e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes e_{44} \otimes \sum_{m=1}^{n} e_{mm} \otimes x_0^*(T_m)_{j,k} \;,
\end{aligned}
\tag{3.12}
$$

$\widetilde{\mathrm{QISO}}^+_J(D_F)$ *coacts trivially on the two summands* $\mathbb{C}$ *of* $B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$*, while on the remaining summands the coaction is*

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0,0,e_{ii},0\rangle) = \langle 0,0,e_{ii},0\rangle \otimes 1 \;, \tag{3.13a}$$

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0,0,e_{12},0\rangle) = \langle 0,0,e_{12},0\rangle \otimes x_0 \;, \tag{3.13b}$$

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0,0,e_{21},0\rangle) = \langle 0,0,e_{21},0\rangle \otimes x_0^* \;, \tag{3.13c}$$

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0,0,0,e_{ij}\rangle) = \sum\nolimits_{k,l=1,2,3} \langle 0,0,0,e_{kl}\rangle \otimes (T_1)^*_{ki}(T_1)_{lj} \;. \tag{3.13d}$$

Let us denote by Q_n the amalgamated free product of n copies of $A_u(3)$ over the common Woronowicz C^*-subalgebra $A_{\mathrm{aut}}(M_3(\mathbb{C}))$ (cf. Theorem 3.4 of [36]).

Corollary 3.4. $\widetilde{\mathrm{QISO}}^+_J(D_F)$ *is a quantum subgroup of the free product*

$$\underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1} * Q_n * A_u(n) \;.$$

The Woronowicz C^**-ideal of this CQG giving* $\widetilde{\mathrm{QISO}}^+_J(D_F)$ *is determined by the relations* (3.11a)*.*

From (3.13), it is clear that $\mathrm{QISO}^+_J(D_F)$ is the free product of $C(U(1))$, with generator x_0, and the CQG $PA_u(3) \simeq A_{\mathrm{aut}}(M_3(\mathbb{C}))$ with generators $(T_1)^*_{ki}(T_1)_{lj}$ (cf. Def. 2.7 and Prop. 2.9).

Corollary 3.5. *The quantum isometry group of the internal space of the Standard Model is*

$$\mathrm{QISO}^+_J(D_F) = C(U(1)) * A_{\mathrm{aut}}(M_3(\mathbb{C})) \;.$$

Its abelianization is given by (complex functions on) the classical group $U(1) \times PU(3)$*.*

Although $\widetilde{\mathrm{QISO}}^+_J(D_F)$ depends on Υ_ν and Υ_R (cf. (3.11a)), the quantum group $\mathrm{QISO}^+_J(D_F)$ does not depend on the explicit form of these two matrices. We stress the importance of this results, since neutrino masses are not known (at the moment, we only know that they cannot be all zero [27, 1]). Also, $\mathrm{QISO}^+_J(D_F)$ is independent on the number of generations.

Let us conclude this section by explaining how elementary particles transform under the corepresentation U in physics notation. As explained in Sec. 3.1, we have

$$\begin{aligned}
\nu_{L,k} &:= e_1 \otimes e_1 \otimes e_1 \otimes e_k \;, && \text{(left-handed neutrino, generation } k) \\
\nu_{R,k} &:= e_1 \otimes e_1 \otimes e_4 \otimes e_k \;, && \text{(right-handed neutrino, generation } k) \\
e_{L,k} &:= e_2 \otimes e_1 \otimes e_1 \otimes e_k \;, && \text{(left-handed electron, generation } k) \\
e_{R,k} &:= e_2 \otimes e_1 \otimes e_4 \otimes e_k \;, && \text{(right-handed electron, generation } k) \\
u_{L,c,k} &:= e_1 \otimes e_{c+1} \otimes e_1 \otimes e_k \;, && \text{(left-handed up-quark, color } c\text{, generation } k) \\
u_{R,c,k} &:= e_1 \otimes e_{c+1} \otimes e_4 \otimes e_k \;, && \text{(right-handed up-quark, color } c\text{, generation } k) \\
d_{L,c,k} &:= e_2 \otimes e_{c+1} \otimes e_1 \otimes e_k \;, && \text{(left-handed down-quark, color } c\text{, generation } k) \\
d_{R,c,k} &:= e_2 \otimes e_{c+1} \otimes e_4 \otimes e_k \;, && \text{(righ-handed down-quark, color } c\text{, generation } k)
\end{aligned}$$

where $\{e_i\,,\,i=1,\ldots,r\}$ is the canonical orthonormal basis of $\mathbb{C}^r$, $c=1,2,3$ and $k=1,\ldots,n$. These together with the corresponding antiparticles form a linear basis of H_F. A straightforward computation using (3.12) proves that we have the following transformation laws

$$\begin{aligned}
U(\nu_{L,k}) &:= \nu_{L,k} \otimes x_0x_k \;, & U(\nu_{R,k}) &:= \textstyle\sum_{j=1}^n \nu_{R,j} \otimes \overline{V}_{jk} \;, \\
U(e_{L,k}) &:= e_{L,k} \otimes x_k \;, & U(e_{R,k}) &:= e_{R,k} \otimes x_k \;, \\
U(u_{L,c,m}) &:= \textstyle\sum_{c'=1}^3 u_{L,c',m} \otimes (T_m)_{c'c} \;, & U(u_{R,c,m}) &:= \textstyle\sum_{c'=1}^3 u_{R,c',m} \otimes (T_m)_{c'c} \;, \\
U(d_{L,c,m}) &:= \textstyle\sum_{c'=1}^3 d_{L,c',m} \otimes x_0^*(T_m)_{c'c} \;, & U(d_{R,c,m}) &:= \textstyle\sum_{c'=1}^3 d_{R,c',m} \otimes x_0^*(T_m)_{c'c} \;,
\end{aligned}$$

where $U(v)$, $v \in H_F$, is a shorthand notation for $U(v \otimes 1_Q)$. Antiparticles transform according to the conjugate corepresentations.

3.3 $\widetilde{\mathrm{QISO}}^+_J$ for the minimal Standard Model

As we already noticed, $\widetilde{\mathrm{QISO}}^+_J(D_F)$ depends upon the explicit form of Υ_ν and Υ_R. In particular, on one extreme we have the case when Υ_ν is invertible (this is the case of the Dirac operator in the moduli space as in Prop. 1.192 of [20]) and on the other extreme we have the case $\Upsilon_\nu = 0$.

If Υ_ν is invertible, the first equation in (3.11a) is equivalent to $V = \mathrm{diag}(x_1^*x_0^*,\ldots,x_n^*x_0^*)$ and the factor $A_u(n)$ in Corollary 3.4 disappear. The second equation becomes $(\Upsilon_R)_{ij}(x_i^*x_0^* - x_0x_j) = 0$, which implies $x_i^*x_0^* = x_0x_j$ whenever $(\Upsilon_R)_{ij} \neq 0$. We get the following corollary.

Corollary 3.6. *If Υ_ν is invertible, $\widetilde{\mathrm{QISO}}^+_J(D_F)$ is the free product of Q_n with the quotient of*

$$\underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1}$$

by the relations

$$x_i^*x_0^* = x_0x_j \qquad \forall\, i,j \text{ such that } (\Upsilon_R)_{ij} \neq 0 \;.$$

Although disproved by experiment, it is an interesting exercise to study the case of massless ($\Upsilon_\nu = 0$) left-handed neutrinos, that is the so-called *minimal* Standard Model.

Corollary 3.7. *If $\Upsilon_\nu = 0$, $\widetilde{\mathrm{QISO}}^+_J(D_F)$ is isomorphic to*

$$\underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1} * Q_n * A',$$

where $A' := A_u(n)/\sim$, $A_u(n)$ is generated by the $n \times n$ biunitary V and "$\sim$" is the relation $V\Upsilon_R = \Upsilon_R \overline{V}$.

Now we take a closer look at this corollary.

The factor A' coacts only on the subspace $(e_{11} \otimes e_{11} \otimes (e_{22} + e_{44}) \otimes 1)H_F$ of right-handed neutrinos, and can be neglected in the minimal Standard Model (where we consider only left-handed neutrinos). As a consequence of Noether's theorem, there exists a conservation law corresponding to each classical group of symmetries.

It is easy to give an interpretation to the $C(U(1))$ factors generated by x_i, $i = 1, \ldots, n$. Passing from the $C(U(1))$ coaction to the dual $U(1)$ action, one easily sees that for $i > 0$, x_i gives a phase transformation of the i-th generation of ν_L, e_L, e_R (plus the opposite transformation for the antiparticles). In the minimal Standard Model, which has only left-handed (massless) neutrinos, these symmetries give the conservation laws of the total number of leptons in each generation (electron number, muon number, tau number, plus other $n-3$ for the other families of leptons).

