The local properties of the Markov processes of Ornstein-Uhlenbeck type [*]

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Abstract

We prove the existence of a local time, the continuity of the local time about t, and the regular property for *a.e.* $x \in R$ of a Ornstein-Uhlenbeck type $\{X_t, \ t \in R^+\}$ driven by a general Lévy process, under mild regularity conditions. We discuss the asymptotic behaviour of the local time when X is ergodic. We also investigate the first passage problem. These results give precise information about the local properties of the sample functions.

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1 Introduction

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and $M_+(R^d)$ be the totality of real $d \times d$ matrices whose all eigenvalues have positive real parts. The starting at x Markov process of Ornstein-Uhlenbeck (O-U) type $X = \{X_t, t \in R^+, P_x\}$ over $(\Omega, \mathfrak{F}, P)$ is a Feller process

with infinitesimal generator

$$A = G - \sum_{j=1}^{d}\sum_{k=1}^{d} Q_{jk} x_k \frac{\partial}{\partial x_j},$$

where G is the infinitesimal generator of a Lévy process $Z = \{Z_t, t \in R^+\}$ taking values in R^d, $Q \in M_+(R^d)$ and $x \in R^d$. An equivalent definition of this process X is given by the unique solution of the equation

$$X_t = x - \int_0^t Q X_s ds + Z_t,$$

which can be expressed as

$$X_t = e^{-tQ}x + \int_0^t e^{(s-t)Q} dZ_s, \tag{1.1}$$

where the stochastic integral with respect to the Lévy process Z is defined by convergence in probability from integrals of simple functions. When Z is the Brownian motion taking values in R^d, X is the ordinary O-U process.

The study of Markov processes of O-U type keeps receiving much attention both in the physical and the mathematical literature, for example, in climate models to explain the so-called Dansgaard-Oeschger events–see [7] and the references therein. Many authors investigated the recurrence [13] [15], the strong Feller property and the exponential β-mixing property [10]. In [9], the authors used the local time as the kernel function of the empirical likelihood inference, but as to the author's knowledge, there is not paper investigated the existence of the local time of X.

In this paper, we will prove the existence of a local time under mild regularity conditions, and its continuous properties about time t. Here we consider the local time as the Radon-Nikodym derivative of the occupation measure of X relative to a Borel set A. There are some different between this definition and the Blumenthal-Getoor local time. We will consider their connection. Many authors like to consider X on R^d, but when $d \geq 2$, single points are essentially polar even for Brownian motion, that is to say, $L(x,t) = 0$ for $a.e.$ x. Hence, we define X on R. It is well-known that X is ergodic under very mild regularity condition, let F be the unique invariant distribution of X, if there is a density f of the distribution F, then

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{2t\epsilon} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds = f(x).$$

Under some conditions, we have

$$\lim_{t \to \infty} \frac{L(x,t)}{t} = f(x) \qquad L^2(P),$$

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that is to say, $\frac{L(x,t)}{t}$ or $\frac{1}{2t\epsilon}\int_0^t I_{|X_s-x|<\epsilon}ds$ is a unbiased consistent estimator of f, but we do not go any further in this direction in this paper.

The paper is organized as follows. we will give some basic results about the Lévy process and X in section 2. In Section 3, we establish completely general criteria for the existence of the local time of X in terms of Fourier analysis following Berman [2]. In section 4, we shall discuss the continuity and the asymptotic behavior of the local time about time t. In section 5, we study the first passage cross a lever.

Throughout this paper, x is the starting point of the Markov process of O-U type. C always stands for a positive constant, whose value is irrelevant. The expectation operator under P and P_x are denoted by E and E_x. $\varphi_\eta(\cdot)$ stands for the characteristic function of a random variable or a distribution η and $\mathfrak{F}_t$ is the P-completed sigma-field generated by $(X_s, s \leq t)$.

2 Preliminaries

In this section, we collect some basic results about the Lévy process and the Markov process of O-U type which will be used in the following section.

