MATCHINGS, COVERINGS, AND CASTELNUOVO-MUMFORD REGULARITY

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Abstract. We show how co-chordal covers of the edges of a graph give upper bounds on the Castelnuovo-Mumford regularity of its edge ideal. The proof is by an easy application of a deep result of Kalai and Meshulam. We also give a topological proof of the best lower bound and slight improvements to it. Using results from the graph theory literature, we will be able to calculate and/or bound the Castelnuovo-Mumford regularity for edge ideals of several new classes of graphs.

1. Introduction and background

Let Δ be any simplicial complex, and for any subset S of its vertex set $V(\Delta)$ let $\Delta[S]$ be the induced subcomplex on S. If k is any field, we define the *regularity of* Δ *over* k to be

$$\operatorname{reg}_k \Delta = \max\{i : \tilde{H}_{i-1}(\Delta[S]; k) \neq 0 \text{ for some } S \subseteq V(\Delta)\}.$$

Our results will mostly be independent of the choice of k, and in such cases we will drop k from our notation.

Since $\operatorname{reg} \Delta \geq \operatorname{reg} \Delta[S]$ for any $S \subseteq V(\Delta)$, regularity is a reasonable measure of the topological complexity of Δ. For example, $\operatorname{reg} \Delta = 0$ if and only if Δ is a simplex. Complexes Δ with $\operatorname{reg} \Delta \leq d$ have also been called *d-Leray*, and been used to prove Helly-type results [16].

In addition to its role as a measure of topological complexity, we are interested in $\operatorname{reg} \Delta$ via a connection with commutative algebra. The *Stanley-Reisner ring over* k of a simplicial complex Δ with vertex set $\{x_1, \ldots, x_n\}$ is the commutative ring

$$k[\Delta] = k[x_1, \ldots, x_n]/(x^E : E \text{ not a face of } \Delta).$$

Then $\operatorname{reg}_k \Delta$ is the Castelnuovo-Mumford regularity of $k[\Delta]$, and in this form has been the object of some recent interest [11, 14, 20, 21, 22, 24]. We notice that if R is a polynomial ring, then $\operatorname{reg} R = 0$. Since $0 \to \mathcal{I} \to R \to R/\mathcal{I} \to 0$ is an exact sequence for any ideal $\mathcal{I}$, it follows from a standard long exact sequence argument in homological

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algebra that $\operatorname{reg} \mathcal{I} = \operatorname{reg} R/\mathcal{I} + 1$. In particular, studying the regularity of the Stanley-Reisner ring and of the associated ideal are equivalent problems, and both forms appear in the literature.

The *independence complex* of a graph G, denoted $I(G)$, is the family of independent sets of G, that is, of subsets of $V(G)$ containing no edge. Simplicial complexes that can be realized as the independence complex of a graph are called *flag complexes.* The ideal in the Stanley-Reisner ring is called the *edge ideal* in this case, and is generated by square-free monomials of degree 2 corresponding to the edges of the graph. More generally, any simplicial complex can be realized as the independence complex of a more general object called a *clutter* or *Sperner system*, but we restrict ourselves to the graph case in this paper.

A graph G is *chordal* if every induced cycle in G has length 3, and G is *co-chordal* if the complement graph $\overline{G}$ is chordal. We notice that $\operatorname{reg}_k \Delta = 1$ (over any k) if and only if Δ is the independence complex of a co-chordal graph with at least one edge: this can be viewed as equivalent to a similar result on linear resolutions of edge ideals [12], or a direct proof is straightforward.

We present several classes of co-chordal graphs, which we will use in Section 3.

Example 1. Any graph G such that $V(G)$ can be partitioned into a complete subgraph union an (induced) independent set is both chordal and co-chordal. Such graphs are referred to as *split graphs.*

Example 2. A *threshold graph* is recursively defined to be either the single vertex graph, or else a graph obtained from a threshold graph by either adding either a new disjoint vertex, or a new dominating vertex. Threshold graphs are a subclass of split graphs, hence are co-chordal, and are examined at length in [19].

