AH_3-MANIFOLDS OF CONSTANT ANTIHOLOMORPHIC SECTIONAL CURVATURE [1]

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The purpose of this paper is to prove that an AH_3-manifold of constant antiholomorphic sectional curvature is a real space form or a complex space form.

1. Introduction. Let M be a $2m$-dimensional almost Hermitian manifolds with metric tensor g and almost complex structure J. The Riemannian connection and the curvature tensor are denoted by ∇ and R, respectively.

If $\nabla J = 0$, or $(\nabla_X J)X = 0$ or

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0\,,$$

then M is said to be a Kähler, or nearly Kähler, or almost Kähler manifold, respectively. The corresponding classes of manifolds are denoted by K, NK, AK. The general class of almost Hermitian manifold is denoted by AH. If L is a class of almost Hermitian manifolds, its subclass of L_i-manifolds is defined by the identity i), where

1) $R(X, Y, Z, U) = R(X, Y, JZ, JU)$;
2) $R(X, Y, Z, U) = R(X, Y, JZ, JU) + R(X, JY, Z, JU) + R(JX, Y, Z, JU)$;
3) $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$.

It is well known, that

$$K = K_1 \subset NK = NK_2\,, \qquad K \subset AK_2\,,$$
$$K = NK \cap AK\,, \qquad AH_1 \subset AH_2 \subset AH_3\,,$$

see e.g. [4].

A plane α in $T_p(M)$ is said to be holomorphic (resp. antiholomorphic) if $\alpha = J\alpha$ (resp. $\alpha \perp J\alpha$). The manifold M is said to be of pointwise constant holomorphic (respectively, antiholomorphic) sectional curvature ν, if for each point $p \in M$ the curvature of an arbitrary holomorphic (resp. antiholomorphic) plane α in $T_p(M)$ doesn't depend on α: $K(\alpha) = \nu(p)$.

For Kähler manifolds the requirements for constant holomorphic and constant antiholomorphic sectional curvature are equivalent [2]. In [3] it is proved a classification theorem for nearly Kähler manifolds of constant holomorphic sectional curvature.

If M is a $2m$-dimensional AH_3-manifold of pointwise constant antiholomorphic sectional curvature ν, and if $m > 2$, then ν is a global constant [5]. In [1] it is proved a classification theorem for nearly Kähler manifolds of constant antiholomorphic sectional curvature and a corresponding result for AK_3-manifolds is obtained in [6].

In section 3 we shall prove the following theorem:

Theorem. *Let M be a $2m$-dimensional AH_3-manifold, $m > 2$. If M is of pointwise constant antiholomorphic sectional curvature, then M is a real space form or a complex space form.*

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Here a real space form means a Riemannian manifold of constant sectional curvature and a complex space form means a Kähler manifold of constant holomorphic sectional curvature.

2. Basic formulas. If M is an AH_3-manifold, its Ricci tensor S satisfies

$$S(X,Y) = S(Y,X) = S(JX,JY)\,.$$

If moreover M has pointwise constant antiholomorphic sectional curvature ν, its curvature tensor has the form

$$R = \frac{1}{6}\psi(S) + \nu\pi_1 - \frac{2m-1}{3}\nu\pi_2\,, \tag{2.1}$$

where

$$\begin{aligned}\psi(Q)(x,y,z,u) = {}& g(x,Ju)Q(y,Jz) - g(x,Jz)Q(y,Ju) - 2g(x,Jy)Q(z,Ju)\\ &+g(y,Jz)Q(x,Ju) - g(y,Ju)Q(x,Jz) - 2g(z,Ju)Q(x,Jy)\end{aligned}$$

for an arbitrary tensor Q of type (0,2) and

$$\pi_1(x,y,z,u) = g(x,u)g(y,z) - g(x,z)g(y,u)\,,$$

$$\pi_2 = \frac{1}{2}\psi(g)\,,$$

see [1]. According to (2.1), M is an AH_2-manifold.

On the other hand, it is known, that if M is an AK_2-manifold,

$$R(x,y,z,u) - R(x,y,Jz,Ju) = \frac{1}{2}g((\nabla_x J)y - (\nabla_y J)x, (\nabla_z J)u - (\nabla_u J)z)\,, \tag{2.2}$$

holds good [4].