Let $\{Z_t,\ t \in R^+\}$ be a Lévy process taking values in R, whose characteristic function is given by

$$E(e^{i\theta Z_t}) = \exp\{-t\psi(\theta)\},$$

where

$$\psi(\theta) = ib\theta + \frac{\sigma^2\theta^2}{2} - \int_R (e^{i\theta u} - 1 - \frac{i\theta u}{1+|u|^2})\rho(du) \tag{2.1}$$

is called the Lévy exponent, $b \in R$, $\sigma \geq 0$ and ρ is a measure on R satisfying that $\rho(\{0\}) = 0$ and the integrability condition

$$\int_{R^d} (1 \wedge |u|^2)\rho(du) < \infty.$$

Certainly, the process Z is characterized by the generating triplet $(b, \sigma, \rho(\cdot))$.

Let $X = \{X_t, t \in R^+, P^x\}$ be a Markov process of O-U type defined by (1.1). The next proposition specifies the characteristic function of the transition probability of X.

Proposition 2.1. (Sato and Yamazato 1984, Theorem 3.1)

Let $P(t, x, \cdot)$ be the transition probability of X. The characteristic function of $P(t, x, \cdot)$ is

$$\varphi_{P(t,x,\cdot)}(\theta) = \exp\left\{ixe^{-tQ}\theta - \int_0^t \psi(e^{-sQ}\theta)ds\right\}, \tag{2.2}$$

where ψ is given in (2.1). In particular, the generating triplet of $P(t, x, \cdot)$ is given by $(b_{t,x}, \sigma_t, \rho_t)$, where

$$
b_{t,x} = e^{-tQ}x + \int_0^t e^{-sQ}bds + \int_R \int_0^t e^{-sQ}z\{I_{\{|e^{-sQ}z|\leq 1\}} - I_{\{|z|\leq 1\}}\}ds\rho(dz),
$$

$$
\sigma_t^2 = \int_0^t e^{-2sQ}\sigma^2 ds,
$$

$$
\rho_t(E) = \int_0^t \rho(e^{sQ}E)ds.
$$

Now assume that

$$
\int_{|z|>1} \log|z|\rho(dz) < \infty, \tag{2.4}
$$

or, equivalently, $E[\log\{max(1, |Z_1|)\}] < \infty$.

Proposition 2.2. (Sato and Yamazato 1984, Theorem 4.1 and 4.2)

(a) If (2.4) holds, there exists a limit distribution F such that

$$
P(t, x, A) \to F(A), \quad as \quad t \to \infty
$$

for any $x \in R$ and $A \in \mathfrak{B}(R)$. This F is the unique invariant distribution of X. Moreover, the characteristic function of F is given by

$$
\varphi_F(\theta) = \exp\left\{ \int_0^\infty \psi(e^{-sQ}\theta)ds \right\},
$$

where ψ is given in (2.1). In particular, the generating triplet of $P(t, x, \cdot)$ is given by $(b_\infty, \sigma_\infty, \rho_\infty)$, where

$$
b_\infty = Q^{-1}b + \int_R \int_0^\infty e^{-sQ}z\{I_{\{|e^{-sQ}z|\leq 1\}} - I_{\{|z|\leq 1\}}\}ds\rho(dz)
$$

$$
\sigma_\infty^2 = \int_0^\infty e^{-2sQ}\sigma^2 ds
$$

$$
\rho_\infty(E) = \int_0^\infty \rho(e^{sQ}E)ds, \quad E \in \mathfrak{B}(R).
$$

(b) If (2.4) fails to hold, then X has no invariant distribution, and moreover, for any $x \in R$, $P(t, x, \cdot)$ does not converge to any probability measure as $t \to \infty$.

According to Proposition 2.2, under condition (2.4), X is ergodic. We shall use this in the section 4.

To begin, we introduce some definitions following [3].

Definition 2.1. (Occupation measure)   *For every $t > 0$, the occupation measure on the time $[0, t]$ is the measure μ_t given for every measurable function $f : R \to [0, \infty)$ by*

$$\int_R f(x)\mu_t(dx) = \int_0^t f(X_s)ds.$$

When the occupation measure is absolutely continuous, Lebesque's differentiation theorem enables us to define a particular version of the density of the occupation measure, called the local time.

Definition 2.2. (Local time)   *For every $t \geq o$ and $x \in R$, the quantity*

$$\limsup \frac{1}{2\epsilon} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds$$

denoted by $L(x, t)$ and called the local time at level x and time t.

