Edge-intersection graphs of grid paths: the bend-number

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Abstract

We investigate edge-intersection graphs of paths in the plane grid regarding a parameter called the bend-number. The bend-number is related to the interval-number and the track-number of a graph. We provide new upper and lower bounds of the bend-number of any given simple graph in terms of the coloring number, edge clique covers and the maximum degree. We show that the bend-number of an outerplanar graph is at most two and that several subclasses of planar graphs have a bend-number of at most three or four. Moreover we determine the bend-numbers of several complete bipartite graphs. Finally, we prove that recognizing single-bend graphs is NP-complete, providing the first such result in this field.

1 Introduction

Golumbic, Lipshteyn and Stern [11] introduced *edge-intersection graphs of paths on a grid* (EPG graphs), a concept arising from VLSI grid layout problems [6]. A simple graph G is an EPG graph, if there is an assignment of paths in the plane grid to the vertices, such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid edge. The paths may not have u-turns but may intersect themselves even in grid edges. EPG graphs generalize *edge-intersection graphs of paths on degree 4 trees* as considered by Golumbic, Lipshteyn and Stern in [12]. A proof is contained in [11], showing that every graph is an EPG graph, however a certain parameter of EPG representations has awoken some interest. The *bend-number* of G (written $b(G)$) is the minimum k, such that G has an EPG representation, with each path having at most k bends. Here a *bend of a grid path* is a switch in its direction between horizontal and vertical. Figure 1 shows $K_{3,3}$ after the removal of one edge and an EPG representation where each path has at most one bend, hence $b(K_{3,3}\backslash e) \leq 1$. Generally, a graph G with $b(G) \leq k$ is referred to as a *k-bend graph*.

Remark. Most of the literature concerning this topic, including [2, 4, 11], is considering B_k, the *class of k-bend graphs*. Clearly $b(G) \leq k$ just paraphrases $G \in B_k$. However, we prefer to use $b(G)$ rather than B_k.

Graphs with bend-number at most 1, called *single-bend graphs*, already aroused interest in several respects, as seen in [11, 19]. In [2, 4] it has been shown that the bend-number of a simple graph can be arbitrary large. Hence it is interesting to determine graphs or graph classes with bounded bend-number. Asinowski and

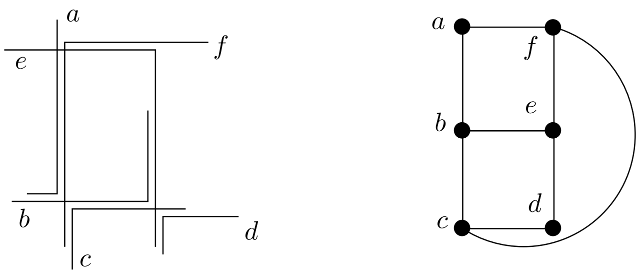

Figure 1: The $K_{3,3}\backslash e$ is a *single-bend* graph.

Suk [2] give bounds on the bend-number of complete bipartite graphs. Biedl and Stern [4] show $b(G) \leq 5$ for planar G and $b(G) \leq 3$ for outerplanar G. They also give upper bounds on $b(G)$ in terms of tree-width, path-width, coloring number and maximum degree of G.

In this paper we do the following:
In Section 2 we compare the bend-number with other graph parameters. In particular, a graph's bend-number is closely related to its interval-number $i(G)$, see [21], and its track-number $t(G)$, see [16]. Although not apparent in the literature so far, this connection seems to be natural to us. We also introduce some notation. The end of the section contains a first important lemma that will be used now and then in the paper, the Lower-Bound-Lemma.

Section 3 covers complete bipartite graphs. We prove a lower bound on $b(K_{m,m})$, which equals the upper bound given in [2]. In particular, this will prove that for every $k \geq 0$ there is a G with $b(G) = k$, which answers a question of [2, 11]. Moreover we derive upper and lower bounds on the bend-number of $K_{m,n}$ for $m \neq n$. We determine $b(K_{m,n})$ in a couple of cases: if n is a specific quadratic function of m and if n is bigger than some degree 4 polynomial in m (improving a result of [4]).

Section 4 focuses on the relation between the coloring number and the bend-number of a graph. We improve the upper bound from [4] and show that it is tight even for bipartite graphs.

In Section 5 we provide a general method to bound $b(G)$ from above depending on edge clique covers of G. In particular we improve a result of [4] concerning the maximum degree.

Section 6 deals with planar and outerplanar graphs. We prove a conjecture of [4], namely that the bend-number of an outerplanar graph is at most two. This bound is best possible. We also improve the bounds of other classes of planar graphs.

In Section 7 we show that recognizing single-bend graphs is NP-complete, which answers a question of [11, 19].

We conclude the paper with some open problems in Section 8.

2 Comparing parameters and a first lemma

Interval graphs are intersection graphs of intervals on the real line. Every vertex is associated with an interval, in such a way that two intervals overlap if and only if the corresponding vertices are adjacent. This subject has been extended to intersection graphs of systems of intervals. In a *k-interval representation* of a graph G every vertex is associated with a set of at most k intervals on the real line, such that vertices are adjacent iff any of their intervals intersect. The *interval-number* $i(G)$ is then defined as the minimum k, such that G has a k-interval representation.

In a *k-track representation* of a simple graph G there are k parallel lines, called *tracks*. Every vertex is associated with one interval from each track. Again vertex adjacency is equivalent to interval intersection and the *track-number* $t(G)$ is the minimum k, such that G has a k-track representation. It is easy to see that $i(G) \leq t(G)$, since a k-track representation can be transferred into a k-interval representation by putting the tracks in any order on a single real line.

A *k-bend representation* associates every vertex with at most $k+1$ intervals, called *segments* in this context. Each segment lies on either a horizontal or a vertical grid line. Additionally, the segments of a vertex have to form a grid path, that is they are ordered such that horizontal and vertical segments alternate and the endpoint of a segment equals the starting point of the subsequent segment. The common point of successive segments is called a *bend*. Clearly the number of bends of a path is one less than the number of segments. We say that two segments *intersect* if they have a grid edge in common, *touch* if they have a grid point in common, and *cross* if they don't intersect and the touching point is not an endpoint of either segment. We say that two grid paths intersect, touch, or cross if any of their segments intersect, touch, or cross. This notation is illustrated in Figure 2. Now two vertices in G are adjacent if and only if their paths intersect. For convenience we will sometimes refer to a vertex and the path that represents this vertex with the same name.

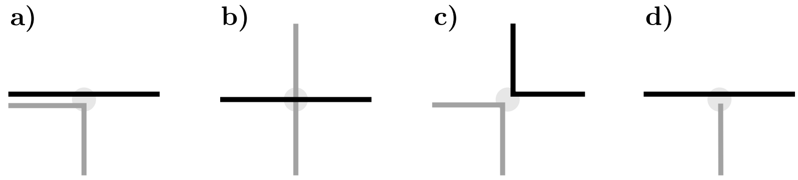

Figure 2: Four examples of two grid paths that touch at a grid point. The paths intersect in case **a)** and cross in case **b)**. In cases **c)** and **d)** they neither cross, nor intersect.

The bend-number $b(G)$ (the minimum k, such that G has a k-bend representation) can be set in relation to $i(G)$ and $t(G)$: Since the grid lines of a k-bend representation can be stringed together on a single line, $i(G) \leq b(G) + 1$. On the other hand $b(G)$ is only a constant factor away from $i(G)$ and $t(G)$: An interval or track representation may be considered as part of the infinite grid and the intervals of a vertex may be connected without introducing unwanted adjacencies. Using this one easily gets $b(G)+1 \leq 4{\cdot}i(G)$ and $b(G)+1 \leq 4{\cdot}t(G)$. Moreover, note that interval graphs are precisely the graphs with $i(G) = t(G) = b(G) + 1 = 1$.

Many extremal questions about interval-numbers and track-numbers have been studied. In Table 1 we have listed some considered graph classes and the maximum $i(G)$, $t(G)$ and $b(G) + 1$ among all G in this class. In the last two columns (corresponding to the track-number and the bend-number) some values remain unknown. In particular this includes the bend-number of complete

	$i(G)$		$t(G)$		$b(G)+1$	
forest	2	[17]	2	[7]	2	[11]
outerplanar	2	[21]	2	[18]	3	Thm 6.1
planar	3	[21]	4	[13]	≤ 6	[4]
+ bipartite	3	[21]	4	[13, 14]	3	[4]
$K_{m,n}$ $(m \leq n)$	$\lceil \frac{mn+1}{m+n} \rceil$	[17]	$\lceil \frac{mn}{m+n-1} \rceil$	[16]	$\geq \lceil \frac{mn+\sqrt{mn}}{m+n} \rceil$	Lem 2.2
					$\leq \lceil \frac{n}{2} \rceil + 1$	[2]
					$\leq 2m-1$	[2]
line graph	2		–		3	[4]
max degree Δ	$\lceil \frac{\Delta+1}{2} \rceil$	[15]	$\leq \Delta+1$		$\leq \Delta+1$	Cor 5.3

Table 1: Some graph classes and their maximum interval-number, track-number and bend-number.

bipartite graphs (c.f. Section 3), line graphs, graphs with maximum degree Δ (c.f. Section 5 for both) and planar graphs (c.f. Section 6). We are not aware of any result concerning the following question and conjecture.