Example 3. Since the complement of a complete ℓ-partite graph $K_{n_1,\dots,n_\ell}$ is the disjoint union of cliques, $K_{n_1,\dots,n_\ell}$ is co-chordal.

Example 4. Co-chordal graphs that are also bipartite are called *chain graphs* or *difference graphs*, and are exactly the bipartite graphs with no induced $2K_2$ subgraph.

Example 5. An *interval graph* is a graph with vertices corresponding to some set of intervals in $\mathbb{R}$, and edges between two intervals with non-empty intersection; a *co-interval graph* is the complement of an interval graph. Interval graphs are exactly the chordal graphs which can be represented as the incomparability graph of a poset. Equivalently, interval graphs are the incomparability graphs of the *2+2-free* posets, that is those posets with no subposet consisting of 2 disjoint nontrivial chains [3].

The remainder of the paper is organized as follows. In Section 2, we give lower bounds on regularity of independence complexes. We prove the induced matching lower bound by a geometric technique, and give generalizations. In Section 3, we use a theorem of Kalai and Meshulam [16] to bound regularity from above by the co-chordal cover number. Using results from the graph theory literature, we bound and/or exactly calculate the regularity for several new classes of graphs.

For undefined graph theory terms we refer to [3, 19], and for geometric combinatorics terms to [2]. We consider all graphs and simplicial complexes in this paper to be finite.

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2. Lower bounds

Lower bounds for regularity are straightforward to construct: we find a subcomplex with non-vanishing homology in a high dimension.

An *induced matching* in a graph G is a matching which forms an induced subgraph of G, that is, a set of edges of which no two share a vertex or are connected by a third edge. Induced matchings have a considerable literature, see e.g. [1, 5, 6, 10, 13]. We let $\operatorname{im}(G)$ be the number of edges in the largest induced matching. The following theorem is essentially due to Katzman:

Theorem 6. (Katzman [17, Lemma 2.2]) *For any graph G, we have $\operatorname{reg} I(G) \geq \operatorname{im}(G)$.*

We give a short geometric proof: Notice that if G is the disjoint union of subgraphs G_1 and G_2, then $I(G)$ is the join $I(G_1) * I(G_2)$. Thus, the independence complex of the disjoint union of j edges is the j-fold join of 0-spheres, hence a $(j-1)$-sphere. (It is the boundary complex of the $(j-1)$-dimensional cross-polytope.) The result follows. $\square$

A more general result follows immediately from the Künneth formula in algebraic topology [2, 9.12]:

Lemma 7. *Let k be any field. For any simplicial complexes Δ_1 and Δ_2, we have* $\operatorname{reg}_k(\Delta_1 * \Delta_2) = \operatorname{reg}_k \Delta_1 + \operatorname{reg}_k \Delta_2$.

In the context of independence complexes, if G_1 and G_2 are any two graphs then $\operatorname{reg}_k I(G_1 \dot\cup G_2) = \operatorname{reg}_k I(G_1) + \operatorname{reg}_k I(G_2)$.

The inequality of Theorem 6 can be strict. For example, Kozlov calculated the homotopy type of paths and cycles [18, Propositions 4.6 and 5.2], from which the following proposition follows:

Proposition 8. $\operatorname{reg} I(C_n) = \operatorname{reg} I(P_n) = \left\lfloor \frac{n-2}{3} \right\rfloor + 1$ *for* $n \geq 3$.

It is easy to see that the regularity is equal to the lower bound of Theorem 6 in the P_n case, and in the C_n case when $n \not\equiv 2 \pmod 3$, but that $\operatorname{reg} I(C_{3i+2}) = i + 1 = \operatorname{im}(C_{3i+2}) + 1$. Combining with Lemma 7, we get:

Corollary 9. *If a graph G has an induced subgraph H which is the disjoint union of edges and cycles*

$$H \cong \dot{\bigcup}_{i=1}^{m} e \,\dot\cup\, \dot{\bigcup}_{j=1}^{n} C_{3i_j+2}$$

then $\operatorname{reg} G \geq m + n + \sum_{j=1}^{n} i_j$.