To conclude the list of conservation laws, there is still one classical $U(1)$ subgroup of the factor Q_n that should be mentioned. If we denote by y the unitary generator of $C(U(1))$, a surjective CQG homomorphism $\varphi : \widetilde{\mathrm{QISO}}^+_J(D_F) \to C(U(1))$ is given by

$$x_0 \mapsto 1 , \qquad x_i \mapsto 1 , \qquad V_{j,k} \mapsto \delta_{j,k} , \qquad (T_i)_{j,k} \mapsto \delta_{j,k} y ,$$

for all $i = 1, \ldots, n$ and $j, k = 1, 2, 3$. From U we get the following corepresentation of this $U(1)$ subgroup on H_F:

$$\begin{aligned}(id \otimes \varphi)(U) = 1 \otimes e_{11} \otimes 1 \otimes 1 \otimes 1_{C(U(1))} \\ + 1 \otimes (1 - e_{11}) \otimes (e_{11} + e_{44}) \otimes 1 \otimes y \\ + 1 \otimes (1 - e_{11}) \otimes (e_{22} + e_{33}) \otimes 1 \otimes y^* .\end{aligned}$$

The representation of $U(1)$ dual to this corepresentation of $C(U(1))$ is given by a phase transformation on the subspace $\mathbb{C}^2 \otimes (1 - e_{11})\mathbb{C}^4 \otimes (e_{11} + e_{44})\mathbb{C}^4 \otimes \mathbb{C}^n$ of quarks and the inverse transformation on the subspace $\mathbb{C}^2 \otimes (1 - e_{11})\mathbb{C}^4 \otimes (e_{22} + e_{33})\mathbb{C}^4 \otimes \mathbb{C}^n$ of anti-quarks and is called in physics the "baryon phase symmetry". It corresponds to the conservation of the baryon number (total number of quarks minus the number of anti-quarks).

In this section we discussed conservation laws associated to classical subgroups of $\widetilde{\mathrm{QISO}}^+_J(D_F)$ in the massless neutrino case. It would be interesting to extend this study to the full quantum group $\widetilde{\mathrm{QISO}}^+_J(D_F)$ in the sense of a suitable Noether analysis extended to the quantum group framework. If we consider massive neutrinos, we lose a lot of classical symmetries, but we still have many quantum symmetries. A natural question is whether quantum symmetries are suitable for deriving conservation laws (i.e. physical predictions). A first step in this direction is to investigate whether the spectral action is invariant under quantum isometries. We discuss this point in the next section.

4 Quantum isometries of $M \times F$

4.1 Quantum isometries of a product of spectral triples

Before discussing the spectral action, we want to understand whether the quantum isometry group of the finite geometry F is also a quantum group of orientation preserving isometries of the full spectral triple of the Standard Model, that is the product of F with the canonical spectral triple of a compact Riemannian spin manifold M with no boundary. The answer is affirmative and we can prove it in a more general situation:

- Let $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1)$ be any unital real spectral triple ($\gamma_1 = 1$ if the spectral triple is odd).
- Let $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ be a finite-dimensional unital even real spectral triple.
- Let $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ be the product triple, i.e.

$$\mathcal{A} := \mathcal{A}_1 \otimes_{\mathrm{alg}} \mathcal{A}_2 , \qquad \mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2 , \qquad D := D_1 \otimes \gamma_2 + 1 \otimes D_2 ,$$
$$\gamma := \gamma_1 \otimes \gamma_2 , \qquad J := J_1 \otimes J_2 .$$

In the case of the Standard Model, $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ will be the canonical spectral triple of M and the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$ respectively.

We claim that:

Lemma 4.1. $\widetilde{\mathrm{QISO}}^+(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ *coacts by "orientation and real structure preserving isometries" on the product triple* $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$.

Proof. Let Q_0 be the quantum group $\widetilde{\mathrm{QISO}}^+_{J_2}(D_2)$ and U its corepresentation on $\mathcal{H}_2$. Then $\hat{U} := 1 \otimes U$ is a unitary corepresentation on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and we need to prove that it satisfies (2.3a), (2.3b), and (2.3c). The first two conditions are easy to check. Indeed, if U commutes with D_2 and γ_2, clearly $1 \otimes U$ commutes with $D = D_1 \otimes \gamma_2 + 1 \otimes D_2$ and $\gamma = \gamma_1 \otimes \gamma_2$. Moreover, for any vector $\xi = \xi_1 \otimes \xi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$\begin{aligned}
(J \otimes *)\hat{U}(\xi \otimes 1) &= (J_1 \otimes J_2 \otimes *)(1 \otimes U)(\xi_1 \otimes \xi_2 \otimes 1) \\
&= J_1\xi_1 \otimes (J_2 \otimes *)U(\xi_2 \otimes 1) \\
&= J_1\xi_1 \otimes U(J_2\xi_2 \otimes 1) \\
&= (1 \otimes U)(J_1\xi_1 \otimes J_2\xi_2 \otimes 1) \\
&= \hat{U}(J\xi \otimes 1) ,
\end{aligned}$$

and thus (2.3b) is proved.

Any element of $\mathcal{A}$ is a finite sum of tensors $a_1 \otimes a_2$, with $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, and since $\mathcal{A}_2$ is finite dimensional implies $U(a_2 \otimes 1_{Q_0})U^* \in \mathcal{A}_2 \otimes_{\mathrm{alg}} Q_0$, we have

$$\mathrm{Ad}_{\hat{\mathrm{U}}}(a_1 \otimes a_2) = \hat{U}(a_1 \otimes a_2 \otimes 1_{Q_0})\hat{U}^* = a_1 \otimes U(a_2 \otimes 1_{Q_0})U^* \in \mathcal{A}_1 \otimes_{\mathrm{alg}} \mathcal{A}_2 \otimes_{\mathrm{alg}} Q_0$$

which implies (2.3c). ☐

4.2 Invariance of the spectral action

The dynamics of a unital spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is governed by an action functional. According to the spectral action principle [12], the bosonic part of the action is

$$S_b[A] = \mathrm{Tr}_{\mathcal{H}}\, f(D_A/\Lambda) \;,$$

where the trace is on the Hilbert space $\mathcal{H}$, f is a suitable cut-off function (with $\Lambda > 0$), and $D_A := D + A + \epsilon' JAJ^{-1}$ is the gauged Dirac operator, with $A \in \Omega_D^{1,s.a.} \subset \mathcal{B}(\mathcal{H})$ a self-adjoint one-form and ϵ' is the sign in (2.1). More precisely, f is a smooth approximation of the characteristic function of the interval $[-1, 1]$, so that $f(D_A/\Lambda)$ is a trace class operator on $\mathcal{H}$ and $S_b[A]$ is well defined. The full spectral action is

$$S[A, \psi] := S_b[A] + S_f[A, \psi] \;,$$

where the fermionic part

$$S_f[A, \psi] = \langle J\psi, D_A\psi\rangle$$

is a functional $\Omega_D^{1,s.a.} \oplus \mathcal{V} \to \mathbb{C}$, with $\mathcal{V}$ either $\mathcal{H}$ or the eigenspace $\mathcal{H}_+$ corresponding to the eigenvalue $+1$ of the grading γ. While one uses $\mathcal{V} = \mathcal{H}$ in Yang-Mills theories, the reduction to $\mathcal{H}_+$ is employed in the Standard Model to solve the fermion doubling problem [31, 20].

Given a CQG Q with a unitary corepresentation $\hat{U}$ on $\mathcal{V}$, its coaction on $\Omega_D^{1,s.a.} \oplus \mathcal{V}$ is given by,

$$\beta : (A, \psi) \mapsto \big(\hat{U}(A \otimes 1)\hat{U}^*, \hat{U}(\psi \otimes 1)\big) \;.$$

Since β maps $\Omega_D^{1,s.a.} \oplus \mathcal{V}$ into $(\Omega_D^{1,s.a.} \oplus \mathcal{V}) \otimes Q$, to discuss the (co)invariance of the spectral action we need to extend it to the latter space. There is a natural way to do it. The inner product $\langle\,,\,\rangle : \mathcal{V} \otimes \mathcal{V} \to \mathbb{C}$ can be extended in a unique way to an Hermitian structure $\langle\,,\,\rangle_Q : \mathcal{M} \otimes \mathcal{M} \to Q$ on the right Q-module $\mathcal{M} := \mathcal{V} \otimes Q$ by the rule $\langle \psi \otimes q, \psi' \otimes q'\rangle_Q = q^* q' \langle \psi, \psi'\rangle$. Unitary (resp. antiunitary) maps L on $\mathcal{V}$ are extended in a unique way to Q-linear (resp. antilinear) maps on $\mathcal{M}$ as $L \otimes 1$ (resp. $L \otimes *$). The corresponding extension of the spectral action is given by the Q-valued functional

$$\tilde{S}[\tilde{A}, \tilde{\psi}] := \tilde{S}_b[\tilde{A}] + \tilde{S}_f[\tilde{A}, \tilde{\psi}] \;,$$

where

$$\tilde{S}_b[\tilde{A}] := (\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\, f(D_{\tilde{A}}/\Lambda) \;,$$
$$\tilde{S}_f[\tilde{A}, \tilde{\psi}] := \big\langle (J \otimes *)\tilde{\psi}, D_{\tilde{A}}\tilde{\psi}\big\rangle_Q \;,$$

and $\tilde{A} \in \Omega_D^{1,s.a.} \otimes Q$, $\tilde{\psi} \in \mathcal{M} = \mathcal{V} \otimes Q$, $D_{\tilde{A}} := D \otimes 1 + \tilde{A} + \epsilon'(J \otimes *)\tilde{A}(J \otimes *)^{-1}$.

In the remaining part of the section we prove that under reasonable assumptions

$$\tilde{S}[\beta(A, \psi)] = S[A, \psi] \cdot 1_Q. \tag{4.1}$$

We give the proof separately first for the fermionic part and then for the bosonic part.