Question 1. Is $t(G)/b(G)$ (or equivalently $t(G)/i(G)$) bounded by a constant independent of G?

We expect that the answer is 'No'. We even conjecture a statement which, if true, would immediately imply a negative answer to the above question. A good candidate for proving Conjecture 2.1 seems to be the line graph of K_n.

Conjecture 2.1. *The track-number of line graphs is not bounded by a constant.*

Scheinerman and West [21] already considered *depth-r interval representations.* In any such representation the number of intervals with non-trivial common intersection is bounded by r. This notion makes sense for interval-, track- and bend-representations. Clearly every representation of a triangle free graph has depth at most 2. In the construction of representations of any kind it is convenient to restrict to depth-2 representations, which we will call *simple.* Indeed *almost all* EPG representations in the literature so far (except the 2-bend representation of line graphs in [4]) are simple. Also, all upper bounds in this paper, except those in Section 5, rely on simple representations.

In general a simple EPG representation may require many more bends than a non-simple one: A good example is K_n. But when restricted to simple representations we quickly obtain a lower bound on the required number of bends.

Lemma 2.2 (Lower-Bound-Lemma). *Let L denote the set of grid lines, a simple k-bend representation of $G = (V, E)$ uses. Then we have*

$$|E| + |L| \leq (k+1)|V|.$$

Moreover there is a k-bend representation in which

$$|L| \geq \sqrt{k|V|}.$$

Proof. W.l.o.g. consider a simple k-bend representation in which every vertex path has exactly k bends. Look at the rightmost or upmost grid edge of each of the $k+1$ segments of each vertex v. If this grid edge is shared by another vertex w (there can be only one in a simple representation), we assign v to the edge $\{v, w\}$ in the graph. This way

- every vertex is assigned to at most $k+1$ edges,
- every edge is assigned at least once, and
- at the rightmost (upmost) grid edge of every line in L either no assignment is done or an edge is assigned twice.

Hence we have $|E| \leq (k+1)|V| - |L|$.

Moreover, since the representation is simple, at most 4 bends can share a grid point. Hence at least $\frac{1}{4}k|V|$ grid points support a bend. The minimal number of grid lines crossing in that many grid points is at least $2\sqrt{\frac{1}{4}k|V|} = \sqrt{k|V|}$. $\square$

Although the Lower-Bound-Lemma only depends on the number of vertices and edges, it turns out to be very powerful in several cases. Note that a direct analog is used in [15] to derive tight lower bounds on the interval-number of every complete bipartite graph. It can be used for tight lower bounds on the track-number of every such graph as well. When considering the bend-number the Lower-Bound-Lemma is weaker. Nevertheless it is tight for some particular $K_{m,n}$ (c.f. Theorem 3.1 and Theorem 3.2).

3 Complete bipartite graphs

The interval-number of complete bipartite graphs has been determined independently by Harary and Trotter [17] and Griggs and West [15]. The track-number of complete bipartite graphs was determined by Gyárfás and West [16]. Table 1 contains the specific values. In this section we investigate the bend-number of complete bipartite graphs. Throughout this paper we denote the bipartition classes of $K_{m,n}$ by $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ and will always assume $2 \leq m \leq n$. For convenience we depict vertices of A by black paths and vertices of B by grey paths.

In [2] it is shown that $b(K_{m,n}) \leq \lceil \frac{\max\{m,n\}}{2} \rceil = \lceil \frac{n}{2} \rceil$, see Figure 3. They also ask, whether this representation is best-possible in the case $m = n$. We prove this (Theorem 3.1), which solves a conjecture of [11], asking whether for every $k \geq 0$ there is a graph G with $b(G) = k$. Since bipartite graphs are triangle-free, any representation is simple here. Hence we can apply the Lower-Bound-Lemma.

Theorem 3.1. *For all $2 \leq m \leq n$ we have $b(K_{m,n}) \leq \lceil \frac{n}{2} \rceil$. This holds with equality if $m = n$ and if $m+1 = n$ is even.*

Proof. Figure 3 shows a representation of $K_{5,6}$ with $\lceil n/2 \rceil = 3$ bends. This example is due to [2] and is meant to be instructive for general values of m and n, certifying $b(K_{m,n}) \leq \lceil n/2 \rceil$. For equality we will prove that $K_{m,2\lceil \frac{m}{2} \rceil}$ cannot be represented with less than $\lceil \frac{m}{2} \rceil$ bends. Supposing that we could use less, the Lower-Bound-Lemma (Lemma 2.2) gives $2m\lceil \frac{m}{2} \rceil + |L| \leq \lceil \frac{m}{2} \rceil (m + 2\lceil \frac{m}{2} \rceil)$. If m is even we obtain $m^2 + |L| \leq m^2$ which is a contradiction since $|L| \geq 1$. For odd m we calculate $|L| \leq \frac{1}{2}(m+1)$. But by the Lower-Bound-Lemma we can assume $|L| \geq \sqrt{\frac{1}{2}(m-1)(2m+1)}$ which leads to a contradiction for $m \geq 2$. $\square$

When considering $K_{m,n}$ with increasing n compared to m, the bound in Theorem 3.1 gets worse. Being interested in the behavior of $b(K_{m,n})$, we now determine the exact value for a certain $n \in \theta(m^2)$.

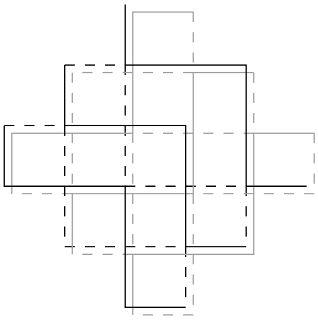

Figure 3: A 3-bend representation of $K_{5,6}$. The smaller bipartition class A is black.

Theorem 3.2. *Let $m \geq 3$. For even m we have $b(K_{m,(m+1)(m-2)}) = m-1$ and for odd m we have $b(K_{m,m(m-2)}) = m-1$.*

Proof. Let m be even. We use a braid-like path P with $m-1$ bends and $m/2-1$ crossings as a template for every path in A. We represent each of the m vertices in A by a copy of P translated by a very small amount along the diagonal. See the left of Figure 4 for an illustration. The vertices of B are represented by small staircases with $m-1$ bends, each interlaced around a bend or a crossing of P. At every bend of P, except the braid's turning point, we interlace m grey staircases and at every crossing another two staircases (see the right of Figure 4). This gives in total the $(m-2) \cdot m + (m/2-1) \cdot 2 = (m-2)(m+1)$ vertices of B.

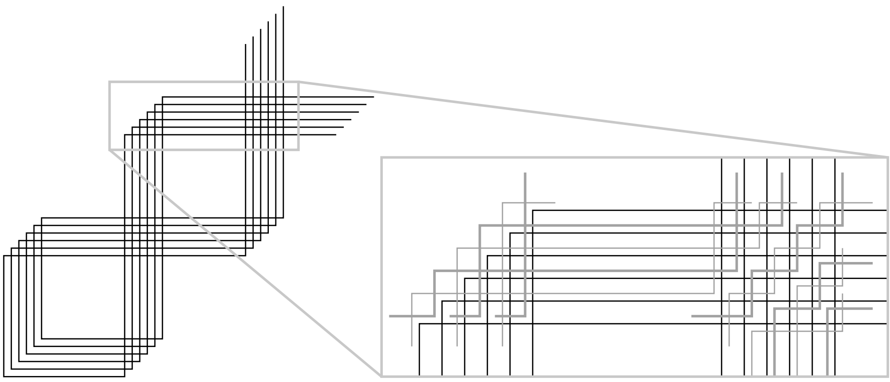

Figure 4: Representing $K_{m,(m+1)(m-2)}$ (m even) with $m-1$ bends: Vertices in A (black) are represented by braid-like paths. Vertices in B (grey) are represented by staircases interlaced at the bends and crossings.

For odd m, first represent $K_{m-1,m(m-3)}$ as described in the even case with $m-2$ bends for each path. Then add the missing vertex $a_m \in A$ by a snake-like $(m-1)$-bend path like the dashed one on the left of Figure 5. The vertical end of the so far existing paths in $B \backslash \{b^*\}$ can be extended to reach a_m and endowed with another bend to connect to it. The special vertex b^* is extended horizontally to reach a_m as depicted in Figure 5.