The local time is defined at last in three different ways, namely via stochastic calculus, via excursion theory, and via additive functions. Definition 2.2 is the first approach. the Blumenthal-Getoor local time is defined as the unique continuous additive function supported by a single point x, and $L(x, t)$ exist if and only if x is a regular point. See [5] and [8]

3   The local time of the Markov process of O-U type

In this section, we will give a completely general criterion for the existence of local time as a density of occupation measure. The proof is based on Fourier analysis approach due to S. M. Berman [1], [2], See also [3].

At first, we calculate Fourier transform of $X_s - X_t$ for $0 < t < s$. For every $\theta \in R$, by the Markov property, the time-homogeneous and (2.2),

$$\varphi_{X_s - X_t}(\theta) = Ee^{i<\theta, X_s - X_t>} = E\left(E[e^{i\theta(X_s - X_t)}|\mathfrak{F}_t]\right)$$

$$= E\left[e^{-i\theta X_t} E_{X_t} e^{i\theta(X_{s-t})}\right]$$

$$= Ee^{-i\theta X_t} \exp\left\{iX_t e^{-(s-t)Q}\theta - \int_0^{s-t} \psi(e^{-uQ}\theta)du\right\}$$

$$= Ee^{iX_t\theta(e^{-(s-t)Q} - 1)} \exp\left\{-\int_0^{s-t} \psi(e^{-uQ}\theta)du\right\}$$

$$= \exp\left\{ixe^{-tQ}(e^{-(s-t)Q} - 1)\theta - \int_0^t \psi(e^{-uQ}\theta(e^{-(s-t)Q} - 1))du\right.$$

$$\left. - \int_0^{s-t} \psi(e^{-uQ}\theta)du\right\}. \tag{3.1}$$

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Hence, we have

$$
\begin{aligned}
|\varphi_{X_s - X_t}(\theta)| &= \exp\left\{ -\int_0^t \mathfrak{Re}\psi(e^{-uQ}\theta(e^{-(s-t)Q} - 1))du - \int_0^{s-t} \mathfrak{Re}\psi(e^{-uQ}\theta)du\right\} \\
&\leq \exp\left\{ -\int_0^{s-t} \mathfrak{Re}\psi(e^{-uQ}\theta)du\right\},
\end{aligned}
\tag{3.2}
$$

where $\mathfrak{Re}\psi$ is the real part of ψ.

Theorem 3.1. *Let X be defined by (1.1) and ψ is the characteristic exponent of Z. Suppose that either of the following conditions holds true for each $t \in R^+$*

(a) $\sigma > 0$.

(b) There exist constants $\alpha \in (0, 2)$ and $c > 0$ such that

$$
\int_{\{z:|vz|\leq 1\}} |vz|^2 \rho(dz) \geq c|v|^{2-\alpha}
\tag{3.3}
$$

for any $v \in R^d$ satisfying $|v| \geq 1$. Then the local time exist in $L^2(dp)$ a.e.

Proof. Introduce the measure μ by

$$
\int_R f(x)\mu(dx) = \int_0^\infty e^{-2Qs} f(X_s)ds = \int_0^\infty dt e^{-2Qt} \int_R f(x)\mu_t(dx),
\tag{3.4}
$$

the occupation measure μ_t is absolutely continuous with respect to μ with density bounded from above by e^t. Now, by Fubini's theorem and Plancherel's theorem, what we have to check is

$$
\int_{-\infty}^\infty E(|\mathfrak{F}\mu(\theta)|^2)d\theta < \infty,
\tag{3.5}
$$

where $\mathfrak{F}\mu(\theta)$ denotes the Fourier transform of μ.