Proposition 4.2. *If $\hat{U}$ satisfies* (2.3a) *and* (2.3b)*, then*

$$\tilde{S}_f[\beta(A, \psi)] = S_f[A, \psi] \cdot 1_Q$$

for all $(A, \psi) \in \Omega_D^{1,s.a.} \oplus \mathcal{V}$.

Proof. This is a simple algebraic identity. Since $\hat{U}$ commutes with D and $J \otimes *$, we have

$$D_{\hat{U}(A\otimes 1)\hat{U}^*} = D \otimes 1 + \hat{U}(A \otimes 1)\hat{U}^* + \epsilon'(J \otimes *)\hat{U}(A \otimes 1)\hat{U}^*(J \otimes *)^{-1} = \hat{U}(D_A \otimes 1)\hat{U}^*. \tag{4.2}$$

Thus,

$$\begin{aligned} \tilde{S}_f[\beta(A,\psi)] &= \big\langle (J \otimes *)\hat{U}(\psi \otimes 1_Q), D_{\hat{U}(A\otimes 1)\hat{U}^*}\hat{U}(\psi \otimes 1_Q)\big\rangle_Q \\ &= \big\langle \hat{U}(J\psi \otimes 1_Q), \hat{U}(D_A\psi \otimes 1_Q)\big\rangle_Q \\ &= \langle J\psi, D_A\psi\rangle \cdot 1_Q = S_f[A,\psi] \cdot 1_Q \ , \end{aligned}$$

by the unitarity of $\hat{U}$. □

For the rest of the subsection, we will assume that $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is the product of two real spectral triples, one of them being even and finite-dimensional. In fact, we will use the notations in Subsection 4.1. Moreover, we assume that $\hat{U} := 1 \otimes U$ where U is a unitary corepresentation of the CQG Q such that (Q, U) coacts by orientation and real structure preserving isometries on the finite dimensional spectral triple $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$. Under these assumptions, we now establish the invariance for the bosonic part.

Lemma 4.3. *For any trace-class operator L on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$*

$$(\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\hat{U}(L \otimes 1)\hat{U}^* = \mathrm{Tr}_{\mathcal{H}}(L) \cdot 1_Q \ .$$

Proof. Let $L = L_1 \otimes L_2$ with $L_1 \in \mathcal{L}^1(\mathcal{H}_1)$ and $L_2 \in \mathcal{B}(\mathcal{H}_2)$. Since

$$\hat{U}(L \otimes 1)\hat{U}^* = L_1 \otimes U(L_2 \otimes 1)U^*,$$

by Lemma 2.6, we have:

$$\begin{aligned} (\mathrm{Tr}_{\mathcal{H}_1\otimes\mathcal{H}_2} \otimes \mathrm{id})\hat{U}(L \otimes 1)\hat{U}^* &= \mathrm{Tr}_{\mathcal{H}_1}(L_1) \cdot (\mathrm{Tr}_{\mathcal{H}_2} \otimes \mathrm{id})\, U(L_2 \otimes 1)U^* \cdot 1_Q \\ &= \mathrm{Tr}_{\mathcal{H}_1\otimes\mathcal{H}_2}(L) \cdot 1_Q \ . \end{aligned}$$

Since $\mathcal{H}_2$ is finite dimensional, any element of $\mathcal{L}^1(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a finite sum of elements of the form $L := L_1 \otimes L_2$, with $L_1 \in \mathcal{L}^1(\mathcal{H}_1)$ and $L_2 \in \mathcal{B}(\mathcal{H}_2)$, and thus by the linearity of the trace, the proof is finished. □

Proposition 4.4. *For any $A \in \Omega^{1,s.a.}_D$, $\tilde{S}_b[\mathrm{Ad}_{\hat{U}}(A)] = S_b[A] \cdot 1_Q$.*

Proof. From (4.2) we have

$$\begin{aligned} \tilde{S}_b[\hat{U}(A \otimes 1)\hat{U}^*] &= (\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\, f(D_{\hat{U}(A\otimes 1)\hat{U}^*}/\Lambda) \\ &= (\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\, f\big(\hat{U}(D_A \otimes 1)\hat{U}^*/\Lambda\big) \ . \end{aligned}$$

By continuous functional calculus,

$$f\big(\hat{U}(D_A \otimes 1)\hat{U}^*/\Lambda\big) = \hat{U} f\big((D_A \otimes 1)/\Lambda\big)\hat{U}^* = \hat{U}\big(f(D_A/\Lambda) \otimes 1\big)\hat{U}^*$$

and applying Lemma 4.3 to the trace-class operator $L := f(D_A/\Lambda)$ we get

$$\begin{aligned} \tilde{S}_b[\hat{U}(A \otimes 1)\hat{U}^*] &= (\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\, \hat{U}(L \otimes 1)\hat{U}^* \\ &= \mathrm{Tr}_{\mathcal{H}}(L) \cdot 1_Q \equiv \mathrm{Tr}_{\mathcal{H}} f(D_A/\Lambda) \cdot 1_Q \\ &= S_b[A] \cdot 1_Q \ , \end{aligned}$$

which concludes the proof. □

Corollary 4.5. *The bosonic and the fermionic part of the spectral action of the Standard Model are preserved by the compact quantum group* $\widetilde{\mathrm{QISO}}^+(B_F, H_F, D_F, \gamma_F, J_F)$.

Proof. For the spectral triple of the Standard Model, $\widetilde{\mathrm{QISO}}^+(B_F, H_F, D_F, \gamma_F, J_F)$ is a valid candidate for Q because of Lemma 4.1 and as its corepresentation preserves $\mathcal{H}_+$. Thus, Proposition 4.2 and Proposition 4.4 taken together proves the desired result. □

5 Some remarks on real $*$-algebras and their symmetries

In Sec. 3.2 we computed the quantum isometry group of the finite part of the Standard Model by replacing the real C^*-algebra A_F with the complex C^*-algebra B_F. Here we explain what happens if we work with A_F.

Any real $*$-algebra $\mathcal{A}$ (i.e. unital, associative, involutive algebra over $\mathbb{R}$) can be thought of as the fixed point subalgebra of its complexification $\mathcal{A}_{\mathbb{C}} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the involutive (conjugate-linear) real $*$-algebra automorphism σ defined by

$$\sigma(a \otimes_{\mathbb{R}} z) = a \otimes_{\mathbb{R}} \overline{z} \quad \forall\, a \in \mathcal{A}, z \in \mathbb{C}\ , \tag{5.1}$$

that is

$$\mathcal{A} = \{a \in \mathcal{A}_{\mathbb{C}} : \sigma(a) = a\}\ .$$

A crucial observation is that we can characterize the automorphisms of $\mathcal{A}$ as those automorphisms of $\mathcal{A}_{\mathbb{C}}$ which commute with σ, as proved in the following lemma.

Lemma 5.1. *For any real $*$-algebra $\mathcal{A}$,*

$$\mathrm{Aut}(\mathcal{A}) \simeq \big\{\phi \in \mathrm{Aut}(\mathcal{A}_{\mathbb{C}})\ :\ \sigma\phi = \phi\sigma\big\}\ . \tag{5.2}$$

Proof. If φ is any (real) $*$-algebra morphism of $\mathcal{A}$, $\phi(a \otimes_{\mathbb{R}} z) := \varphi(a) \otimes_{\mathbb{R}} z$ defines a (complex) $*$-algebra morphism of $\mathcal{A}_{\mathbb{C}}$ clearly satisfying $\sigma\phi = \phi\sigma$. The map $\varphi \mapsto \phi$ gives an inclusion of the left hand side of (5.2) into the right hand side. Conversely, if $\phi \in \mathrm{Aut}(\mathcal{A}_{\mathbb{C}})$ satisfies $\sigma\phi = \phi\sigma$, then it maps the real subalgebra $\mathcal{A} \simeq \mathcal{A} \otimes_{\mathbb{R}} 1 \subset \mathcal{A}_{\mathbb{C}}$ into itself, since

$$\sigma\phi(a \otimes_{\mathbb{R}} 1) = \phi\sigma(a \otimes_{\mathbb{R}} 1) = \phi(a \otimes_{\mathbb{R}} 1)$$

for any $a \in \mathcal{A}$. Therefore, we can define an element $\varphi \in \mathrm{Aut}(\mathcal{A})$ by $\varphi(a) \otimes_{\mathbb{R}} 1 := \phi(a \otimes_{\mathbb{R}} 1)$.

The two group homomorphisms $\varphi \mapsto \phi$ and $\phi \mapsto \varphi$ are the inverses of each other and thus, we have the isomorphism in (5.2). □

From a dual point of view, if $G = \mathrm{Aut}(\mathcal{A})$, the right coaction of $C(G)$ on $\mathcal{A}_{\mathbb{C}}$ is the map $\alpha : \mathcal{A}_{\mathbb{C}} \to \mathcal{A}_{\mathbb{C}} \otimes C(G) \simeq C(G; \mathcal{A}_{\mathbb{C}})$ defined by

$$(\mathrm{id} \otimes \mathrm{ev}_\phi)\alpha(a) := \phi(a),\ \ \phi\ \in G,\ \ a\ \in \mathcal{A}_{\mathbb{C}}\ .$$

We can rephrase Lemma 5.1 as follows.