Finally, every path $a_i \in A \backslash \{a_m\}$ receives a last bend at its vertical or horizontal end, depending on the parity of i. We obtain several new crossings where an additional set of m paths may be threaded in, see the right of Figure 5. This way B contains $m(m-2)$ vertices.

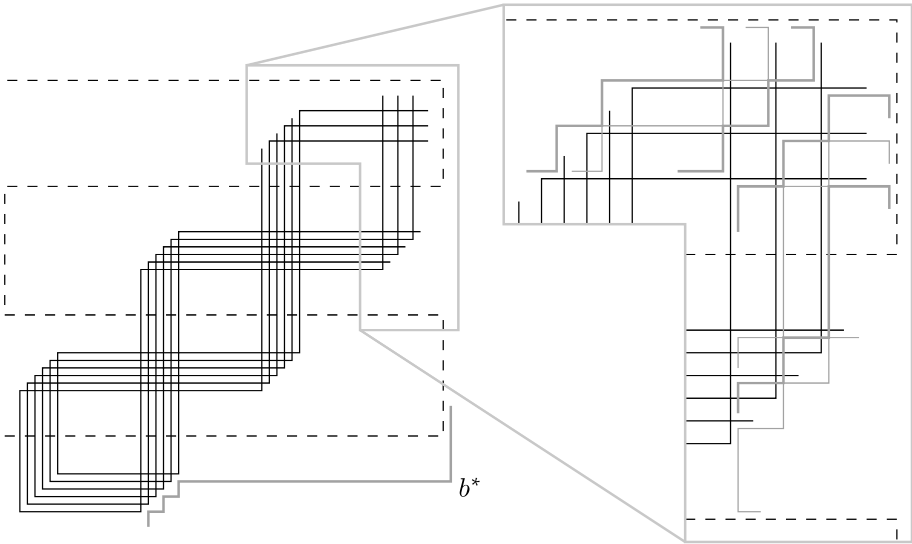

Figure 5: Representing $K_{m,m(m-2)}$ (m odd) with $m-1$ bends: Vertices in $A \backslash \{a_m\}$ (black) are represented by braid-like paths and a_m (dashed) by a snake-like path. The first $m(m-3)$ vertices in B (grey) are interlaced as in the even case and extended vertically to reach a_m. Only b^* is extended as depicted. Another m vertices in B are threaded in as shown on the right.

The above constructions show that both bend-numbers are at most $m-1$. Equality follows from a straightforward application of the Lower-Bound-Lemma. $\square$

Theorem 3.2 only holds if $m \geq 3$. The bend-number of $K_{2,n}$ has been determined for all n in [2]: $b(K_{2,n}) = 2$ iff $n \geq 5$ and $b(K_{2,n}) = 0$ iff $n = 0, 1$.

Now we investigate the extremal case where n gets very large compared to m. Asinowski and Suk [2] showed that $b(K_{m,n}) = 2m-2$ for every $n \geq N$ with $N \in \Omega(m^m)$, see Figure 8 for a representation. Later on, Biedl and Stern [4] improved this to $N = 4m^4 - 8m^3 + 2m^2 + 2m + 1$. In Theorem 3.5 we lower this once more to $N = m^4 - 2m^3 + 5m^2 - 4m + 1$ and show in Theorem 3.6 that this leaves at most a quadratic gap to the true value, disproving a conjecture of [4]. We begin with bounding the number of crossings of two paths which have a given odd number of bends. This bound may be of interest on its own.

Lemma 3.3. *Two $(2m-1)$-bend paths cross in at most $m(m+1)$ points. This is tight.*

Proof. Consider two given $(2m-1)$-bend paths v and w. Both have exactly m horizontal and m vertical segments. We color the vertical segments of v and the horizontal segments of w blue and the remaining segments red. Along each path we index the segments, starting with its blue end, i.e., v_1 and w_1 are blue and

$v = (v_1, \dots, v_{2m})$ and $w = (w_1, \dots, w_{2m})$. Now every crossing is monochromatic, either blue with odd indices or red with even indices. We partition the pairs of segments that have the same color but come from different paths into four sets. Set $\mathcal{B}$ contains all blue pairs that *do* cross and $\overline{\mathcal{B}}$ all blue pairs that *do not* cross. Similarly $\mathcal{R}$ and $\overline{\mathcal{R}}$ are defined for red segments.

Consider a blue crossing $\{v_i, w_j\} \in \mathcal{B}$ and the grid line ℓ containing v_i. Each of $v_{i-1}, v_{i+1}, w_{j-1}, w_{j+1}$, if existent, is red and lies completely on one side of ℓ. Moreover w_{j-1} and w_{j+1} cannot lie on the same side since w_j crosses ℓ. Now consider v_{i-1} (or v_{i+1}) and the w-segment on the other side of ℓ. This pair evidently is in $\overline{\mathcal{R}}$. This way we associate up to two red non-crossings with every blue crossing, even if there are more (see Figure 6).

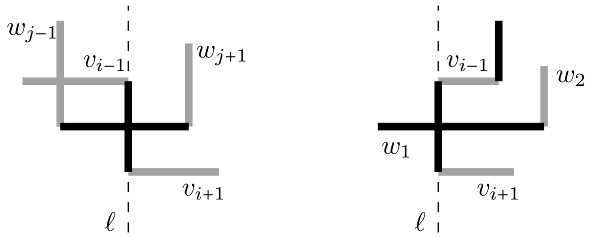

Figure 6: A blue crossing is associated with every pair of red segments from different sides of ℓ: The blue crossing on the left is associated with $\{v_{i-1}, w_{j+1}\}$ and $\{v_{i+1}, w_{j-1}\}$. The blue crossing on the right is associated with no red non-crossing.

Next we partition $\mathcal{B}$ in two ways. Firstly, divide $\mathcal{B}$ into $\mathcal{B}_0, \mathcal{B}_1$, and $\mathcal{B}_2$ according to the number of red non-crossings, the blue crossings are associated with in the above way. Secondly, we write $\mathcal{B}(v_1)$ for the set of blue crossings v_1 participates in and do the same with w_1. We denote $\mathcal{B}_i(w_1) := \mathcal{B}_i \cap \mathcal{B}(w_1)$ for $i = 0, 1, 2$. Then $\mathcal{B}_0 = \mathcal{B}_0(w_1)$ and $\mathcal{B}_1 = (\mathcal{B}(v_1) \backslash \mathcal{B}(w_1)) \cup \mathcal{B}_1(w_1)$. Note that every red non-crossing is associated with at most two blue crossings and hence we have $|\mathcal{B}_1| + 2|\mathcal{B}_2| \le 2|\overline{\mathcal{R}}|$. This leads to:

$$2|\mathcal{B}| \le 2|\overline{\mathcal{R}}| + |\mathcal{B}(v_1) \backslash \mathcal{B}(w_1)| + |\mathcal{B}_1(w_1)| + 2|\mathcal{B}_0(w_1)|$$

Now observe the following: When tracing the path v between any two blue segments contributing to a $\mathcal{B}_0(w_1)$-crossing there must be a blue segment of v that either participates in a $\mathcal{B}_2(w_1)$-crossing or does not cross w_1 at all. Hence, because v has only m blue segments, we have $2|\mathcal{B}_0(w_1)| - 1 + |\mathcal{B}_1(w_1)| \le m$ as well as $|\mathcal{B}(v_1) \backslash \mathcal{B}(w_1)| \le m - 1$.

Plugging both into the above inequality, we calculate $2|\mathcal{B}| \le 2|\overline{\mathcal{R}}| + 2m$. Thus, $|\mathcal{B}| - |\overline{\mathcal{R}}| \le m$. Now adding m^2 on both sides we obtain: $|\mathcal{B}| + |\mathcal{R}| \le m^2 + m$, where $\mathcal{R}$ denotes the set of red crossings.

Figure 7 shows that $m(m+1)$ can indeed be attained. □

Part (i) of the following lemma is due to Biedl and Stern [4].

Lemma 3.4. *Consider a k-bend representation of $K_{m,n}$ and a subset B' of B, such that every vertex in B' establishes two of its incidences with either a single segment or two consecutive segments. Then*

(i) $|B'| \le 2\binom{m(k+1)}{2}$ *and*

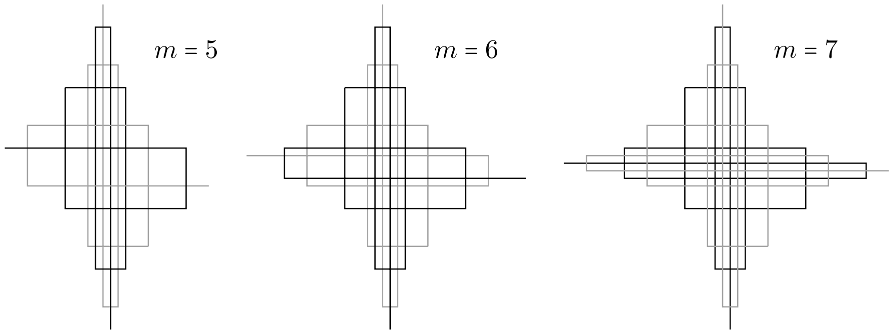

Figure 7: Two $(2m-1)$-bend paths can cross in $m(m+1)$ points.