We shall use also the second Bianchi identity

$$(\nabla_x R)(y,z,u,v) + (\nabla_y R)(z,x,u,v) + (\nabla_z R)(x,y,u,v) = 0\,. \tag{2.3}$$

3. Proof of the theorem.

Lemma. *The conditions of the theorem imply that M is an Einsteinian manifold.*

Proof of Lemma. Let p be an arbitrary point of M and let x, $y \in T_p(M)$. According to the second Bianchi identity,

$$(\nabla_x R)(Jx,y,y,Jx) + (\nabla_{Jx} R)(y,x,y,Jx) + (\nabla_y R)(x,Jx,y,Jx) = 0\,. \tag{3.1}$$

Let $\{e_i,\, Je_i;\, i = 1,\dots,m\}$ be an orthonormal basis of $T_p(M)$ such that $Se_i = \lambda_i e_i$, $i = 1,\dots,m$. Putting in (3.1) $x = e_i$, $y = e_j$ or $x = e_k$, $y = e_i + e_j$ for $i \neq j \neq k \neq i$ and using (2.1), we obtain

$$(\nabla_{e_j} S)(e_i,e_j) + \{\lambda_i + \lambda_j - 2(2m-1)\nu\}g(Je_i, (\nabla_{e_j} J)e_j) = 0\,; \tag{3.2}$$

$$\begin{aligned}&(\nabla_{e_i} S)(e_j,e_k) + \{\lambda_i + \lambda_k - 2(2m-1)\nu\}g(Je_k, (\nabla_{e_j} J)e_i)\\ &+(\nabla_{e_j} S)(e_i,e_k) + \{\lambda_j + \lambda_k - 2(2m-1)\nu\}g(Je_k, (\nabla_{e_i} J)e_j) = 0\,,\end{aligned} \tag{3.3}$$

respectively. Analogously from

$$(\nabla_{e_i} R)(Je_j,e_j,e_j,Je_k) + (\nabla_{Je_j} R)(e_j,e_i,e_j,Je_k) + (\nabla_{e_j} R)(e_i,Je_j,e_j,Je_k) = 0$$

we find

$$3(\nabla_{e_i}S)(e_j,e_k)+6\{\lambda_j-(2m-1)\nu\}g((\nabla_{e_i}J)e_j,Je_k)$$
$$(3.4)\qquad -(\nabla_{e_j}S)(e_i,e_k)-\{\lambda_i+\lambda_j-2(2m-1)\nu\}g((\nabla_{e_j}J)e_i,Je_k)=0$$

and hence

$$8(\nabla_{e_i}S)(e_j,e_k)+\{17\lambda_j-\lambda_i-16(2m-1)\nu\}g((\nabla_{e_i}J)e_j,Je_k)$$
$$(3.5)\qquad +3(\lambda_i-\lambda_j)g((\nabla_{e_j}J)e_i,Je_k)=0\,.$$

In (3.5) we change j and k and we add the result with (3.5)

$$16(\nabla_{e_i}S)(e_j,e_k)+17(\lambda_j-\lambda_k)g((\nabla_{e_i}J)e_j,Je_k)$$
$$(3.6)\qquad +3(\lambda_i-\lambda_j)g((\nabla_{e_j}J)e_i,Je_k)+3(\lambda_i-\lambda_k)g((\nabla_{e_k}J)e_i,Je_j)=0\,.$$

On the other hand, (3.3) and (3.4) imply

$$(3.7)\qquad \{3\lambda_j-\lambda_i-2\lambda_k\}g((\nabla_{e_i}J)e_j,Je_k)+\{3\lambda_i-\lambda_j-2\lambda_k\}g((\nabla_{e_j}J)e_i,Je_k)=0\,.$$

Hence it is not difficult to find

$$3(\lambda_j-\lambda_k)g((\nabla_{e_i}J)e_j,Je_k)+(\lambda_i-\lambda_j)g((\nabla_{e_j}J)e_i,Je_k)+(\lambda_i-\lambda_k)g((\nabla_{e_k}J)e_i,Je_j)=0$$

and by using (3.6) this implies

$$(3.8)\qquad 2(\nabla_{e_i}S)(e_j,e_k)=(\lambda_k-\lambda_j)g((\nabla_{e_i}J)e_j,Je_k)\,.$$