3. Upper bounds

The principle tool that we will use to find upper bounds for regularity is the following deep result proved by Kalai and Meshulam [16], answering a conjecture of Terai [22].

Theorem 10. (Kalai-Meshulam [16, Theorem 1.2])
If $\Delta_1, \ldots, \Delta_s$ are simplicial complexes on the same vertex set and k is any field, then

$$\operatorname{reg}_k \bigcap_{i=1}^{s} \Delta_i \leq \sum_{i=1}^{s} \operatorname{reg}_k \Delta_i.$$

In the context of independence complexes, if $G_1, \ldots, G_s$ are graphs on the same vertex set, then $\operatorname{reg}_k I(\bigcup_{i=1}^{s} G_i) \leq \sum_{i=1}^{s} \operatorname{reg}_k I(G_i)$.

Let G be a graph, and $\mathcal{F}$ be a family of graphs. The *$\mathcal{F}$-cover number* of G is the minimum number of subgraphs $H_1, \ldots, H_s$ of G such that every H_i is in $\mathcal{F}$ and $\bigcup E(H_i) = E(G)$.

Let $\operatorname{cochord}(G)$ denote the co-chordal cover number of G. Then the following is an immediate consequence of Theorem 10 and the fact that $\operatorname{reg} I(G) \leq 1$ for a co-chordal graph G.

Theorem 11. *For any graph G, we have* $\operatorname{reg} I(G) \leq \operatorname{cochord}(G)$.

Although the proof of Theorem 11 is easy, it connects the study of regularity with other problems studied in the graph theory and computer science literature. We use this connection to give new proofs of upper bounds on regularity, improving some of the best previously known.

In particular, the $\mathcal{F}$-cover number of G for any family $\mathcal{F}$ from Examples 1-5 is an upper bound on $\operatorname{reg} G$. We examine these in turn, giving references to the literature and drawing consequences for regularity.

Remark 12. The inequality of Theorem 11 can be strict. For example, since the graph formed by two disjoint edges is not co-chordal, we get that co-chordal subgraphs of C_n ($n \geq 5$) are paths with at most 3 edges. Thus $\operatorname{cochord}(C_{3k+1}) = k+1$, but by Proposition 8 we have that $\operatorname{reg} I(C_{3k+1}) = k$.

Proposition 13. *If G is a graph such that $V(G)$ is covered by an independent set J_0 together with s complete subgraphs $J_1, \ldots, J_s$, then $\operatorname{reg} I(G) \leq s$.*

Proof. Let J_i' be the subgraph consisting of all edges incident to at least one vertex in $V(J_i)$. Since J_i' can be decomposed as the complete subgraph J_i union an independent set, J_i' is a split graph. Then $J_1', J_2', \ldots, J_s'$ is a split graph covering (hence a co-chordal covering) of G, and the result follows by Theorem 11. □

Remark 14. If in the situation of Proposition 13 we have $J_0 = \emptyset$, then $J_1, \ldots, J_s$ is exactly an s-coloring of $\overline{G}$. In this case, however, the bound is trivial, since $\chi(\overline{G}) \geq \alpha(G) = \dim I(G) + 1$, and $\tilde{H}_i(\Delta)$ always vanishes above $\dim \Delta$.

Remark 15. Hà and Van Tuyl [14, Theorem 6.7] showed that $\operatorname{reg}_k I(G)$ is at most the matching number of G, that is, the maximum size of a matching. Proposition 13 is a strong generalization of their result, since any maximal (not necessarily maximum) matching gives the required covering.