(ii) $|B'| \leq (k+1)(\frac{k+3}{2}\binom{m}{2} + 2m)$ *if k is odd.*

Proof. Let s and s' be segments of distinct vertices in A and $b \in B'$ a path intersecting s and s' with either the same or consecutive segments. In the first case, the corresponding segment of b must contain an endpoint of each s and s'. Since B is an independent set, b is the only vertex that intersects s and s' with the same segment. In the second case b must have a bend where the grid lines through s and s' intersect. Here, beside b at most one other vertex in B can intersect s and s' with consecutive segments.

Since there are at most $\binom{m(k+1)}{2}$ pairs of segments of vertices in A, there are at most twice as many vertices in B'. This concludes part (i).

For a more careful analysis, in the case of odd k, observe that *two* paths in B can intersect s and s' with consecutive segments only if s and s' *cross*. By Lemma 3.3, two k-bend paths can cross in at most $(k+1)(k+3)/4$ points if k is odd. Hence there are at most $\binom{m}{2}(k+1)(k+3)/4$ crossings between vertices in A.

If s and s' are perpendicular but *do not cross*, at most *one* vertex $b \in B$ can intersect both with consecutive segments and at least one such segment of b must contain an endpoint of s or s'. Since there are at most $2m(k+1)$ endpoints of segments of vertices in A, we conclude that $|B'| \leq 2\binom{m}{2}(k+1)(k+3)/4 + 2m(k+1) = (k+1)(\binom{m}{2}(k+3)/2 + 2m)$. □

Theorem 3.5. *We have $b(K_{m,n}) = 2m-2$ for all $n > m^4 - 2m^3 + 5m^2 - 4m$.*

Proof. If $m = 1$, then $K_{m,n}$ is a star and thus an interval graph, i.e., $b(K_{1,n}) = 0$ for all $n > 0$.

For $m > 1$ suppose $b(K_{m,n}) \leq 2m-3$. Then applying Lemma 3.4–(ii) with $k = 2m-3$ yields, that at most $N := (2m-2)(\binom{m}{2}m + 2m) = m^4 - 2m^3 + 5m^2 - 4m$ vertices in B can establish two of its incidences with the same or consecutive segments. Hence if $|B| > N$, there must be a vertex $b \in B$ with at least one "empty" segment between any two "non-empty" segments. Moreover every segment of b establishes at most one incidence. Since b has degree m, we conclude that b must have at least $2m-1$ segments.

Figure 8 shows that $b(K_{m,n}) \leq 2m-2$, regardless of n. □

Biedl and Stern [4] conjectured that Theorem 3.5 is already true for all $n \geq N$ with $N \in \mathcal{O}(m^2)$. We disprove this and show that Theorem 3.5 is not far from

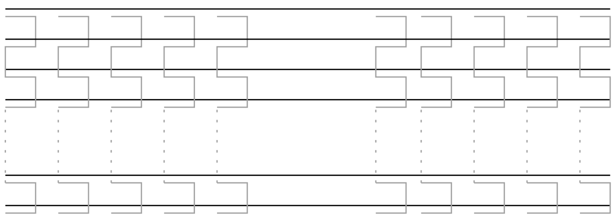

Figure 8: A representation verifying $b(K_{m,n}) \leq 2m-2$.

being tight.

Theorem 3.6. *If* $n \leq m^4 - 2m^3 + \frac{5}{2}m^2 - 2m - 4$ *then* $b(K_{m,n}) \leq 2m-3$*. Note that this leaves only a quadratic discrepancy to the bound of the preceeding theorem.*

Proof. We equally divide A into A_1 and A_2. We use the two $(2m-3)$-bend paths P_1 and P_2 from the tight example in Figure 7 as templates for vertices in A_1 and A_2, respectively. Note that each P_i has $\binom{m-1}{2}$ crossings and P_1 and P_2 cross $m(m-1)$ times.

For $i = 1, 2$ every $a \in A_i$ runs within a small distance along P_i. We ensure that at every bend of P_i every pair of paths in A_i crosses. This way, every such pair crosses $2\binom{m-1}{2} + 2m - 3 = m(m-1) - 1$ times. A pair of vertices, one from A_1 and the other from A_2, crosses $m(m-1)$ times. Hence the total number of crossings between vertices in A is given by $\lfloor \frac{m}{2} \rfloor \lceil \frac{m}{2} \rceil m(m-1) + (\binom{\lceil m/2 \rceil}{2} + \binom{\lfloor m/2 \rfloor}{2})(m(m-1)-1)$. At every crossing we can interlace two vertices of B.

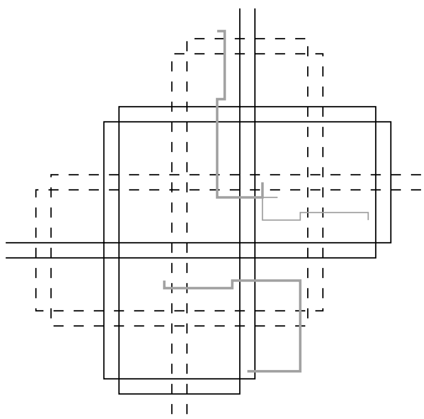

Figure 9: A $(2m-3)$-bend representation of $K_{m,n}$, where $n \leq m^4 - 2m^3 + \frac{5}{2}m^2 - 2m - 4$: Every vertex in A (black) runs within a small distance along one of two paths with the maximum number of crossings. Every vertex in B (grey) is interlaced around one crossing or two endpoints of black segments.

Moreover every endpoint of a segment from A (except the ends of the paths and the eight endpoints of the topmost, rightmost, bottommost, and leftmost segment) can be used to interlace vertices of B. There are $2m(2m-3) - 8$ suitable endpoints. Interlacing one vertex uses two of them.

Figure 9 suggests how to insert one vertex $b \in B$ with two suitable endpoints and two vertices $b, b' \in B$ at one crossing. By doing this we can insert $n = \lfloor m^4 - 2m^3 + \frac{5}{2}m^2 - 2m - 4 \rfloor$ vertices into B. $\square$

Question 2. What is the maximal n for which $b(K_{m,n}) \leq 2m - 3$?

The representations we have given for complete bipartite graphs can naturally be extended to different values of n, though they might not be optimal anymore. For instance the first and the second representation in Theorem 3.2 yields that $b(K_{5,24}) \leq 5$ and $b(K_{5,25}) \leq 6$, respectively. Moreover Figure 11 certifies $b(K_{3,10}) \leq 2$, i.e., $b(K_{3,5}) \leq 2$. In Figure 10 we sketch a region which contains the graph of $b(K_{5,n})$.

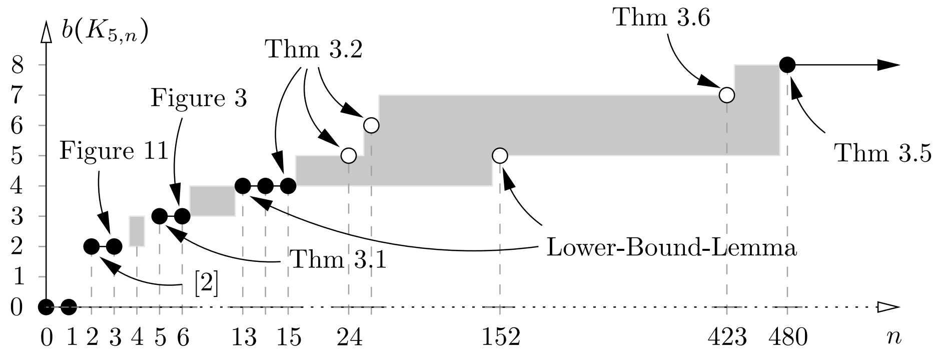

Figure 10: Upper and lower bounds for $b(K_{5,n})$. The filled circles and the straight line are values of $b(K_{5,n})$.

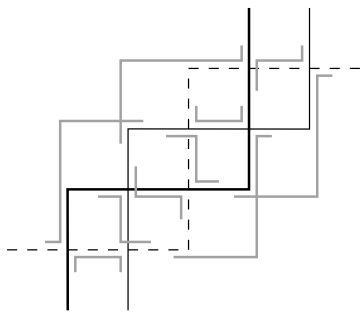

Figure 11: A 2-bend representation of $K_{3,10}$.