Noted that $E(|\mathfrak{F}\mu(\theta)|^2)$ is a non-negative real function, from the definition of μ (3.4), (3.2), and Fubini's theorem,

$$
\begin{aligned}
E(|\mathfrak{F}\mu(\theta)|^2) &= E[\mathfrak{F}\mu(\theta)\mathfrak{F}\mu(-\theta)] \\
&= E[(\int_0^\infty e^{-2Qs}\exp\{i\theta X_s\}ds)(\int_0^\infty e^{-2Qt}\exp\{-i\theta X_t\})] \\
&= E(\int_0^\infty \int_0^\infty e^{-2Q(s+t)}\exp\{i\theta(X_s - X_t)\}dtds) \\
&\leq \int_0^\infty \int_0^\infty e^{-2Q(s+t)}\exp\left\{ -\int_0^{s-t}\mathfrak{Re}\psi(e^{-uQ}\theta)du\right\}dtds.
\end{aligned}
\tag{3.6}
$$

When $\sigma > 0$, by (3.2) and the definition of ψ,

$$
\exp\left\{ -\int_0^{s-t}\mathfrak{Re}\psi(e^{-uQ}\theta)du\right\} \leq \exp\{-\frac{1}{2}(\theta\sigma)^2\int_0^{s-t}e^{-2uQ}du\},
\tag{3.7}
$$

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By (3.6), (3.7) and Fubini's theorem,

$$
\begin{aligned}
\int_R E(|\mathfrak{F}\mu(\theta)|^2)d\theta &\leq 2\int_R d\theta \int_0^\infty dt e^{-2Qt} \int_t^\infty ds e^{-2Qs} \exp\{-\frac{(\theta\sigma)^2}{2}\int_0^{s-t} e^{-2uQ}du\} \\
&= 2\int_R d\theta \int_0^\infty dt e^{-2Qt} \int_t^\infty ds e^{-2Qs} \exp\{-\frac{(\sigma\theta)^2}{4Q}(1-e^{-2(s-t)Q})\}d\theta \\
&= 2\int_0^\infty dt e^{-2Qt} \int_t^\infty ds e^{-2Qs} \int_R \exp\{-\frac{(\sigma\theta)^2}{4Q}(1-e^{-2(s-t)Q})\} \\
&= C\int_0^\infty dt e^{-2Qt} \int_t^\infty e^{-2Qs}(1-e^{-2(s-t)Q})^{-\frac{1}{2}}ds \\
&\leq C\Gamma(\frac{1}{2}) < \infty. \tag{3.8}
\end{aligned}
$$

So that, whenever $\sigma > 0$, assertion (a) follows (3.8).

Turing to (b), by (3.2),

$$
\exp\left\{-\int_0^{s-t}\mathfrak{Re}\psi(e^{-uQ}\theta)du\right\} \leq \exp\left\{-\int_0^{s-t}\int_R[1-\cos(e^{-uQ}\theta z)]\rho(dz)du\right\} \tag{3.9}
$$

Let

$$
J(\theta) = \exp\left\{-\int_0^{s-t}\int_R[1-\cos(e^{-uQ}\theta z)]\rho(dz)du\right\}
$$

Using the inequality $1-\cos \geq 2(x/\pi)^2$ for $|x| \leq \pi$ and assumption (3.3), when $e^{-sQ}\theta \geq 1$,

$$
J(\theta) \leq \exp\{-C\int_0^{s-t}|e^{-uQ}\theta|^{2-\alpha}du\}. \tag{3.10}
$$

By (3.6) and (3.10), we have

$$
\begin{aligned}
&\int_R E(|\mathfrak{F}\mu(\theta)|^2)d\theta \\
&\leq 2\int_R \int_0^\infty \int_t^\infty J(\theta)e^{-2(s+t)Q}dsdtd\theta \\
&= 4\int_0^\infty \int_t^\infty \int_{e^{sQ}}^\infty J(\theta)e^{-2(s+t)Q}d\theta dsdt + 4\int_0^\infty \int_t^\infty \int_0^{e^{sQ}} J(\theta)e^{-2(s+t)Q}dsd\theta dsdt \\
&\leq 4\int_0^\infty \int_t^\infty \int_{e^{sQ}}^\infty \exp\{-C\int_0^{s-t}|e^{-uQ}\theta|^{2-\alpha}du\}e^{-2(s+t)Q}d\theta dsdt \\
&\qquad +4\int_0^\infty dt e^{-2Qt}\int_t^\infty \int_0^{e^{sQ}} J(\theta)e^{-2(s+t)Q}d\theta dsdt \\
&\leq C\Gamma(\frac{1}{2-\alpha}) + 4\int_0^\infty \int_t^\infty \int_0^{e^{sQ}} e^{-2(s+t)Q}d\theta dsdt \tag{3.11} \\
&\leq \infty.
\end{aligned}
$$