Lemma 5.2. *For a finite dimensional real C^*-algebra $\mathcal{A}$, the condition $\sigma\,\phi = \phi\,\sigma\ \forall\ \phi \in G$ is equivalent to*

$$(\sigma \otimes *_{C(G)})\alpha = \alpha\sigma\ .$$

Proof. Let $\alpha_\phi\sigma = (\sigma \otimes ev_\phi *_{C(G)})\alpha$ and $\phi \in G,\ a \in \mathcal{A}_\mathbb{C}$. Let us suppose that $(\sigma \otimes *_{C(G)})\alpha = \alpha\sigma$. Then $\sigma\phi(a) = (\mathrm{id} \otimes ev_\phi)\alpha\sigma(a) = (\sigma \otimes ev_\phi *_{C(G)})\alpha(a) = (\sigma \otimes *_\mathbb{C}\ \mathrm{ev}_\phi)\alpha(a) = \phi\sigma(a)$ by the antilinearity of σ. Conversely, if $\sigma\,\phi = \phi\,\sigma\ \forall\ \phi \in G$ then for all ϕ, $(\mathrm{id} \otimes \mathrm{ev}_\phi)\alpha(\sigma(a)) = (\sigma \otimes \mathrm{ev}_\phi)(\alpha(a))$. Thus, $(\sigma \otimes \mathrm{ev}_\phi *_{C(G)})\alpha(a) = (\sigma \otimes *_\mathbb{C}\ \mathrm{ev}_\phi)\alpha(a) = \sigma((\mathrm{id} \otimes \mathrm{ev}_\phi)\alpha(a)) = \sigma\phi(a) = \phi\sigma(a) = (\mathrm{id} \otimes \mathrm{ev}_\phi)\alpha(\sigma(a))$. As $\{\mathrm{ev}_\phi : \phi \in G\}$ separates points on G, this proves $(\sigma \otimes *_{C(G)})\alpha = \alpha\sigma$. $\square$

Motivated by this lemma, we consider the category $\mathfrak{C}_{J,\mathbb{R}}$ of CQGs coacting by orientation and real structure preserving isometries via a unitary corepresentation U (in the sense of Def. 2.10) on the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$ whose adjoint coaction $\mathrm{Ad_U}$ can be extended to a coaction α on $(A_F)_\mathbb{C} = A_F \otimes_\mathbb{R} \mathbb{C}$ satisfying

$$(\sigma \otimes *)\alpha = \alpha\sigma\ . \tag{5.3}$$

We notice that it is a subcategory of $\mathfrak{C}_J$: objects of $\mathfrak{C}_{J,\mathbb{R}}$ are those objects of $\mathfrak{C}_J$ compatible with σ in the sense explained above, and the morphisms in the two categories are the same.

Thus any object, say Q, of $\mathfrak{C}_{J,\mathbb{R}}$ satisfies the relations of the universal object $\widetilde{\mathrm{QISO}}^+_J(D_F)$ of $\mathfrak{C}_J$ in Prop. 3.3. In the rest of this subsection, with a slight abuse of notation, we will continue to denote the generators of Q by the same symbols as in Prop. 3.3.

Theorem 5.3. *A CQG Q is an object in $\mathfrak{C}_{J,\mathbb{R}}$ if and only if the generators satisfy*

$$(T_m)_{jk}(T_m)^*_{j'k'}(T_m)_{j''k''} = (T_m)_{j''k''}(T_m)^*_{j'k'}(T_m)_{jk} \tag{5.4}$$

for all $m = 1, \ldots, n$ and all $j, j', j'', k, k', k'' \in \{1, 2, 3\}$.

Proof. The real algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is the fixed point subalgebra of $(A_F)_\mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ with respect to the automorphism

$$\sigma(\lambda, \lambda', q, m, m') = (\overline{\lambda}', \overline{\lambda}, \sigma_2\overline{q}\sigma_2, \overline{m}', \overline{m})\ ,$$

where σ_2 is the second Pauli matrix:

$$\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

It is easy to check that $q \in M_2(\mathbb{C})$ satisfies $\sigma_2\overline{q}\sigma_2 = q$ if an only if it is of the form (3.2), and that under the isomorphism (3.8) $\mathbb{C}$ is identified with the real subalgebra of $\mathbb{C} \oplus \mathbb{C}$ with elements $(\lambda, \overline{\lambda})$ and $M_3(\mathbb{C})$ with the real subalgebra of $M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ with elements $(m, \overline{m})$.

The coaction on the factor $B_F \subset (A_F)_\mathbb{C}$ is given by (3.13), and an extension $\widetilde{\mathrm{Ad_U}}$ to $(A_F)_\mathbb{C}$ satisfying (5.3) exists if and only if

$$\begin{aligned}
\widetilde{\mathrm{Ad_U}}(0,0,0,0,e_{ij}) &= (\sigma \otimes *)\widetilde{\mathrm{Ad_U}}\,\sigma(0,0,0,0,e_{ij}) \\
&= (\sigma \otimes *)\widetilde{\mathrm{Ad_U}}(0,0,0,\overline{e}_{ij},0) \\
&= (\sigma \otimes *)\big(\mathrm{Ad_U}(\langle 0,0,0,e_{ij}\rangle),0\big) \\
&= (\sigma \otimes *)\sum\nolimits_{k,l=1,2,3}(0,0,0,e_{kl},0) \otimes (T_1)^*_{ki}(T_1)_{lj} \\
&= \sum\nolimits_{k,l=1,2,3}(0,0,0,0,e_{kl}) \otimes (T_1)^*_{lj}(T_1)_{ki}\ .
\end{aligned}$$

The only conditions left to impose is that this extension is a coaction of a CQG. As it is already a coaction on B_F, we need to impose it for the coaction on the second copy of $M_3(\mathbb{C})$, which has to be preserved by $\widetilde{\mathrm{Ad}_\mathrm{U}}$. At this point, we note that as $\widetilde{\mathrm{Ad}_\mathrm{U}}$ is an extension of Ad_U, which preserves the trace on the first copy of $M_3(\mathbb{C})$, the formula $\widetilde{\mathrm{Ad}_\mathrm{U}}(0,0,0,0,e_{ij}) = \sum_{k,l=1,2,3}(0,0,0,0,e_{kl}) \otimes (T_1)^*_{lj}(T_1)_{ki}$ forces $\widetilde{\mathrm{Ad}_\mathrm{U}}$ to preserve the trace on the second copy of $M_3(\mathbb{C})$. Thus, by Theorem 4.1 of [37], it suffices to impose the conditions (4.1-4.5) in that paper with a^{kl}_{ij} replaced by $(T_m)^*_{lj}(T_m)_{ki}$. It is easy to check that (4.3-4.5) are automatically satisfied. The only non trivial conditions come from (4.1) and (4.2).

From (4.1), we get

$$\sum\nolimits_{v=1}^{3}(T_m)^*_{vj}(T_m)_{ki}(T_m)^*_{ls}(T_m)_{vr} = \delta_{jr}(T_m)^*_{ls}(T_m)_{ki} \tag{5.5}$$

From (4.2), we get the same relation with $(T_m)^t$ instead of T_m. Now we show that (5.5) and (5.4) are equivalent, which will finish the proof since if T_m satisfies (5.4), then $(T_m)^t$ satisfies it too.

If we multiply both sides of (5.5) by $(T_m)_{qj}$ from the left and sum over j, we get

$$\sum\nolimits_{v=1}^{3}\delta_{vq}(T_m)_{ki}(T_m)^*_{ls}(T_m)_{vr} = \sum\nolimits_{j=1}^{3}\delta_{jr}(T_m)_{qj}(T_m)^*_{ls}(T_m)_{ki}$$

using biunitarity of T_m. The last equation is clearly equivalent to (5.4). To prove that (5.4) implies (5.5), it is enough to multiply both sides by $(T_m)_{j''k'''}$ from the left, then sum over j'' and use the biunitarity of T_m again. □

It is easy to check that (5.4) defines a Woronowicz C^*-ideal, and hence the quotient of $\widetilde{\mathrm{QISO}}^+_J(D_F)$ by (5.4) is a CQG. This leads to the following corollary.

Corollary 5.4. *Let $\widetilde{\mathrm{QISO}}^+_\mathbb{R}(D_F)$ be the quantum subgroup of the CQG $\widetilde{\mathrm{QISO}}^+_J(D_F)$ in Prop. 3.3 defined by the relations* (5.4)*. Then $\widetilde{\mathrm{QISO}}^+_\mathbb{R}(D_F)$ is the universal object in the category $\mathfrak{C}_{J,\mathbb{R}}$.*

Motivated by (5.4), we give the following definition.

Definition 5.5. *For a fixed N, we call $A^*_u(N)$ the universal unital C^*-algebra generated by a $N \times N$ biunitary $u = ((u_{ij}))$ with relations*

$$ab^*c = cb^*a\ , \qquad \forall\, a,b,c \in \{u_{ij},\, i,j = 1,\dots,N\}\ . \tag{5.6}$$

$A^*_u(N)$ is a CQG with coproduct given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

We will call $A^*_u(N)$ the N-dimensional *half-liberated unitary group.* This is similar to the half-liberated orthogonal group $A^*_o(N)$, that can be obtained by imposing the further relation $a = a^*$ for all $a \in \{u_{ij},\, i,j,= 1,\dots,N\}$ (cf. [5]).

Remark 5.6. *We notice that there are two other possible ways to "half-liberate" the free unitary group. Instead of $ab^*c = cb^*a$ (which by adjunction is equivalent to $a^*bc^* = c^*ba^*$), one can consider respectively the relation $a^*bc = cba^*$ (which is equivalent to $abc^* = c^*ba$ and to the adjoints $ab^*c^* = c^*b^*a$ and $a^*b^*c = cb^*a^*$) or $abc = cba$ (equivalent to $a^*b^*c^* = c^*b^*a^*$) for any triple $a,b,c \in \{u_{ij},\, i,j = 1,\dots,N\}$.*

Like $A^*_o(N)$, the projective version of $A^*_u(N)$ is also commutative, as proved in the next proposition.