Question 3. What is the behavior of $b(K_{m,n})$ for $n \in \theta(m^3)$?

4 Coloring number

In [4] an acyclic orientation of G with maximum indegree k is referred to as a *k-regular acyclic orientation.* The *coloring number* $\mathrm{col}(G)$ of G is the smallest number k, such that G has a $(k-1)$-regular acyclic orientation. It was introduced by Erdős and Hajnal in [9].

In this section we provide a tight upper bound on the bend-number of graphs with a fixed coloring number. In [4] the following result was suspected to be true.

Theorem 4.1. *Every graph G has a simple $(2\mathrm{col}(G)-3)$-bend representation.*

Proof. Take a $(\mathrm{col}(G)-1)$-regular acyclic orientation of G in the above sense. A topological ordering then gives us a building recipe for G, where every new vertex will be connected to at most $\mathrm{col}(G)-1$ vertices of the already constructed part. We construct a simple $(2\mathrm{col}(G)-3)$-bend representation simultaneously to the building process of G. We maintain one private vertical part for every already inserted vertex, all running parallel. At the segment containing the vertical part (either above or below it) ends another, horizontal, private part. In Figure 12 we explain how to insert a new vertex while maintaining this invariant. □

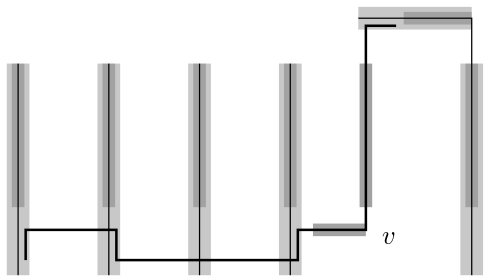

Figure 12: Building a $(2\mathrm{col}(G)-3)$-bend representation of G, a vertex v is inserted. Private parts before and after the insertion are highlighted with light-grey and dark-grey, respectively.

Theorem 4.2. *For every m there is a bipartite G with $\mathrm{col}(G) = m+1$ and $b(G) = 2\mathrm{col}(G)-3$.*

Proof. The graph G is huge and arises from a $K_{m,n}$ with very large $n = |B|$. Firstly, we connect $m(2m-2)+1$ new vertices with every m-subset of B. The set of the $(m(2m-2)+1)\binom{n}{m}$ added vertices is denoted by C. Moreover, we connect $m(2m-2)+1$ new vertices with every m-subset of C. This set of vertices is denoted by D.

Clearly $\mathrm{col}(G) = m+1$. Now suppose $b(G) \le 2m-2$ and consider the induced $K_{m,n}$ the construction started with. By Lemma 3.4–(i) all but roughly $4m^4$ vertices in B are represented by paths in which exactly every second segment is establishing one incidence. By this with increasing n an arbitrary large subset $B' \subset B$ must look like in Figure 8, i.e., vertices in B' can establish further incidences only with vertical segments. Additionally, there are many pairs of m-subsets of B' such that every path in one subset lies to the left of every path in the other subset.

We fix m distinct m-subsets $B_1, \ldots, B_m$ of B', such that every path in B_1 lies to the left of every path in $B_2 \cup \ldots \cup B_m$. Hence every vertex in C that is connected to B_1 lies completely to the left of every vertex in C that is connected to one of $B_2, \ldots, B_m$.

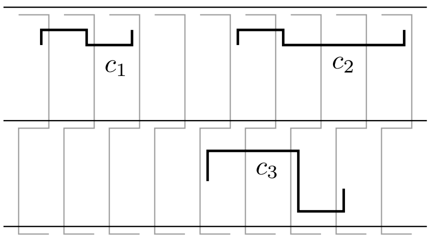

Figure 13: A part of a hypothetical $(2\mathrm{col}(G)-4)$-bend representation of G. The sets B' and $C' = \{c_1, c_2, c_3\}$ are depicted grey and thick, respectively.

We connected $m(2m-2)+1$ vertices with every B_i, that is, one more than there are bends in B_i. Hence for every $i \in \{1,\dots,m\}$ there is at least one vertex $c_i \in C$ that is connected to B_i and whose vertical segments are contained in segments of B_i (see Figure 13 for an example). In G a set $D' \subset D$ of $m(2m-2)+1$ vertices is connected to $C' := \{c_1,\dots,c_m\}$; again one more than there are bends in C'. Hence of at least one vertex in D' all *horizontal* segments are contained in segments of C'. But this is impossible since the path $c_1 \in C'$ lies to the left of all other paths in C'. □

5 The Clique Cover Argument

In this section we present a general method to represent any graph with a number of bends depending on an edge clique cover. Note, that the acquired representations are not simple in general.

Theorem 5.1. *Let $\mathcal{C}$ be a cover of the edges of a graph G with not necessarily disjoint cliques. If we can color the cliques in $\mathcal{C}$ with k colors, such that every color class is a set of vertex disjoint cliques, then we have $b(G) \leq k-1$.*

Proof. Let $\mathcal{C}_i$ be the set of cliques with color i, extended by 1-cliques, such that each $\mathcal{C}_i$ covers all vertices. We use an arbitrary, not self-intersecting $(k-1)$-bend path P as a template (In Figure 14 we have chosen a snake-like path P). Every vertex in G runs within a small distance of P. Along the i-th segment, vertices run through the same grid line if and only if they are in the same clique in $\mathcal{C}_i$. □

As illustrated in Figure 15, every induced subgraph of the triangular plane grid admits an edge clique cover $\mathcal{C}$ with cliques of size at most three, which can be 3-colored as in Theorem 5.1. We conclude:

Corollary 5.2. *The bend-number of the triangular plane grid and all of its induced subgraphs is at most 2.*

Note that the remark at the end of Section 6 implies that the triangular plane grid has no *simple* 2-bend representation.

A proper edge coloring of G is a special case of a colored edge clique cover of G as in Theorem 5.1. Hence a graph's bend-number $b(G)$ is bounded by its edge chromatic number $\chi'(G)$. This implies a strengthening of a result of [4] concerning the maximal degree Δ of G:

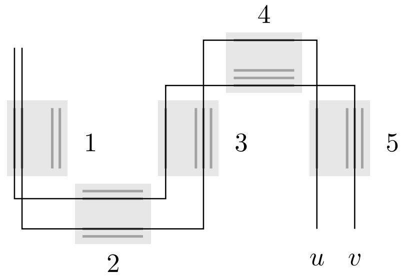

Figure 14: A $(k-1)$-bend representation based on an edge clique cover with k colors: The grey blocks correspond to the colors of the clique cover. Every clique of color i is assigned a grid line within the i-th block. Paths are inserted as demonstrated by u and v according to the cliques they are in.

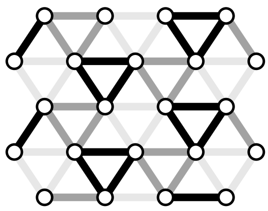

Figure 15: A subgraph G of the triangular plane grid with a 3-colored edge clique cover. Every color class is a set of vertex disjoint cliques. Hence $b(G) \leq 2$.

Corollary 5.3. *If $\chi'(G)$ denotes the edge chromatic number of G, then $b(G) \leq \chi'(G) - 1$. In particular Vizing's Theorem [24] yields $b(G) \leq \Delta$ for G with maximum degree Δ.*

Question 4. What is the maximum bend-number of a graph with maximum degree Δ? By Theorem 3.1 and Corollary 5.3 it is between $\lceil \frac{\Delta}{2} \rceil$ and Δ.

The *degree* of an edge clique cover $\mathcal{C}$ of G is defined to be the maximum number of cliques in $\mathcal{C}$ that a vertex of G participates in.

Theorem 5.4. *If G has an edge clique cover of degree k, then $b(G) \leq 2k - 2$.*

Proof. Consider an edge clique cover $\mathcal{C}$ of G of degree k. Reserve parallel parts of vertical grid lines, one for each clique in $\mathcal{C}$. We represent every vertex v by a snake-like path with at most $2k-2$ bends whose (at most k) vertical segments contain the reserved parts corresponding to the cliques v participates in. All horizontal segments are chosen without introducing unwanted adjacencies. See Figure 16 for an illustration. $\square$

Line graphs are exactly the graphs that admit an edge clique cover of degree at most 2. Hence Theorem 5.4 generalizes a result of [4], stating that line graphs have bend-number at most 2. Since triangle graphs (see [3]) admit edge clique covers of degree at most 3, we conclude in this case from Theorem 5.4 that $b(G) \leq 4$. Some classes of triangle graphs are listed in [3].