The proof is complete. $\qquad\qquad\square$

Remark 3.1. *The local time could be expressed at the "sum of times spent at x up to time t". To avoid fixed t, μ is defined. For the $2Q$ in e^{-2Q} of μ, it is used in (3.11).*

Remark 3.2. *As above, Theorem 3.1 is not true a.s. at every x. From the point of view of occupation densities, such aberrant behavior at a single state is irrelevant. the local time as occupation density is different from the Blumenthal-Getoor local time, as a continuous additive function of some point. For example, when Z is a Poisson process, the start point is 0, then there is the Blumenthal-Getoor local time at 0 about X. In fact, 0 is a holding point, so a regular point. But there do not exit a occupation density about X.*

Remark 3.3. *Meyer [11] has proved: let the process $Y = (Y_t)$, adapted to the natural $\sigma-$fields of a Brownian motion $W = (W_t)$, have trajectories of bounded variation; then there exit an occupation density of $W_t + Y_t$. Theorem 3.1 asserts that there are local time when W is a general O-U process, and Y is O-U type of pure jump.*

4   Some properties of local time

In this section, we shall obtain the smoothness of the local time in the time variable, when the level has been fixed. At the end of the section, we shall discuss the limit property of the local time at $t \to \infty$ when X is ergodic.

We assume that the conditions of Theorem 3.1 is satisfied in this section. Lebesgue's differentiation theorem enables us to define a particular version of the occupation density, called the local time as

$$\limsup \frac{1}{2\epsilon} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds$$

for every $t \leq 0$ and $x \in R$. Now, we can replace "$\limsup$" by "$\lim$" in the definition of local time. Before proving it, we have some lemmas. The following Lemma come from Masuda [10].

Lemma 4.1. *The following statements hold true for each $t \in R_+$.*

(a) If $\sigma > 0$, then $P(t, x, \cdot)$ admits a C_b^∞ density.

(b) If there exist constants $\alpha \in (0, 2)$ and $c > 0$ such that (3.3) satisfy, then $P(t, x, \cdot)$ admits a C_b^∞ density.

For Lévy process Z, because it has stationary independent increments, $p(t, x, y) = p(t, 0, y - x)$ for every $t \in R_+$ and $x \in R$. Unfortunately, there is not this property for

X, but there is a similar property as following:

Lemma 4.2. *Let $p(t, x, y)$ be the density of $P(t, x, \cdot)$, then*

$$p(t, x, y) = p(t, 0, y - xe^{-tQ}). \tag{4.1}$$

Moreover, $p(t, x, y)$ is continuous about x and tends to 0 as $x \to \infty$.

Proof. This is immediate from the inversion formula and (2.2). The last assertion stems from Lemma 4.1 and the property of probability density function. $\square$

Theorem 4.1. *For a.e. $x \in R$*

$$\lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds = L(x, t)$$

uniformly on compact intervals of time, in $L^2(P)$. As a consequence, the process $L(x, \cdot)$ is continuous a.s.

Proof. By Theorem 3.1, there exists a local time $L(x, \tau)$ in $L^2(dy \otimes dP)$, where τ is an independent random time with an exponential distribution of parameter 1. Mimicking the argument of Bertoin [3], for $a.e. y \in R$, the following convergence holds in $L^2(P)$:

$$\lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_0^\tau I_{\{|X_s - y| < \epsilon\}} ds = \lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_{y-\epsilon}^{y+\epsilon} L(v, \tau) dv = L(y, \tau). \tag{4.2}$$

Pick any y for which (4.2) is fulfilled and for every $\epsilon > 0$, consider the martingale

$$M_t^\epsilon = E\left(\frac{1}{2\epsilon} \int_0^\tau I_{\{|X_s - y| < \epsilon\}} ds \middle| \mathfrak{F}_t'\right), \quad t \le 0, \tag{4.3}$$

where $\mathfrak{F}_t' = \mathfrak{F}_t \vee \sigma(t \wedge \tau)$. By (4.2), and Doob's maximal inequality, M_t^ϵ converges as $\epsilon \to 0+$, uniformly on $t \in [0, \infty)$, in $L^2(P)$.