Proposition 5.7. *The CQG $PA^*_u(N)$ is isomorphic to $C(PU(N))$.*

Proof. We recall (Rem. 2.8) that for a CQG Q generated by a biunitary $u = ((u_{ij}))$, the projective version is the C^*-subalgebra generated by products $u^*_{ij}u_{kl}$.

Clearly $C(U(N))$ is a quantum subgroup of $A^*_u(N)$, and the latter is a quantum subgroup of $A_u(N)$. Thus, $C(PU(N))$ is a quantum subgroup of $PA^*_u(N)$, which is a quantum subgroup of $PA_u(N)$. Since the abelianization of $PA_u(N)$ is exactly $C(PU(N))$, any commutative (as a C^*-algebra) quantum subgroup of $PA_u(N)$ containing $C(PU(N))$ coincides with $C(PU(N))$. Thus, the proof will be over if we can show that the C^*-algebra of $PA_u(N)$ is commutative, i.e. $PA_u(N)$ is the space of continuous functions on a compact group. This is a simple computation. Using first (5.6) and then its adjoint we get:

$$\begin{aligned}(u^*_{ij}u_{kl})(u^*_{pq}u_{rs}) &= u^*_{ij}(u_{kl}u^*_{pq}u_{rs}) = u^*_{ij}(u_{rs}u^*_{pq}u_{kl}) \\ &= (u^*_{ij}u_{rs}u^*_{pq})u_{kl} = (u^*_{pq}u_{rs}u^*_{ij})u_{kl} \\ &= (u^*_{pq}u_{rs})(u^*_{ij}u_{kl}) \ .\end{aligned}$$

This proves that the generators of $PA_u(N)$ commute, which concludes the proof. □

In complete analogy with Corollary 3.4 we have:

Corollary 5.8. $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ *is a quantum subgroup of the free product*

$$\underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1} * Q^*_n * A_u(n)$$

where Q^*_n is the amalgamated free product of n copies of $A^*_u(3)$ over the common Woronowicz C^*-subalgebra $C(PU(3))$. The Woronowicz C^*-ideal of this CQG defining $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ is determined by the relations (3.11)*.*

As in the complex case, let us denote by $\mathrm{QISO}^+_{\mathbb{R}}(D_F)$ the C^*-subalgebra of $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ generated by $\{(\varphi \otimes \mathrm{id})\mathrm{Ad}_{U_{\mathbb{R}}} : \varphi \in (B_F)^*\}$, where $U_{\mathbb{R}}$ is the corepresentation of $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$. An immediate corollary of Prop. 5.7 and Corollary 5.8 is the following.

Corollary 5.9. $\mathrm{QISO}^+_{\mathbb{R}}(D_F) = C(U(1)) * C(PU(3))$*.*

Remark 5.10. *Since $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ is a quantum subgroup of $\widetilde{\mathrm{QISO}}^+_J(D_F)$, its coaction still preserves the spectral action.*

A detailed study of quantum automorphisms for finite-dimensional real C^*-algebras, along the lines of the discussion in this section, will be reported elsewhere.

6 Proof of Proposition 3.3

In this section, we prove the main result, that is, Proposition 3.3. Throughout this section, (Q, U) will denote an object in $\mathfrak{C}_J$. We start by exploiting the conditions regarding γ_F and J_F, then we use the conditions regarding D_F and Ad_U to get a neater expression for U in Lemma 6.2, 6.3 and 6.4 and then using these simplified expressions in the next Lemmas, we derive the desired form of U from which we can identify the quantum isometry group.

Lemma 6.1. *$U \in M_{32n}(\mathbb{C}) \otimes Q$ satisfies $(\gamma_F \otimes 1)U = U(\gamma_F \otimes 1)$ and $(J_0 \otimes 1)\overline{U} = U(J_0 \otimes 1)$ iff*

$$\begin{aligned} U = &\sum\nolimits_{IJ} (e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes e_{i_3 j_3} \otimes e_{i_4 j_4}) \otimes u_{IJ} \\ &+ \sum\nolimits_{IJ} (e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes e_{i_3+2, j_3+2} \otimes e_{i_4 j_4}) \otimes u^*_{IJ} , \end{aligned} \tag{6.1}$$

where the multi-indices $I = (i_1, \dots, i_4)$, $J = (j_1, \dots, j_4)$, etc. run in $\{1,2\} \times \{1,2,3,4\} \times \{1,2\} \times \{1,2,\dots,n\}$.

Proof. $(\gamma_F \otimes 1)U = U(\gamma_F \otimes 1)$ implies that $u_{i_1,j_1,i_2,j_2,i_3,j_3,i_4,j_4} = 0$ unless (i_3, j_3) belongs to $\{1,2\} \times \{1,2\}$ or $\{3,4\} \times \{3,4\}$. Using the reduced form of U obtained from this observation, the relation $(J_0 \otimes 1)\overline{U} = U(J_0 \otimes 1)$ is applied to reach the desired expression. □

Let V_1, V_2, V_3, V_4 denote the subspaces $(e_{11} \otimes e_{11} \otimes 1 \otimes 1)\mathcal{H}$, $(e_{22} \otimes e_{11} \otimes 1 \otimes 1)\mathcal{H}$, $(e_{11} \otimes (1 - e_{11}) \otimes 1 \otimes 1)\mathcal{H}$, and $(e_{22} \otimes (1 - e_{11}) \otimes 1 \otimes 1)\mathcal{H}$ respectively.

Lemma 6.2. *If U commutes with D_F, the subspaces V_i, $i = 1,2,3,4$ are kept invariant by U and thus (6.1) becomes*

$$\begin{aligned} U = &\sum_{i=1,2} e_{ii} \otimes e_{11} \otimes \begin{pmatrix} \alpha^i_{11} & \alpha^i_{12} & 0 & 0 \\ \alpha^i_{21} & \alpha^i_{22} & 0 & 0 \\ 0 & 0 & \overline{\alpha}^i_{11} & \overline{\alpha}^i_{12} \\ 0 & 0 & \overline{\alpha}^i_{21} & \overline{\alpha}^i_{22} \end{pmatrix} \\ &+ \sum_{\substack{i=1,2 \\ j,k=1,2,3}} e_{ii} \otimes e_{j+1,k+1} \otimes \begin{pmatrix} \beta^{i,j,k}_{11} & \beta^{i,j,k}_{12} & 0 & 0 \\ \beta^{i,j,k}_{21} & \beta^{i,j,k}_{22} & 0 & 0 \\ 0 & 0 & \overline{\beta}^{i,j,k}_{11} & \overline{\beta}^{i,j,k}_{12} \\ 0 & 0 & \overline{\beta}^{i,j,k}_{21} & \overline{\beta}^{i,j,k}_{22} \end{pmatrix} \end{aligned} \tag{6.2}$$

where, as in (3.5) we identify $M_4(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes Q$ with $M_{4n}(Q)$, we called $\alpha^i_{j_1 k_1}$ is the $n \times n$ matrix with entries $(\alpha^i_{j_1 k_1})_{j_2 k_2} := u_{JK}$ with $J = (i, 1, j_1, j_2)$ and $K = (i, 1, k_1, k_2)$ and we called $\beta^{i,j_0,k_0}_{j_1 k_1}$ the $n \times n$ matrix with entries $(\beta^{i,j_0,k_0}_{j_1 k_1})_{j_2 k_2} := u_{JK}$ with $J = (i, j_0 + 1, j_1, j_2)$ and $K = (i, k_0 + 1, k_1, k_2)$ (all the other elements u_{IJ} being zero).

Proof. The subspaces $V_i, i = 1,2,3,4$ are D_F-invariant and correspond to distinct sets of eigenvalues (masses of the generations of ν, e, u and d respectively). Since $(D_F \otimes 1)U = U(D_F \otimes 1)$ these four subspaces must be preserved by U and this completes the proof of the lemma. □

Lemma 6.3. *Let (Q, Δ) be any CQG satisfying (2.3a), (3.10a) and (3.10b). Then each one of the four summands in $B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ is a coinvariant subalgebra under the adjoint coaction $\mathrm{Ad_U}(a) = U(a \otimes 1)U^*$ of Q.*

Proof. We start with the basis element $\langle 0,1,0,0 \rangle$ of the second copy of $\mathbb{C}$. Equation (3.10b) means that

$$\begin{aligned} \mathrm{Ad_U}(\langle 0,1,0,0 \rangle) = &\langle 1,0,0,0 \rangle \otimes a^{\langle 1,0,0,0 \rangle} + \langle 0,1,0,0 \rangle \otimes a^{\langle 0,1,0,0 \rangle} \\ &+ \sum_{i,j=1,2} \langle 0,0,e_{ij},0 \rangle \otimes a^{\langle 0,0,e_{ij},0 \rangle} + \sum_{i,j=1,2,3} \langle 0,0,0,e_{ij} \rangle \otimes a^{\langle 0,0,0,e_{ij} \rangle} \end{aligned} \tag{6.3}$$

where $a^{\langle \cdot \rangle}$ are some elements of Q.