Let us first assume that $g((\nabla_{e_i}J)e_j,Je_k)\neq 0$. Using three times (3.7), we obtain

$$(3\lambda_i-\lambda_k-2\lambda_j)(3\lambda_j-\lambda_i-2\lambda_k)(3\lambda_k-\lambda_j-2\lambda_i)$$
$$-(3\lambda_i-\lambda_j-2\lambda_k)(3\lambda_j-\lambda_k-2\lambda_i)(3\lambda_k-\lambda_i-2\lambda_j)=0$$

or equivalently

$$(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)=0\,.$$

Hence it follows $\lambda_i=\lambda_j=\lambda_k$. Indeed we have to consider two cases:

C a s e 1. $\lambda_i=\lambda_j$. In (3.7) we made a cyclic change of i, j, k and we use $\lambda_i=\lambda_j$:

$$(\lambda_i-\lambda_k)\{3g((\nabla_{e_j}J)e_k,Je_i)+g((\nabla_{e_k}J)e_i,Je_j)\}=0\,,$$
$$(3.9)\qquad (\lambda_i-\lambda_k)\{g((\nabla_{e_k}J)e_i,Je_j)+3g((\nabla_{e_i}J)e_j,Je_k)\}=0\,.$$

If $g((\nabla_{e_k}J)e_i,Je_j)=0$ the last equation implies $\lambda_i=\lambda_k$, i.e. $\lambda_i=\lambda_j=\lambda_k$. So we assume $g((\nabla_{e_k}J)e_i,Je_j)\neq 0$. In (3.5) we change i and k and we use $\lambda_i=\lambda_j$ and (3.8):

$$\{17\lambda_i-\lambda_k-16(2m-1)\nu\}g((\nabla_{e_k}J)e_j,Je_i)+3(\lambda_k-\lambda_i)g((\nabla_{e_j}J)e_k,Je_i)=0\,.$$

Hence, using (3.9), we obtain $\lambda_i=(2m-1)\nu$. On the other hand, (3.5) and (3.8) result

$$3\lambda_i+\lambda_k-4(2m-1)\nu=0$$

and so we find $\lambda_k=(2m-1)\nu$, i.e. $\lambda_i=\lambda_j=\lambda_k$.

C a s e 2. $\lambda_j=\lambda_k$. From (3.7)we obtain

$$(\lambda_i-\lambda_j)\{g((\nabla_{e_i}J)e_j,Je_k)-3g((\nabla_{e_j}J)e_i,Je_k)\}=0\,.$$

If $g((\nabla_{e_j}J)e_i,Je_k)=0$ this implies $\lambda_i=\lambda_j$, so $\lambda_i=\lambda_j=\lambda_k$. But $g((\nabla_{e_j}J)e_i,Je_k)\neq 0$ is the Case 1.

So we have $\lambda_i = \lambda_j = \lambda_k$ and using (3.5) and (3.8), we find $\lambda_i = (2m-1)\nu$. If $m = 3$ M is Einsteinian in p. Let $m > 3$. For $s \neq i,\, j,\, k$ we have

$$(\nabla_{e_i} R)(e_s, Je_s, e_j, Je_k) + (\nabla_{e_s} R)(Je_s, e_i, e_j, Je_k) + (\nabla_{Je_s} R)(e_i, e_s, e_j, Je_k) = 0\,.$$

Because of (2.1) this implies

$$(\nabla_{e_i} S)(e_j, e_k) + \{\lambda_j + \lambda_s - 2(2m-1)\nu\} g((\nabla_{e_i} J)e_j, Je_k) = 0\,.$$

Hence, using $\lambda_j = \lambda_k = (2m-1)\nu$ and (3.8), we derive $\lambda_s = (2m-1)\nu$. Consequently M is Einsteinian in p.