By the Markov property and the lack of memory of the exponential law, we have a.s.

$$M_t^\epsilon = \frac{1}{2\epsilon} \int_0^{t \wedge \tau} I_{\{|X_s - y| < \epsilon\}} ds + I_{\{t < \tau\}} f_\epsilon(X_t), \tag{4.4}$$

where

$$f_\epsilon(x) = E_x\left(\frac{1}{2\epsilon} \int_0^\tau I_{\{|X_s - y| < \epsilon\}} ds\right).$$

Now, what we have to do is proving $f_\epsilon(X_t)$ convergence uniformly on $t \in [0, \infty)$. Applying Fubini's theorem,

$$f_\epsilon(x) = \frac{1}{2\epsilon} \int_0^\infty e^{-t} P_x(|X_t - y| < \epsilon) dt.$$

Applying Lemma 4.2, we get our assertion. $\square$

By Theorem 4.1, *a.e.* $x \in R$, $L(x,t)$ is continuous additive function about $t \in R+$, $L(x,t)$ also is the Blumenthal-Getoor local time. Hence we have the following corollary:

Corollary 4.1. *Under the conditions of Theorem 3.1, a.e. x in the range of X are regular.*

Recalling Proposition 2.2, assume that

$$\int_{|z|>1} \log |z| \rho(dz) < \infty, \tag{4.5}$$

then there exists a limit distribution F such that

$$P(t,x,A) \to F(A) \quad \text{as} \quad t \to \infty$$

for any $x \in R$ and Borol set A. This F is the unique invariant distribution of X. Hence under (4.5), X is ergodic. By the ergodic theorem, we have

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds = \mu_F(B(x, \epsilon)), \quad \text{in} \quad L^2(P). \tag{4.6}$$

If the conditions of Theorem 3.1 is holding, F has a density f by Lemma 4.1, hence,

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{2t\epsilon} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds = \lim_{\epsilon \to 0} \frac{\mu_F(B(x, \epsilon))}{2\epsilon} = f(x). \tag{4.7}$$

On the other hand, by Theorem 4.1,

$$\lim_{t \to \infty} \lim_{\epsilon \to 0} \frac{1}{2t\epsilon} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds = \lim_{t \to \infty} \frac{L(x,t)}{t}. \tag{4.8}$$

We can get

$$\lim_{t \to \infty} \frac{L(x,t)}{t} = f(x), \quad \text{in} \quad L^2(P),$$

if the limits in (4.7) can commute. But this is obvious, because

$$\lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_0^t I_{\{|X_s - x| < \epsilon\}} ds = L(x,t)$$

uniformly on $[0,t]$ for any $t \in R^+$. More precisely, we have the following:

Theorem 4.2. *Assume that the conditions of Theorem 3.1 and (4.5) hold true, then a.e. x in the range of X,*

$$\lim_{t \to \infty} \frac{L(x,t)}{t} = f(x), \quad \text{in} \quad L^2(P).$$

5 The first passage cross a lever

Let $X = \{X_t, t \in R^+, P^x\}$ be one dimensional Markov process of O-U type defined by (1.1) taking values in R. Given a real number $a > x$, let us introduce the first passage time strictly above a, $T_a = \inf\{t \geq 0 : X_t > a\}$, and let $\sigma_a = \inf\{t \geq 0 : X_t = a\}$ provided that the sets in braces is not empty, and $+\infty$ otherwise.

When Z is a Lévy process with non-positive jumps, $\Delta X_t = \Delta Z_t \leq 0$. If $T_a < \infty$, one gets immediately

$$X_{T_a} = a. \tag{3.1}$$

Using martingle technique, Hadjiev [6] proved that

$$E \exp\{-\theta T_a\} = \frac{\int_0^\infty y^{\theta/Q-1} \exp\{xy + g(y)\}dy}{\int_0^\infty y^{\theta/Q-1} \exp\{ay + g(y)\}dy}, \quad \theta > 0, \tag{3.2}$$

where

$$g(y) = Q^{-1} \int_1^y u^{-1}\psi(iu)du, \quad y > 0.$$

When Z is a Lévy process with positive jumps, does the similar property (3.1) hold? We will prove that the answer is negative.