By (6.2), $U(\langle 0,1,0,0\rangle\otimes 1)U^*$ has e_{22} in the first position and e_{jk} in the third, with $j,k=3,4$. Therefore, $U(\langle 0,1,0,0\rangle\otimes 1)U^*$ vanishes on the subspaces $(e_{11}\otimes 1\otimes e_{44}\otimes 1)H_F$, $(1\otimes 1\otimes e_{11}\otimes 1)H_F$ and $(1\otimes(1-e_{11})\otimes(e_{22}+e_{33})\otimes 1)H_F$. Applying (6.3) on these three subspaces and using (3.9) we get respectively:

$$\begin{aligned}
0 &= (e_{11}\otimes 1\otimes e_{44}\otimes 1)\otimes a^{\langle 1,0,0,0\rangle}+0+0+0\;,\\
0 &= 0+0+\sum\nolimits_{i,j=1,2}\langle 0,0,e_{ij},0\rangle\otimes a^{\langle 0,0,e_{ij},0\rangle}+0\;,\\
0 &= 0+0+0+\sum\nolimits_{i,j=1,2,3}\langle 0,0,0,e_{ij}\rangle\otimes a^{\langle 0,0,0,e_{ij}\rangle}\;.
\end{aligned}$$

Therefore $a^{\langle 1,0,0,0\rangle}=a^{\langle 0,0,e_{ij},0\rangle}=a^{\langle 0,0,0,e_{ij}\rangle}=0$ and $\mathrm{Ad_U}(\langle 0,1,0,0\rangle)\subset\langle 0,1,0,0\rangle\otimes Q$. The proof for the other three factors is similar.

For the rest of the proof, let $\lambda\in\mathbb{C},\ q\in M_2(\mathbb{C}), m\in M_3(\mathbb{C})$ be arbitrary.

$U(\langle 1,0,0,0\rangle\otimes 1)U^*$ vanishes on the subspaces $(e_{22}\otimes(1-e_{11})\otimes e_{44}\otimes 1)H_F$, $(1\otimes e_{11}\otimes e_{11}\otimes 1)H_F$ and $(1\otimes(1-e_{11})\otimes e_{22}\otimes 1)H_F$ and hence this implies respectively that the coefficients of $\langle 0,\lambda,0,0\rangle$, $\langle 0,0,q,0\rangle$, $\langle 0,0,0,m\rangle$ in $\mathrm{Ad_U}(\langle 1,0,0,0\rangle)$ are zero.

$U(\langle 0,0,q,0\rangle\otimes 1)U^*$ vanishes on the subspaces $(e_{11}\otimes 1\otimes e_{44}\otimes 1)H_F$, $(e_{22}\otimes 1\otimes e_{44}\otimes 1)H_F$ and $(1\otimes(1-e_{11})\otimes e_{33}\otimes 1)H_F$ and hence this implies respectively that the coefficients of $\langle\lambda,0,0,0\rangle$, $\langle 0,\lambda,0,0\rangle$, $\langle 0,0,0,m\rangle$ in $\mathrm{Ad_U}(\langle 0,0,q,0\rangle)$ are zero.

Finally, $U(\langle 0,0,0,m\rangle\otimes 1)U^*$ vanishes on the subspaces $(e_{11}\otimes e_{11}\otimes e_{44}\otimes 1)H_F$, $(e_{22}\otimes e_{11}\otimes e_{44}\otimes 1)H_F$ and $(1\otimes e_{11}\otimes e_{11}\otimes 1)H_F$ which implies respectively that the coefficients of $\langle\lambda,0,0,0\rangle$, $\langle 0,\lambda,0,0\rangle$, $\langle 0,0,q,0\rangle$ in $\mathrm{Ad_U}(\langle 0,0,0,m\rangle)$ are zero. □

Lemma 6.4. *If* (3.10b) *is satisfied, the matrices* $\alpha^i_{j_1k_1}$ *and* $\beta^{i,j_0,k_0}_{j_1k_1}$ *in* (6.2) *are zero for all* $j_1\neq k_1$.

Proof. We are under the hypothesis of Lemma 6.3. Since $\mathrm{Ad_U}(\langle 1,0,0,0\rangle)\subset\langle 1,0,0,0\rangle\otimes Q$, it is easy to see that $(e_{ii}\otimes e_{11}\otimes e_{11}\otimes 1\otimes 1_Q)\mathrm{Ad_U}(\langle 1,0,0,0\rangle)$ equals zero for all $i=1,2$. On the other hand, straightforward computation gives

$$(e_{ii}\otimes e_{11}\otimes e_{11}\otimes 1\otimes 1_Q)\mathrm{Ad_U}(\langle 1,0,0,0\rangle)=e_{ii}\otimes e_{11}\otimes e_{11}\otimes\alpha^i_{12}(\alpha^i_{12})^*=0\;,$$

from which it follows that $\alpha^i_{12}=0$ for all $i=1,2$. Similarly,

$$(1\otimes e_{11}\otimes e_{22}\otimes 1\otimes 1_Q)\mathrm{Ad_U}(\langle 0,0,e_{ii},0\rangle)=e_{ii}\otimes e_{11}\otimes e_{22}\otimes\alpha^i_{21}(\alpha^i_{21})^*=0$$

gives $\alpha^i_{21}=0$ for all $i=1,2$. Finally, $\mathrm{Ad_U}(\langle 0,0,0,e_{k_0l_0}\rangle)$ applied to the projections $1\otimes 1\otimes e_{11}\otimes 1$ and $1\otimes 1\otimes e_{44}\otimes 1$, we get the conditions

$$\beta^{i,j_0,k_0}_{12}(\beta^{i,l_0,n_0}_{12})^*=\overline{\beta}^{i,j_0,k_0}_{21}(\beta^{i,l_0,n_0}_{21})^t=0$$

for all i,j_0,k_0,l_0,n_0. In particular setting $j_0=l_0$ and $k_0=n_0$ we get $\beta^{i,j_0,k_0}_{12}=\beta^{i,j_0,k_0}_{21}=0$. □

Now we impose $U(D_F\otimes 1)=(D_F\otimes 1)U$, with D_F as in (3.6) and the reduced form of U as in (6.2) by imposing Lemma 6.4, that is, $\alpha^i_{j_1k_1}$ and $\beta^{i,j_0,k_0}_{j_1k_1}$ are zero for all $j_1\neq k_1$.

Lemma 6.5. *Any* U *of the form as in* (6.2) *satisfies* $U(D_F\otimes 1)=(D_F\otimes 1)U$ *if and only if*

1. *all* α^2_{ss} *and* $\beta^{i,j,k}_{rr}$ *are diagonal* $n\times n$ *matrices,*

2. $\alpha^2_{22} = \overline{\alpha}^2_{11}$, $\beta^{i,j,k}_{22} = \overline{\beta}^{i,j,k}_{11}$,

3. $\alpha^1_{11}\Upsilon_\nu = \Upsilon_\nu\,\overline{\alpha}^1_{22}$, $\alpha^1_{22}\Upsilon_\nu = \Upsilon_\nu\,\overline{\alpha}^1_{11}$, $\alpha^1_{22}\Upsilon_R = \Upsilon_R\,\overline{\alpha}^1_{22}$.

Proof. The above condition is equivalent to the following sets of equations:

$$\alpha^1_{11}\Upsilon_\nu = \Upsilon_\nu\,\overline{\alpha}^1_{22}\;, \qquad \alpha^1_{22}\Upsilon_\nu = \Upsilon_\nu\,\overline{\alpha}^1_{11}\;, \qquad \alpha^1_{22}\Upsilon_R = \Upsilon_R\,\overline{\alpha}^1_{22}\;, \tag{6.4a}$$

$$\alpha^2_{11}\Upsilon_e = \Upsilon_e\,\overline{\alpha}^2_{22}\;, \qquad \alpha^2_{22}\Upsilon_e = \Upsilon_e\,\overline{\alpha}^2_{11}\;, \qquad \beta^{1,j,k}_{11}\Upsilon_u = \Upsilon_u\,\overline{\beta}^{1,j,k}_{22}\;, \tag{6.4b}$$

$$\beta^{1,j,k}_{22}\Upsilon_u = \Upsilon_u\,\beta^{1,j,k}_{11}\;, \qquad \beta^{2,j,k}_{11}\Upsilon_d = \Upsilon_d\,\overline{\beta}^{2,j,k}_{22}\;, \qquad \beta^{2,j,k}_{22}\Upsilon_d = \Upsilon_d\,\overline{\beta}^{2,j,k}_{11}\;, \tag{6.4c}$$

Actually, there are additional 9 relations that — recalling that Υ_x $(x = \nu, e, u, d)$ are positive and diagonal and Υ_R is symmetric — turn out to be the "bar" of previous ones and hence they do not give any new information.