Now we assume that

$$g((\nabla_x J)y, z) = 0$$

whenever $x,\, y,\, z$ are choosen among the basic vectors $e_i,\, Je_i;\; i = 1, \ldots, m$ and $x \neq y, z, Jy, Jz$. In (2.3) we put $x = Je_i$, $y = v = e_j$, $z = -Ju = e_k$ for $i \neq j \neq k \neq i$. Using (2.1), we obtain

$$(\nabla_{e_i} S)(e_i, e_j) + \{\lambda_j + \lambda_k - 2(2m-1)\nu\} g(Je_i, (\nabla_{e_j} J)e_j,) = 0\,.$$

From this equality and (3.2) it follows that if $g(Je_i, (\nabla_{e_j} J)e_j) \neq 0$ for some $i,\, j$, then $\lambda_s = \lambda_k$ for $s,\, k \neq j$. Consequently if $(\nabla_{e_s} J)e_s \neq 0$ for any $s \neq j$ then M is Einsteinian in p.

Let us assume that M is not Einsteinian in p. Then M is not Einsteinian in a neighbourhoohd U of p. We shall prove that M is an AK_2-manifold in U. Let $q \in U$. If M is a Kähler manifold in q, M is an AK_2-manifold in U. Let M is not Kähler in q. Let $\{f_i,\, Jf_i,\, i = 1, \ldots, m\}$ be an orthonormal basis of $T_p(M)$, such that $Sf_i = \mu_i f_i$, $i = 1, \ldots, m$. Since M is non Kähler and non Einsteinian in q we may assume that $(\nabla_{f_1} J)f_1 \neq 0$, $\mu_2 = \ldots = \mu_m = \mu$ and

$$(\nabla_x J)y = 0\,, \qquad g((\nabla_{f_1} J)x, y) = 0 \tag{3.10}$$

whenever $x,\, y$ are choosen among $f_i,\, Jf_i$ for $i > 1$. Analogously to (3.2)

$$(\nabla_{f_j} S)(f_i, f_j) + \{\mu_i + \mu_j - 2(2m-1)\nu\} g(Jf_i, (\nabla_{f_j} J)f_j) = 0 \tag{3.2$'$}$$

holds good and according to (3.10) this implies

$$(\nabla_{f_j} S)(f_i, f_j) = (\nabla_{Jf_j} S)(f_i, Jf_j) = 0 \quad \text{for} \;\; j > 1,\;\, j \neq i\,. \tag{3.11}$$

In (2.3) we put $x = f_i$, $y = -Jv = f_j$, $z = -Ju = f_1$ for $i \neq j \neq 1 \neq i$ and using (2.1), (3.10) and (3.11) we obtain

$$\begin{gathered}(\nabla_{f_i} S)(f_j, f_j) + (\nabla_{f_i} S)(f_1, f_1) - (\nabla_{f_1} S)(f_i, f_1) \\ +2\{\mu - (2m-1)\nu\} g(Jf_i, (\nabla_{f_1} J)f_1) = 0\,.\end{gathered} \tag{3.12}$$

Now let $k \neq i$. From

$$(\nabla_{f_i} R)(f_k, Jf_k, Jf_k, f_k) + (\nabla_{f_k} R)(Jf_k, f_i, Jf_k, f_k) + (\nabla_{Jf_k} R)(f_i, f_k, Jf_k, f_k) = 0$$

it follows

$$\begin{gathered}2(\nabla_{f_i} S)(f_k, f_k) - (\nabla_{f_k} S)(f_i, f_k) + \{\mu_i + \mu_k - 2(2m-1)\nu\} g(Jf_i, (\nabla_{f_k} J)f_k) \\ -(\nabla_{Jf_k} S)(f_i, Jf_k) + \{\mu_i + \mu_k - 2(2m-1)\nu\} g(Jf_i, (\nabla_{Jf_k} J)Jf_k) = 0\,.\end{gathered}$$

Hence using (3.2′) we derive

$$(\nabla_{f_i} S)(f_k, f_k) = (\nabla_{f_k} S)(f_i, f_k) + (\nabla_{Jf_k} S)(f_i, Jf_k)\,. \tag{3.13}$$

Now (3.11) and (3.13) imply

$$(\nabla_{f_i} S)(f_j, f_j) = 0 \quad \text{for} \;\; i,\, j > 1,\, i \neq j\,.$$