Lemma 5.1. Let $X = \{X_t, t \in R^+, P^x\}$ be a Markov process of O-U type defined by (1.1). Then for every $x \neq 0$ and $y \in R^d$, the potential measure of X is diffuse, that is,

$$U(x, \{y\}) = 0.$$

Proof. Since

$$X_t = e^{-tQ}x + \int_0^t e^{(s-t)Q}dZ_t$$

and the distribution of Z is a diffuse except when Z is a compound Poisson process for every $x \neq 0$,

$$P^x\{X_t = y\} = 0,$$

which implies

$$U(x, \{y\}) = \int_0^\infty P^x\{X_t = y\}dt = 0.$$

Theorem 5.1. Let $X = \{X_t, t \in R^+, P^x\}$ be a Markov process of O-U type defined by (1.1). If $\rho(-\infty, 0) = 0$, we have

$$P^x\{X_{T(a)-} < a = X_{T(a)}\} = 0.$$

Proof.   Let $f, g \geq 0$ be two Borel functions with $f(a) = 0$. Applying the compensation formula and recalling $\Delta X_t = \Delta Z_t \geq 0$, we have

$$\int_{0 \leq y < a \leq z} f(y)g(z)P^x\{X_{T(a)-} \in dy, X_{T(a)} \in dz\}$$

$$= E^x\big(f(X_{T(a)-})g(X_{T(a)})\big)$$

$$= E^x\big(\sum_{t \geq 0} f(X_{t-})g(X_{t-} + \Delta X_t)I_{\{a - \Delta X_t \leq X_{t-} < a\}}\big)$$

$$= \int_0^\infty dt E^x\big(f(X_{t-})I_{\{X_{t-} < a\}} \int_0^\infty g(X_{t-} + s)I_{\{s \geq a - X_{t-}\}}\rho(ds)\big)$$

$$= \int_{0 \leq y < a \leq z} f(y)g(z)P^x\{X_{T(a)-} \in dy, X_{T(a)} \in dz\}$$

$$= \int_0^\infty dt \int_{0 \leq y < a, s \geq a - y} f(y)g(y + s)P^x\{X_t \in dy\}\rho(ds)$$

$$= \int_{0 \leq y < a \leq z} f(y)g(z)U(x, dy)\rho(dz - y).$$

Taking $f = I_{[0,a)}$ and $g = I_{\{a\}}$, we obtain

$$P^x\{X_{T(a)-} < a = X_{T(a)}\} = \int_{[0,a)} U(x, dy)\rho(\{a - y\}). \tag{3.3}$$

There are at most countably many $y \in [0, a)$ with $\rho(\{a - y\}) > 0$ and $\rho(\{0\}) = 0$. Moreover the potential measure is diffuse by Lemma 3.1. Hence the right-hand side of (3.3) is zero.

We deduce from Theorem 3.1 that X is a.s. continuous at time $T(a)$ on the event $X_{T(a)} = a$, so $P\{X_{T(a)-} = a | X_{T(a)} = a\} = 1$ on $P\{X_{T(a)} = a\} > 0$ and $P\{X_{T(a)} > a | X_{T(a)-} < a\} = 1$ on $P\{X_{T(a)-} < a\} > 0$.

It is well known from [3] that we can write $Z_t = at + \sigma W_t + Z_s^1$, where at is a drift, W_t is the Brownian motion and Z_s^1 is a Lévy process of pure jumps type. Hence X has the following decomposition:

$$X_t = e^{-tQ}x + a \int_0^t e^{(s-t)Q}ds + \sigma \int_0^t e^{(s-t)Q}dW_s + \int_0^t e^{(s-t)Q}dZ_s^1. \tag{3.4}$$

Theorem 5.2.   *Assume that $x = a = \sigma = 0$ in (3.4), if*

$$\rho(-\infty, 0) = 0 \ and \ \int_0^1 x\rho(dx) = C < +\infty, \tag{3.5}$$

then $X_{T(a)} > a$ a.s.

Proof.   Note that $0 < e^{(s-t)Q} < 1$ for $s < t$, X gets its supremum just by jumping, that is, $P\{X_{T(a)-} < a\} > 0$ by Theorem 3.1, the theorem is proved.

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