From $\alpha^2_{22}\Upsilon_e = \Upsilon_e\,\overline{\alpha}^2_{11}$ and $\alpha^2_{11}\Upsilon_e = \Upsilon_e\,\overline{\alpha}^2_{22}$, since Υ_e is positive diagonal, we deduce

$$\alpha^2_{22}\Upsilon^2_e = \Upsilon_e\overline{(\alpha^2_{11}\Upsilon_e)} = \Upsilon_e\overline{(\Upsilon_e\,\overline{\alpha}^2_{22})} = \Upsilon^2_e\alpha^2_{22}\;.$$

In a similar way we find that all α^2_{ss} commute with Υ^2_e, and all $\beta^{2,j,k}_{rr}$ commute with Υ^2_d. Since Υ^2_x $(x = e, u, d)$ are diagonal with distinct eigenvalues, we deduce that all α^2_{ss} and $\beta^{i,j,k}_{rr}$ must be diagonal $n \times n$ matrices. Conversely, if these matrices are diagonal and since Υ_x $(x = e, u, d)$ are invertible, then the relations in (6.4b) are satisfied if $\alpha^2_{22} = \overline{\alpha}^2_{11}$ and $\beta^{i,j,k}_{22} = \overline{\beta}^{i,j,k}_{11}$ which proves the lemma. ☐

In view of Lemma 6.5, we introduce the notation

$$\alpha^2_{11} = \sum_{k=1}^{n} e_{kk} \otimes x_k,\;\; \beta^{i,j,k}_{11} = \sum_{m=1}^{n} e_{mm} \otimes z^{i,j,k}_m$$

Hence,

$$\beta^{i,j,k}_{22} = \sum_{m=1}^{n} e_{mm} \otimes (z^{i,j,k}_m)^*.$$

Moreover, we define 3×3 matrices $(T(i,m))_{j,k} = z^{i,j,k}_m$ and $(T'(i,m))_{j,k} = (z^{i,j,k}_m)^*$.

Lemma 6.6. *U being a unitary corepresentation implies that the matrices α^i_{rr} and $T(i,m)$ are biunitaries. In particular, $\{x_1, x_2, \ldots.., x_n\}$ are unitary elements.*

Proof. The condition $UU^* = 1 \otimes 1$ implies that for $r = 1, 2$,

$$\alpha^i_{rr}(\alpha^i_{rr})^* = \overline{\alpha}^i_{rr}(\overline{\alpha}^i_{rr})^* = 1,$$

$$\sum_k \beta^{i,j,k}_{rr}(\beta^{i,l,k}_{rr})^* = \sum_k \overline{\beta}^{i,j,k}_{rr}(\overline{\beta}^{i,j,k}_{rr}) = \delta_{jl}.$$

Similarly, from $U^*U = 1 \otimes 1$ we get the relations

$$(\alpha^i_{rr})^*\alpha^i_{rr} = (\overline{\alpha}^i_{rr})^*\overline{\alpha}^i_{rr} = 1,$$

$$\sum_k (\beta^{i,l,k}_{rr})^*\beta^{i,j,k}_{rr} = \sum_k (\overline{\beta}^{i,l,k}_{rr})^*\overline{\beta}^{i,j,k}_{rr} = \delta_{jl}..$$

Thus, the matrices α^i_{rr}, $(T(i,m))$ and $(T'(i,m))$ are biunitaries. In fact, the biunitarity of $T'(i,m)$ follows from the biunitarity of $T(i,m)$. This proves the result. ☐

Lemma 6.7. *From the condition* (3.10b), *i.e. that the coaction* Ad_{U} *preserves the subalgebra* B_F, *we derive that there exists a unitary* b *such that*

$$\alpha^1_{11} = \mathrm{diag}(bx_1, .., bx_n) , \qquad \alpha^2_{22} = \sum_{k=1}^n e_{kk} \otimes x_k^* , \tag{6.5a}$$

$$T(1,r)(T(2,r))^* = \mathrm{diag}(b,\ b,\ b) , \quad (z_m^{1ki})^* z_m^{1lj} = (z_{m'}^{1ki})^* z_{m'}^{1lj} , \quad \forall\, r, m, m' = 1, 2, ..., n , \tag{6.5b}$$

and the adjoint coaction is

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0, 0, e_{12}, 0\rangle) = \langle 0, 0, e_{12}, 0\rangle \otimes b, \tag{6.6a}$$

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0, 0, e_{21}, 0\rangle) = \langle 0, 0, e_{21}, 0\rangle \otimes b^*, \tag{6.6b}$$

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0, 0, 0, e_{ij}\rangle) = \sum\nolimits_{kl} \langle 0, 0, 0, e_{kl}\rangle \otimes (z_1^{1ki})^* z_1^{1lj} . \tag{6.6c}$$

Moreover, $\langle 1, 0, 0, 0\rangle$, $\langle 0, 1, 0, 0\rangle$ *and* $\langle 0, 0, e_{ii}, 0\rangle$ *(*$i = 1, 2$*) are coinvariant.*

Proof. We use the notations of the previous lemmas. The coinvariance of $\langle 1, 0, 0, 0\rangle$, $\langle 0, 1, 0, 0\rangle$ and $\langle 0, 0, e_{ii}, 0\rangle$ ($i = 1, 2$) follows automatically from unitarity of U. Since

$$\mathrm{Ad}_{\mathrm{U}}(\langle 0, 0, e_{12}, 0\rangle) = e_{12} \otimes e_{11} \otimes e_{11} \otimes \alpha^1_{11}(\alpha^2_{11})^* + \sum\nolimits_{ijk} e_{12} \otimes e_{i+1,k+1} \otimes e_{11} \otimes \beta^{1,i,j}_{11}(\beta^{2,k,j}_{11})^* ,$$

condition (3.10b) implies that there exists $b \in Q$ such that

$$\alpha^1_{11}(\alpha^2_{11})^* = \sum_{i=1}^n e_{ii} \otimes b , \qquad \sum_j \beta^{1,i,j}_{11}(\beta^{2,k,j}_{11})^* = \delta_{i,k}(\sum_{i=1}^n e_{ii} \otimes b) . \tag{6.7}$$

Unitarity of α^i_{rr} implies unitarity of b. Moreover, we have that $\alpha^1_{11} = \mathrm{diag}(bx_1, .., bx_n)$.
Using the relation $\alpha^2_{11} = \overline{\alpha^2_{22}}$ in Lemma 6.5, we deduce that $\alpha^2_{22} = \sum_{k=1}^n e_{kk} \otimes x_k^*$.
We get $\mathrm{Ad}_{\mathrm{U}}(\langle 0, 0, e_{12}, 0\rangle) = \langle 0, 0, e_{12}, 0\rangle \otimes b$ and $\mathrm{Ad}_{\mathrm{U}}(\langle 0, 0, e_{21}, 0\rangle) = \langle 0, 0, e_{21}, 0\rangle \otimes b^*$.
From the second equation of (6.7), we deduce that $T(1,r)(T(2,r))^* = \mathrm{diag}(b,\ b,\ b)$.
From coinvariance of $M_3(\mathbb{C})$ we deduce that for all m, m', $(z_m^{1ki})^* z_m^{1lj} = (z_{m'}^{1ki})^* z_{m'}^{1lj}$. Hence, the adjoint coaction is given by (6.6c). □

Definition 6.8.
1. We will denote by Q_n *the amalgamated free product of* n *copies of* $A_u(3)$ *over the common Woronowicz* C^**-subalgebra* $A_{\mathrm{aut}}(3)$ *(cf. Theorem 3.4 of [36]).*
2. We will denote by Q' *be the universal unital* C^* *algebra generated by elements*

i. $\{x'_i : i = 0, 1, 2, ..., n\}$, *each of which is a unitary,*

ii. matrix entries of a $n \times n$ *biunitary* V',

iii. matrix entries of n 3×3 *biunitaries* $\{T'_m : m = 1, 2, .., n\}$.

satisfying the conditions (3.11) *with* x_i, V, T_m *replaced by* x'_i, V', T'_m *respectively.*

We will denote the matrix entries of V' *and* T'_m *by* V'_{ij} *and* $(T'_m)_{kl}$ *respectively, where* i, j *belong to* $\{1, 2, ..., n\}$ *and* k, l *belong to* $\{1, 2, 3\}$. *Finally, let* U' *be the element in* $B(H_F) \otimes Q'$ *obtained from* (3.12) *by a similar change in symbols for* x_i, V, T_m.

Proposition 6.9. Q' *is a compact quantum group.*

Proof. Noting that for each $i = 0, 1, 2, ..., n,\ x'_i$ generate a $*$-subalgebra of $C(U(1))$, while matrix entries of V' and those of T'_m generate $*$-subalgebras of $A_u(n)$ and $A_u(3)$ respectively, it follows that Q' is a $*$-subalgebra of

$$Z = \underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1} * A_u(n) * \underbrace{A_u(3) * A_u(3) * \ldots * A_u(3)}_{n}.$$

Z has a CQG structure by [36] with the coproduct Δ' on the generators given by

$$\Delta'(x_k) = x'_k \otimes x'_k,\ \Delta'(V'_{ij}) = \sum_{k=1}^{n} V'_{ik} \otimes V'_{kj},\ \Delta'((T'_m)_{ij}) = \sum_{k=1}^{3} (T'_m)_{ik} \otimes (T'_m)_{kj}.$$

By (3.11b) and Theorem 3.4 of [36], $C^*\{(T'_m)_{j,k} : j,k = 1,2,3,\ m = 1,2,...,n\}$ is isomorphic to Q_n. Hence, the proposition follows by checking that the C^*-ideal generated by the relations $V'\Upsilon_\nu = V'^t\Upsilon_\nu = \mathrm{diag}(x_1'^* x_0'^*, ..., x_n'^* x_0^*)\Upsilon_\nu,\ V'\Upsilon_R = \Upsilon_R \overline{V'}$ in (3.11a) is actually a Woronowicz C^*-ideal of Z. The check is quite routine remembering that the matrix Υ_ν is a diagonal matrix and hence we omit its proof. ☐

Proposition 6.10. *(Q', U') is an object of the category $\mathfrak{C}_J$.*

Proof. It is an easy check that $(\mathrm{id} \otimes \Delta')U' = U'_{(12)}U'_{(13)}$, that is, U' is a unitary representation of Q' on H_F. From the formula of U', it can be checked that U' commutes with D_F. Moreover, the conditions with γ and J follow by comparing the formula of U in Lemma 6.1. Finally, by (3.11a) and (3.11b), it follows that $\mathrm{Ad}_{U'}$ coacts trivially on the two summands $\mathbb{C}$ of B_F, while on the remaining summands the coaction is given by (3.13), with the obvious change in notations. Thus $\mathrm{Ad}_{U'}$ preserves B_F and this finishes the proof of the proposition. ☐

We are now in a position to prove Proposition 3.3.