Proposition 13 also allows us to recover a result of Hà and Van Tuyl on chordal graphs:

Corollary 16. (Hà-Van Tuyl [14, Corollary 1.7]) *If G is a chordal graph, then $\operatorname{reg} I(G) = \operatorname{im}(G)$.*

Proof. Cameron [5] observed that the edges of a chordal graph G can be covered by $\operatorname{im}(G)$ cliques. □

Definition 17. The split graph cover number of G (as in the proof of Proposition 13) has also been referred to as the *split dimension* of G

[7]. Although it gives weaker results for our purposes, the threshold graph cover number (or *threshold dimension*) has been more studied: see [19] and its references.

The technique to calculate $\operatorname{reg} I(G)$ by proving $\operatorname{im}(G) = \operatorname{cochord}(G)$ is more broadly useful. A graph is *weakly chordal* if every induced cycle in both G and $\overline{G}$ has length ≤ 4. A weakly chordal graph that is also bipartite is called *chordal bipartite*.

Proposition 18. *If G is a weakly chordal graph, then* $\operatorname{reg} I(G) = \operatorname{im}(G)$.

Proof. Busch, Dragan, and Sritharan show [4] that $\operatorname{im}(G) = \operatorname{cochord}(G)$ for any weakly chordal graph G. (Abueida, Busch, and Sritharan earlier showed the same result for a chordal bipartite graph [1, Corollary 1].) □

Definition 19. The *biclique cover number* of G is the smallest number of complete bipartite graphs K_{n_1,n_2} required to cover the edge of G. The biclique number has also been referred to as the *bipartite dimension*. Unfortunately (for our purposes), there seem to be stronger results about lower bounds than for upper bounds on the biclique cover number, e.g. [8, 15]. I'm not aware of any study of the analogous cover problem for complete k-partite graphs, although this might give interesting bounds on regularity.

Definition 20. The *boxicity* of G, denoted $\operatorname{box} G$, is the co-interval cover number of $\overline{G}$, that is, the minimum number of co-interval subgraphs required to cover the edges of $\overline{G}$. (The original formulation of boxicity was somewhat different, and the connection with covering is made in [9].) Theorem 11 gives that $\operatorname{reg} I(G) \leq \operatorname{box}(\overline{G})$.

If G is a planar graph, then by Proposition 8 we see that $\operatorname{reg} I(G)$ may be unbounded. On the other hand, since a planar graph G contains no K_5 subgraph, we have that $\operatorname{reg} I(G) \leq \dim I(G) + 1 = \alpha(G) \leq 4$ (as in Remark 14). The literature on boxicity yields a stronger result:

Proposition 21. *If G is a planar graph, then* $\operatorname{reg} I(\overline{G}) \leq 3$.

Proof. Thomassen [23] proves that $\operatorname{box}(G) \leq 3$. □

The complement of $3K_2$ is the 1-skeleton of the octohedron, which is well-known to be planar. Hence Corollary 21 gives the best possible regularity bound on complements of planar graphs.

Remark 22. The $\mathcal{F}$-cover number for any interesting subfamily $\mathcal{F}$ of the co-chordal graphs seems to be difficult to compute. Yannakakis shows [25] that determining whether $\operatorname{cochord}(G) \leq k$ is NP-complete, even when we restrict to bipartite graphs and the chain graph cover problem. Moreover he shows that for a bipartite graph G we have $\operatorname{cochord}(G) = \operatorname{box}(\overline{G})$, hence determining whether boxicity of a graph is $\leq k$ is also NP-complete. The corresponding problem for the split graph cover number was shown to be NP-complete in [7]. An easily-accessible account of these complexity results can be found in [19, Chapter 7].

We close with a question. Nevo [21] shows that if G is a ($2K_2$, claw)-free graph then $\operatorname{reg} I(G) \leq 2$.

Question 23. *If G is ($2K_2$, claw)-free, then is* $\operatorname{cochord}(G) \leq 2$*?*

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