Then (3.12) takes the form

$$(\nabla_{f_i} S)(f_1, f_1) - (\nabla_{f_1} S)(f_i, f_1) + 2\{\mu - (2m-1)\nu\} g(Jf_i, (\nabla_{f_1} J) f_1) = 0$$

and using (3.13), we obtain

$$(\nabla_{Jf_1} S)(f_i, Jf_1) + 2\{\mu - (2m-1)\nu\} g(Jf_i, (\nabla_{f_1} J) f_1) = 0$$

which implies

$$\begin{gathered} (\nabla_{f_1} S)(f_i, f_1) + (\nabla_{Jf_1} S)(f_i, Jf_1) \\ +2\{\mu - (2m-1)\nu\} g(Jf_i, (\nabla_{f_1} J) f_1 + (\nabla_{Jf_1} J) Jf_1) = 0\,. \end{gathered} \tag{3.14}$$

Since M is not Einsteinian in q the first equation of (3.2) and (3.14) result

$$(\nabla_{f_1} J) f_1 + (\nabla_{Jf_1} J) Jf_1 = 0\,. \tag{3.15}$$

From (3.10) and (3.15) it follows easily that M is an almost Kähler maniflod in q. Consequently it is an almost Kähler manifold in U and hence an AK_2-manifold in U. If M is a Kähler manifold in U it is of constant holomorphic sectional curvature [2] and hence Einsteinian in U which contradicts our assumption. Let M is non Kähler in q (we shall use the above notations for the basis of $T_q(M)$) and let

$$(\nabla_{f_1} J) f_i = \alpha_i f_1 + \beta_i Jf_1 \quad \text{for} \;\; i > 1\,.$$

In (2.2) we put $x = u = f_i$, $y = z = f_1$:

$$\nu - \frac{1}{6}(\mu + \mu_1) + \frac{2m-1}{3}\nu = -\frac{1}{2}(\alpha_i^2 + \beta_i^2)$$

for $i > 1$ which implies

$$\alpha_i^2 + \beta_i^2 = \alpha_j^2 + \beta_j^2 \quad \text{for} \;\; i,\, j > 1\,. \tag{3.16}$$

Now we put in (2.2) ($x = f_i$, $y = z = f_1$, $u = f_1$), ($x = f_i$, $y = z = f_j$, $u = Jf_j$) respectively and we obtain

$$\begin{gathered} \alpha_i \alpha_j + \beta_i \beta_j = 0\,, \\ \alpha_i \beta_j - \alpha_j \beta_i = 0\,, \end{gathered} \tag{3.17}$$

respectively. But (3.16) and (3.17) imply $\alpha_i = \beta_i = 0$ for $i > 1$ which is a contradiction. This proves the Lemma.

Now we prove the Theorem. Since M is Einsteinian (2.1) takes the form

$$R = \nu \pi_1 + \lambda \pi_2$$

with a constant λ. Consequently M is a real space form or a complex space form [7].

REFERENCES

1. G. Ganchev, O. Kassabov. Nearly Kähler manifolds of constant antiholomorphic sectional curvature. *C. R. Acad. bulg. Sci.*, **35**, 1982, 145-147.
2. B.-Y. Chen, K. Ogiue. Some characterizations of complex space forms. *Duke Math. J.,* **40**, 1973, 797-799.
3. A. Gray. Classification des variétés approximativement kähleriennes de courbure sectionnelle holomorphe constante. *C. R. Acad. Sci. Paris, Sér. A*, **279**, 1974, 797-800.
4. A. Gray. Curvature identities for Hermitian and almost hermitian manifolds. *Tôhoku Math. J.,* **28**, 1976, 601-612.
5. O. Kassabov. Sur le théorème de F. Schur pour une variété presque hermitienne. *C. R. Acad. bulg. Sci.,* **35**, 1982, 905-908.
6. O. Kassabov. Almost Kähler manifolds of constant antiholomorphic sectional curvature. *Serdica*, **9**, 1983, 373-376.
7. F. Tricerri, L. Vanhecke. Curvature tensors on almost Hermitian manifolds. *Trans. Amer. Math. Soc.*, **267**, 1981, 365-398.

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