Proof of Proposition 3.3.
We redefine $T(1, m)$ as T_m, b as x_0 and α^1_{22} as V. By collecting all the results in this section, it follows that U has to be of the form given in (3.12) and the only conditions that the generators need to satisfy are those listed in Proposition 3.3. The relation $(\mathrm{id} \otimes \Delta)U = U_{(12)}U_{(13)}$ implies that the coproduct Δ on Q has to satisfy

$$\Delta(x_k) = x_k \otimes x_k,\ \Delta(V_{ij}) = \sum_{k=1}^{n} V_{ik} \otimes V_{kj},\ \Delta((T_m)_{ij}) = \sum_{k=1}^{3} (T_m)_{ik} \otimes (T_m)_{kj}.$$

From (3.12), we note that for each $i = 1, 2; m = 1, 2, 3, 4$ and $k = 1, 2, ..n$, U keeps the following sets of subspaces invariant: $(e_{ii} \otimes e_{11} \otimes e_{mm} \otimes e_{rr})H_F$ and $(e_{ii} \otimes (1 - e_{11}) \otimes e_{mm} \otimes e_{rr})H_F$. Since U is a unitary corepresentation on H_F, it follows that it restricts to a unitary representation on these subspaces. Using the fact that T_m, V are biunitaries, by the universality of the CQG $A_u(n)$, we conclude that for each m, $C^*\{(T_m)_{kl} : k, l = 1, 2, 3\}$ and $C^*\{V_{ij} : i, j = 1, 2, ..., n\}$ are quantum subgroups of $A_u(3)$ and $A_u(n)$ respectively. The exact structure of V depends on the matrices Υ_ν and Υ_R as can be seen from (3.11a), while from (3.11b) (as previously noted in Proposition 6.9), $C^*\{(T_m)_{kl} : k, l = 1, 2, 3,\ m = 1, 2, ..., n\}$ is a quantum subgroup of Q_n. It is clear that the map π sending x'_k, V'_{ij} and $(T'_m)_{kl}$ to x_k, V_{ij} and $(T_m)_{kl}$ extends to a CQG morphism from Q' to Q. Thus, any object of $\mathfrak{C}_J$ is a sub-object of (Q', U'). Conversely, as Q' is an object of $\mathfrak{C}_J$ by Proposition 6.10, the proof is finished. ☐

Acknowledgments

We would like to thank Prof. A.H. Chamseddine for useful conversations and comments.

References

[1] Q.R. Ahmad et al. (SNO Collaboration), *Direct Evidence for Neutrino Flavor Transformation from Neutral-Current Interactions in the Sudbury Neutrino Observatory*, Phys. Rev. Lett. **89** (2002), 011301.

[2] T. Banica, *Le groupe quantique compact libre $U(n)$*, Commun. Math. Phys. **190** (1997), 143–172.

[3] T. Banica, *Quantum automorphism groups of small metric spaces*, Pacific J. Math. **219** (2005), 27–51.

[4] T. Banica, *Quantum automorphism groups of homogeneous graphs*, J. Funct. Anal. **224** (2005), 243–280.

[5] T. Banica and R. Vergnioux, *Invariants of the half-liberated orthogonal group*, to appear in Ann. Inst. Fourier, arxiv:0902.2719v2 [math.QA].

[6] J. Bhowmick, D. Goswami and A. Skalski, *Quantum Isometry Groups of 0-Dimensional Manifolds*, to appear in Trans. AMS, arXiv:0807.4288v1 [math.OA].

[7] J. Bhowmick and D. Goswami, *Quantum Group of Orientation preserving Riemannian Isometries*, J. Funct. Anal. **257** (2009), 2530–2572.

[8] J. Bhowmick and D. Goswami, *Quantum isometry groups of the Podles spheres*, J. Funct. Anal. **258** (2010), 2937–2960.

[9] J. Bhowmick and D. Goswami, *Some counterexamples in the theory of quantum isometry groups*, to appear in Lett. Math. Phys., arXiv:0910.4713v1 [math.OA].

[10] J. Bhowmick and A. Skalski, *Quantum isometry groups of noncommutative manifolds associated to group C^*-algebras*, to appear in J. Geom. Phys., arXiv:1002.2551v2 [math.OA].

[11] J. Bichon, *Quantum automorphism groups of finite graphs*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 665–673.

[12] A.H. Chamseddine and A. Connes, *The Spectral Action Principle*, Commun. Math. Phys. **186** (1997), 731–750.

[13] A.H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. **58** (2008), 38–47.

[14] A.H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. **11** (2007), 991–1090.

[15] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[16] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. **36** (1995), 6194–6231.

[17] A. Connes, *Noncommutative differential geometry and the structure of space-time*, Proceedings of the Symposium on Geometry, Huggett, S.A. (ed.) et al., pp. 49-80, Oxford Univ. Press, Oxford UK 1998.

[18] A. Connes, *Noncommutative geometry and the Standard Model with neutrino mixing*, JHEP **11** (2006), 081.

[19] A. Connes, *On the spectral characterization of manifolds*, 2008, arXiv:0810.2088v1 [math.OA].

[20] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*, Colloquium Publications, vol. 55, AMS, 2008.

[21] R. Coquereaux, *On the finite dimensional quantum group $M_3 \oplus (M_{2|1}(\Lambda^2))_0$*, Lett. Math. Phys. **42** (1997), 309–328.

[22] F. D'Andrea, L. Dąbrowski, G. Landi and E. Wagner, *Dirac operators on all Podleś spheres*, J. Noncomm. Geom. **1** (2007), 213–239.

[23] F. D'Andrea, L. Dąbrowski and G. Landi, *The Isospectral Dirac Operator on the 4-dimensional Orthogonal Quantum Sphere*, Commun. Math. Phys. **279** (2008), 77–116.

[24] L. Dąbrowski, G. Landi, M. Paschke and A. Sitarz, *The spectral geometry of the equatorial Podleś sphere*, Comptes Rendus Acad. Sci. Paris **340** (2005), 819–822.

[25] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J.C. Várilly, *The Dirac operator on $SU_q(2)$*, Comm. Math. Phys. **259** (2005), 729–759.

[26] L. Dabrowski, F. Nesti and P. Siniscalco, *A Finite Quantum Symmetry of $M(3, \mathbb{C})$*, Int. J. Mod. Phys. **A13** (1998), 4147–4162.

[27] Y. Fukuda et al. (Super-Kamiokande Collaboration), *Evidence for Oscillation of Atmospheric Neutrinos*, Phys. Rev. Lett. **81** (1998), 1562–1567.

[28] D. Goswami, *Quantum Group of Isometries in Classical and Noncommutative Geometry*, Commun. Math. Phys. **285** (2009), 141–160.

[29] D. Goswami, *Quantum Isometry Group for Spectral Triples with Real Structure*, SIGMA **6** (2010), 007.

[30] D. Kastler, *Regular and adjoint representation of $SL_q(2)$ at third root of unit*, CPT internal report (1995).

[31] F. Lizzi, G. Mangano, G. Miele and G. Sparano, *Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories*, Phys. Rev. **D55** (1997), 6357–6366.

[32] A. Maes and A. Van Daele, *Notes on compact quantum groups*, Nieuw Arch. Wisk. **16** (1998), 73–112.

[33] P. Podles, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups*, Comm. Math. Phys. **170** (1995), 1-20.

[34] P.M. Sołtan, *Quantum $SO(3)$ groups and quantum group actions on M_2*, J. Noncommut. Geom. **4** (2010), 1–28.

[35] A. Van Daele and S. Wang, *Universal quantum groups*, International J. Math. **7** (1996), 255–264.

[36] S. Wang, *Free products of compact quantum groups*, Comm. Math. Phys. **167** (1995), no. 3, 671–692.

[37] S. Wang, *Quantum Symmetry Groups of Finite Spaces*, Commun. Math. Phys. **195** (1998), 195–211.

[38] S. Wang, *Structure and Isomorphism Classification of Compact Quantum Groups $A_u(Q)$ and $B_u(Q)$*, J. Operator Theory **48** (2002), 573–583.

[39] S. Wang, *Ergodic actions of universal quantum groups on operator algebra*, Comm. Math. Phys. **203** (1999), no. 2, 481–498.

[40] S.L. Woronowicz, *Compact matrix pseudogroups*, Commun. Math. Phys. **111** (1987), 613–665.

[41] S.L. Woronowicz, *Compact quantum groups*, in Symétries quantiques (Les Houches, 1995), pp. 845–884, edited by A. Connes et al., Elsevier, Amsterdam, 1998.

Email addresses: `jyotishmanb@gmail.com`, `dandrea@sissa.it`, `dabrow@sissa.it`.