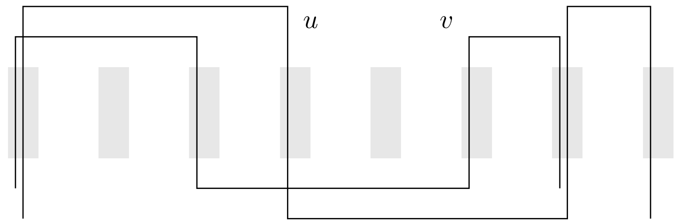

Figure 16: A $(2k-2)$-bend representation based on an edge clique cover of degree k: Each vertical grey part is reserved for a clique in $\mathcal{C}$. Paths are inserted as demonstrated by u and v according to the cliques they are in.

6 Planar and outerplanar graphs

The maximum interval-number of a planar graph is 3, see [21], and its maximum track-number is 4, see [13]. Biedl and Stern [4] showed that the maximum bend-number of a planar graph is at most 5. In this secton we present several classes of planar graphs, with bend-number less than 5.

The maximum interval-number of an outerplanar graph is 2, see [18], and so is its maximum track-number, see [21]. In [4] Biedl and Stern conjectured, that the maximum bend-number of an outerplanar graph is 2 as well. We confirm this conjecture in Theorem 6.1. That 2 is best possible follows from the graph in Figure 17.

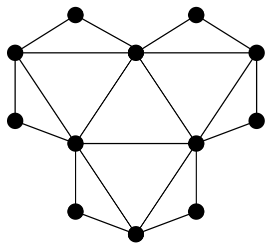

Figure 17: An outerplanar graph with bend-number 2. This example is due to [4].

A 2-bend representation of an outerplanar G can be deduced from the 2-track representation of G given in [18]. But we can even prove:

Theorem 6.1. *Every outerplanar graph has a simple 2-bend representation.*

Proof. We prove the statement for an edge-maximal outerplanar graph G. But observe that any adjacency in the construction sequence can be omitted. The graph G can be constructed starting with an edge. Then every new vertex v is connected to the two vertices a, b of an edge of the already constructed graph G'. Every edge will be used at most once.

We build a simple 2-bend representation step by step as the graph is built. We use the following invariant: Adjacent vertices $a, b \in G'$, which have not been connected to a new vertex yet, have private parts, positioned as in one of the cases that are illustrated in the top row of Figure 18. The second row shows how to introduce a vertex v and maintain the invariant for the new edges $x = \{a, v\}$ and $y = \{b, v\}$.

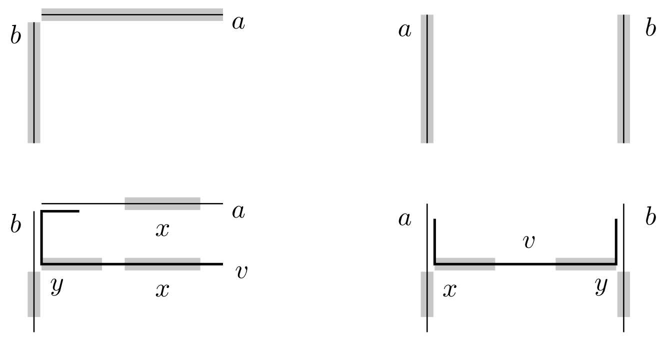

Figure 18: Building a 2-bend representation of an outerplanar graph, a vertex v is attached to the edge $\{a, b\}$: The private parts of $\{a, b\}$ and $x = \{a, v\}, y = \{b, v\}$ are highlighted in the upper and lower row respectively.

□

Theorem 6.2. *Every planar graph of tree-width at most* 3 *has a simple* 4*-bend representation.*

Proof. By a result of El-Mallah and Colbourn [8], planar graphs of tree-width at most 3 are exactly the subgraphs of planar 3-trees. Planar 3-trees, also known as *stacked triangulations*, can be constructed starting with a triangle. Every further vertex v is inserted into an inner facial triangle $\{a, b, c\}$ in the so far constructed G' and connected to its vertices a, b and c. As in the proof for outerplanar graphs we will produce a simple 4-bend representation on the fly, maintaining an invariant for some private parts of a, b and c. See Figure 19 for an explanation. Note that the construction works, even if we only want to build a subgraph of a planar 3-tree.

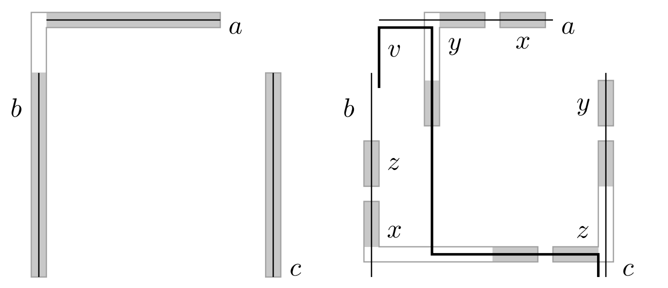

Figure 19: Building a 4-bend representation of a planar graph of tree-width at most 3, a vertex v is attached to the triangle $\{a, b, c\}$: The private parts of $\{a, b, c\}$ and $x = \{a, b, v\}, y = \{a, c, v\}, z = \{b, c, v\}$ are highlighted on the left and the right respectively. The part of an L-shape that is not filled may or may not contain the path within this leg, but does *not* contain any other path.

□

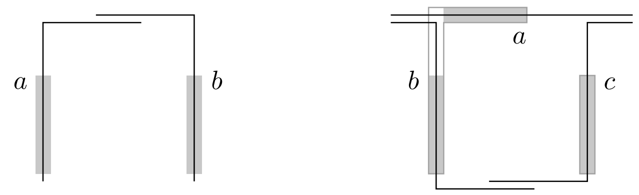

Figure 20: The representation of the starting edge in an outerplanar graph and the starting triangle in a stacked triangulation.

Theorem 6.3. *Planar* 4*-connected triangulations and their subgraphs have simple* 3*-bend representations.*

Proof. In [23] Thomassen proves, that every proper subgraph of a 4-connected planar triangulation can be represented as the contact graph of axes aligned boxes. Moreover every contact is realized as a line segment. The boundary of each box can be seen as a closed path with 3 bends starting and ending at the same corner of the box.

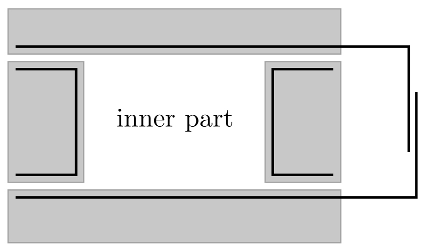

Figure 21: A 3-bend representation of a planar 4-connected triangulation: This shows how to represent the missing edge e.

If G is a 4-connected triangulation itself, take a rectangle representation of $G \backslash \{e\}$ for some e. It looks as depicted in Figure 21, i.e., the facial quadrilateral obtained by deleting e is represented by the four grey boxes; every other box lies *inside*. We replace the outer boxes by the black paths (as shown in the figure) and each inner box again by its boundary path. □

It is easy to see that Theorem 6.3 is best possible. For instance in every plane triangulation $|E| = 3|V| - 6$. Hence by the Lower-Bound-Lemma every *simple* 2-bend representation may use at most 6 grid lines, which is obviously not possible when the graph is large.

Remark. Indeed one can argue that no plane triangulation with 9 or more vertices admits a simple 2-bend representation. On the other hand Figure 22 shows a plane triangulation on 8 vertices with a simple 2-bend representation. Moreover Euler's Formula together with the Lower-Bound-Lemma immediately shows that no higher-genus surface triangulation has a simple 2-bend representation.

Due to [4] the bend-number of a planar graph is at most 5. In contrast we do not even know of an example of a planar graph having no simple 3-bend representation. This leads to the:

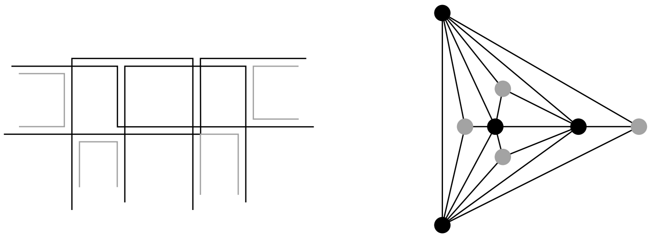

Figure 22: A planar (stacked) triangulation on 8 vertices with a simple 2-bend representation.

Conjecture 6.4. *The bend-number of planar graphs is at most* 4.

By Corollary 5.3, Theorem 4.1, and Theorem 6.2, a counter-example G to the above conjecture should have $\chi'(G) \geq 6$, $\mathrm{col}(G) \geq 3$, treewidth ≥ 4. Theorem 6.3 and [23] yield, that such a G would contain one of the subgraphs listed in [23]. In particular, G must contain a triangle. Moreover Theorem 6.2 with [1] or [8] gives, that G must have either the octahedral graph (Figure 23) or $K_2 \times C_5$ (the prism over a 5-cycle) as a minor.

7 Complexity

In [19] it is asked for the complexity of recognizing k-bend graphs. In general, the bend-number of a graph can be computed by solving a mixed integer program (MIP). Unfortunately the problem instance becomes so huge, that this approach is inapplicable even for graphs with only 10 vertices. It is well-known that interval graphs, that is 0-bend graphs, can be recognized in polynomial time [5]. In this section we prove that recognizing single-bend graphs (1-bend graphs) is NP-complete. In [16] it was shown that recognizing 2-track graphs is NP-complete and [22] proves that recognizing k-interval graphs is NP-complete for every fixed $k \geq 2$. One easily sees that every single-bend graph is a 2-track graph as well as a 2-interval graph. But the converse is not true. For example every outerplanar G has $t(G) \leq 2$ [18] and $i(G) \leq 2$ [21], but is not necessarily a single-bend graph (see Figure 17).

It is easy to verify a single-bend representation, so SINGLE-BEND-RECOGNITION is in NP. For NP-hardness we set up a reduction from ONE-IN-THREE 3-SAT, i.e., we are given a formula $\mathcal{F} = (\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_n)$ that is a conjunction of clauses $\mathcal{C}_1, \ldots, \mathcal{C}_n$. Each clause is the *exclusive disjunction* of exactly three literals $\mathcal{C}_i = (x_{i1} \veebar x_{i2} \veebar x_{i3})$ which are in turn either negated or non-negated boolean variables. Given such a formula $\mathcal{F}$, it is NP-complete [10, 20] to decide, whether there is an assignment of the variables fulfilling $\mathcal{F}$, that is in each clause there is *exactly one true literal.* Moreover ONE-IN-THREE 3-SAT remains NP-complete if each literal is a *non-negated variable* and each clause consists of three *distinct literals.* We will use both additional assumptions on $\mathcal{F}$, even though the first is just for convenience. The distinctness assumption is crucial in the following reduction.

Given a ONE-IN-THREE 3-SAT formula $\mathcal{F}$ we will define a graph $G_{\mathcal{F}}$, such that $b(G_{\mathcal{F}}) = 1$ if and only if $\mathcal{F}$ can be fulfilled. The graph consists of an induced subgraph $G_{\mathcal{C}}$ for every clause $\mathcal{C}$ with 13 vertices, called the *clause gadget*, a vertex v_j for every variable x_j and 31 additional vertices.

7.1 Clause Gadgets

Constructing a clause gadget $G_{\mathcal{C}}$ starts with an induced octahedral graph O. Label the vertices by $\{a, A, b, B, c, C\}$ as in Figure 23. This way $\{a, A\}$, $\{b, B\}$ and $\{c, C\}$ are the three *non*-edges and their complements $\{b, C, B, c\}$, $\{a, C, A, c\}$ and $\{a, B, A, b\}$ are the three induced 4-cycles in O.

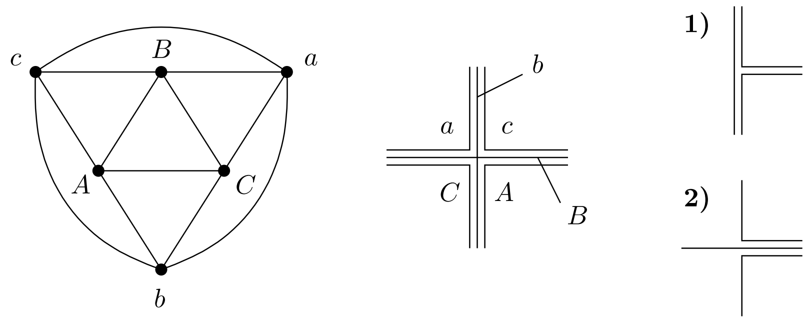

Figure 23: The labeled octahedral graph O, a single-bend representation of O, and the two possible ways a triangle of O is represented.

Lemma 7.1. *We have $b(O) = 1$ and in every single-bend representation*

(i) there is a unique grid point, called the center*, that is contained in every path,*

(ii) each intersection between vertices in O lies on a half ray starting at the center, called a center ray*,*

(iii) for every pair of center rays, there is a unique vertex in O intersecting exactly these two center rays, and

(iv) every triangle in O is represented in one of the two ways on the right of Figure 23.

Proof. Figure 23 shows $b(O) \leq 1$ and since O contains induced 4-cycles it is not an interval graph. Hence $b(O) = 1$.

By a result of [11], every induced 4-cycle in a single-bend representation is either a *frame*, a *true pie* or a *false pie*. These terms are illustrated in Figure 24. If an induced 4-cycle is represented by a frame, then the bends of its vertices are distinct. Thus in a single-bend representation no other vertex can be adjacent to all of them. Since for each induced 4-cycle in O there is a vertex that is adjacent to all of its vertices, $\{a, B, A, b\}$, $\{a, C, A, c\}$ and $\{b, C, B, c\}$ are pies. So all pies share the middle point, the claimed center, and every vertex intersects exactly two center rays. Since every edge in O is part of an induced 4-cycle, no two vertices in O can intersect the same pair of center rays. This concludes (i)–(iii). Part (iv) is easily obtained from this. ☐

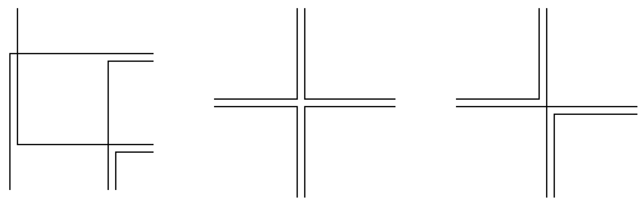

Figure 24: Single-bend representations of an induced 4-cycle: A frame, a true pie and a false pie.

To complete a clause gadget $G_{\mathcal{C}}$ seven vertices are added: W_{ABC} is connected to $\{A, B, C\}$, w_{abC}, w_{aBc} and w_{Abc} are connected to $\{a, b, C\}$, $\{a, B, c\}$ and $\{A, b, c\}$, respectively, and s_{ab}, s_{ac} and s_{bc} are connected to $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$, respectively. The resulting graph is depicted in Figure 25.

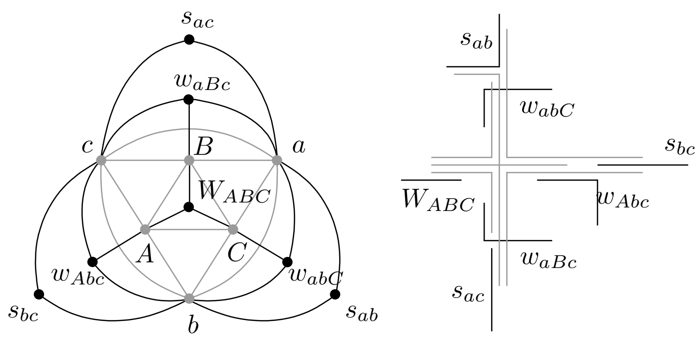

Figure 25: The clause gadget $G_{\mathcal{C}}$ with a single-bend representation.

Lemma 7.2. *We have $b(G_{\mathcal{C}}) = 1$ and in every single-bend representation*

(i) every center ray contains a segment of exactly one of W_{ABC}, w_{abC}, w_{aBc}, and w_{Abc} and

(ii) every such segment, except the one of W_{ABC}, is contained in a segment from $\{a, b, c\}$.

Proof. Let $w \in \{W_{ABC}, w_{abC}, w_{aBc}, w_{Abc}\}$. Then w is connected to every vertex of the triangle Δ in O. By Lemma 7.1–(iv), Δ is represented in one of the two ways that are illustrated on the right of Figure 23. In case 1), w has to be contained in two center rays in order to intersect with all three vertices. But then, by Lemma 7.1–(iii), w would intersect vertices in O that are not adjacent to w. Hence Δ is represented as in case 2) and one segment of w is necessarily contained in the center ray that carries all three vertices in Δ. This concludes part (i).

Now consider a pair (w, s) in $\{(w_{abC}, s_{ab}), (w_{aBc}, s_{ac}), (w_{Abc}, s_{bc})\}$. Both, w and s, are contained in at most one center ray. Moreover it is the same center ray and it contains one of the capitalized vertices, that is adjacent to w but not s. Hence the segment of s lies further away from the center than the segment of w. Thus the segment of w is contained in a segment of each neighbor of s. □

7.2 The reduction

Given a formula $\mathcal{F} = (\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_n)$ with clauses $\mathcal{C}_i = (x_{i1} \vee x_{i2} \vee x_{i3})$ for $i = 1, \ldots, n$ we are now ready to define the graph $G_\mathcal{F}$ as follows. See Figure 26 for an example.

1. For each clause $\mathcal{C}$ there is a clause gadget $G_\mathcal{C}$.
2. For each variable x_j there is a vertex v_j that is adjacent to w_{Abc}, w_{aBc}, or w_{abC}, whenever x_j is the first, second, or third variable in $\mathcal{C}$, respectively.
3. There is a vertex V adjacent to every W in the clause gadgets.
4. There is a $K_{2,4}$ with a specified vertex T of the larger part, called the *truth-vertex*. T is adjacent to every v_j and V.
5. There are two octahedral graphs O_1 and O_2. The vertex T is connected to the vertices of a triangle of each.
6. There are two more octahedral graphs O_3 and O_4. The vertex V is connected to the vertices of a triangle of each.

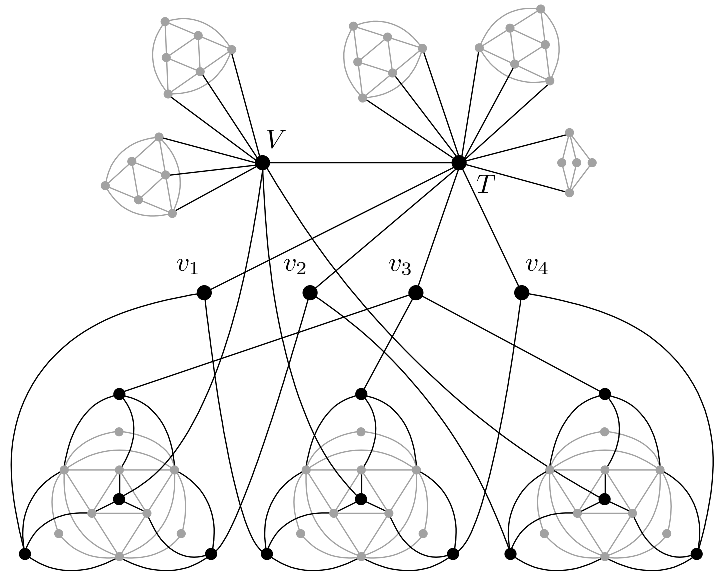

Figure 26: The graph $G_\mathcal{F}$ for $\mathcal{F} = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee x_3 \vee x_4)$.

We will prove that a ONE-IN-THREE 3-SAT-formula $\mathcal{F}$ can be fulfilled if and only if $b(G_\mathcal{F}) = 1$.

Theorem 7.3. *SINGLE-BEND-RECOGNITION is NP-hard.*

Proof. First suppose $b(G_\mathcal{F}) = 1$ and consider a single-bend representation of $G_\mathcal{F}$. W.l.o.g. assume, that V and T intersect with their horizontal segments. We set a variable x_j true if v_j intersects the truth-vertex T with its horizontal segment and false if v_j intersects T with its vertical segment.

Note that in every single-bend representation of a $K_{2,4}$, every vertex of the larger part, in particular T here, has its bend in a false pie (see [2] for a reasoning). The truth-vertex T is attached to a triangle of O_1 and O_2. The proof of Lemma 7.2–(i) shows that a segment of T is contained in exactly one center ray of each, O_1 and O_2. As the bend of T is in a false pie of $K_{2,4}$, the endpoints of T are contained in O_1 and O_2, respectively. Hence every further segment that intersects T is necessarily contained in a segment of T. Consequently, a vertex v_j intersects the lower-case w-vertex in each clause gadget with its vertical segment if and only if x_j is true.

For the same reason V intersects every neighbor other than T with its vertical segment. Since V is attached to a triangle of O_3 and O_4, the two endpoints of the vertical segment of V are contained in O_3 and O_4, respectively. Thus, the vertical segment of the upper-case W-vertex in each clause gadget is contained in the vertical segment of V. In consequence, the horizontal segment of every such W by Lemma 7.2–(i) is contained in a horizontal center ray. Hence of the other three center rays, two are vertical and one is horizontal. Together with Lemma 7.2–(ii) this yields, that in every clause gadget exactly two of $\{w_{abC}, w_{aBc}, w_{Abc}\}$ intersect the corresponding v_j with their horizontal segment and exactly one with its vertical segment. In other words every clause contains exactly one true variable.

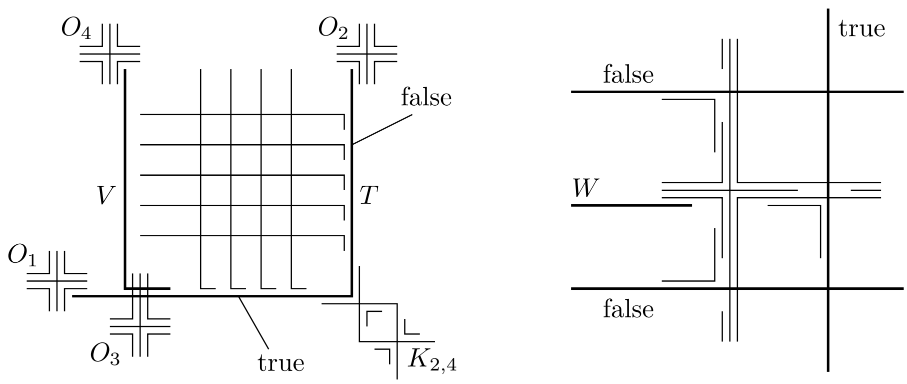

Figure 27: On the left: A single-bend representation of $G_{\mathcal{F}}$. The vertex V and the truth-vertex T are drawn bold. The clause gadgets are omitted. On the right: A single-bend representation of a clause gadget $G_{\mathcal{C}}$. The vertices v_j that correspond to the variables in the clause $\mathcal{C}$ and the vertex W of the clause gadget are drawn bold.

Now given a truth assignment fulfilling $\mathcal{F}$, we can construct a single-bend representation of $G_{\mathcal{F}}$. First represent all of $G_{\mathcal{F}}$ but the clause gadgets as on the left side in Figure 27. A vertex v_j is connected to the truth-vertex T horizontally if x_j is true and vertically if x_j is false.

To interlace a clause gadget $G_{\mathcal{C}}$, introduce a horizontal grid line l_h between the horizontal grid lines used by the two false variables in $\mathcal{C}$. Then connect the W-vertex in $G_{\mathcal{C}}$ to V vertically with its bend on l_h. Furthermore introduce a vertical grid line l_v between the vertical grid lines used by V and the true variable in $\mathcal{C}$. Where l_h and l_v cross, introduce the center of the clause gadget as illustrated on the right side in Figure 27. Note that the clause gadget is symmetric in A, B and C and hence it can be represented with every center ray pointing into the desired direction. □

To the end of this section there is an obvious question. Note that the complexity of recognizing k-track graphs for $k \geq 3$ is open.

Question 5. What is the complexity of recognizing k-bend graphs for $k \geq 2$?

8 Open Problems and Concluding Remarks

Still, the maximum bend-number of many graph classes is unknown. It is known that the interval-number of planar graphs is at most 3, see [21], and the track-number of planar graphs is at most 4, see [13]. We conjecture that the bend-number of planar graphs is at most 4 and can be achieved even with simple representations. Also the general relation between track-number and bend-number seems interesting to us. Is it true that the track-number is not bounded by a constant multiple of the bend- or interval-number?

A large gap appears when putting bend-number of G in relation to its maximum degree Δ. Corollary 5.3 implies that $b(G) \leq \Delta$. On the other hand $K_{m,m}$ requires $(m+1)/2$ bends if m is odd. So the maximum bend-number among graphs with maximum degree Δ lies between $\lceil \Delta/2 \rceil$ and Δ. What is the exact value?

Looking at bipartite graphs, we ask for the bend-number of $K_{m,n}$ with fixed m and different values of n. How does it grow with n? In Section 3 we give some explicit intermediate values from which lower and upper bounds can be derived. In particular the behavior for $n \in \theta(m^3)$ is unknown.

In Section 7 we prove NP-hardness of SINGLE-BEND-RECOGNITION. Hence computing $b(G)$ is NP-hard. But the complexity of recognizing k-bend graphs for $k \geq 2$ remains open.

Modifying the global settings yields interesting questions. As done in [21] and mentioned in Section 2, one can parameterize k-bend representations by the maximal number of paths that are allowed to share a grid edge. Introducing simple representations (a first step in this procedure) already allowed us to derive optimal representations for triangle-free graphs. The main tool, the Lower-Bound-Lemma, can be adapted to depth-r bend representations with $r > 2$ in a straightforward way. There might be interesting results for K_{r+1}-free graphs at reach.

Instead of the plane grid as the host graph, the grid on the torus may be worthwhile to consider. We obtain a generalization of circular arc graphs. Can the number of bends be reduced in this case? This seems to be the case for most of the representations of complete bipartite graphs. How does the bend-number of a graph relate to its *toroidal* bend-number?

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